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Research article

$N(2,2,0)$-Algebras and dimonoids

Fang-An Deng[1,*], **Qiao Xin**[1], **Qian Yang**[1], **Suixin He**[1] **and Yichuan Yang**[2]

[1] School of Mathematics and Statistics, Yili Normal University, Yili 835000, China
[2] School of Mathematical Sciences, Beihang University, Shahe Campus, Beijing 102206, China

* **Correspondence:** Email: dengfangans@126.com.

Abstract: J.-L. Loday introduced the concept of dimonoids, which serve as fundamental tools for investigating Leibniz algebras. $N(2,2,0)$-algebras, abbreviated as NA-algebras, are bisemigroup algebraic systems equipped with two associative binary operations subject to extra identities. This work constructs a novel research paradigm to advance theoretical and applied studies of NA-algebras from the perspective of dimonoid theory. By thoroughly exploring the left and right commutative semigroup properties of NA-algebras, we reveal the inherent relationships between NA-algebras and a range of related algebraic structures: dimonoids, doppelsemigroups, g-dimonoids, restrictive bisemigroups, and digroups. We derive sufficient conditions under which these structures may carry an NA-algebra structure, and prove that every NA-algebra is an abelian dimonoid with a bar-unit denoted "1".

Keywords: $N(2,2,0)$-algebras; dimonoid; semigroup; digroups; doppelsemigroups
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1. Introduction

J.-L. Loday introduced the concept of dimonoids [1]. Dimonoids generalize semigroups and serve as core tools for investigating Leibniz algebras. The Cayley representation theorem for dimonoids states that every dimonoid is isomorphic to a transformation dimonoid [2]. Multiple construction methods for dimonoids have been proposed, and it was proven that any dimonoid can be isomorphically embedded into a dimonoid built from a semigroup [3].

Furthermore, by imposing specific interassociativity axioms, A.V. Zhuchok defined the notion of a doppelsemigroup [4]. A key structural difference separates these two objects: Every dimonoid is a doppelsemigroup, yet not every doppelsemigroup satisfies all dimonoid axioms, particularly the idempotency conditions [5, 6].

T. Pirashvili introduced duplexes as a generalization of dimonoids and constructed free duplexes [7]. In [8], A. V. Zhuchok proved that every commutative dimonoid decomposes into a semilattice of archimedean subdimonoids. Reference [9] constructed free commutative dimonoids and characterized the least idempotent congruence on such dimonoids. Y. V. Zhuchok [10] built free abelian dimonoids and described the least abelian congruence on free dimonoids; he also demonstrated that free abelian dimonoids are uniquely determined by their endomorphism semigroups.

V. M. Gavrylkiv completed isomorphism classifications of all two-element dimonoids, all commutative three-element dimonoids, and all abelian three-element dimonoids [11]. As two-binary-operation algebraic systems, dimonoids have acted as an important bridge connecting nonassociative algebra and higher-dimensional algebra over recent decades. Notably, dimonoids share close structural ties with restrictive bisemigroups first introduced by B. M. Schein [12]. References [7, 13] formalized the concepts of interassociativity and strong interassociativity, alongside related semigroups and doppelalgebras, all of which exhibit natural connections to dimonoids [14].

The concept of $N(2,2,0)$-algebras (NA-algebras) was first proposed by F. A. Deng and Y. Xu in 1996 [15]. In our previous work, we investigated the fundamental properties of an NA-algebra $(S, *, \triangle, 1)$ and proved that if the binary operation $*$ is idempotent, then $(S, *, \triangle, 1)$ forms a rewriting system whose unification problem is undecidable [16]. Similarly, suppose the operation $\triangle$ is nilpotent; then $(S, \triangle, 1)$ becomes an associative BCI-algebra [15, 17]. Our prior research covers the study of basic properties of regular semigroups, inner regular semigroups, E-inverse semigroups, zero divisors, middle units, ideals, and stabilizers within NA-algebras. We constructed congruence relations on NA-algebras and built quotient NA-algebras via these congruences [16]. We also derived several structural connections between nilpotent NA-algebras and other (2,0)-type algebraic systems [16–18].

As a special class of bisemigroup structures $(S, *, \triangle, 1)$, NA-algebras satisfy all the fundamental axioms defining dimonoids. Accordingly, the well-established research techniques and theoretical outcomes of dimonoids can further enrich the theoretical and applied framework of NA-algebras, offering new perspectives and analytical tools for subsequent studies of NA-algebras. In the present paper, we prove that every NA-algebra is a dimonoid equipped with a bar-unit "1". We characterize the left-commutative and right-commutative semigroup properties inherent to NA-algebras and point out that interassociative NA-algebras possess broad application prospects across multiple mathematical disciplines.

As a generalization of implication algebras, NA-algebras provide a unified framework for fuzzy implication operators, which facilitates the analysis of implication relations and their algebraic properties in fuzzy logic. Existing literature also demonstrates that this bisemigroup structure exhibits promising applications in algebraic geometry and representation theory, particularly for describing representations of certain nonassociative algebraic systems.

The rest of this paper is structured as follows. Section 2 reviews fundamental definitions and known results concerning NA-algebras. We also establish several new propositions to further clarify the internal structure of NA-algebras. Section 3 elaborates the intrinsic connections between NA-algebras and dimonoids. Section 4 explores the correspondence between NA-algebras and related bisemigroup-type structures, including doppelsemigroups, g-dimonoids, restrictive bisemigroups, and digroups. We derive sufficient conditions for NA-algebras to satisfy interassociativity identities. Finally, Section 5 concludes the paper and outlines several directions for future research.

2. Preliminaries

First, recall some useful facts that we shall use in this paper.

2.1. Concepts for semigroups

Let S be a semigroup. An element e of a semigroup $(S, *)$ is called a left identity (respectively, a right identity) in S if $e * a = a$ (respectively, $a * e = a$) for any $a \in S$. An element e is called an identity if e is a left identity and a right identity [19].

An element z of S is a left (respectively, right) zero of S if $zs = z$ (respectively, $sz = z$) for all $s \in S$; z is a zero of S if it is both a left and right zero of S. A semigroup all of whose elements are left (respectively, right) zeros is a left (respectively, right) zero semigroup [20].

Elements a and b of S commute if $ab = ba$; S is commutative if any two of its elements commute.

An element e of S is idempotent if $e * e = e$. The set of idempotents of a subset A of S is denoted by $E(A)$. The semigroup is a band if all its elements are idempotents [21]. Commutative bands are called semilattices [19, 22, 23].

If $(S, \star)$ is a semigroup, then the semigroup $(S, \star^d)$ with operation $x \star^d y = y \star x$ is called dual to $(S, \star)$ [11].

Recall that an isomorphism between $(S, *)$ and $(S', \circ)$ is a bijective function $\psi : S \to S'$ such that $\psi(x * y) = \psi(x) \circ \psi(y)$ for all $x, y \in S$. If there exists an isomorphism between $(S, *)$ and $(S', \circ)$, then $(S, *)$ and $(S', \circ)$ are said to be isomorphic, denoted $(S, *)\cong(S', \circ)$. An isomorphism between $(S, *)$ and $(S, *)$ is called an automorphism of a semigroup $(S, *)$. By $\mathrm{Aut}(S, *)$, we denote the automorphism group of a semigroup $(S, *)$ [19, 23].

2.2. Concepts for NA-algebras

In 1996, we first introduce the notion of the $N(2, 2, 0)$-algebra and study it in detail. In this article, we will abbreviate it as NA-algebra.

