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Research article

Finite-time stability analysis of fractional-order Cohen–Grossberg neural networks with time-varying delays and fuzzy logics

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Abstract: This paper addresses the finite-time stability problem for a class of fractional-order Cohen–Grossberg neural networks characterized by fuzzy logic operators and time-varying delays. The coexistence of variable delays and fuzzy feedback structures poses considerable challenges in deriving effective stability conditions. To tackle these difficulties, a delay-dependent estimation strategy based on a weighted envelope framework is developed to characterize the influence of delayed states. Subsequently, a novel fractional Gronwall-type inequality is established and utilized to derive a sufficient finite-time stability criterion for the investigated neural network model. The obtained criterion explicitly captures the effects of fuzzy min/max operations and time-varying delays. Finally, three simulation examples are provided to verify the applicability and effectiveness of the proposed theoretical results.

Keywords: fractional-order Cohen–Grossberg neural networks; fuzzy logics; finite-time stability; time-varying delays; Gronwall-type inequality
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1. Introduction

Neural networks have received considerable research interest due to their strong ability in information processing, associative memory, pattern recognition, optimization, and parallel computation. Since the pioneering works on Hopfield neural networks and Cohen–Grossberg neural networks (CGNNs), many neural network models have been proposed to describe complex nonlinear dynamics in artificial and biological systems [1, 2]. Among them, CGNNs form a general and flexible class of neural models because the amplification functions and behavior functions are explicitly incorporated into their dynamic structure. This modeling framework includes several well-known neural networks as particular cases and provides a useful platform for studying stability, synchronization, bifurcation, and other dynamical behaviors.

In many practical systems, integer-order differential equations are not sufficient to characterize memory effects and hereditary properties. Fractional calculus offers an effective mathematical tool to describe such nonlocal temporal behavior [3–5]. Compared with integer-order models, fractional-order neural networks can better reflect the dependence of the present state on the past evolution of the system. Fractional-order neural networks have attracted considerable attention because of their ability to describe memory and hereditary effects. Chen et al. investigated global Mittag–Leffler stability and synchronization of memristor-based fractional-order neural networks [6]. Yang et al. developed a theoretical framework for fuzzy cellular neural networks [7]. Pu et al. studied exponential synchronization of fuzzy cellular neural networks with mixed delays [8]. Li et al. established Mittag–Leffler stability results for fractional-order nonlinear systems [9]. Bhat and Bernstein investigated finite-time stability of continuous autonomous systems [10]. These studies provide important theoretical foundations for the stability analysis of fractional-order and fuzzy neural systems. In particular, fractional-order Cohen–Grossberg neural networks (FOCGNNs) combine the structural generality of CGNNs with the memory property of fractional calculus, making them suitable for modeling neural systems with long-term dependence.

Stability analysis is a fundamental issue in the study of neural networks. Most classical stability concepts focus on the behavior of a system over an infinite time horizon. However, in many engineering and biological applications, one is more concerned with whether the system state remains within a prescribed bound during a given finite interval. This requirement leads to the concept of finite-time stability (FTS). Different from asymptotic stability, finite-time stability emphasizes transient performance within a fixed finite-time interval. Such a property is important in applications where large transient deviations may cause undesirable system responses even if the system is ultimately stable. Yang et al. investigated the finite-time stability of fractional-order neural networks with delays [11]. Chen et al. established finite-time stability criteria for a class of delayed fractional-order neural networks [12]. Wu et al. derived related finite-time stability conditions by using fractional differential inequalities [13]. Finite-time stability of fractional delay differential equations and Caputo-type fractional-order neural networks with delays and uncertain terms was investigated in [14, 15]. Regarding fractional-order neural network models, finite-time stability analysis is more difficult than that for integer-order systems because the standard chain rule and some classical differential inequalities cannot be directly applied to fractional derivatives.

Time delay is another important factor that cannot be ignored in neural network modeling. Signal transmission, synaptic communication, the finite switching speed of amplifiers, and information processing all inevitably introduce delays into neural systems. In earlier studies, many results were developed for neural networks with constant delays [12, 13, 16]. Nevertheless, in realistic systems, delays are often not fixed. They may vary with time due to changes in the communication channels, external disturbances, hardware implementation, or uncertain signal transmission. Therefore, time-varying delays provide a more accurate description than constant delays. At the same time, they also bring significant mathematical difficulties. In particular, the delayed state $x(t - \varphi(t))$ cannot be simply estimated from the current state x(t), and improper treatment of the time-varying delay terms may lead to conservative or even invalid stability criteria. Hence, it is necessary to develop suitable inequality techniques to handle the effect of time-varying delays in fractional-order neural networks [14, 17, 18].

