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Research article

Large deviations of small uniform spacings

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Abstract: The k-th smallest spacing $m_n(k)$ induced by n uniformly distributed order statistics on $(0, 1)$ was considered in this paper. It has been known that $n^2 m_n(k)$ possesses a limiting distribution as n tends to infinity. In this paper (large) deviation probabilities of $n^2 m_n(k)$ to zero or infinity are studied, with k possibly depending on n. This study generalized a result obtained by Devroye in their 1982 paper.

Keywords: uniform spacing; minimal spacing; large deviations
Mathematics Subject Classification: 60F10, 62G30, 60G50

1. Background and main result

Let $n-1$ points totally randomly drop on a line with unit length, dividing the line into n pieces (which are called *uniform spacings*). The study of uniform spacings has a fruitful and rich history due to their various applications. Whitworth (see [16]) was among the first in the 1880s to study such a classical topic. Mathematically speaking, suppose that $U_1, U_2, \ldots, U_{n-1}$ are independent and identically distributed (i.i.d.) uniform random variables on $(0, 1)$ with their corresponding order statistics denoted as $U_{(1)} < U_{(2)} < \ldots < U_{(n-1)}$. Then, uniform spacings are $Y_k, 1 \leq k \leq n$ defined as: $Y_k = U_{(k)} - U_{(k-1)}$ for $1 \leq k \leq n$ with convention $U_{(0)} = 0$ and $U_{(n)} = 1$. Some early works dealing with uniform spacings can be found in [13, 14] and the references therein.

Extreme uniform spacings play important roles because of their applications. For any given integer $k \geq 1$, the k-th smallest spacing and k-th largest spacing are denoted as $m_n(k)$ and $M_n(k)$, respectively. In particular, the minimal spacing is $m_n := \min_{1\leq k\leq n} Y_k = m_n(1)$ and the maximal spacing is $M_n := \max_{1\leq k\leq n} Y_k = M_n(1)$. The joint distribution of the order statistics $(Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)})$ of $(Y_1, Y_2, \ldots, Y_n)$ in different forms has been studied in [16]. The asymptotic distribution theory of uniform spacings have been developed extensively in the literature. In particular, the limiting distribution of M_n (and more generally $M_n(k)$) has been achieved in [8, 11]. Almost sure convergence of $nM_n/\ln n$ to 1 is proved in [5], while large deviations of $nM_n/\ln n$ away from 1 have been studied in [5, 12].

Explicit but a little involved expression of the distribution function of $m_n(k)$ is derived in [1] (see also [2]). Law of the iterated logarithm of $m_n(k)$ is studied in [6, 15]. For any given integer $k \geq 1$, it is shown in [8] that a limiting distribution of $n^2 m_n(k)$ as $n \to \infty$ exists: $n^2 m_n(k) \to \Gamma_k$ in distribution, where Γ_k is a *Gamma*$(k, 1)$ random variable. Such weak limit naturally suggests the study of large deviation probabilities of the forms: $\mathbb{P}(n^2 m_n(k) > a_n)$ and $\mathbb{P}(n^2 m_n(k) < b_n)$ for suitable $a_n \to \infty$ and $b_n \to 0^+$ as $n \to \infty$. Throughout this paper, $f(n) \sim g(n)$ stands for $f(n) = g(n)(1 + o(1))$, meaning $f(n)/g(n) \to 1$ as $n \to \infty$. The main result of the paper is formulated as follows.

Theorem 1.1. *Let $k \geq 1$ be a fixed integer or $k = k(n)$ such that $k \ln k = o(n^{1/3})$.*

(i) For any $a_n \to \infty$ such that $a_n = o(n^{1/3})$, it holds that

$$\mathbb{P}(n^2 m_n(k) > a_n) = (1 + o(1)) a_n^{k-1} e^{-a_n}/(k-1)!. \tag{1.1}$$