Definition 2.1. [15] An NA-algebra $(S, *, \triangle, 1)$ is a set S together with two binary operations $*$ and $\triangle$ and with a nullary operation "1" satisfying the following axioms. For all $a, b, c \in S$,

$(N_1) a * (b \triangle c) = c * (a * b)$;

$(N_2) (a \triangle b) * c = b * (a * c)$;

$(N_3) 1 * a = a$.

The following propositions give some basic properties of the NA-algebra.

Theorem 2.1. Let $(S, *, \triangle, 1)$ be an NA-algebra. Then for all $a, b, c \in S$, the following hold:

$(N_4) a * b = b \triangle a$;

$(N_5) (a * b) * c = a * (b * c)$ ($*$ associative law); $a * (b * c) = b * (a * c)$ ($*$ left commutative law);

$(N_6) (a \triangle b) \triangle c = a \triangle (b \triangle c)$ ($\triangle$ associative law); $(a \triangle b) \triangle c = (a \triangle c) \triangle b$ ($\triangle$ right commutative law).

Proof: (N_4): $a \triangle b \overset{(N_3)}{=} 1 * (a \triangle b) \overset{(N_1)}{=} b * (1 * a) \overset{(N_3)}{=} b * a$;

(N_5): $(a * b) * c \overset{(N_4)}{=} (b \triangle a) * c \overset{(N_2)}{=} a * (b * c)$; $a * (b * c) \overset{(N_4)}{=} a * (c \triangle b) \overset{(N_1)}{=} b * (a * c)$;

(N_6): $(a \triangle b) \triangle c \overset{(N_4)}{=} c * (a \triangle b) \overset{(N_1)}{=} b * (c * a) \overset{(N_5)}{=} c * (b * a) \overset{(N_5)}{=} (c * b) * a \overset{(N_4)}{=} (b \triangle c) * a \overset{(N_4)}{=} a \triangle (b \triangle c)$;

$(a \triangle b) \triangle c \overset{(N_4)}{=} c * (a \triangle b) \overset{(N_1)}{=} b * (c * a) \overset{(N_4)}{=} (c * a) \triangle b \overset{(N_4)}{=} (a \triangle c) \triangle b$.

This completes the proof.

Corollary 2.2. [15, 17, 18] If $(S, *, \triangle, 1)$ is an NA-algebra, then $(S, *, 1)$ and $(S, \triangle, 1)$ are a pair of dual semigroups.

Theorem 2.3. Let $(S,*,\triangle,1)$ be an NA-algebra, and for all $x,y\in S$, the following hold:

(1) If $x*x=1$, then $x*1=x$. In this case, $x*y=x\triangle y=y*x$, i.e., $(S,*,1)$ and $(S,\triangle,1)$ are a commutative semigroup, and the operations of the NA-algebra coincide.

(2) If $x*1=1$, then $x\in E(S)$. Semigroup $(S,*,1)$ is a right zero semigroup and $(S,\triangle,1)$ is a left zero semigroup.

Proof: (1) Since $x*x=1$, then $x*1=x*(x*x)=(x*x)*x=1*x=x$. So, $x*1=1*x=x$. Thus, 1 is a unit element. Since $1'=1*1'=1'*1=1$, hence 1 is a unique unit in S.

$y*x=y*(x*1)\overset{(N_5)}{=}x*(y*1)=x*y=y\triangle x$. Hence, $(S,*,1)$ is a commutative semigroup.

Further, from $x*x=1$, we have that x is an inverse x. If there exists an inverse element y of x such that $x*y=1$, then $x=1*x=(x*y)*x=x*(y*x)=y*(x*x)=y*1=y$. Hence, every element $x\in S$ is its own unique inverse.

So, for NA-algebras, if $x*x=1$, then $x*1=x$, and $(S,*,1)$ is an abelian group.

(2) For all $x\in S$, if $x*1=1$, then

$$x^2=x*x=x*(1*x)=(x*1)*x=1*x=x.$$

We get $x*1=1\Rightarrow x^2=x$, i.e., $x\in E(S)$.

Moreove, for any $x,y\in S$, we have $x*y=x*(1*y)=(x*1)*y=1*y=y$. Hence, $(S,*,1)$ is a right zero semigroup. Using (N_4), we can get that $(S,\triangle,1)$ is a left zero semigroup.

From Theorem 2.3(2), we have the following conclusion.

Remark 2.4. (1) For an NA-algebra $(S,*,\triangle,1)$, two binary operations $*$ and $\triangle$ satisfy the associative property but are not necessarily commutative semigroups. In case $x*x=1,\forall x\in S$, the NA-algebra must be a commutative semigroup.

(2) For an NA-algebra $\forall x,y\in S$, if $x\neq y, x*1=1$, using Theorem 2.3(2), we have $x*y=y\neq x=y*x$. Hence, semigroup $(S,*)$ and $(S,\triangle)$ are non-commutative semigroups. In this case, if $(S,*)$ is a commutative semigroup, then the NA-algebra degenerates to a trivial semigroup of a singleton set.

We use the following Example 1 to illustrate Case (1).

Example 1. Let $X=\{1,a,b,c\}$. We define binary operations $\triangle$ on X with the following Tables 1 and 2:

Table 1. $\triangle$ table.

$\triangle$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Table 2. $*$ table.

$*$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

It is not complicated to check that $(X; \triangle, 1)$ is also a Klein four-group. Define $x * y = y \triangle x$, for all $x, y \in X$, we can immediately write down the dual group $(X; *, 1)$ with the above Table 2.

Obviously, for all $\forall x, y \in X, x \triangle y = y * x, x * x = x \triangle x = 1$. Therefore, $(X; *, \triangle, 1)$ is a special NA-algebra, and the operations $*$ and $\triangle$ coincide, as they are both abelian groups. This example shows the trivial NA-algebra of the Klein four-group.

Theorem 2.5. Let $(S, *, \triangle, 1)$ be an NA-algebra and $a, b, c, d \in S$. Then the following equation holds:

$$(a * b) \triangle (c * d) = (a \triangle c) * (b \triangle d). \tag{2.1}$$

Equation (2.1) is called the cross-commutative law.

Proof: Let $(S, *, \triangle, 1)$ be an NA-algebra. By Theorem 2.1, we have
$(a*b)\triangle(c*d) \overset{(N_4)}{=} (c*d)*(a*b) \overset{(N_5)}{=} a*((c*d)*b) \overset{(N_5)}{=} a*(c*(d*b)) \overset{(N_5)}{=} c*(a*(d*b) \overset{(N_5)}{=} (c*a)*(d*b)$,
$(a \triangle c) * (b \triangle d) \overset{(N_4)}{=} (c * a) * (d * b)$.
Hence, $(a * b) \triangle (c * d) = (a \triangle c) * (b \triangle d)$ for all $a, b, c, d \in S$.

Theorem 2.6. Let $(S, \circ)$ be an abelian monoid with a unit 1 and let $\psi : S \to S$ be an idempotent endomorphism. Define operations $*$ and $\triangle$ on S as follows:

$$x * y := \psi(x) \circ y, x \triangle y := x \circ \psi(y),$$

for all $x, y \in S$. Then $(S, *, \triangle, 1)$ is an NA-algebra.

Proof: For all $x, y, z \in S$, using the definitions of $*$ and $\triangle$, we have
$x * (y \triangle z) = \psi(x) \circ (y \triangle z) = \psi(x) \circ (y \circ \psi(z))$,
$z * (x * y) = \psi(z) \circ (x * y) = \psi(z) \circ (\psi(x) \circ y) = \psi(x) \circ (y \circ \psi(z))$.
Thus, $x * (y \triangle z) = z * (x * y)$. So, the axiom (N_1) holds.