Fuzzy logic is also an important tool for modeling uncertainty and imprecision in neural systems. Since the introduction of fuzzy cellular neural networks, fuzzy AND and fuzzy OR operations have been widely used to describe nonlinear feedback mechanisms involving uncertain or vague information [19]. Compared with conventional neural networks, fuzzy neural networks can better represent complex decision processes and nonlinear interactions under uncertain environments. When fuzzy min and max templates are incorporated into FOCGNNs, the resulting model becomes more general [20]. However, the fuzzy feedback terms are usually coupled with delayed states, activation functions, and amplification functions. This coupling makes the stability analysis more complicated, especially in the presence of time-varying delays and fractional-order derivatives.

Recently, several important works have been devoted to finite-time stability and the finite-time synchronization of fractional-order delayed neural networks. Du and Lu developed new finite-time criteria for FOCGNNs with constant delays by using a fractional-order delayed Gronwall inequality [21]. They also proposed a new sufficient condition for finite-time synchronization of fractional-order memristive neural networks with time delay, where a Gronwall-type inequality was used to avoid an invalid estimation of the delayed error terms [22]. Fang et al. studied finite-time stability criteria for fractional-order fuzzy neural networks with inertial effects, and pointed out that special care is needed when constructing finite-time differential inequalities for fractional-order systems [23]. In addition, finite-time stabilization and synchronization problems have been investigated for fractional-order neural networks subject to time-varying delays and fuzzy structures [24–26]. These studies provide useful theoretical foundations. However, most existing results mainly focus on constant delays, ordinary delayed neural networks, memristive neural networks, fuzzy cellular neural networks, or inertial neural networks [20, 27]. It should be emphasized that the present work is essentially different from the existing finite-time stability results on fractional-order neural networks. In [21], finite-time stability was investigated for FOCGNNs with constant delays, while the delay considered in this paper is time-varying and cannot be handled by directly applying the delayed Gronwall inequality used therein. In [22], the finite-time synchronization of fractional-order memristive neural networks with time delay was studied, but the model's structure is different from the Cohen–Grossberg framework and does not involve fuzzy min/max feedback templates. The results in [23] considered fractional-order fuzzy neural networks with inertial terms and fuzzy logics, whereas the amplification functions and behavior functions of CGNNs were not included. Moreover, some related works on time-varying delays, such as [24, 25], mainly focused on stabilization or controller design for fractional-order neural networks, but they did not simultaneously address Cohen–Grossberg-type amplification functions and fuzzy logic operators.

Therefore, the main difficulty of the present paper lies in the simultaneous treatment of fractional-order Cohen–Grossberg dynamics, time-varying delays, and fuzzy min/max feedback terms. The time-varying delayed state cannot be estimated simply from the current state, and the fuzzy feedback terms are coupled with the amplification functions and delayed activation functions. As far as we know, finite-time stability analysis for FOCGNNs simultaneously involving time-varying delays and fuzzy logic has not been sufficiently explored. In the existing literature, the term finite-time stability is used in different senses. One commonly used concept focuses on the boundedness of a system's trajectories over a prescribed finite-time interval. More precisely, if the initial state is bounded by a given constant, then the corresponding state trajectory remains within a prescribed bound during the interval under consideration. Another concept of finite-time stability is related to finite-time convergence, which requires the system's trajectory to reach an equilibrium point within a finite settling time. In this paper, finite-time stability is understood in the first sense. That is, the objective is to guarantee that the error trajectory remains bounded within the finite interval $[0, T]$, rather than to prove that the error converges to zero in finite time.