(ii) For any $b_n \to 0^+$ such that $k \ln(b_n) = o(n^{1/3})$, it holds that

$$\mathbb{P}(n^2 m_n(k) < b_n) = (1 + o(1)) b_n^k / k!. \tag{1.2}$$

Theorem 1.1 generalizes a result in [6] in which (1.1) and (1.2) are studied with k being a fixed integer (i.e., for a fixed $k \geq 1$, (1.1) and (1.2) are proved to hold in [6] for suitable sequences $a_n \to \infty$ and $b_n \to 0^+$, respectively). When k depends on n, special attention is given to the interactive relations between k and n during asymptotic estimates, which result in the restrictions on k, a_n, and b_n formulated in Theorem 1.1. The $k(n)$-th extreme spacings with diverging rank arise naturally in the description of random partitions (see [10] in the framework of Poisson-Dirichlet limit) and coverage problems ([9]), but appear to have received comparatively little direct statistical exploitation. It seems that the restriction $k \ln k = o(n^{1/3})$ cannot be weakened as it is needed in Lemmas 2.5 and 2.8.

2. Proof of Theorem 1.1

2.1. Preliminaries

The proof ideas are based on classical exponential representations of uniform spacings (see Lemma 2.1) together with various tail estimates of Gamma and Binomial random variables. This approach is used in [12] to study large deviations of large uniform spacings $M_n(k)$. Both small and large uniform spacings utilize extremes of i.i.d. exponential random variables. However, unlike small uniform spacings $n^2 m_n(k)$, which converge to a limiting distribution Γ_k, large uniform spacings $n M_n(k)/\ln n$ converge to 1 almost surely. This distinguishes the choice of the order y_n of controlling error used in (2.1) and (2.2). Let us now recall the following exponential representation of uniform spacings.

Lemma 2.1 ([13, 14])**.** *Let $X_1, X_2, \ldots, X_n$ be i.i.d. exponential random variables with mean 1 and $T_n = \sum_{i=1}^n X_i$. Then, it holds that $(Y_1, Y_2, \ldots, Y_n) \stackrel{d}{=} (X_1/T_n, X_2/T_n, \ldots, X_n/T_n)$ with $\stackrel{d}{=}$ denoting equality in distribution.*

Lemma 2.2 ([5])**.** *For every $\omega > 0$, it holds that*

$$\mathbb{P}(T_n/n > 1 + \omega) \leq e^{-n\omega^2(1-\omega)/2}, \qquad \mathbb{P}(T_n/n < 1 - \omega) \leq e^{-n\omega^2/2}.$$

Lemma 2.3 ([7], Theorem 2 in VI.3)**.** *Let $B \sim B(n,p)$ be a Binomial random variable with the parameter p that is possibly dependent on n. Then, for any $k \geq 1$, it follows that, as $n \to \infty$,*

$$\mathbb{P}(B(n,p) \geq k) \sim \mathbb{P}(B(n,p) = k), \text{ if } np \to 0,$$
$$\mathbb{P}(B(n,p) < k) \sim \mathbb{P}(B(n,p) = k-1), \text{ if } np \to \infty.$$

2.2. Upper deviations (1.1)

Let ℓ_n^k denote the k-th smallest in the sequence of i.i.d. exponential random variables $X_1, X_2, \ldots, X_n$ with $T_n = \sum_{i=1}^n X_i$, then it follows from Lemma 2.1 that $m_n(k)$ is identical in distribution as ℓ_n^k/T_n. Let us take $y_n = n^{-1/3}$,

$$\begin{aligned}\mathbb{P}(\ell_n^k > a_n(1+y_n)/n) - \mathbb{P}(T_n > n(1+y_n)) &\leq \mathbb{P}(n^2 m_n(k) > a_n) \\ &\leq \mathbb{P}(\ell_n^k > a_n(1-y_n)/n) + \mathbb{P}(T_n \leq n(1-y_n)).\end{aligned} \tag{2.1}$$

Lemma 2.4. *Under the assumptions of Theorem 1.1, it holds that*

$$\mathbb{P}(\ell_n^k > a_n/n) = (1+o(1)) a_n^{k-1} e^{-a_n}/(k-1)!.$$

Proof. The event $\{\ell_n^k > a_n/n\}$ indicates that there are less than k of $X_1, X_2, \ldots, X_n$ which are less than a_n/n; therefore, it is clear that $\mathbb{P}(\ell_n^k > a_n/n) = \mathbb{P}(B(n,p) < k)$ with p given as $p = p(n) = P(X_i < a_n/n) = 1 - e^{-a_n/n}$ (which equals $a_n/n - (a_n/n)^2 e^{\theta(n)}/2$ for some $\theta(n) \in [-a_n/n, 0]$). In this case, $np = a_n - e^{\theta(n)} a_n^2/2n \to \infty$ since $a_n^2/n = o(n^{2/3}/n) = o(n^{-1/3}) = o(1)$. Lemma 2.3 implies that, with Stirling's approximation $n! \sim \sqrt{2\pi n}(n/e)^n$ for large n,