Using identity (2.1), we obtain
$(x \triangle y) * z = (x \circ \psi(y)) * z = (\psi(x \circ \psi(y))) \circ z) = (\psi(x) \circ \psi(\psi(y))) \circ z = (\psi(x) \circ \psi(y)) \circ z$,
$y * (x * z) = \psi(y) \circ (x * z) = \psi(y) \circ (\psi(x) \circ z) = (\psi(y) \circ (\psi(x)) \circ z = (\psi(x) \circ \psi(\psi(y))) \circ z$.
Hence, $(x \triangle y) * z = y * (x * z)$, thus the axiom (N_2) holds.

Further, according to abelian monoid $(S, \circ)$ with a unit 1, for all $x \in S$, we get $1 * x = \psi(1) \circ x = 1 \circ x = x(= x \circ 1 = x \circ \psi(1) = x \triangle 1)$. So, the axiom (N_3) holds. The proof is complete.

It is clear that a left commutative semigroup satisfies any identity of the form

$$x_1 x_2 \ldots x_n a = x_{1\pi} x_{2\pi} \ldots x_{n\pi} a. \tag{2.2}$$

Dually, a right commutative semigroup satisfies any identity of the form

$$a x_1 x_2 \ldots x_n = a x_{1\pi} x_{2\pi} \ldots x_{n\pi}, \tag{2.3}$$

where π is a permutation of $\{1, 2, \ldots, n\}$ [24].

If $(S, \circ)$ is dual to $(S, *)$, then it is clear that $(S, *)$ is anti-isomorphic to $(S, \circ)$ [25].

Obviously, a binary relation ϱ on a semigroup $(S, *)$ is a congruence if and only if ϱ is a congruence on the semigroup $(S, *)$ dual to $(S, \circ)$.

From Theorem 2.1, it is not difficult to obtain the following:

(1) The NA-algebra is an algebra system with a pair of dual semigroups.

(2) In NA-algebra $(S, *, \triangle, 1)$, $(S, *)$ is a left commutative semigroup that satisfies Eq (2.2), and $(S, \triangle)$ is a right commutative semigroup that satisfies Eq (2.3). $(S, *)$ is anti-isomorphic to $(S, \triangle)$.

Several interesting properties of NA-algebras have been discussed earlier [15–18].

3. Relation between NA-algebras and dimonoids

3.1. Properties of dual NA-algebras

V. M. Gavrylkiv is one of the leading researchers in the field of dimonoids. His work is on the structural classification of non-commutative dimonoids, duality, and self-duality. Gavrylkiv's research is not limited to simple classification, but delves deeper into the algebraic properties of duality [11, 20, 26]:

Definition 3.1. [1] An algebra $(D, \dashv, \vdash)$ with two binary associative operations $\dashv$ and $\vdash$ is called a dimonoid if for all $x, y, z \in D$, the following conditions hold:

$(D_1)(x \dashv y) \dashv z = x \dashv (y \vdash z)$,

$(D_2)(x \vdash y) \dashv z = x \vdash (y \dashv z)$,

$(D_3)(x \dashv y) \vdash z = x \vdash (y \vdash z)$.

A dimonoid $(D, \dashv, \vdash)$ is called abelian [11] if the equality

$$x \dashv y = y \vdash x \tag{3.1}$$

holds for all $x, y \in D$.

A dimonoid $(D, \dashv, \vdash)$ is called commutative if both its operations are commutative, i.e.,

$$x \dashv y = y \dashv x, x \vdash y = y \vdash x \tag{3.2}$$

holds for all $x, y \in D$.

A dimonoid $(D, \dashv, \vdash)$ is called self-dual if $(D, \dashv, \vdash)^d = (D, \dashv, \vdash)$.

Non-abelian dimonoids are divided into pairs of dual dimonoids [20].

This section considers properties of dual NA-algebras similar to V. M. Gavrylkiv's study of the duality of dimonoids.

Let $(S, *, \triangle, 1)$ be an NA-algebra. Define new operations $*^d$ and $\triangle^d$ on S by

$$x *^d y = y * x,$$

and

$$x \triangle^d y = y \triangle x.$$

It is immediate to check that $(S, *^d, \triangle^d)$ is a new NA-algebra, which we call the dual NA-algebra of $(S, *, \triangle)$ that we denote by $(S, *, \triangle)^d$. It follows that the unary duality operation is involutive in the sense that $((S, *, \triangle)^d)^d = (S, *, \triangle)$. In fact, $(S, *, \triangle)^d$ is an NA-algebra if and only if $(S, *, \triangle)$ is an NA-algebra. As usual, an NA-algebra $(S, *, \triangle, 1)$ is said to be self-dual if $(S, *, \triangle, 1)^d = (S, *, \triangle, 1)$.

If $(S, *)$ is dual to $(S, \triangle)$, then $(S, *)$ is anti-isomorphic to $(S, \triangle)$.

Definition 3.2. An NA-algebra $(S, *, \triangle, 1)$ is called abelian if $x * y = y \triangle x$ for all $x, y \in S$. If $x * y = y * x$ for all $x, y \in S$, then $(S, *, 1)$ is a commutative NA-algebra (degenerate NA-algebra).

Obviously, in an NA-algebra, the abelian property is different from the operation commutativity.

Proposition 3.1. Let $(S, *, \triangle, 1)$ be an NA-algebra. Then the following conditions are equivalent:

(1) $(S, *)$ and $(S, \triangle)$ are dual semigroups;

(2) $(S, *, \triangle)$ is abelian;

(3) $(S, *, \triangle)$ is self-dual.

Proof: (1) $\Rightarrow$ (2). If $(S, *)$ and $(S, \triangle)$ are dual semigroups, then $x * y = y \triangle x$ for all $x, y \in$, and hence, $(S, *, \triangle, 1)$ is an abelian NA-algebra.

(2) $\Rightarrow$ (3). Let $(S, *, \triangle)$ be an abelian NA-algebra. Taking into account that $x *^d y = y \triangle x = x * y$ and $x \triangle^d y = y * x = x \triangle y$, we conclude that $*^d = *$ and $\triangle^d = \triangle$, and hence $(S, *, \triangle, 1)$ is a self-dual NA-algebra.

(3) $\Rightarrow$ (1). If $(S, *, \triangle, 1)$ is a self-dual NA-algebra, then $x * y = x *^d y = y \triangle x$ for all $x, y \in$, and hence, $(S, *)$ and $(S, \triangle)$ are dual semigroups.

Proposition 3.1 implies that if the semigroups $(S, *)$ and $(S, \triangle)$ of NA-algebra $(S, *, \triangle, 1)$ do not coincide, then NA-algebra $(S, \triangle, 1)$ is non-commutative.

3.2. Relation with dimonoids

In recent years, dimonoids have become standard tools in the study of various structures, particularly in the theory of Leibniz algebras. One of the earliest foundational results on dimonoids is due to J.-L. Loday [1]. A wide range of classes of dimonoids were systematically investigated by A. V. Zhuchok [8] and Y. V. Zhuchok [10, 23]. In [11, 20, 26], V. M. Gavrylkiv presented complete classifications, up to isomorphism, of all two-element dimonoids, all commutative three-element dimonoids, and all abelian three-element dimonoids. In this section, we study the structure of NA-algebras and present another approach to constructing dimonoids based on the operations of NA-algebras.

Here is another version of the definition of dimonoid.