Based on the discussion above, this paper studies the finite-time stability of FOCGNNs with time-varying delays and fuzzy logic. The considered model contains Caputo fractional derivatives, Cohen–Grossberg-type amplification functions, time-varying delay terms, and fuzzy min/max feedback templates. To handle the challenges introduced by time-varying delays, a weighted envelope function technique is introduced to estimate delayed states. Based on this treatment, a new Gronwall-type inequality is established and then applied to derive a finite-time stability criterion for the addressed FOCGNNs. Compared with existing results for constant-delay FOCGNNs, the proposed criterion is applicable to a more general model with time-varying delays and fuzzy logic. Finally, three numerical examples are provided to validate the effectiveness of the proposed theoretical results.

The distinctive contributions of this work are summarized below.

(1) A class of FOCGNNs with both time-varying delays and fuzzy logic is formulated, which extends the existing delayed FOCGNN models.
(2) The influence of time-varying delays is estimated through a weighted envelope function method, avoiding the direct and rough replacement of delayed states by the current states, and a sufficient condition for the finite-time stability of FOCGNNs with time-varying delays and fuzzy logics is obtained through a new Gronwall inequality.
(3) A finite-time stability criterion is obtained by combining the proposed inequality technique with the error-system estimation. The obtained result provides a useful approach for studying the transient boundedness of fractional-order neural networks with more complex delays and fuzzy structures.

The rest of this paper is organized as follows. In Section 2, some basic definitions and useful lemmas on fractional calculus are recalled, and the FOCGNNs with time-varying delays and fuzzy logics are formulated. In Section 3, a new Gronwall-type inequality is first derived. Based on this inequality, a sufficient condition is established to guarantee the finite-time stability of the considered FOCGNNs. In Section 4, three numerical examples are presented to demonstrate the effectiveness of the proposed theoretical findings. Finally, Section 5 summarizes the main findings of this paper.

Notations: In this paper, $\mathbb{R}$ represents the set of real numbers, $\mathbb{R}^n$ stands for the n-dimensional Euclidean space, and $\mathbb{R}^{n\times n}$ denotes the set of all real matrices of order $n \times n$. Let $\mathbb{R}_+ = (0, +\infty)$ and $\mathbb{R}_0^+ = [0, +\infty)$. For a positive integer n, $\mathbb{N}_1^n$ denotes the set $\{1, 2, \dots, n\}$. For a vector $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, its norm is defined by

$$\|x\| = \sum_{i=1}^{n} |x_i|, \quad \text{and} \quad |x| = (|x_1|, |x_2|, \dots, |x_n|)^T.$$

For a matrix

$$A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}, \quad |A| = (|a_{ij}|)_{n\times n} \quad \text{and} \quad \|A\| = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|.$$

The notation $\text{diag}(a_1, a_2, \ldots, a_n)$ represents a diagonal matrix with the diagonal entries $a_1, a_2, \ldots, a_n$. For two vectors $x, y \in \mathbb{R}^n$, $x \leq y$ means $x_i \leq y_i$ for all $i \in N_1^n$. The symbol $C(I, \mathbb{R}^n)$ denotes the space of all continuous functions from an interval I to $\mathbb{R}^n$, and $C^m(I, \mathbb{R}^n)$ denotes the space of all m-order continuously differentiable functions. The superscript T represents the transpose operation for a vector or matrix. The gamma function is denoted as $\Gamma(\cdot)$. The operators $\bigwedge$ and $\bigvee$ stand for the fuzzy AND and fuzzy OR operations, respectively. For an initial error function $\omega(s)$ defined on $[-\rho, 0]$, its norm is defined as

$$\|\omega\| = \sup_{s\in[-\rho,0]} \|\omega(s)\|.$$

$T > 0$ denotes a prescribed finite-time horizon, and $\varphi_j(t)$ denotes the time-varying delay satisfying $0 \leq \varphi_j(t) \leq \varphi < \rho$.

2. Preliminaries

In this section, several fundamental definitions and useful lemmas are reviewed to support the following analysis. Lemma 2 was derived to prepare for the derivation of Theorem 2 in Section 3. Subsequently, we introduce a class of FOCGNNs involving time-varying delays and fuzzy logic terms. Several necessary assumptions on the amplification functions, activation functions, fuzzy logic functions, and time-varying delays are also presented. Finally, we present the definition of finite-time stability for the considered system.

