Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(8): 24913–24927.
DOI:10.3934/math.20261001
Received: 26 March 2026
Revised: 21 July 2026
Accepted: 29 July 2026
Published: 13 August 2026

Research article

General coefficient bounds of q-close-to convex functions associated with a vertical strip domain

Tuğba Yavuz*

Department of Mathematics, Istanbul Beykent University, Faculty of Arts and Sciences, Istanbul 34500, Türkiye

* **Correspondence:** Email: tugbayavuz@beykent.edu.tr.

Abstract: In the present study, we define a new subclass of q-close-to-convex functions associated with a vertical strip domain within the framework of q- calculus. For functions belonging to this class, sharp upper bounds for the initial Taylor Maclaurin coefficients are established. The analysis is carried out by employing the principle of subordination together with properties of functions with a positive real part. Several coefficient inequalities are obtained, thereby extending and refining the known results for classical close-to-convex functions and their q-analogues.

Keywords: analytic functions; q-calculus; close-to-convex functions; coefficient bounds; principle of subordination
Mathematics Subject Classification: 30C45, 05A30

1. Introduction

Let D be the unit disc $\{z : |z| < 1\}$, and let A be the class of analytic functions in D, satisfying the conditions

$$f(0) = 0 \text{ and } f'(0) = 1.$$

Then, each $f \in A$ has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Moreover, if $f \in A$ is univalent, we say that f belongs to class S. Furthermore, $S^*(\alpha)$ denotes the class of starlike functions of order α, defined by analytic condition

$$S^*(\alpha) = \left\{ f(z) \in S : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in D \right\}.$$

Similarly, $S^c(\alpha)$ denotes the class of convex functions of order α with analytic condition

$$S^c(\alpha) = \left\{ f(z) \in S : \Re\left(1 + \frac{zf''(z)}{f(z)}\right) > \alpha,\ z \in D \right\}.$$

It should be noted that $S^*(0) = S^*$ and $S^c(0) = C$, where S^* and C represent the classes of classical starlike and classical convex functions, respectively.

The class $\mathcal{P}$ consists of all functions p analytic in D and having positive real part with $p(0) = 1$. The class $\mathcal{P}$ is also called "Carathedory Class". Carathedory Class satify a one-to-one relationship between $S^*(\alpha)$ and $S^c(\alpha)$. In particular, the Alexander's relation

$$\text{If } f(z) \in S^*(\alpha), \text{ then } zf'(z) \in S^c(\alpha)$$

is satisfied. The class of close-to-convex functions includes both starlike and convex functions. Close-to-convex functions were introduced and studied by Kaplan [8]. A function $f \in A$ belongs to the class of close-to-convex functions, denoted by K, if and only if there exist a function $g \in S^*$ and a real number $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that

$$\Re\left(e^{i\theta}\frac{zf'(z)}{g(z)}\right) > 0 \text{ for } z \in D.$$

It is clear that $S^c(\alpha) \subset S^*(\alpha) \subset K$. In recent years, numerous studies on close-to-convex functions have appeared in the literature ([21, 22]). For, two functions $f, g \in A$, we say that the function f is subordinate to g in D, denoted by $f(z) \prec g(z)\ (z \in D)$, if there exists a Schwarz function w, that is analytic in D with $w(0) = 0$ and $|w(z)| < 1\ (z \in D)$ such that $f(z) = g(w(z))$. Furthermore, it is well known that

$$f(z) \prec g(z)\,(z \in D) \Longrightarrow f(0) = g(0) \text{ and } f(D) \subset g(D). \tag{1.2}$$

In particular, if the function g is univalent in D, then we have the following equivalence:

$$f(z) \prec g(z)\,(z \in D) \Longleftrightarrow f(0) = g(0) \text{ and } f(D) \subset g(D). \tag{1.3}$$

Quantum calculus can be regarded as a generalization of classical calculus that bypasses the conventional notion of limits. It primarily encompasses q-calculus and h-calculus, where the parameter h is typically associated with Planck's constant and q represents the quantum parameter.

Over the past few decades, q-calculus has attracted considerable attention from researchers due to its diverse applications across various branches of mathematics and physics. The systematic development of this field, particularly the formal definitions of the q-derivative and q-integral, was initiated by Jackson [12, 13]. Subsequently, the geometric interpretation of q-analysis has been extensively explored through studies on quantum groups, revealing profound connections between q-analysis and integrable systems.

Numerous applications of q-calculus have been documented in various mathematical disciplines, including numerical analysis, fractional calculus, the theory of special polynomials and analytic number theory. Recently, researchers in geometric function theory (GFT) have utilized the framework of q-calculus to introduce and investigate novel subclasses of analytic functions. One of the pioneering contributions in this direction was made by Ismail et al. [7], who introduced a generalized class of starlike functions, commonly referred to as q-starlike functions. A broad spectrum of research has focused applying q-calculus to various families of analytic functions. For instance, coefficient bounds and estimates for the second Hankel determinant associated with certain subclasses of q-starlike functions were investigated by Çağlar et al. [5], Seoudy et al. [16] and Srivastava et al. [17]. Moreover, Al-Shbeil et al. [3], Srivastava et al. [18] and Srivastava and El-Deeb [19] introduced several classes of analytic and bi-univalent functions defined via q-calculus framework and derived corresponding coefficient estimates. Furthermore, Ali et al. [2] defined a noteworthy subclass of analytic and multivalent functions by employing q-derivative operator and investigated various geometric properties such as coefficient bounds, distortion inequalities, convex linear combinations, as well as the radii of starlikeness, convexity, and close-to-convexity. In their recent work, Shah et al. [9] derived upper bounds for the coefficient estimates, the Hankel determinant, and the Fekete–Szegő functional associated with the class of p-valent and q-starlike functions. Furthermore, Khan et al. [10] investigated the second Hankel determinant for a subclass of q-bounded turning functions associated with the q-Rogers–Ramanujan polynomial, introducing this new function class and establishing the corresponding results.