$$\begin{aligned}\mathbb{P}(\ell_n^k > a_n/n) &= \mathbb{P}(B < k-1) \sim \mathbb{P}(B = k-1) = \binom{n}{k-1} p^{k-1}(1-p)^{n-k+1} \\ &= (1+o(1)) \frac{\sqrt{2\pi n}\,(n/e)^n}{(k-1)!\,\sqrt{2\pi(n-k+1)}\,((n-k+1)/e)^{n-k+1}} \cdot p^{k-1}(1-p)^{n-k+1} \\ &= (1+o(1)) \frac{(n/e)^{k-1}\,(n/e)^{n-k+1}}{(k-1)!\,((n-k+1)/e)^{n-k+1}} \cdot p^{k-1}(1-p)^{n-k+1} \\ &= (1+o(1)) \frac{n^{k-1}}{(k-1)!} e^{-(k-1)} \left(\frac{n}{n-k+1}\right)^{n-k+1} p^{k-1}(1-p)^{n-k+1}.\end{aligned}$$

With $N = \frac{n-k+1}{k-1}$, the following term can be written as

$$\left(\frac{n}{n-k+1}\right)^{n-k+1} = \left(1 + \frac{k-1}{n-k+1}\right)^{n-k+1} = \left(\left(1 + \frac{1}{\frac{n-k+1}{k-1}}\right)^{\frac{n-k+1}{k-1}}\right)^{k-1} = e^{(k-1)\ln\left[\left(1+\frac{1}{N}\right)^N\right]}.$$

Using simple estimates $\left(1+\frac{1}{N}\right)^N < e < \left(1+\frac{1}{N}\right)^{N+1}$, it follows that $\frac{N}{N+1}e < \left(1+\frac{1}{N}\right)^N < e$. Therefore,

$$e^{(k-1)\ln\left(\frac{N}{N+1}e\right)} < \left(\frac{n}{n-k+1}\right)^{n-k+1} < e^{(k-1)\ln(e)},$$

$$e^{-(k-1)} e^{(k-1)\ln\left(\frac{N}{N+1}e\right)} \leq e^{-(k-1)} \left(\frac{n}{n-k+1}\right)^{n-k+1} \leq 1.$$

With $k = o(n^{1/2})$, it holds that

$$e^{-(k-1)} e^{(k-1)\ln\left(\frac{N}{N+1}e\right)} = \left(\frac{N}{N+1}\right)^{k-1} \to 1.$$

Therefore,

$$\begin{aligned}
\mathbb{P}(\ell_n^k > a_n/n) &= (1+o(1)) \frac{n^{k-1}}{(k-1)!} \left(\frac{a_n}{n}\left(1 - \frac{a_n}{2n} e^{\theta(n)}\right)\right)^{k-1} (1-p)^{n-k+1} \\
&= (1+o(1)) \frac{a_n^{k-1}}{(k-1)!} \left(1 - \frac{a_n}{2n} e^{\theta(n)}\right)^{k-1} e^{(n-k+1)\ln(1-p)}.
\end{aligned}$$

Let $\frac{1}{m} = \frac{a_n}{2n} e^{\theta(n)}$ with $k = o(n^{1/2})$, then

$$\left(1 - \frac{a_n}{2n} e^{\theta(n)}\right)^{k-1} = \left(\left(1 - \frac{1}{m}\right)^m\right)^{\frac{k-1}{m}} = \left[(1+o(1))e^{-1}\right]^{\frac{k-1}{m}} = \left[(1+o(1))e^{-1}\right]^{o(1)} = 1 + o(1).$$