2.1. Preliminary definitions and lemmas

Definition 1. [3, 28] The definition of the α-order Caputo derivative is given below:

$${}^{C}_{t_0}\mathcal{D}^{\alpha}_{t} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{y^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau,$$

where $1 < \alpha < 2$, and $n = 2$ is adopted in this paper, with

$$y(t) \in C^2(\mathbb{R}^0_+, \mathbb{R}), \quad \alpha \in (1, 2).$$

The gamma function $\Gamma(\alpha)$, introduced by Euler, can be expressed as the following integral

$$\Gamma(\alpha) = \int_0^{+\infty} \hbar^{\alpha-1} \exp\{-\hbar\} d\hbar.$$

Lemma 1. [29] Let $y(t) \in \mathbb{R}$ be a continuous and differentiable function. Then for any $t \geq t_0$, the inequality below holds: let $I = [t_0, T]$. Suppose that $a, b \in C(I, \mathbb{R})$ and $y \in C^1(I, \mathbb{R})$ satisfy

$$\begin{cases} y'(t) \leq a(t)y(t) + b(t), & t \in I, \\ y(t_0) \leq y_0. \end{cases} \tag{1}$$

Then, for any $t \in I$, one has

$$y(t) \leq y_0 \exp\left(\int_{t_0}^{t} a(s)\, ds\right) + \int_{t_0}^{t} \exp\left(\int_{s}^{t} a(r)\, dr\right) b(s)\, ds. \tag{2}$$

Lemma 2. Let $0 < T < \infty$, $u(t)$, $g(t)$, $h(t)$, $b(t)$, $l(t)$, $m(t)$, and $k(t)$ are continuous and non-negative on $[t_0, T]$, and suppose

$$u(t) \leq g(t) + b(t)\left\{\int_{t_0}^{t} [h(s)u(s) + u(s)m(s)l(s) + m(s)k(s)]^{\wp} \,\mathrm{d}s\right\}^{\frac{1}{\wp}}, \tag{3}$$

where $\wp \geq 1$. Then

$$\begin{aligned} u(t) \leq g(t) + b(t)\bigg\{\int_{t_0}^{t} & 2^{\wp-1}\,[h(s)g(s) + m(s)k(s) + g(s)m(s)l(s)]^{\wp} \\ & \times \exp\left(\int_{s}^{t} 2^{\wp-1}\,(h(r) + m(r)l(r))^{\wp}\, b^{\wp}(r)\,dr\right) ds\bigg\}^{\frac{1}{\wp}}. \end{aligned} \tag{4}$$

Furthermore, if $g(t)$, $b(t)$ are nondecreasing, $g(t) > 0$, then

$$u(t) \leq g(t) + g(t)\left\{\exp\left(\int_{t_0}^{t} 2^{\wp-1} b^{\wp}(t)\left[h(s) + m(s)\left(l(s) + \frac{k(s)}{g(s)}\right)\right]^{\wp} ds\right) - 1\right\}^{\frac{1}{\wp}}. \tag{5}$$

Proof. Let $v(t)$ be defined as follows:

$$v(t) = \int_{t_0}^{t} [h(s)u(s) + u(s)m(s)l(s) + m(s)k(s)]^{\wp} ds, \qquad t \in [t_0, T]. \tag{6}$$

According to Eqs (3) and (6), it follows that

$$u(t) \leq g(t) + b(t)v^{\frac{1}{\wp}}(t), \qquad t \in [t_0, T]. \tag{7}$$

We then have

$$\begin{aligned} v'(t) &= [h(t)u(t) + u(t)m(t)l(t) + m(t)k(t)]^{\wp} \\ &\leq \left[(h(t) + m(t)l(t))\left(g(t) + b(t)v^{\frac{1}{\wp}}(t)\right) + m(t)k(t)\right]^{\wp} \\ &\leq 2^{\wp-1}\left[(h(t)g(t) + m(t)k(t) + g(t)m(t)l(t))^{\wp} + (h(t) + m(t)l(t))^{\wp} b^{\wp}(t)v(t)\right] \\ &= 2^{\wp-1}\,(h(t)g(t) + m(t)k(t) + g(t)m(t)l(t))^{\wp} + 2^{\wp-1}(h(t) + m(t)l(t))^{\wp} b^{\wp}(t)v(t). \end{aligned}$$