Now, we recall the basic definitions of q-calculus ([12, 13]).

For $0 < q < 1$, the q-number and the q-factorial of n are defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & n \in \mathbb{C} \\ 1 + q + q^2 + ... + q^{n-1}, & n \in \mathbb{N} \end{cases}$$

and

$$[n]_q! = \begin{cases} 1, & n = 0 \\ \prod\limits_{r=1}^{n} [r]_q\,, & n \in \mathbb{N}, \end{cases}$$

respectively. As $q \to 1^-$, then $[n]_q \to n$ and $[n]_q! \to n!$. For a function f defined on a subset of $\mathbb{C}$, Jackson's $q-$derivative $\partial_q f$ is defined by

$$\partial_q f = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$$

provided that $f'(0)$ exists. Jackson [13] introduced the $q-$integral defined by

$$\int_0^z f(t)d_q t = z(1-q)\sum_{k=0}^{\infty} q^k f\left(zq^k\right),$$

as long as the series converges. If the function $f \in A$ is of the form (1.1), then we obtain

$$\partial_q f(z) = 1 + \sum_{n=1}^{\infty} [n]_q\, a_n z^{n-1} \ \ (z \in \mathcal{D})\,. \tag{1.4}$$

For a function $f \in A$ given by (1.4) and (1.1), we derive that

$$\partial_q^{(1)} f(z) \ \ = \ \ 1 + \sum_{n=1}^{\infty} [n]_q\, a_n z^{n-1} =: \partial_q f(z),$$

$$\begin{aligned}\partial_q^{(2)} f(z) &= 1+\sum_{n=2}^{\infty}[n]_q\,[n-1]_q\,a_n z^{n-2},\\ &\vdots\\ \partial_q^{(m)} f(z) &= 1+\sum_{n=j}^{\infty}\frac{[n]_q!}{[n-j]_q!}a_n z^{n-m},\end{aligned}$$

where $\partial_q^{(m)} f(z)$ is the m^{th} q-derivative of $f(z)$.

2. Preliminaries

This paper introduces a broad class of starlike functions associated with a vertical strip domain. Building upon this class, a new subclass of close-to-convex functions is defined. Moreover, sharp bounds for the Taylor coefficients of functions in both classes are established.

First, we introduce a new subclass of starlike functions through the following definition. The class of analytic functions defined below may be regarded as a generalization of the class of starlike functions. Consequently, this class is nonempty.

Definition 1. *Let γ be a non-zero complex number, $0 \le \alpha < 1 < \beta$ and let f be a univalent function of the form* (1.1). *We say that f belongs to $S_q^*(\alpha,\beta,\gamma)$ if it satisfies the following inequality*

$$\alpha < \Re\left\{1+\frac{1}{\gamma}\left(\frac{z\partial_q f(z)}{f(z)}-1\right)\right\} < \beta, \tag{2.1}$$

where $z \in D$.

Lemma 1. *Let $f \in A$ be defined by* (1.1) , $0 \le \alpha < 1 < \beta$ *and* $0 < q < 1$. *Then*

$$f \in S_q^*(\alpha,\beta,\gamma) \Longleftrightarrow \frac{1}{\gamma}\left(\frac{z\partial_q f(z)}{f(z)}-1\right) \prec \frac{\beta-\alpha}{\pi}i\log\left(\frac{1-qe^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-qz}\right),\ z \in D. \tag{2.2}$$

Proof. Let us consider the function $f_{q,\alpha,\beta}(z)$ defined by

$$f_{q,\alpha,\beta}(z) = 1+\frac{\beta-\alpha}{\pi}i\log\left(\frac{1-qe^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-qz}\right) \tag{2.3}$$

with $\alpha < 1$ and $\beta > 1$. Then, it is not difficult to see that the function $f_{q,\alpha,\beta}(z)$ is analytic and univalent in D with $f_{q,\alpha,\beta}(0) = 1$. Additionally, noting that

$$1+\frac{\beta-\alpha}{\pi}i\log\left(\frac{1-qe^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-qz}\right) = \frac{\alpha+\beta}{2}+\frac{\beta-\alpha}{2}i\log\left(\frac{ie^{-i\pi\frac{1-\alpha}{1-\beta}z}-ie^{i\pi\frac{1-\alpha}{1-\beta}z}}{1-z}\right), \tag{2.4}$$

simple calculations give us that $f_{q,\alpha,\beta}(z)$ maps the unit disk D onto the strip domain $\Omega_{\alpha,\beta}$, where

$$\Omega_{\alpha,\beta} = \{w \in \mathbb{C} : \alpha < \Re(w) < \beta\}. \tag{2.5}$$