Using the Taylor series, $f(x) = f(x_0) + f^{'}(x_0)(x - x_0) + \frac{f^{''}(\xi)}{2!}(x - x_0)^2$, for some $\xi \in [x_0, x]$, we have $\ln(1 - p) = -p - p^2/2\xi^2$, for some $\xi \in [1 - p, 1]$. Therefore,

$$\begin{aligned}
\mathbb{P}(\ell_n^k > a_n/n) &= (1+o(1)) \frac{a_n^{k-1}}{(k-1)!} e^{(n-k+1)(-p-p^2/2\xi^2)} \\
&= (1+o(1)) \frac{a_n^{k-1}}{(k-1)!} e^{-np - \frac{n}{2\xi^2}\left(\frac{a_n}{n}\right)^2\left(1 - \frac{a_n}{2n}e^{\theta(n)}\right)^2 + (k-1)\frac{a_n}{n}\left(1 - \frac{a_n}{2n}e^{\theta(n)}\right) + \frac{k-1}{2\xi^2}\left(\frac{a_n}{n}\right)^2\left(1 - \frac{a_n}{2n}e^{\theta(n)}\right)^2} \\
&= (1+o(1)) \frac{a_n^{k-1}}{(k-1)!} e^{-np + \left(-\frac{1}{2\xi^2}\frac{a_n^2}{n} + \frac{(k-1)a_n}{n} + \frac{k-1}{2\xi^2}\left(\frac{a_n}{n}\right)^2\right)(1+o(1))} \\
&= (1+o(1)) \frac{a_n^{k-1}}{(k-1)!} e^{-np+o(1)} = (1+o(1)) \frac{a_n^{k-1}}{(k-1)!} e^{-a_n},
\end{aligned}$$

where the facts $a_n = o(n^{1/2})$ and $ka_n = o(n)$ are used. $\square$

Lemma 2.5. *Let $y_n = n^{-1/3}$. Under the assumptions of Theorem 1.1, it holds that*

$$\mathbb{P}(|T_n - n| > ny_n)/\mathbb{P}(\ell_n^k > a_n/n) \to 0.$$

Proof. It follows from Lemma 2.2 that $\mathbb{P}(|T_n - n| > ny_n) \leq 2e^{-ny_n^2/4}$ for large enough n. Then

$$\begin{aligned}
&\frac{\mathbb{P}(|T_n - n| > ny_n)}{\mathbb{P}(\ell_n^k > a_n/n)} \leq \frac{2e^{-ny_n^2/4}}{\mathbb{P}(\ell_n^k > a_n/n)} = 2(1+o(1))(k-1)!\,a_n^{-(k-1)} e^{-ny_n^2/4 + a_n} \\
&= 2(1+o(1))\sqrt{2\pi(k-1)}\left(\frac{k-1}{e}\right)^{k-1} a_n^{-(k-1)} e^{-ny_n^2/4 + a_n} \\
&= 2\sqrt{2\pi}(1+o(1)) a_n^{-(k-1)} e^{-(k-1)} e^{(k-1/2)\ln(k-1)} e^{-ny_n^2/4 + a_n} \\
&= 2\sqrt{2\pi}(1+o(1)) a_n^{-(k-1)} \exp\left\{-n^{1/3}\left(\frac{1}{4} + \frac{k-1}{n^{1/3}} - \frac{a_n}{n^{1/3}} - \frac{(k-1/2)\ln(k-1)}{n^{1/3}}\right)\right\} \to 0,
\end{aligned}$$

where the facts $k\ln k = o(n^{1/3})$ and $a_n = o(n^{1/3})$ are applied in the last line. $\square$

Lemma 2.6. *Let $y_n = n^{-1/3}$. Under the assumptions of Theorem 1.1, it holds that*

$$\mathbb{P}(\ell_n^k > a_n(1+y_n)/n) \sim \mathbb{P}(\ell_n^k > a_n(1-y_n)/n) \sim \mathbb{P}(\ell_n^k > a_n/n).$$

Proof. To prove $\mathbb{P}(\ell_n^k > (1+y_n)a_n) \sim \mathbb{P}(\ell_n^k > (1-y_n)a_n)$, we replace on each side $(1+y_n)a_n$ or $(1-y_n)a_n$ for a_n in Lemma 2.4 to obtain

$$\begin{aligned}
&\mathbb{P}(\ell_n^k > a_n(1+y_n)/n)/\mathbb{P}(\ell_n^k > a_n(1-y_n)/n)\\
&= (1+o(1))\frac{(a_n(1+y_n))^{k-1}}{(k-1)!}e^{-a_n(1+y_n)} \cdot (k-1)!(a_n(1-y_n))^{-(k-1)}e^{a_n(1-y_n)}\\
&= (1+o(1))\left(\frac{1+y_n}{1-y_n}\right)^{k-1} e^{-2a_ny_n} = (1+o(1))\left(\frac{1+n^{-1/3}}{1-n^{-1/3}}\right)^{k-1} e^{-2a_ny_n}\\
&= (1+o(1))\left(1+\frac{2n^{-1/3}}{1-n^{-1/3}}\right)^{k-1} e^{-2a_ny_n}.
\end{aligned}$$