Definition 3.3. [9] A set D equipped with two binary associative operations $\vdash$ and $\dashv$ satisfying the following axioms:

$(D_1')(x \vdash y) \vdash z = x \vdash (y \dashv z)$,

$(D_2')(x \dashv y) \vdash z = x \dashv (y \vdash z)$,

$(D_3')(x \vdash y) \dashv z = x \dashv (y \dashv z)$

for all $x, y, z \in D$ is called a dimonoid.

Note: A dimonoid $(D, \dashv, \vdash)$ is abelian and different from dimonoid $(D, \dashv, \vdash)$, which is commutative.

An element e of a dimonoid $(D, \dashv, \vdash)$ is a bar-unit if the equality

$$x \dashv e = x = e \vdash x \tag{3.3}$$

holds for every $x \in D$ [8, 14].

The class of all abelian dimonoids satisfies the conditions of Birkhoff's theorem, and therefore it is a variety. A dimonoid that is free in the variety of abelian dimonoids will is called a free abelian dimonoid [10].

If operations of a dimonoid coincide, the dimonoid becomes a semigroup.

Define the notion of the diband of dimonoids as follows:

A dimonoid $(D, \dashv, \vdash)$ is called an idempotent dimonoid or diband, if the equality

$$x \dashv x = x = x \vdash x \tag{3.4}$$

holds for all $x, y \in D$.

Theorem 3.2. Let $(S, *, \triangle, 1)$ be an NA-algebra. We define two binary operations $\dashv$ and $\vdash$ by

$$x \vdash y = x * y, x \dashv y = x \triangle y$$

for all $x, y \in S$, then $(S, \vdash, \dashv, 1)$ is an abelian dimonoid with a bar-unit "1".

Proof: Let $x, y, z \in S$. By the definition of operations $\vdash$ and $\dashv$, we obtain

$$(x \dashv y) \dashv z) = (x\triangle y)\triangle z \overset{(N_4)}{=} z * (y * x) \overset{(N_5)}{=} y * (z * x) \overset{(N_5)}{=} (y * z) * x \overset{(N_4)}{=} x \triangle (y * z) = x \dashv (y \vdash z).$$

Therefore, the axiom (D_1) holds.

Further, we show that axioms $(D_2) - (D_3)$ hold:

$$(x \vdash y) \dashv z = (x * y) \triangle z \overset{(N_4)}{=} z * (x * y) \overset{(N_5)}{=} x * (z * y) \overset{(N_4)}{=} x * (y \triangle z) = x \vdash (y \dashv z),$$

which completes the verification of (D_2).

Analogously,

$$(x \dashv y) \vdash z = (x\triangle y) * z \overset{(N_4)}{=} (y * x) * z \overset{(N_5)}{=} y * (x * z) \overset{(N_5)}{=} x * (y * z) = x \vdash (y \vdash z).$$

Hence, (D_2) holds.

From the Definition 3.1 and (N_4), for any $x, y \in S$, we have $x \vdash y = x * y = y \triangle x = y \dashv x$, $x = 1 * x = 1 \vdash x, x = x \triangle 1 = x \dashv 1$, so "1" is a bar-unit.

This shows that $(S, \vdash, \dashv)$ is an abelian dimonoid and there exists a bar-unit "1".

Note: In contrast to monoids, a dimonoid may have many bar-units. Meanwhile, there also exist dimonoids without bar-units [10].

Remark 3.3. NA-algebra $(S, *, \triangle, 1)$ is an important subclass of associative dialgebra, but it is not necessarily commutative. If the operations $*$ and $\triangle$ coincide, the semigroup becomes a monoid.

From Lemma 2 [8], it follows that the operations $\dashv$ and $\vdash$ of a commutative dimonoid $(D, \dashv, \vdash)$ are indistinguishable for three and more multipliers, and the product of these elements doesn't depend on the parenthesizing.

Similar to Example 2.4 in [14], let G be a semigroup, and let X be a right G-set. Defining

$$(g, x) \triangle (h, y) = (gh, x), \tag{3.5}$$

and

$$(g, x) * (h, y) = (gh, y) \tag{3.6}$$

on $G \times X$ yields NA-algebra $(G \times X, \triangle, *)$, i.e., the NA-algebra of the action (X, G).

Proposition 3.4. Let G be a commutative semigroup with an identity element "1", and let X be a right G-set. We define two operations on $G \times X$ with Eqs (3.5) and (3.6), then $(G \times X, *, \triangle)$ is an NA-algebra.

Proof: For all $a, b, c \in G$, and $\forall x, y, z \in X$,
$(a, x) * ((b, y) \vartriangle (c, z)) = (a, x) * (bc, y) = (abc, y)$,
$(c, z) * ((a, x) * (b, y)) = (c, z) * (ab, y) = (cab, y) = (abc, y)$,
hence, $(a, x) * ((b, y) \vartriangle (c, z)) = (c, z) * ((a, x) * (b, y))$, i.e., (N_1) holds.
$((a, x) \vartriangle (b, y)) * (c, z) = (ab, x) * (c, z) = (abc, z)$,
$(b, y) * ((a, x) * (c, z)) = (b, y) * (ac, z) = (abc, z)$,
thus, we have $((a, x) \vartriangle (b, y)) * (c, z) = (b, y) * ((a, x) * (c, z))$, so (N_2) holds.
$(1, x) * (a, y) = (a, y)$.
Here, $(1, x)$ is the left identity element of the operation $*$ on $G \times X$. So, we verified that (N_3) holds.

Thus, we prove that $(G \times X, *, \vartriangle)$ is an NA-algebra.

Example 2. Let $S = \{1, a, b, c, d\}$, and define the operations $*$ and $\vartriangle$ with Cayley Tables 3 and 4:

Table 3. $*$ table.

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	a	b	c	d
b	1	a	b	c	d
c	1	a	b	c	d
d	1	a	b	c	d

Table 4. $\vartriangle$ table.

$\vartriangle$	1	a	b	c	d
1	1	1	1	1	1
a	a	a	a	a	a
b	b	b	b	b	b
c	c	c	c	c	c
d	d	d	d	d	d

Next, we verify that $(S, *, \vartriangle, 1)$ is an NA-algebra and also a dimonoid.

Step 1: Verify that it is an NA-algebra.

Verification 1: $x * (y \vartriangle z) = z * (x * y), \forall x, y, z \in S$.

As can be seen from Tables 3 and 4 above, $x * (y \vartriangle z) = x * y = y$, $z * (x * y) = z * y = y$. So, (N_1) holds.

Verification 2: $(x \vartriangle y) * z = y * (x * z), \forall x, y, z \in S$.

From Tables 3 and 4 above, we have $(x \vartriangle y) * z = x * z = z, y * (x * z) = y * z = z$. Hence, (N_2) holds.

It is evident from Tables 3 and 4 above that $1 * x = x, \forall x, y, z \in S$, so (N_3) holds true.

Therefore, $(S, *, \vartriangle, 1)$ is an NA-algebra.

Similar to verification, $(S, *, \vartriangle, 1)$ is a dimonoid.

Below is an example of a three-element dimonoid, but it is not an NA-algebra.

Example 3. Let $S = \{1, a, b\}$ be a set. We define the operations $\vdash$ and $\dashv$ as follows: for any $x, y \in S$,

$$x \dashv y = x,$$

$$x \vdash y = \begin{cases} 1, & x = 1, \\ a, & x = a, \\ b, & x = b, \end{cases} \tag{3.7}$$

that is, $x \vdash y = x$.

We can verify that $(S, \vdash, \dashv)$ is a dimonoid. However, it is not an NA-algebra. Here, by (3.7), $1 * a = 1 \neq a$, so (N_3) not satisfied.