Thus it follows from the fact given by (1.3) that the subordination (2.2) is equivalent to the inequality (2.1) in Definition 1. $\square$

Let rearrange the function $f_{q,\alpha,\beta}(z)$ as follows:

$$\begin{aligned} f_{q,\alpha,\beta}(z) &= 1+\frac{\beta-\alpha}{\pi} i \log\left(\frac{1-qe^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-qz}\right) \\ &= 1+\frac{\beta-\alpha}{\pi} i\left\{\log\left(1-qe^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z\right)-\log(1-qz)\right\}. \end{aligned}$$

If we substitute $\kappa = qe^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z$, then we get

$$\log\left(1-qe^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z\right)=\log(1-\kappa)=-\kappa-\sum_{j=2}^{\infty}\frac{\kappa^j}{j}.$$

Thus, we obtain the following series expansions

$$\log\left(1-qe^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z\right)=-qe^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z-\sum_{n=2}^{\infty}\frac{\left(qe^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z\right)^n}{n} \tag{2.6}$$

and

$$-\log(1-qz)=qz+\sum_{n=2}^{\infty}\frac{(qz)^n}{n}. \tag{2.7}$$

By combining Eqs (2.6) and (2.7), the following series expansion for $f_{q,\alpha,\beta}(z)$ is obtained

$$f_{q,\alpha,\beta}(z)=1+\sum_{n=1}^{\infty}\frac{\beta-\alpha}{n\pi} iq^n\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right)z^n=1+\sum_{n=1}^{\infty}B_nz^n. \tag{2.8}$$

It follows immediately that $f_{q,\alpha,\beta}(z)$ belongs to the class $\mathcal{P}$.

Recently, Sun et al. [20] investigated applications of the vertical strip domain to the class of starlike functions. They established integral representations, convolution properties, and coefficient estimates for functions belonging to this class. In addition, they studied radius problems and inclusion relationships involving several subclasses of starlike functions, including strongly starlike and parabolic starlike functions. In a recent paper, M. Algahani et al. [1] obtained comprehensive results for certain classes of analytic functions associated with the vertical strip. Subsequently, Bulut [6] extended the use of the vertical strip domain to the class of close-to-convex functions.

Using the class of starlike functions defined in the vertical strip introduced in Definition 1, we define the following subclass of close-to-convex functions.

Definition 2. *Let γ be a non-zero complex number,* $0 \le \alpha < 1 < \beta$ *and let f be a univalent function of the form* (1.1). *We say that f belongs to $S_{q,g}(\alpha,\beta,\gamma)$ if it satisfies the following inequality*

$$\alpha < \Re\left\{1+\frac{1}{\gamma}\left(\frac{z\partial_q f(z)}{g(z)}-1\right)\right\}<\beta, \tag{2.9}$$

where $g(z)\in S_q^*(\delta,\beta,\gamma)$ with $0\le\delta<1<\beta$.

Using Definition 2 and the principle of the subordination, we can immediately obtain following Lemma.

Lemma 2. *Let γ be a non-zero complex number and α, β, δ real numbers such that* $0 \le \alpha, \delta < 1 < \beta$. *Assume that the function $f \in A$ is defined by* (1.1) *. Then,*

$$f \in S_{q,g}(\alpha, \beta, \gamma) \Longleftrightarrow 1 + \frac{1}{\gamma}\left(\frac{z\partial_q f(z)}{g(z)} - 1\right) \prec f_{q,\alpha,\beta}(z), \ (z \in D), \tag{2.10}$$

where $f_{q,\alpha,\beta}(z)$ is defined by (2.3) *.*

Proof. The proof of Lemma 2 is similar to the proof of Lemma 1 and is therefore omitted. □

Let consider the function $l_1(z) = \frac{1-(1-2\beta)z}{1+z}$. Since

$$l_1\left(e^{i\theta}\right) = \beta + i(\beta - 1)\frac{\sin\theta}{1+\cos\theta} \text{ and } \beta > 1.$$

If $\theta \to \pi^+$, we obtain $\Im\left(l_1\left(e^{i\theta}\right)\right) \to -\infty$. If $\theta \to \pi^-$, we obtain $\Im\left(l_1\left(e^{i\theta}\right)\right) \to \infty$. Then, we see that $l_1(z)$ maps unit circle onto the vertical line with $\{w : \Re(w) = \beta\}$. From $l_1(0) = 1$, we conclude that $l_1(z)$ maps the unit disk onto the region $\{w : \Re(w) < \beta\}$.

Similarly, for the function $l_2(z) = \frac{1+(1+2\alpha)z}{1-z}$, we obtain

$$l_2\left(e^{i\theta}\right) = \alpha + i(1 - 2\alpha)\frac{\sin\theta}{1-\cos\theta} \text{ and } \alpha < 1.$$

For this reason, $l_2(z)$ maps unit circle on the vertical line with $\{w : \Re(w) = \alpha\}$. Since $l_2(0) = 1$, we can infer that $l_2(z)$ maps the unit disk onto the region $\{w : \Re(w) > \alpha\}$.