By Lemma 1, we obtain

$$\begin{aligned} v(t) \leq \int_{t_0}^{t} & 2^{\wp-1}\,(h(s)g(s) + m(s)k(s) + g(s)m(s)l(s))^{\wp} \\ & \times \exp\left(\int_{s}^{t} 2^{\wp-1}\,(h(r) + m(r)l(r))^{\wp}\, b^{\wp}(r)\,dr\right) ds. \end{aligned} \tag{8}$$

It follows from (7) and (8) that (4) holds. If $g(t)$, $b(t)$ are nondecreasing, and

$$g(t) > 0,$$

then for $t \in [t_0, T]$, we obtain

$$\begin{aligned}
u(t) &\le g(t) + b(t)\left\{\int_{t_0}^{t} 2^{\wp-1}\left[h(s)g(s) + m(s)k(s) + g(s)m(s)l(s)\right]^{\wp}\right.\\
&\quad \left.\times \exp\left(\int_{s}^{t} 2^{\wp-1}\left(h(r) + m(r)l(r)\right)^{\wp} b^{\wp}(r)\,dr\right) ds\right\}^{\frac{1}{\wp}}\\
&= g(t) + b(t)\left\{\int_{t_0}^{t} 2^{\wp-1} g^{\wp}(s)\left(h(s) + \frac{m(s)k(s)}{g(s)} + m(s)l(s)\right)^{\wp}\right.\\
&\quad \left.\times \exp\left(\int_{s}^{t} 2^{\wp-1}\left(h(r) + m(r)l(r)\right)^{\wp} b^{\wp}(r)\,dr\right) ds\right\}^{\frac{1}{\wp}}\\
&\le g(t) + g(t)\left\{\int_{t_0}^{t} 2^{\wp-1} b^{\wp}(t)\left(h(s) + \frac{m(s)k(s)}{g(s)} + m(s)l(s)\right)^{\wp}\right.\\
&\quad \left.\times \exp\left(\int_{s}^{t} 2^{\wp-1}\left(h(r) + m(r)l(r)\right)^{\wp} b^{\wp}(r)\,dr\right) ds\right\}^{\frac{1}{\wp}}\\
&\le g(t) + g(t)\left\{\int_{t_0}^{t} 2^{\wp-1} b^{\wp}(t)\left[h(s) + m(s)\left(l(s) + \frac{k(s)}{g(s)}\right)\right]^{\wp}\right.\\
&\quad \left.\times \exp\left(\int_{s}^{t} 2^{\wp-1} b^{\wp}(t)\left[h(r) + m(r)\left(l(r) + \frac{k(r)}{g(r)}\right)\right]^{\wp} dr\right) ds\right\}^{\frac{1}{\wp}}\\
&= g(t) + g(t)\left\{\exp\left(\int_{t_0}^{t} 2^{\wp-1} b^{\wp}(t)\left[h(s) + m(s)\left(l(s) + \frac{k(s)}{g(s)}\right)\right]^{\wp} ds\right) - 1\right\}^{\frac{1}{\wp}}.
\end{aligned} \tag{9}$$

This completes the proof. □

2.2. Model description

Consider a class of FOCGNNs with time-varying delays and fuzzy logics, whose dynamic equation is described by the following Caputo fractional-order differential equation:

$$\begin{aligned}
{}^{C}D_0^{\alpha} x_i(t) = &- f_i(x_i(t))\left(g_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} p_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} q_j(x_j(t - \varphi_j(t)))\right.\\
&\left. - \bigwedge_{j=1}^{n} e_{ij} G_j(x_j(t - \varphi_j(t))) - \bigvee_{j=1}^{n} k_{ij} G_j(x_j(t - \varphi_j(t))) - I_i\right),
\end{aligned} \tag{10}$$

where $i \in \mathbb{N}_1^n$ and $1 < \alpha < 2$ indicates the fractional derivative order; $x_i(t)$ represents the state variable of the i-th neuron; a_{ij} and b_{ij} denote the synaptic connection weight from the j-th neuron to the i-th neuron; $f_i(\cdot)$ is the amplification function of the i -th neuron; $g_i(\cdot)$ is the well-behaved function; I_i is the external constant input; $p_j(\cdot)$, $q_j(\cdot)$, and $G_j(\cdot)$ denote the activation function; and φ_j: $\mathbb{R} \to \mathbb{R}_0^+$ describes the time-varying delay associated with the j-th neuron at time t. Moreover, there is a positive constant φ such that $\varphi_j(t) \le \varphi$. In addition, e_{ij} and k_{ij} denote the corresponding entries in the fuzzy feedback min and max templates.