Let $\gamma = \frac{n^{1/3}-1}{2}$, then we have

$$\begin{aligned}
&\mathbb{P}(\ell_n^k > a_n(1+y_n)/n)/\mathbb{P}(\ell_n^k > a_n(1-y_n)/n)\\
&= (1+o(1))\left(\left(1+\frac{1}{\gamma}\right)^{\gamma}\right)^{\frac{k-1}{\gamma}} e^{-2a_ny_n} = (1+o(1))((1+o(1))e)^{\frac{k-1}{\gamma}} e^{-2a_ny_n} = 1+o(1),
\end{aligned}$$

where the facts $k = o(n^{1/3})$ and $a_n = o(n^{1/3})$ are used. The proof of $\mathbb{P}(\ell_n^k > a_n(1-y_n)/n) \sim \mathbb{P}(\ell_n^k > a_n/n)$ follows using the same arguments as above. □

Dividing (2.1) throughout by $\mathbb{P}(\ell_n^k > a_n/n)$, we have

$$\begin{aligned}
\frac{\mathbb{P}(\ell_n^k > a_n(1+y_n)/n)}{\mathbb{P}(\ell_n^k > a_n/n)} - \frac{\mathbb{P}(T_n > n(1+y_n))}{\mathbb{P}(\ell_n^k > a_n/n)} &\le \frac{\mathbb{P}(n^2m_n(k) > a_n)}{\mathbb{P}(\ell_n^k > a_n/n)}\\
&\le \frac{\mathbb{P}(\ell_n^k > a_n(1-y_n)/n)}{\mathbb{P}(\ell_n^k > a_n/n)} + \frac{\mathbb{P}(T_n \le n(1-y_n))}{\mathbb{P}(\ell_n^k > a_n/n)}.
\end{aligned}$$

The above inequalities together with Lemmas 2.4–2.6 complete the proof of (1.1).

2.3. *Lower deviations* (1.2)

Just as (2.1), $P(n^2m_n(k) < b_n)$ can be controlled as follows, with $y_n = n^{-1/3}$,

$$\begin{aligned}
\mathbb{P}(\ell_n^k < b_n(1-y_n)/n) - \mathbb{P}(T_n < n(1-y_n)) &\le \mathbb{P}(n^2m_n(k) < b_n)\\
&\le \mathbb{P}(\ell_n^k < b_n(1+y_n)/n) + \mathbb{P}(T_n \ge n(1+y_n)).
\end{aligned} \tag{2.2}$$

Lemma 2.7. *Under the assumptions of Theorem 1.1, it holds that*

$$\mathbb{P}(\ell_n^k < b_n/n) = (1+o(1))b_n^k/k!.$$

Proof. The event $\{\ell_n^k < xb_n/n\}$ indicates that there are at least k of $X_1, X_2, \dots, X_n$ which are less than b_n/n. Therefore, $\mathbb{P}(\ell_n^k < b_n/n) = \mathbb{P}(B(n,p) \geq k)$ with $p = p(n) = 1 - e^{-b_n/n}$ (which equals $b_n/n - (b_n/n)^2 e^{\theta(n)}/2$, where $\theta(n) \in [-b_n/n, 0]$). Since $np = b_n - e^{\theta(n)} b_n^2/2n \to 0$, it follows from Lemma 2.3 that

$$
\begin{aligned}
\mathbb{P}(\ell_n^k < b_n/n) &= \mathbb{P}(B \geq k) \sim \mathbb{P}(B = k) = \binom{n}{k} p^k (1-p)^{n-k} \\
&= \frac{\sqrt{2\pi n}\,(n/e)^n}{k!\,\sqrt{2\pi(n-k)}\,((n-k)/e)^{n-k}} \cdot p^k (1-p)^{n-k} \\
&= (1+o(1)) \frac{(n/e)^k\,(n/e)^{n-k}}{k!\,((n-k)/e)^{n-k}} \cdot p^k (1-p)^{n-k} \\
&= (1+o(1)) \frac{1}{k!} n^k e^{-k} \left(\frac{n}{n-k}\right)^{n-k} p^k (1-p)^{n-k}.
\end{aligned}
$$