In view of above discussion, we can establish the following subordination inequalities

$$1 + \frac{1}{\gamma}\left(\frac{z\partial_q f(z)}{g(z)} - 1\right) \prec \frac{1+(1+2\alpha)z}{1-z}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{z\partial_q f(z)}{g(z)} - 1\right) \prec \frac{1-(1-2\beta)z}{1+z},$$

where $0 \le \alpha, \delta < 1 < \beta$, $f \in S_{q,g}(\alpha, \beta, \gamma)$ and $g \in S_q(\alpha, \beta, \gamma)$.

Remark 1. *The function class introduced in Definition 2 contains several previously studied subclasses as special cases. Consequently, it serves as a unified and more general framework for the investigation of these function classes and their geometric properties.*

(i) If we let $q \to 1^-$, then we have the class of close-to-convex functions with complex order γ on vertical strip domain $S^*(\alpha, \beta, \gamma)$ which consists of functions satisfying

$$\alpha < \Re\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{g(z)} - 1\right)\right\} < \beta, z \in D,$$

where $g \in S(\delta, \beta)$.

(ii) If we let $q \to 1^-$and $\gamma = 1$, we obtain the class of close-to-convex functions on vertical strip domain $\Omega_{\alpha,\beta}$ which consists of functions satisfying

$$\alpha < \Re\left\{\frac{zf'(z)}{g(z)}\right\} < \beta, \ z \in D,$$

where $g \in S(\delta,\beta)$. *This class is introduced by Bulut* [4].

(iii) If we let $\beta \to \infty$, *then the class* $S_{q,g}(\alpha,\beta,\gamma)$ *reduces to the class* $C_q(\delta,\alpha,\gamma)$ *of* $q-$*close-to-convex functions of complex order* γ *and type* α $(0 \leq \alpha < 1)$ *which consists of functions satisfying*

$$\Re\left\{1+\frac{1}{\gamma}\left(\frac{z\partial_q f(z)}{g(z)}-1\right)\right\} > \alpha,$$

where $g \in S_q^*(\delta)$ $(0 \leq \delta < 1)$.

(iv) If we let $\beta \to \infty$ *and* $\gamma = 1$, *then the class* $S_{q,g}(\alpha,\beta,\gamma)$ *reduces to the class* $C_q(\delta,\alpha)$ *of* $q-$*close-to-convex functions of type* α $(0 \leq \alpha < 1)$ *which consists of functions satisfying*

$$\Re\left\{\frac{z\partial_q f(z)}{g(z)}\right\} > \alpha,$$

where $g \in S_q^*(\delta)$ $(0 \leq \delta < 1)$ *introduced by Bulut [6]* .

Another objective of this paper is to establish coefficient inequalities for the class of functions $\mathcal{B}_{q,g}(\alpha,\beta,\gamma;\mu,m)$ defined via the following nonhomogeneous Cauchy-Euler fractional q-differential equation. Accordingly, we introduce this subclass by following definition.

Definition 3. *Let* γ *be a non-zero complex number and let* f *be a univalent function of the form* (1.1) . *We say that* f *belongs to* $\mathcal{B}_{q,g}(\alpha,\beta,\gamma;\mu,m)$ *if it satisfies the following Cauchy Euler fractional* $q-$*differential equation*

$$z^m\partial_q^{(m)}f(z)+\binom{m}{1}(\mu+m-1)\,z^{m-1}\partial_q^{(m-1)}f(z)+\ldots+\binom{m}{m}f(z)\prod_{j=0}^{m-1}(\mu+j)$$
$$=h(z)\prod_{j=0}^{m-1}(\mu+j+1)\,, \quad (2.11)$$

where $\mu \in \mathbb{R}\backslash(-\infty,-1]$, $m \in \mathbb{N}\backslash\{1\}$ *and* $h(z) \in S_{q,g}(\alpha,\beta,\gamma)$.

3. Coefficient estimates for the functions in the class $S_q^*(\alpha,\beta,\gamma)$

The following lemma will be used in the proofs of our main results.

Lemma 3. *[15] Let the function* $g(z)$ *given by*

$$g(z)=\sum_{n=1}^{\infty}d_nz^n,\ z \in D,$$

be a convex function in D. *Also, let the the function*

$$f(z)=\sum_{n=1}^{\infty}c_nz^n,\ z \in D,$$

be analytic in D. *If* $f(z) \prec g(z)$ *then,* $|c_n| \leq |d_1|$, $n \in \mathbb{N}$.

The following result can be established by applying Lemmas 2 and 3.

Theorem 1. *Let* α *and* β *be real numbers and* γ *non zero complex number. Let the function* $f \in A$ *defined by* (1.1) *. If* $f \in S_q^*(\alpha,\beta,\gamma)$ *then*

$$|a_n| \leq |\gamma| \prod_{k=2}^{n} \frac{\left(\left|q\,[k-2]_q + \gamma - 1\right| + \frac{2|\gamma|(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}\right)}{\left|q\,[k-1]_q + \gamma - 1\right|}, \; n = 2, 3, ... \tag{3.1}$$

Proof. Let the function $f \in S_q^*(\alpha,\beta,\gamma)$ be of the form (1.1) . Let us define $p(z)$ by

$$p(z) = 1+\frac{1}{\gamma}\left(\frac{z\partial_q f(z)}{f(z)} - 1\right) \;\; (z \in D)\,. \tag{3.2}$$