$x \vdash (y \dashv z) = x \vdash y = x$, $z \vdash (x \vdash y) = z \vdash x = z$; when $x \neq z$, (N_1) does not hold.

Hence, $(S, *, \triangle, 1)$ is not an NA-algebra.

4. Relation between NA-algebras and other related bialgebra systems

4.1. Connections of n-tuple semigroups

In this section, we establish connections between n-tuple semigroups and NA-algebraic structures. As usual, N denotes the set of all positive integers. For $n \in N$, denote the set $\{1, 2, \cdots, n\}$ by $\overline{n}$. Let us recall that an n-tuple semigroup is a nonempty set G equipped with n binary operations $\star_1, \star_2, ..., \star_n$, satisfying the axioms

$$(x \star_r y) \star_s z = x \star_r (y \star_s z)$$

for all $x, y, z \in G$ and $r, s \in \overline{n}$.

A. V. Zhuchok gave examples of n-tuple semigroups and established connections between n-tuple semigroups and some other algebraic structures [25, 28]:

(a) Obviously, every semigroup is a 1-tuple semigroup.

(b) Every semigroup $(S, *)$ can be considered as an n-tuple semigroup T if we assume that $T = (S, \underbrace{\cdot, \cdot, \cdots, \cdot}_{n})$.

Let B be an arbitrary nonempty set, and let $(*, \circ)$ be an ordered pair of binary operations defined on B. Following B. M. Schein [12], the pair$(*, \circ)$ is associative if

$$(x * y) \circ z = x * (y \circ z), \tag{4.1}$$

for any $x, y, z \in B$. Associative pairs of operations were considered in [20].

Proposition 4.1. Every NA-algebra $(S, *, \triangle)$ is a 2-tuple semigroup, and the pair $(*, \triangle)$ is associative.

Proof: Let $(S, *, \triangle)$ be an NA-algebra. For any $x, y, z \in S$, we have

$(x * y) \triangle z \overset{(N_4)}{=} z * (x * y), x * (y \triangle z) \overset{(N_1)}{=} z * (x * y)$.

Hence, $(x * y) \circ z = x * (y \circ z)$. So, the pair $(*, \triangle)$ is associative.

4.2. Interassociativity for NA-algebras

The role of interassociativity is key to structures with two operations (such as rings, modules, vector spaces), which satisfies $(a * b) \circ c = a * (b \circ c)$ [29]. If $*$ and $\circ$ are the same operation, then the interassociativity degenerates into associativity itself $(a*b)*c = a*(b*c)$. Therefore, interassociativity can be seen as a more general form [30].

Definition 4.1. [30] Given a semigroup $(S, \dashv)$, consider a semigroup $(S, \vdash)$ defined on the same set. We say that the semigroups $(S, \dashv)$ and $(S, \vdash)$ are interassociative provided

$$(x \dashv y) \vdash z = x \dashv (y \vdash z), \tag{4.2}$$

and

$$(x \vdash y) \dashv z = x \vdash (y \dashv z) \tag{4.3}$$

for all $x, y, z \in S$.

When this occurs, we say that $(S, \dashv)$ is an interassociate of $(S, \vdash)$ or that the semigroups are interassociates of each other. If the semigroups $(S, \dashv)$ and $(S, \vdash)$ are interassociative, then rearranging the parentheses in any expression involving only the operations $\dashv, \vdash$ and elements of S will not change the result [30].

Proposition 4.2. Let $(S, *, \triangle, 1)$ be an NA-algebra. If the equality holds, for all $x, y, z \in S$,

$$x * (y * z) = z * (y * x), \tag{4.4}$$

or

$$(x \triangle y) \triangle z = (z \triangle y) \triangle x, \tag{4.4$'$}$$

then the semigroups $(S, *)$ and $(S, \triangle)$ are interassociative.

Proof: Let $(S, *, \triangle, 1)$ be an NA-algebra. For all $x, y \in S$, we put $\dashv = \triangle$ and $\vdash = *$, then

$(x \dashv y) \vdash z = (x \triangle y) * z \overset{(N_2)}{=} y * (x * z)$,

$x \dashv (y \vdash z) = x \triangle (y * z) \overset{(N_4)}{=} (y * z) * x \overset{(N_5)}{=} y * (z * x) \overset{(4.4)}{=} x * (z * y) \overset{(N_4)}{=} x * (y \triangle z) \overset{(N_1)}{=} z * (x * y) \overset{(4.4)}{=} y * (x * z)$.

So, $(x \dashv y) \vdash z = x \dashv (y \vdash z)$, i.e., the equality (4.2) holds.

Next, $(x \vdash y) \dashv z = (x * y) \triangle z \overset{(N_4)}{=} z * (x * y) \overset{(N_1)}{=} x * (y \triangle z) = x \vdash (y \dashv z)$. Hence, the equality (4.3) holds.

To sum up, we draw the following conclusions:

If NA-algebra $(S, *, \triangle, 1)$ satisfies Eq (4.4), then the semigroups $(S, *)$ and $(S, \triangle)$ are interassociative.

4.3. NA-algebras and strong doppelsemigroups

The study of doppelsemigroups was initiated by A.V. Zhuchok. The idea of doppelsemigroups is based on the study of dimonoids in the sense of J.-L.Loday [1]. Doppelsemigroups are a generalization of semigroups, and they have relationships with such algebraic structures as doppelalgebras [6], duplexes [7], interassociative semigroups, restrictive bisemigroups, dimonoids, and trioids. Doppelsemigroups are closely related to bisemigroups considered in the work of [4, 30]. In this section, we will discuss the relationship between NA-algebras and doppelsemigroups.

Definition 4.2. [4] A doppelsemigroup is an algebraic structure $(D, \dashv, \vdash)$ consisting of a nonempty set D equipped with two associative binary operations $\dashv$ and $\vdash$ satisfying the following axioms:

$(D_1)(x \dashv y) \vdash z = x \dashv (y \vdash z)$,

$(D_2)(x \vdash y) \dashv z = x \vdash (y \dashv z)$.

A doppelsemigroup $(D, \dashv, \vdash)$ is called strong [25] if it satisfies the equality

$$x \dashv (y \vdash z) = x \vdash (y \dashv z). \tag{4.5}$$

A doppelsemigroup $(D, \dashv, \vdash)$ is called abelian [10] if for all $x, y \in D$,

$$x \dashv y = y \vdash x. \tag{4.6}$$

Note: A doppelsemigroup $(D, \dashv, \vdash)$ is abelian and different from doppelsemigroup $(D, \dashv, \vdash)$, which is commutative.

Proposition 4.3. Let $(S, *, \triangle, 1)$ be an NA-algebra. If the operation $*$ satisfies equality (4.4), then $(S, *, \triangle, 1)$ is a strong doppelsemigroup. Conversely, if a strong doppelsemigroup $(S, \dashv, \vdash)$ adds a constant "1" satisfying Eqs (4.4) and (4.6), and $1 \vdash x = x = x \dashv 1$, then $(S, \dashv, \vdash, 1)$ is an NA-algebra.

Proof: For all $x, y, z \in S$, we put $\dashv = \triangle$ and $\vdash = *$, then we get

$(x \dashv y) \vdash z = (x \triangle y) * z \overset{(N_2)}{=} y * (x * z) \overset{(3.4)}{=} z * (x * y) \overset{(N_5)}{=} x * (z * y)$,

$x \dashv (y \vdash z) = x \triangle (y * z) \overset{(N_4)}{=} (y * z) * x \overset{(N_5)}{=} y * (z * x) \overset{(4.4)}{=} x * (z * y)$.