With $\lambda = \frac{n-k}{k}$ and the estimates $\frac{\lambda}{\lambda+1} e < \left(1 + \frac{1}{\lambda}\right)^\lambda < e$, the term $\left(\frac{n}{n-k}\right)^{n-k}$ is controlled as

$$
\begin{aligned}
e^{k \ln\left(\frac{\lambda}{\lambda+1} e\right)} &< \left(\frac{n}{n-k}\right)^{n-k} < e^{k \ln(e)}, \\
e^{-k} e^{k \ln\left(\frac{\lambda}{\lambda+1} e\right)} &\leq e^{-k} \left(\frac{n}{n-k}\right)^{n-k} \leq 1.
\end{aligned}
$$

For $k = o(n^{1/2})$, it holds that $\left(\frac{\lambda}{\lambda+1}\right)^k = \left(1 - \frac{k}{n}\right)^k \sim e^{-\frac{k^2}{n}} \to 1$, therefore,

$$
\begin{aligned}
\mathbb{P}(\ell_n^k < b_n/n) &= (1+o(1)) \frac{n^k}{k!} \left(\frac{b_n}{n}\left(1 - \frac{b_n}{2n} e^{\theta(n)}\right)\right)^k (1-p)^{n-k} \\
&= (1+o(1)) \frac{b_n^k}{k!} \left(1 - \frac{b_n}{2n} e^{\theta(n)}\right)^k e^{(n-k)\ln(1-p)}.
\end{aligned}
$$

Let $\frac{1}{\eta} = \frac{b_n}{2n} e^{\theta(n)}$, with $k = o(n^{1/3})$ and $b_n = o(1)$, then it holds that

$$
\left(1 - \frac{b_n}{2n} e^{\theta(n)}\right)^k = \left(\left(1 - \frac{1}{\eta}\right)^\eta\right)^{\frac{k}{\eta}} = ((1+o(1))e^{-1})^{\frac{k}{\eta}} = (e^{-1})^{o(1)} = 1 + o(1).
$$

Using the Taylor series, $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi)}{2!}(x - x_0)^2$, for some $\xi \in [x_0, x]$, and we have $\ln(1-p) = -p - p^2/2\xi^2$, for some $\xi \in [1-p, 1]$. Therefore,

$$
\begin{aligned}
\mathbb{P}(\ell_n^k < b_n/n) &= (1+o(1)) \frac{b_n^k}{k!} e^{(n-k)(-p-p^2/2\xi^2)} \\
&= (1+o(1)) \frac{b_n^k}{k!} \exp\left\{-np - \frac{n}{2\xi^2}\left(\frac{b_n}{n}\right)^2 (1-o(1))^2 + k\frac{b_n}{n}(1-o(1)) + \frac{k}{2\xi^2}\left(\frac{b_n}{n}\right)^2 (1-o(1))^2\right\} \\
&= (1+o(1)) \frac{b_n^k}{k!} \exp\left\{-np + \left(-\frac{1}{2\xi^2}\frac{b_n^2}{n} + \frac{kb_n}{n} + \frac{k}{2\xi^2}\left(\frac{b_n}{n}\right)^2\right)(1-o(1))\right\} = (1+o(1)) \frac{b_n^k}{k!},
\end{aligned}
$$

where the facts $k = o(n^{1/3})$ and $kb_n = o(n^{1/3})$ are used in the final step. □