Then, according to the assertion of Lemma 1, we get

$$p(z) \prec f_{q,\alpha,\beta}(z)$$

where $f_{q,\alpha,\beta}(z)$ is defined by (2.3) . Hence, using Lemma 3, we obtain

$$\left|\frac{p^{(m)}(0)}{m!}\right| = |c_m| \leq |B_1|$$

where $p(z) = 1 + c_1 z + c_2 z^2....,\ (z \in D)$ and

$$|B_1| = \left|\frac{\beta-\alpha}{n\pi} qi\left(1 - e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right)\right| = 2q\frac{(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}. \tag{3.3}$$

Also from (3.2) , we find

$$z\partial_q f(z) = (1-\gamma)\, f(z) + \gamma p(z) f(z). \tag{3.4}$$

Since $a_1 = 1$, in view of (3.4) , we obtain

$$\left([n]_q - 2 + \gamma\right) a_n = \gamma \sum_{j=1}^{n-1} c_j a_{n-j}. \tag{3.5}$$

Applying (3.2) into (3.5), we get

$$\left|[n]_q - 2 + \gamma\right| |a_n| \leq |\gamma|\, |B_1| \sum_{k=1}^{n-1} |a_{n-k}| = |\gamma|\, |B_1| \sum_{k=1}^{n-1} |a_{k-1}| \quad (n = 2, 3, ...)\,. \tag{3.6}$$

For $n = 2, 3, 4$, we have

$$|a_2| \leq \frac{|\gamma|\, |B_1|}{\left|[2]_q - 2 + \gamma\right|}, \tag{3.7}$$

$$|a_3| \leq \frac{|\gamma|\, |B_1|}{\left|[3]_q - 2 + \gamma\right|} \left\{1 + \frac{|\gamma|\, |B_1|}{\left|[2]_q - 2 + \gamma\right|}\right\}, \tag{3.8}$$

$$|a_4| \leq \frac{|\gamma|\,|B_1|\left(|\gamma||B_1| + \left|[2]_q - 2 + \gamma\right|\right)\left(|\gamma||B_1| + \left|[3]_q - 2 + \gamma\right|\right)}{\left|[4]_q - 2 + \gamma\right|\left|[3]_q - 2 + \gamma\right|\left|[2]_q - 2 + \gamma\right|}, \tag{3.9}$$

respectively. We now utilize the mathematical induction for the proof of theorem. From (3.7), it is clear that the assertion of holds true for $n = 2$.

Let us assume that inequality in (3.1) is true for $n = m$. Then, necessarily calculations give us that

$$\begin{aligned}
|a_{m+1}| &\leq \frac{|\gamma|\,|B_1|}{\left|[m+1]_q + \gamma - 2\right|} \sum_{k=2}^{m+1} |a_{k-1}| \\
&\leq \frac{|\gamma|\,|B_1|}{\left|[m+1]_q + \gamma - 2\right|} \left(\sum_{k=2}^{m} |a_{k-1}| + |a_m| \right) \\
&\leq \frac{|\gamma|\,|B_1|}{\left|[m+1]_q + \gamma - 2\right|} \left(1 + \frac{|\gamma|\,|B_1|}{\left|[m]_q + \gamma - 2\right|} \right) \sum_{k=2}^{m} |a_{k-1}| \\
&\leq \frac{|\gamma|\left(\left|[m]_q + \gamma - 2\right| + |\gamma|\,|B_1|\right)}{\left|[m+1]_q + \gamma - 2\right|} \prod_{k=2}^{m} \frac{\left(\left|q\,[k-2]_q + \gamma - 1\right| + |\gamma|\,|B_1|\right)}{\left|q\,[k-1]_q + \gamma - 1\right|}.
\end{aligned}$$

Since, $q\,[k-1]_q = [k]_q - 1$, then we get

$$|a_{m+1}| \leq |\gamma| \prod_{k=2}^{m+1} \frac{\left(\left|q\,[k-2]_q + \gamma - 1\right| + |\gamma|\,|B_1|\right)}{\left|[k]_q + \gamma - 2\right|},$$

which implies (3.1) is true for $n = m + 1$. Hence, we have

$$|a_n| \leq |\gamma| \prod_{k=2}^{n} \frac{\left(\left|q\,[k-2]_q + \gamma - 1\right| + |\gamma|\,|B_1|\right)}{\left|[k]_q + \gamma - 2\right|}, \; n = 2, 3, ...,$$

B_1 is given in (3.3) . This completes the proof of Theorem 1. □

By assigning particular values to the parameters and the fact that

$$[k]_q - 1 = q\,[k-1]_q \;\; (k \geq 2),$$

the following results are derived.

Corollary 1. *If we take $\gamma = 1$, we have following result for functions $S_q^*(\alpha, \beta, 1) = S_q(\alpha, \beta)$*

$$\begin{aligned}
|a_n| &\leq \prod_{k=2}^{n} \frac{\left(q\,[k-2]_q + \frac{2q(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}\right)}{q\,[k-1]_q} \\
&\leq \prod_{k=2}^{n} \frac{\left(q\,[k-2]_q - 1 + \frac{2q(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}\right)}{[k]_q - 1} \\
&= \frac{\prod_{k=2}^{n} \left(q\,[k-2]_q + \frac{2q(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}\right)}{q^{n-1}\,[n-1]_q!}
\end{aligned}$$

$$= \frac{\prod_{k=2}^{n}\left([k-2]_q + \frac{2(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right)}{[n-1]_q!},\ n = 2,3,...$$

This result coincide with the result given by Bulut [6].