Let the vectors x(t) and y(t) as $x(t)=[(x_1(t), x_2(t), \ldots, x_n(t)]^T$ and $y(t)=[y_1(t), y_2(t), \ldots, y_n(t)]^T$ be two arbitrary solutions of system (10) corresponding to different initial functions. Since the considered system contains time-varying delays, the initial functions are defined on the historical interval $[-\rho, 0]$, namely $x(k) = \phi_x(k)$, $y(k) = \phi_y(k)$, and $k \in [-\rho, 0]$, where ϕ_x, $\phi_y \in C^1([-\rho, 0], \mathbb{R}^n)$. Define the error vector as $\varepsilon(t) = x(t) - y(t)$, and define the corresponding initial error function as $\omega(k) = \phi_x(k) - \phi_y(k)$, $k \in [-\rho, 0]$. Then $\varepsilon(k) = \omega(k)$, $k \in [-\rho, 0]$. Moreover, since $1 < \alpha < 2$, the initial derivative error is given by $\varepsilon'(0) = \phi'_x(0) - \phi'_y(0)$. Before deriving the main results, the following assumptions are imposed.

Assumption 1. The amplification function $f_j(\cdot)$ satisfies the Lipschitz condition, a constant exists such that $f^* > 0$ for any $x, y \in \mathbb{R}$, the following holds:

$$0 < \check{f}_i \leq f_i(\cdot) \leq \hat{f}_i, \quad |f_i(x) - f_i(y)| \leq f^*|x - y|, \quad i \in \mathbb{N}_1^n,$$

with $\check{f}_i, \hat{f}_i > 0$.

Assumption 2. The time-varying delay $\varphi_j(t)$ is continuous and bounded, i.e.,

$$0 \leq \varphi_j(t) \leq \varphi, \qquad j = 1, 2, \ldots, n,$$

where $\varphi > 0$ is a known constant. Moreover, there is a constant $\rho > \varphi$ such that

$$\frac{t - t_0 + \rho}{t - \varphi_j(t) - t_0 + \rho} \leq \frac{\rho}{\rho - \varphi}, \qquad t \geq t_0.$$

Remark 1. Assumption 1 is imposed to guarantee that the amplification functions are positive, bounded, and Lipschitz continuous. The lower bound $0 < \check{f}_i \leq f_i(\cdot)$ prevents the amplification function from degenerating, while the upper bound $\hat{f}_i$ avoids an unbounded amplification effect. The Lipschitz condition is used to estimate the difference $f_i(x_i(t)) - f_i(y_i(t))$ in the error system. From the modeling viewpoint, this assumption means that the response gain of each neuron is positive and finite, which is reasonable for CGNNs.

Remark 2. Assumption 2 is used to describe bounded time-varying delays. Compared with constant delays, time-varying delays are more suitable for practical neural networks, where transmission delays may change with the communication conditions, hardware implementation, or external disturbances. The condition $0 \leq \varphi_j(t) \leq \varphi < \rho$ ensures that the delayed state $x_j(t - \varphi_j(t))$ is well defined on the historical interval. Moreover, the constant

$$\mu = \left(\frac{\rho}{\rho - \varphi}\right)^\alpha$$

provides an explicit upper bound for estimating the delayed terms through the weighted envelope function. Therefore, the influence of time-varying delays can be transformed into a tractable form in the subsequent stability analysis.

Assumption 3. For each $j \in \{1, 2, \ldots, n\}$, the function $G_j(\cdot)$ is assumed to be Lipschitz continuous on $\mathbb{R}$ and satisfies $G_j(0) = 0$. That is, there is a constant $L_j > 0$ such that

$$|G_j(u) - G_j(v)| \leq L_j|u - v|, \quad \forall\, u, v \in \mathbb{R}.$$

Assumption 4. A ϑ exists such that

$$|f_i(\kappa)g_i(\kappa) - f_i(s)g_i(s)| \leq \vartheta\,|\kappa - s|\,, \quad i \in \mathbb{N}_1^n$$

with $\kappa, s \in \mathbb{R}$.