Lemma 2.8. *Let $y_n = n^{-1/3}$. Under the assumptions of Theorem 1.1, it holds that*

$$\mathbb{P}(|T_n - n| > ny_n)/\mathbb{P}(\ell_n^k < b_n/n) \to 0.$$

Proof. It follows from Lemmas 2.2 and 2.7 that

$$\begin{aligned}
&\frac{\mathbb{P}(|T_n - n| > ny_n)}{\mathbb{P}(\ell_n^k < b_n/n)} \le \frac{2e^{-ny_n^2/4}}{\mathbb{P}(\ell_n^k < b_n/n)} = 2(1+o(1))k!b_n^{-k}e^{-ny_n^2/4}\\
&= 2(1+o(1))\sqrt{2\pi k}(k/e)^k b_n^{-k}e^{-ny_n^2/4}\\
&= 2\sqrt{2\pi}(1+o(1))e^{-k}e^{(k+1/2)\ln k}e^{-k\ln(b_n)}e^{-n^{1/3}/4}\\
&= 2\sqrt{2\pi}(1+o(1))\exp\left\{-n^{1/3}\left(\frac{1}{4} + \frac{k}{n^{1/3}} - \frac{(k+1/2)\ln k}{n^{1/3}} - \frac{k\ln(b_n)}{n^{1/3}}\right)\right\} \to 0,
\end{aligned}$$

where the facts $k\ln(b_n) = o(n^{1/3})$ and $k\ln k = o(n^{1/3})$ are used in the final step. □

Lemma 2.9. *Let $y_n = n^{-1/3}$.Under the assumptions of Theorem 1.1, it holds that*

$$\mathbb{P}(\ell_n^k < b_n(1+y_n)/n) \sim \mathbb{P}(\ell_n^k < b_n(1-y_n)/n) \sim \mathbb{P}(\ell_n^k < b_n/n).$$

Proof. Lemma 2.7 implies that

$$\begin{aligned}
&\mathbb{P}(\ell_n^k < b_n(1+y_n)/n)/\mathbb{P}(\ell_n^k < b_n(1-y_n)/n)\\
&= (1+o(1))(b_n(1+y_n))^k \cdot (b_n(1-y_n))^{-k}\\
&= (1+o(1))\left(\frac{1+y_n}{1-y_n}\right)^k = (1+o(1))\left(\frac{1+n^{-1/3}}{1-n^{-1/3}}\right)^k = 1+o(1).
\end{aligned}$$

Slight modifications in the above arguments yield $\mathbb{P}(\ell_n^k < b_n(1-y_n)/n) \sim \mathbb{P}(\ell_n^k < b_n/n)$. □

Now (1.2) follows from Lemmas 2.7–2.9 and dividing (2.2) throughout by the term $\mathbb{P}(\ell_n^k < b_n(1 + y_n)/n)$ as follows:

$$\begin{aligned}
\frac{\mathbb{P}(\ell_n^k < b_n(1-y_n)/n)}{\mathbb{P}(\ell_n^k < b_n(1+y_n)/n)} - \frac{\mathbb{P}(T_n < n(1-y_n))}{\mathbb{P}(\ell_n^k < b_n(1+y_n)/n)} &\le \frac{\mathbb{P}(n^2 m_n(k) < b_n)}{\mathbb{P}(\ell_n^k < b_n(1+y_n)/n)}\\
&\le 1 + \frac{\mathbb{P}(T_n \ge n(1-y_n))}{\mathbb{P}(\ell_n^k < b_n(1+y_n)/n)}.
\end{aligned}$$

3. Conclusions

The main purpose of this paper is to study asymptotics of large deviation probabilities of $n^2 m_n(k)$ as $n \to \infty$, allowing k to depend on n. We have successfully identified the exact orders of both lower and upper deviations. This study recovers and generalizes the results obtained by Devroye in [6], where k was assumed to be fixed. The technical restriction $k\ln k = o(n^{1/3})$ comes from the method used in the proof of Theorem 1.1 and cannot be weakened.

k-th extreme spacings have natural extensions to allow ℓ consecutive spacings (called k-th extreme ℓ spacings; see [3, 4]), where ℓ is a positive integer. When $\ell = 1$, k-th extreme ℓ spacings become usual k-th smallest/largest spacings. As remarked in [3, 4], in the study of several statistical applications based on uniform spacings such as consistency of density estimators and goodness-of-fit tests, it seems better to take ℓ as a large constant or a slowly growing quantity. Limiting behaviors of the k-th extreme ℓ spacings as $n \to \infty$ can be studied when ℓ (possibly k as well) tends to infinity with n.

Use of Generative-AI tools declaration

The author declare no use of Artificial Intelligence (A.I) in writing this paper.

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Conflict of interest

The author declares no conflict of interest in this paper.

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