Corollary 2. *If we take $\gamma = 1$ and $\beta \to \infty$, we have following result for functions $f \in S_q^*(\alpha)$*

$$|a_n| \leq \frac{\prod_{k=2}^{n}\left([k-2]_q + 2(1-\alpha)\right)}{[n-1]_q!},\ n = 2,3,... \tag{3.10}$$

This result coincide with the result given by Bulut [4].

Corollary 3. *If we let $\gamma = 1$ and $q \to 1^-$ in Theorem 1, we have the same outcome with Kuroki and Owa [11].*

4. Coefficient estimates for the functions in the class $S_{q,g}(\alpha,\beta,\gamma)$

Theorem 2. *Let α,β and δ be real numbers with $0 \leq \alpha, \delta < 1 < \beta$. γ is non zero complex number and $f \in A$ defined by (1.1). If $f \in S_{q,g}(\alpha,\beta,\gamma)$ then*

$$|a_2| \leq \frac{2q|\gamma|}{[2]_q\left|[2]_q + \gamma - 2\right|}\frac{(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta} + \frac{2q|\gamma|}{[2]_q}\frac{(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha} \tag{4.1}$$

and for $n = 3,4,\ldots$

$$\begin{aligned}|a_n| \leq{}& \frac{|\gamma|}{[n]_q}\prod_{k=2}^{n}\frac{\left(\left|q[k-2]_q+\gamma-1\right| + |\gamma|\frac{2(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta}\right)}{\left|q[k-1]_q+\gamma-1\right|}\\ &+\frac{|\gamma|^2}{[n]_q}\frac{2q(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\prod_{k=1}^{n-2}\frac{\left(\left|q[k]_q+\gamma-1\right| + |\gamma|\frac{2(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta}\right)}{\left|q[k+1]_q+\gamma-1\right|},\end{aligned} \tag{4.2}$$

where $g \in S_q^*(\delta,\beta,\gamma)$.

Proof. Let the function $f \in S_{q,g}(\alpha,\beta,\gamma)$ be of the form (1.1). Therefore, there exists a function

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S_q^*(\delta,\beta,\gamma) \tag{4.3}$$

so that

$$\alpha < \Re\left\{1 + \frac{1}{\gamma}\left(\frac{z\partial_q f(z)}{g(z)} - 1\right)\right\} < \beta. \tag{4.4}$$

Note that by Theorem 1, we have

$$|b_n| \leq |\gamma|\prod_{k=2}^{n}\frac{\left(\left|q[k-2]_q+\gamma-1\right| + |\gamma|\frac{2(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta}\right)}{\left|q[k-1]_q+\gamma-1\right|},\ n = 2,3,...$$

Let us define the function

$$\widetilde{p}(z) = 1+\frac{1}{\gamma}\left(\frac{z\partial_q f(z)}{g(z)} - 1\right) \ (z \in \mathcal{D}). \tag{4.5}$$

Then, according to the assertion of Lemma 1, we get

$$\widetilde{p}(z) \prec f_{q,\alpha,\beta}(z), \tag{4.6}$$

where $f_{q,\alpha,\beta}(z)$ is defined by (2.3). Hence, using Lemma 3, we obtain

$$\left|\frac{\widetilde{p}^{(m)}(0)}{m!}\right| = |d_m| \le |B_1|, \tag{4.7}$$

where $\widetilde{p}(z) = 1 + d_1 z + d_2 z^2 ....,$ $(z \in \mathcal{D})$ and

$$|B_1| = \left|\frac{\beta-\alpha}{n\pi} qi\left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}}\right)\right| = 2q\frac{(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}.$$

Also from (4.5) , we find

$$z\partial_q f(z) = \gamma\widetilde{p}(z) f(z) + (1-\gamma)\, g(z)$$

and

$$[n]_q\, a_n - b_n = \gamma\left[d_{n-1} + d_{n-2}b_2 + \ldots + d_1 b_{n-1}\right]. \tag{4.8}$$

Now from (4.7) and (4.9), we get

$$|a_n| \le \frac{|b_n|}{[n]_q} + \frac{|B_1|\,|\gamma|}{[n]_q}\sum_{j=1}^{n-1}\left|b_{n-j}\right|. \tag{4.9}$$

Using the fact that

$$\sum_{j=1}^{n-1}\left|b_{n-j}\right| = 1 + |b_2| + \ldots + |b_{n-1}| \ \le |\gamma|\prod_{k=1}^{n-2}\frac{\left(\left|q\,[k]_q + \gamma - 1\right| + |\gamma|\,\frac{2(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta}\right)}{\left|q\,[k+1]_q + \gamma - 1\right|},$$

the desired inequalities (4.1) and (4.2) are immediately obtained.

□

Letting $\gamma = 1$ in Theorem 2, we have following result.