Assumption 5. For each $j \in \mathbb{N}_1^n$, it is assumed that $p_j(\cdot)$ and $q_j(\cdot)$ satisfy

$$|p_j(r)| \leq P_j, \quad |q_j(r)| \leq Q_j,$$
$$|p_j(r) - p_j(s)| \leq p_j^*|r - s|, \quad |q_j(r) - q_j(s)| \leq q_j^*|r - s|,$$

where $r, s \in \mathbb{R}$, and $P_j, Q_j, p_j^*, q_j^* > 0$.

Lemma 3. [26] Assume $u, v \in \mathbb{R}$. We then have

$$\left|\bigwedge_j e_{ij}G_j(u) - \bigwedge_j e_{ij}G_j(v)\right| \leq \sum_j |e_{ij}|\,|G_j(u) - G_j(v)|,$$
$$\left|\bigvee_j k_{ij}G_j(u) - \bigvee_j k_{ij}G_j(v)\right| \leq \sum_j |k_{ij}|\,|G_j(u) - G_j(v)|.$$

Remark 3. The assumptions on the activation functions are used to obtain explicit and verifiable estimates for the error system. Although the global Lipschitz conditions on $G_j(\cdot)$, $p_j(\cdot)$, and $q_j(\cdot)$ may seem restrictive, they play a key role in the present finite-time stability analysis. Indeed, these conditions make it possible to convert the nonlinear differences between two solutions into linear error bounds. Without such estimates, the fuzzy feedback terms and delayed activation terms could not be incorporated into the closed integral inequality used in Theorem 2.

Remark 4. Lemma 3 is of great importance in dealing with the fuzzy feedback terms. The fuzzy min and max operators usually make the estimation of the error system more complicated. By using the inequalities in Lemma 3, the differences in the fuzzy logic terms can be bounded by the weighted sums of activation-function differences. Hence, the fuzzy feedback terms can be incorporated into the matrix inequalities together with the ordinary delayed connection terms. If $E = 0$ and $K = 0$, the considered model reduces to a fractional-order CGNN without fuzzy logic. The stability and synchronization problems of fuzzy neural networks with complex structures, such as discontinuities, inertial terms, and parameter uncertainties, have attracted increasing attention [30].

Definition 2. The FOCGNNs with time-varying delays and fuzzy logics described by (10) are finite-time stable in terms of $\{\delta, \eta, T\}$ if, for any initial function ω, the condition $\|\omega\| \leq \delta$, $\max\{\|\omega\|, \|\varepsilon'(0)\|\} \leq \delta$, and $\|\varepsilon(t)\| \leq \eta, t \in [0, T]$ holds, where $\|\omega\| = \sup_{t\in[-\rho,0]} \|\omega(t)\|, 0 < \delta < \eta$.

Lemma 4. Suppose that Assumption 2 holds. Let

$$\varepsilon_i(t) = x_i(t) - y_i(t), \qquad i = 1, 2, \ldots, n,$$

and define

$$M_\varepsilon(t) = \max_{1\leq i\leq n} |\varepsilon_i(t - \varphi_j(t))|, \qquad M_x(t) = \max_{1\leq j\leq n} |x_j(t - \varphi_j(t))|.$$

For $t \in [t_0 - \rho, T]$, set

$$W_\varepsilon(t) = (t - t_0 + \rho)^\alpha M_\varepsilon(t), \qquad \overline{W}_\varepsilon(t) = \sup_{t_0-\rho\le s\le t} W_\varepsilon(s)$$

and

$$W_x(t) = (t - t_0 + \rho)^\alpha M_x(t), \qquad \overline{W}_x(t) = \sup_{t_0-\rho\le s\le t} W_x(s).$$

Furthermore, define

$$\mathcal{M}_{\varepsilon\rho}(t) = \frac{\overline{W}_\varepsilon(t)}{(t - t_0 + \rho)^\alpha}, \qquad \mathcal{M}_{x\rho}(t) = \frac{\overline{W}_x(t)}{(t - t_0 + \rho)^\alpha}$$

and

$$\mu = \left(\frac{\rho}{\rho - \varphi}\right)^\alpha.$$

Then, for any $t \in [t_0, T]$, $i, j = 1, 2, \ldots, n$, one has

$$|\varepsilon_i(t - \varphi_j(t))| \le \mu \mathcal{M}_{\varepsilon\rho}(t)$$

and

$$|x_j(t - \varphi_j(t))| \le \mu \mathcal{M}_{x\rho}(t).$$

Proof. For $t \in [t_0, T]$, since $0 \le \varphi_j(t) \le \varphi < \rho$, we have

$$t - \varphi_j(t) \ge t_0 - \rho. \tag{11}$$

Hence, the delayed terms are well defined on the historical interval.