So, $(x \dashv y) \vdash z = x \dashv (y \vdash z)$, i.e., (D_1) holds.

Since $(x \vdash y) \dashv z = (x * y) \triangle z \overset{(N_4)}{=} z * (x * y)$,

$x \vdash (y \dashv z) = x * (y \triangle z) \overset{(N_1)}{=} z * (x * y)$.

Hence, $(x \vdash y) \dashv z = x \vdash (y \dashv z)$, so (D_2) holds.

Thus, we prove that an NA-algebra satisfying condition (4.4) is a doppelsemigroup.

Using similar arguments, $x \dashv (y \vdash z) = x \triangle (y * z) \overset{(N_4)}{=} (y * z) * x \overset{(N_5)}{=} y * (z * x) \overset{(4.4)}{=} x * (z * y)$,

$x \vdash (y \dashv z) = x * (y \triangle z) \overset{(N_1)}{=} z \triangle (x * y) \overset{(N_5)}{=} x * (z * y)$.

Thus, equality (4.5) holds.

Therefore, we show that an NA-algebra satisfying condition (4.4) is also a strong doppelsemigroup.

From Propositions 4.2 and 4.3, we can see that Eq (4.4) is a sufficient condition for the semigroup $(S, *)$ and $(S, \triangle)$ of the NA-algebra to be interassociative and form a doppelsemigroup.

Conversely, a strong doppelsemigroup $(S, \dashv, \vdash)$ adds a constant "1" satisfying the condition (4.4). Put $\dashv = \triangle$ and $\vdash = *$, and we have

$x * (y \triangle z) = x \vdash (y \dashv z) \overset{(D_2)}{=} (x \vdash y) \dashv z = z \vdash (x \vdash z) = z * (x * y)$. So, (N_1) holds.

$(x \triangle y) * z = (x \dashv y) \vdash z \overset{(D_1)}{=} x \dashv (y \vdash z) \overset{(4.6)}{=} (z \dashv y) \vdash x \overset{(D_1)}{=} z \dashv (y \vdash x) \overset{(4.6)}{=} z \vdash (y \dashv x) \overset{(4.6)}{=} z \vdash (x \vdash y) = z * (x * y) \overset{(4.4)}{=} y * (x * z)$. Hence, (N_2) holds.

It follows from the assumption that $1 \vdash x = 1 * x = x$. Thus, (N_3) holds.

Therefore, $(S, \dashv, \vdash, 1)$ is an NA-algebra.

4.4. NA-algebras and left (right) regular bands

Recall some notions for the idempotent semigroup.

Definition 4.3. [19,32] Let $(S, \cdot)$ be an idempotent semigroup S. For all $a, b, x, y \in S$,

(1) if $a \cdot b \cdot a = a \cdot b$ holds, then $(S, \cdot)$ is a left regular band;

(2) if $a \cdot b \cdot a = b \cdot a$ holds, then $(S, \cdot)$ is a right regular band.

Below, we show that operations of an idempotent NA-algebra $(S, *, \triangle, 1)$ with a left (respectively, right) regular band $(S, *)$ (respectively, $(S, \triangle)$) coincide.

Proposition 4.4. Operations of an NA-algebra $(S, *, \triangle, 1)$ with an idempotent operation $\triangle$ (respectively, $*$) coincide if $(S, *)$ (respectively, $(S, \triangle)$) is a left (respectively, right) regular band.

Proof: For all $x, y \in S$, according to the idempotent property of operations, the properties of an NA-algebra, and the identity (1) in Definition 4.3, we obtain

$x \vartriangle y = (x \vartriangle y) * (x \vartriangle y) = (y * x) * (y * x) = y * ((x * y) * x) = (x * y) * (y * x) = x * (y * (y * x)) = x * ((y * y) * x) = x * y * x \overset{(1)}{=} x * y$.

For all $x, y \in S$, according to the idempotent property of operations, the properties of an NA-algebra, and the identity (2) in Definition 4.3, we have

$x*y = (x*y)\vartriangle(x*y) = (y\vartriangle x)\vartriangle(y\vartriangle x) = ((y\vartriangle x)\vartriangle y)\vartriangle x = (y\vartriangle x\vartriangle y)\vartriangle x \overset{(2)}{=} (x\vartriangle y)\vartriangle x \overset{(N_6)}{=} (x\vartriangle x)\vartriangle y = x\vartriangle y$.

From Proposition 4.4 it follows that an idempotent NA-algebra $(S, *, \vartriangle, 1)$ with left (respectively, right) regular bands $(S, \vartriangle)$ and $(S, *)$ is a left (respectively, right) regular band.

If $(S, \vartriangle)$ is a semigroup, then S with an operation $*$, defined by $x * y = y \vartriangle x$ for all $x, y \in S$, is a semigroup. The semigroup $(S, *)$ is called dual to the semigroup $(S, \vartriangle)$.

Let $(S, \vartriangle, 1)$ be an arbitrary semigroup with nullary operation "1" and $(S, *, 1)$ be a dual semigroup to $(S, \vartriangle, 1)$.

The following theorem gives necessary and sufficient conditions under which an arbitrary semigroup with nullary operation "1" is an NA-algebra.

Theorem 4.5. An algebra $(S, *, \vartriangle, 1)$ is an NA-algebra if and only if $(S, \vartriangle)$ is a right commutative semigroup with right unit element "1".

Proof: Let $(S, *, \vartriangle, 1)$ be an NA-algebra and $(S, *, 1)$ be dual to $(S, \vartriangle, 1)$. According to the axiom of the NA-algebra and the property of operation $\vartriangle$, for all $x, y, z \in S$, we obtain

$$x \vartriangle y \vartriangle z = (x \vartriangle y) \vartriangle z \overset{(N_6)}{=} (x \vartriangle z) \vartriangle y.$$

Hence, $(S, \vartriangle, 1)$ is a right commutative semigroup.

Conversely, let $(S, \vartriangle, 1)$ be a right commutative semigroup with right unit element "1" and $x, y, z \in S$. Then $(S, \vartriangle, 1)$ is a right commutative semigroup and we obtain

$$x * (y \vartriangle z) = (y \vartriangle z) \vartriangle x = (y \vartriangle x) \vartriangle z = z * (y \vartriangle x) = z * (x * y).$$

So, (N_1) holds.

According to the property of operation $*$ and $(S, \vartriangle, 1)$ being a right commutative semigroup, we have

$$(x \vartriangle y) * z = z \vartriangle (x \vartriangle y) = y * (x * z).$$

Hence, (N_2) holds.

Since "1" is the right identity element of the right commutative semigroup $(S, \vartriangle, 1)$, then we have $1 * x = x = x \vartriangle 1$. So, (N_3) holds.

Hence, $(S, *, \vartriangle, 1)$ is an NA-algebra.

4.5. NA-algebras and g-dimonoids

In 2014, the concept of a g-dimonoid (a generalized dimonoid) was proposed by Y. Movsisyan [33]. A g-dimonoid, which is isomorphic to the free g-dimonoid, was given. A free n-nilpotent g-dimonoid was constructed by Y.V. Zhuchok [14].

Definition 4.4. [33] An algebra $(B; \dashv, \vdash)$ is called a g-dimonoid if it satisfies the following identities:

$(B_1)(x \dashv y) \dashv z = x \dashv (y \dashv z)$,

$(B_2)(x \dashv y) \dashv z = x \dashv (y \vdash z)$,

$(B_3)(x \dashv y) \vdash z = x \vdash (y \vdash z)$,

$(B_4)(x \vdash y) \vdash z = x \vdash (y \vdash z)$.