Corollary 4. *If we take $\gamma = 1$, we have following result for function $f \in S_{q,g}(\alpha,\beta,1) = S_{q,g}(\alpha,\beta)$,*

$$|a_2| \le \frac{2}{[2]_q}\frac{(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta} + \frac{2q}{[2]_q}\frac{(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha} \tag{4.10}$$

and for $n = 3, 4, \ldots$

$$|a_n| \le \frac{1}{[n]_q}\prod_{k=2}^{n}\frac{\left(q\,[k-2]_q + \frac{2q(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta}\right)}{q\,[k-1]_q}$$

$$+\frac{2q}{[n]_q}\frac{(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\prod_{k=1}^{n-2}\frac{q\,[k]_q + \frac{2q(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta}}{q\,[k+1]_q}, \tag{4.11}$$

where $g \in S_q^*(\delta,\beta)$. Simple computations reveal that the derived result coincides with that of Bulut [6].

Let consider the case $\gamma = 1$ and $\beta \to \infty$ in Theorem 2, this leads to the following result.

Corollary 5. *If we take $\gamma = 1$, we have following result for functions $f \in S_{q,g}(\alpha,\beta,1) = S_{q,g}(\alpha,\beta)$*

$$|a_2| \le \frac{2(1-\delta)}{[2]_q} + \frac{2q(1-\alpha)}{[2]_q} \tag{4.12}$$

and for $n = 3, 4, \ldots$

$$|a_n| \le \frac{1}{[n]_q}\prod_{k=2}^{n}\frac{\left(q[k-2]_q + 2q(1-\delta)\right)}{q[k-1]_q} + \frac{2q(1-\alpha)}{[n]_q}\prod_{k=1}^{n-2}\frac{\left(q[k]_q + 2q(1-\delta)\right)}{q[k+1]_q}, \tag{4.13}$$

where $g \in S_q^*(\delta)$. After simple computations, we see that this coefficient estimates coincide with result of Bulut [6].

Corollary 6. *If we let $\gamma = 1$ and $q \to 1^-$ in Theorem 2, we have the same outcome with Kuroki and Owa [11].*

Corollary 7. *If we let $\gamma = 1$ $q \to 1^-$ and $\beta \to 1$ in Theorem 2, we have the following result obtained by Libera [14]*

$$|a_n| \le \frac{2(3-2\delta)(4-2\delta)\ldots(n-2\delta)}{n!}\left[n(1-\alpha) + (\alpha-\delta)\right],\ n = 2, 3, \ldots \tag{4.14}$$

5. Coefficient estimates for the functions in the class $\mathcal{B}_{q,g}(\alpha,\beta,\gamma;\mu,m)$

Theorem 3. *Let α,β and δ be real numbers with $0 \le \alpha,\delta < 1 < \beta$. γ is non zero complex number and $f \in A$ defined by* (1.1). *If $f \in \mathcal{B}_{q,g}(\alpha,\beta,\gamma;\mu,m)$ then*

$$|a_2| \le \frac{\prod_{j=0}^{m-1}(\mu+j+1)}{\prod_{j=0}^{m-1}(m+j+2)}\left\{\frac{2q|\gamma|}{[2]_q\left|[2]_q+\gamma-2\right|}\frac{(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta} + \frac{2q|\gamma|}{[2]_q}\frac{(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right\} \tag{5.1}$$

and for $n = 3, 4, \ldots$

$$|a_n| \le \frac{\prod_{j=0}^{m-1}(\mu+j+1)}{\prod_{j=0}^{m-1}(m+j+n)}\left\{\frac{|\gamma|}{[n]_q}\prod_{k=2}^{n}\frac{\left(\left|q[k-2]_q+\gamma-1\right| + |\gamma|\frac{2(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta}\right)}{\left|q[k-1]_q+\gamma-1\right|} + \frac{|\gamma|^2}{[n]_q}\frac{2q(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\prod_{k=1}^{n-2}\frac{\left(\left|q[k]_q+\gamma-1\right| + |\gamma|\frac{2(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta}\right)}{\left|q[k+1]_q+\gamma-1\right|}\right\}, \tag{5.2}$$

where $g \in S_q^*(\delta,\beta,\gamma)$.

Proof. Since $f(z) \in \mathcal{B}_{q,g}(\alpha,\beta,\gamma;\mu,m)$ and defined by (1.1), then there exists $h(z) \in S_{q,g}(\alpha,\beta,\gamma)$ such that (2.11) holds. Thus, it follows that

$$a_n = \frac{\prod\limits_{j=0}^{m-1}(\mu + j + 1)}{\prod\limits_{j=0}^{m-1}(m + j + n)} b_n, \ \mu \in \mathbb{R} \backslash (-\infty, -1].$$

By using the result of Theorem 2, we immediately obtain the desired inequalities (5.1) and (5.2). □

It is also possible to see that Theorem 3 coincides with the result given by Bulut [6], where our parameters are $\gamma = 1$ and $m = 2$.

6. Conclusions

In this paper, we introduced a new subclass of q-close-to-convex functions associated with a vertical strip domain within the framework of q-calculus. By employing the principle of subordination together with the properties of functions having positive real part, we established sharp estimates for the initial Taylor–Maclaurin coefficients of functions belonging to this class.

The proposed function class provides a unified setting that extends several previously studied subclasses of close-to-convex functions. Consequently, the obtained coefficient estimates generalize and refine a number of existing results available in the literature for both classical close-to-convex functions and their q-analogues.

The approach developed in this work may also be applied to investigate other geometric properties of analytic functions, such as Fekete–Szegő inequalities, Hankel determinants, logarithmic coefficients, and radius problems. This work could also be inspiring for classes formed by functions that are meromorphic or multivalent in the vertical strip. These directions constitute interesting topics for future research.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author would like to express sincere gratitude to the anonymous reviewers for their valuable comments and constructive suggestions, which have significantly improved the quality of this paper.