A g-dimonoid $(B; \dashv, \vdash)$ is called a dimonoid if it satisfies the following additional identity:

$$(x \vdash y) \dashv z = x \vdash (y \dashv z).$$

Proposition 4.6. An NA-algebra $(S, *, \triangle, 1)$ is a g-dimonoid. Conversely, it is a g-dimonoid $(S, \vdash, \dashv)$ if there is an element "1" satisfying $1 \vdash x = x = x \dashv 1$. Then $(S, \vdash, \dashv, 1)$ is an NA-algebra.
Proof: Let $(S, *, \triangle, 1)$ be an NA-algebra. We replace the operation $\vdash$ with $*$ and the operation $\dashv$ with $\triangle$ from Theorem 2.1, then
$(x \dashv y) \dashv z = (x \triangle y) \triangle z \overset{(N_6)}{=} x \dashv (y \dashv z) = x \dashv (y \dashv z)$, hence, (B_1) holds.

Analogously, we have
$(x \dashv y) \dashv z = (x \triangle y) \triangle z \overset{(N_6)}{=} (x \triangle z) \triangle y \overset{(N_6)}{=} x \triangle (z \triangle y) \overset{(N_6)}{=} x \triangle (y * z) = x \dashv (y \vdash z)$, so (B_2) holds.
$(x \dashv y) \vdash z = (x \triangle y) * z \overset{(N_6)}{=} y * (x * z) \overset{(N_5)}{=} x * (y * z) = x \vdash (y \vdash z)$, hence (B_3).
$(x \vdash y) \vdash z = (x * y) * z = x * (y * z) = x \vdash (y \vdash z)$, so (B_4) holds.

Therefore, the proposition is valid.

Conversely, the other part is also easy to prove.

From Remark 3.2, we know that every NA-algebra is a dimonoid. From Proposition 4.6, we obtain that every NA-algebra is a g-dimonoid. [33] gives an example of a g-dimonoid, which is not a dimonoid. Therefore, the NA-algebra is an intersection of a g-dimonoid and dimonoid.

4.6. NA-algebras and restrictive bisemigroups

Restrictive bisemigroups have applications in the theory of binary relations [12, 31, 34]. In this section, we characterize connections between NA-algebras and restrictive bisemigroups.
Definition 4.5. [12] Let B be an arbitrary nonempty set and $\dashv, \vdash$ be binary operations on B. An ordered triplet $(B, \dashv, \vdash)$ is called a restrictive bisemigroup if the axioms

$(I)(x \dashv y) \dashv z = x \dashv (y \dashv z)$,

$(II)(x \vdash y) \vdash z = x \vdash (y \vdash z)$,

$(III)x \dashv y \dashv z = y \dashv x \dashv z$,

$(IV)x \vdash y \vdash z = x \vdash z \vdash y$,

$(V)(x \dashv y) \vdash z = x \dashv (y \vdash z)$,

$(VI)x \dashv x = x$,

$(VII)x \vdash x = x$

hold for all $x, y, z \in B$.
Proposition 4.7. Let $(S, *, \triangle, 1)$ be an NA-algebra, and if for any $x \in S$, the equality

$$x * 1 = 1$$

holds, then $(S, *, \triangle, 1)$ is a restrictive bisemigroup. Conversely, if a restrictive bisemigroup $(S, \dashv, \vdash)$ adds a constant "1" satisfying the condition

$$1 \dashv x = x, x \dashv y = y \vdash x, \tag{4.7}$$

then $(S, \dashv, \vdash, 1)$ is an idempotent NA-algebra.

Proof: Suppose that $(S, *, \triangle, 1)$ is an NA-algebra. Replace the operation $\dashv$ with $*$ and the operation $\vdash$ with $\triangle$ from Theorem 2.1. We have that equalities $(I), (II), (III), (IV), (V)$ hold. From Theorem 2.3(2), we get that $(VI), (VII)$ hold. If $x * 1 = 1$, then $(S, *, \triangle, 1)$ is idempotent. Thus, an idempotent NA-algebra is a restrictive bisemigroup.

Conversely, let $(S, \dashv, \vdash)$ be a restrictive bisemigroup. Putting $\dashv = *$, $\vdash = \triangle$, we obtain

$x * (y \triangle z) = x \dashv (y \vdash z) \overset{(V)}{=} (x \dashv y) \vdash z \overset{(4.7)}{=} z \dashv (x \dashv y) = z * (x * y)$, so (N_1) holds.

$(x \triangle y) * z = (x \vdash y) \dashv z = (y \dashv x) \dashv z \overset{(I)}{=} y \dashv (x \dashv z) = y * (x * z)$, so (N_2) holds.

Using the condition $1 \dashv x = x$, we have $1 * x = 1 \dashv x = x$, i.e., (N_3) holds.

Hence, $(S, \dashv, \vdash, 1)$ is an idempotent NA-algebra.

4.7. NA-algebras and digroups

Digroups play an important role in an open problem in the theory of Leibniz algebras. An idea of the notion of a digroup generalizes groups and has close relationships with the dimonoids, trioids, and Leibniz algebras [13, 31, 35, 36].

Definition 4.6. [31] A digroup is a pair $(G, 1)$ equipped with two binary operations $\vdash$ and $\dashv$, a unary operation †, and a nullary operation "1", where "1" is called the unit of G, satisfying each of the following six axioms:

$(G_1)(G, \vdash)$ and $(G, \dashv)$ are both semigroups,

$(G_2) a \dashv (b \vdash c) = a \dashv (b \dashv c)$,

$(G_3)(a \dashv b) \vdash c = (a \vdash b) \vdash c$,

$(G_4) a \vdash (b \dashv c) = (a \vdash b) \dashv c$,

$(G_5) 1 \vdash a = a = a \dashv 1$,

$(G_6) a \vdash a^{\dagger} = 1 = a^{\dagger} \dashv a$.

Later, J.D.Phillips gave a simple basis of independent axioms for the variety of digroups ([13], Theorem 2).

Definition 4.7. [13] A nonempty set G equipped with two binary operations $\vdash$ and $\dashv$, a unary operation $^{-1}$, and a nullary operation "1", is called a digroup if the following conditions hold:

$(\overline{G}_1)(G, \vdash)$ and $(G, \dashv)$ are semigroups,

$(\overline{G}_2) x \vdash (x \dashv z) = (x \vdash x) \dashv z$,

$(\overline{G}_3) 1 \vdash x = x = x \dashv 1$,

$(\overline{G}_4) x \vdash x^{-1} = 1 = x^{-1} \dashv x$.

Element "1" is called a bar-unit of the digroup, and x^{-1} is said to be inverse to x with respect to "1". Other than in groups, a digroup can have many bar-units. If binary operations of a digroup coincide, the digroup becomes a group. Thus, digroups are generalizations of groups [3, 27, 37].

Example 4. Let $S = \{1, a, b\}$. Define the operations $*$ and $\triangle$ on S by the following Tables 5 and 6:

Table 5. $*$ table.

$*$	1	a	b
1	1	a	b
a	1	a	b
b	b	b	b

Table 6. $\triangle$ table.

$\triangle$	1	a	b
1	1	1	b
a	a	a	b
b	b	b	b

It is easy to verify that $(S,*,\triangle,1)$ is an NA-algebra and $(S,*,\triangle,1)$ is a digroup. Here, the digroup has two identities "1" and a such that $a*x=x=x\triangle a$ and $1*x=x=x\triangle 1$ for all $x\in S$.

Definition 4.8. [38] A digroup is a quadruple $(D,\vdash,\dashv,e)$, where D is a set $\vdash$ and $\dashv$ are two binary operations, which are associative and satisfy the compatibility conditions:

$$x\dashv(y\dashv z)=x\dashv(y\vdash z)$$

$$(x\vdash y)\vdash z=(x\dashv y)\vdash z$$

$$(x\vdash y)\dashv z=x\vdash(y\dashv z),$$

and the distinguished element $e\in D$ is such that $\forall x\in D, e\vdash x=x\dashv e=x$ (these two equalities define a so-called bar-unit). Moreover, for each $x\in D$, there exists a unique element x^{-1} such that $x\vdash x^{-1}=x^{-1}\dashv x=e$; x^{-1} is called the inverse of x.

Proposition 4.8. Let $(S,*,\triangle,1)$ be an NA-algebra, and if for any $x\in S$, there exists an element $y\in S$ such that

$$x*y=1=y\triangle x,$$

then $(S,*,\triangle,1)$ is a digroup. The element "1" is called a bar-unit of the digroup $(S,*,\triangle)$, and y is said to be inverse to x with respect to "1", i.e., $y=x^{-1}$.

Proof: Suppose that $(S,*,\triangle,1)$ is an NA-algebra. From Theorem 2.1, we put $\vdash=*$, $\dashv=\triangle$, then
$x\dashv(y\dashv z)=x\triangle(y\triangle z)\overset{(N_6)}{=}(x\triangle y)\triangle z\overset{(N_6)}{=}(x\triangle z)\triangle y=x\triangle(z\vdash y)\overset{(N_3)}{=}x\triangle(y*z)=x\dashv(y\vdash z)$,
$(x\vdash y)\vdash z=(x*y)*z\overset{(N_5)}{=}x*(y*z)\overset{(N_5)}{=}y*(x*z)\overset{(N_5)}{=}(y*x)*z\overset{(N_3)}{=}(x\triangle y)*z=(x\dashv y)\vdash z$,
$(x\vdash y)\dashv z=(x*y)\triangle z\overset{(N_3)}{=}z*(x*y)\overset{(N_1)}{=}x*(y\triangle z)=x\vdash(y\dashv z)$.

From Definitions 4.6–4.8 and Definition 2.1, we know that $1*x=x\triangle 1=x$ for all $x\in S$, and for each $x\in S$, there exists a unique element x^{-1} such that $x\vdash x^{-1}=x^{-1}\dashv x=1$; x^{-1} is the inverse of x.

In fact, for each $x\in S$ such that $x*x=1$, x is its own inverse. Therefore, NA-algebra $(S,*,\triangle,1)$ is a digroup.

From the above analysis, we summarize the inclusion and correspondence relations between NA-algebras and related structures, namely dimonoids, doppelsemigroups, restrictive bisemigroups, and digroups as follows:

(1) NA-algebras and dimonoids. Every NA-algebra is an abelian dimonoid. Precisely, we may identify the binary operations $*$ and $\triangle$ of an NA-algebra with the operations $*$ and $\triangle$ of a dimonoid, and the element "1" serves as the bar-unit of the NA-algebra.

(2) NA-algebras and doppelsemigroups. If the operation $*$ of an NA-algebra is commutative, then the NA-algebra becomes a strong doppelsemigroup.

(3) NA-algebras and restrictive bisemigroups. For an NA-algebra satisfying $x*1=1$ for all x, this structure is a restrictive bisemigroup. Such NA-algebras include idempotent NA-algebras as a special

case.
(4) NA-algebras and digroups. An NA-algebra forms a digroup provided that for every element x, there exists some element y satisfying $x * y = 1 = y \vartriangle x$. We derive the explicit sufficient conditions for an NA-algebra to admit the structure of a restrictive bisemigroup, digroup, or doppelsemigroup. Moreover, all NA-algebras constitute a proper subclass of dimonoids.

In the diagram (Figure 1), we give the relationships between the NA-algebra and other algebraic structures. By $A \to B(A \overset{(a)}{\to} B)$, we mean that A implies B, respectively. (A implies B with the condition "a").

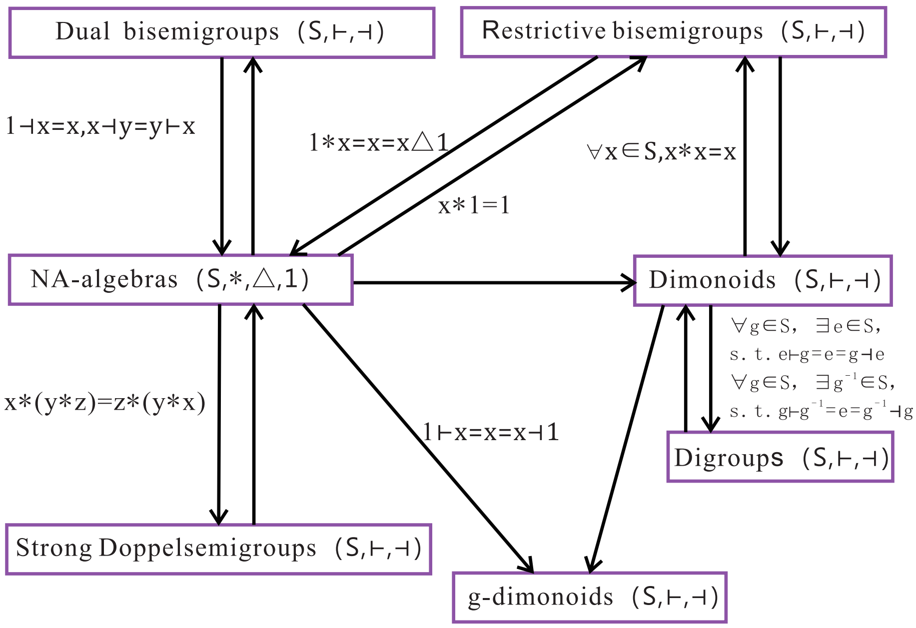

Figure 1. The relationships between NA-algebras and other algebraic structures.

5. Conclusions

Although preliminary results concerning NA-algebras have been obtained, many fundamental algebraic problems remain to be further explored. The intimate structural relationship between dimonoids and NA-algebras, together with mature dimonoid-based research methods, provides a new perspective for the in-depth study of NA-algebras. In particular, the existing classification techniques for finite dimonoids can be naturally extended to NA-algebras, which offers a promising direction for subsequent investigations.

Dimonoids serve not only as a significant algebraic object themselves, but also as an essential bridge connecting classical semigroup theory and modern non-associative algebra. For NA-algebra research, dimonoids provide three major supports. First, they offer structural templates for characterizing the interaction rules between the two binary operations of NA-algebras. Second, they supply effective methodological tools, including regularity judgment, classification problem, and other structural analysis techniques. Third, they provide feasible theoretical extensions, motivating further explorations of fuzzy logic implication operators, potential applications of idempotent NA-algebras in knot theory, and the free construction theory of NA-algebras.

With the increasing intersection of algebra and computer science, the theoretical results of NA-algebras are expected to support broader applications in emerging fields such as artificial intelligence reasoning and cryptographic structure design.

Author contributions

Fang-An Deng: Conceptualization, Methodology, Investigation, Formal analysis, Writing–original draft, Writing–review and editing; Qiao Xin: Supervision, Writing–review and editing; Qian Yang: Validation; Suixin He: Visualization; Yichuan Yang: Supervision, Formal analysis. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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