Conflict of interest

The author declares no conflicts of interest.

References

1. A. M. Alqahtani, R. Murtaza, S. Akmal, A. Abbasi, I. Khan, Generalized q-convex functions characterized by q-calculus, *AIMS Math.*, **8** (2023), 9385–9399 http://dx.doi.org/10.3934/math.2023472
2. E. E. Ali, H. M. Srivastava, A. M. Albalahi, Subclasses of p-valent k-uniformly convex and starlike functions defined by the q-derivative operator, *Mathematics*, **11** (2023), 1–19. https://doi.org/10.3390/math11112578
3. I. Al-Shbeil, A. Cataş, H. M. Srivastava, N. Aloraini, Coefficient estimates of new families of analytic functions associated with q-Hermite polynomials, *Axioms*, **12** (2023), 1–14. https://doi.org/10.3390/axioms12010052
4. S. Bulut, Coefficient bounds for close-to-convex functions associated with vertical strip domain, *Commun. Korean Math. Soc.,* **35** (2020), 789–797. https://doi.org/10.4134/CKMS.c190268
5. M. Çağlar, H. Orhan, H. M. Srivastava, Coefficient bounds for q-starlike functions associated with q-Bernoulli numbers, *J. Appl. Anal. Comput.,* **13** (2023), 2354–2364. http://dx.doi.org/10.11948/20220566
6. S. Bulut, Coefficient bounds for q-close-to-convex functions associated with vertical strip domain, *Filomat,* **38** (2024), 6003–6015. https://doi.org/10.2298/FIL2417003B
7. M. E. H. İsmail, E. Merkes, D. Styer, A generalization of starlike functions, *Complex Var. Theory Appl.,* **14** (1990), 77–84. https://doi.org/10.1080/17476939008814407
8. W. Kaplan, Close-to-convex schlicht functions, *Michigan Math. J.*, **1** (1952), 169–185. https://doi.org/10.1307/mmj/1028988895
9. M. U. Shah, B. Khan, M. Oliveira, S. Araci, Characterization of p-valent q-starlike functions through Hadamard product and Poisson distribution, *Appl. Math. Sci. Eng.*, **33** (2025). https://doi.org/10.1080/27690911.2025.2478040
10. B. Khan, T. G. Shaba, S. Araci, B. O. Adebesin, F. Usta, A q-Rogers–Ramanujan-based characterization of q-bi-bounded turning functions, *Demonstr. Math.*, **59** (2026), 2025–2050. https://doi.org/10.1515/dema-2025-0225
11. K. Kuroki, S. Owa, Notes on new class for certain analytic functions, *RIMS Kokyuroku,* **1772** (2011), 21–25.
12. F. H. Jackson, On q-definite integrals, *Quart. J. Pure Appl. Math.,* **41** (1910), 193–203.
13. F. H. Jackson, On q-functions and a certain difference operator, *Trans. Royal Soc. Edinburgh*, **46** (1908), 253–281. https://doi.org/10.1017/S0080456800002751
14. R. J. Libera, Some radius of convexity problems, *Duke Math. J.,* **31** (1964), 143–158.
15. W. Rogosinksi, On the coefficient of subordinate functions, *London Math. Soc. Ser 2,* **48** (1943), 48–82.
16. T. M. Seoudy, M. K. Aouf, Coefficient estimates of new classes of q-starlike functions of complex order, *J. Math. Inequal.,* **10** (2016), 135–145. http://dx.doi.org/10.7153/jmi-10-11

17. H. M. Srivastava, K. Alshammari, M. Darus, A new q-fractional integral operator and its applications to the coefficient problem involving the second Hankel determinant for q-starlike and q-convex functions, *Nonlinear Var. Anal.,* **7** (2023), 985–994. http://dx.doi.org/10.23952/jnva.7.2023.6.07

18. H. M. Srivastava, I. Al-Shbeil, Q. Xin, F. Tchier, S. Khan, S. N. Malik, Faber polynomial coefficient estimates for bi-close-to-convex functions defined by the q-fractional derivative, *Axioms*, **12** (2023), 1–19. https://doi.org/10.3390/axioms12060585

19. H. M. Srivastava, S. M. El-Deeb, The Faber polynomial expansion method and the Taylor-Maclaurin coefficient estimates of bi-close-toconvex functions connected with the q-convolution, *AIMS Math.*, **5** (2020), 7087–7106. https://doi.org/10.3934/math.2020454

20. Y. Sun, Z. G. Wang, A. Rasila, J. Sokoł, On a subclass of starlike functions associated with a vertical strip domain, *J. Inequal. Appl.,* **35** (2019), 1–14. https://doi.org/10.1186/s13660-019-1988-8

21. W. Ul-Haq, A. Nazneen, N. Rehman, Coefficient estimates for certain subfamilies of close-to-convex functions of complex order, *Filomat*, **28** (2014), 1139–1142. https://doi.org/10.2298/FIL1406139U

22. W. Ul-Haq, A. Nazneen, M. Arif, N. Rehman, Coefficient bounds for certain subclasses of close-to-convex functions of Janowski type, *J. Comput. Anal. Appl.*, **16** (2014), 133–138.