By the definition of $M_\varepsilon(t)$, one obtains

$$|\varepsilon_i(t - \varphi_j(t))| \le M_\varepsilon(t - \varphi_j(t)).$$

From the definition of $W_\varepsilon(t)$, it follows that

$$M_\varepsilon(t - \varphi_j(t)) = \frac{W_\varepsilon(t - \varphi_j(t))}{(t - \varphi_j(t) - t_0 + \rho)^\alpha}. \tag{12}$$

Since $t - \varphi_j(t) \le t$, by the definition of $\overline{W}_\varepsilon(t)$, we have

$$W_\varepsilon(t - \varphi_j(t)) \le \overline{W}_\varepsilon(t).$$

Therefore,

$$M_\varepsilon(t - \varphi_j(t)) \le \frac{\overline{W}_\varepsilon(t)}{(t - \varphi_j(t) - t_0 + \rho)^\alpha}. \tag{13}$$

Equivalently, we have

$$M_\varepsilon(t - \varphi_j(t)) \le \left(\frac{t - t_0 + \rho}{t - \varphi_j(t) - t_0 + \rho}\right)^\alpha \frac{\overline{W}_\varepsilon(t)}{(t - t_0 + \rho)^\alpha}. \tag{14}$$

Let $X = t - t_0 \geq 0$, then

$$\frac{t - t_0 + \rho}{t - \varphi_j(t) - t_0 + \rho} = \frac{X + \rho}{X - \varphi_j(t) + \rho}. \tag{15}$$

Since $0 \leq \varphi_j(t) \leq \varphi$, we have

$$\frac{X + \rho}{X - \varphi_j(t) + \rho} \leq \frac{X + \rho}{X - \varphi + \rho}.$$

Moreover, the function $\frac{X+\rho}{X-\varphi+\rho}$ is non-increasing with respect to $X \geq 0$. Thus,

$$\frac{X + \rho}{X - \varphi + \rho} \leq \frac{\rho}{\rho - \varphi}.$$

Consequently,

$$\frac{t - t_0 + \rho}{t - \varphi_j(t) - t_0 + \rho} \leq \frac{\rho}{\rho - \varphi}. \tag{16}$$

Combining the inequalities above gives

$$|\varepsilon_i(t - \varphi_j(t))| \leq \left(\frac{\rho}{\rho - \varphi}\right)^{\alpha} \frac{\overline{W}_{\varepsilon}(t)}{(t - t_0 + \rho)^{\alpha}} = \mu \mathcal{M}_{\varepsilon\rho}(t). \tag{17}$$

Similarly, by replacing $M_{\varepsilon}(t)$, $W_{\varepsilon}(t)$, and $\overline{W}_{\varepsilon}(t)$ with $M_x(t)$, $W_x(t)$, and $\overline{W}_x(t)$, respectively, one obtains

$$|x_j(t - \varphi_j(t))| \leq \mu \mathcal{M}_{x\rho}(t). \tag{18}$$

The proof is completed. □

Remark 5. The abovementioned assumptions are sufficient rather than necessary. They are adopted to obtain explicit and easily verifiable finite-time stability criteria. In future work, some of these conditions may be relaxed. For instance, the global Lipschitz conditions may be replaced by local Lipschitz or sector-bounded conditions. Moreover, unbounded but linearly growing activation functions, distributed delays, or state-dependent delays may also be considered to reduce the conservativeness of the present results.

3. Finite-time stability theorem

This section is devoted to the finite-time stability analysis of the considered FOCGNNs. In Section 3.1, a new Gronwall-type inequality is derived, which provides an effective tool for estimating the fractional integral inequality with nonlinear terms. In Section 3.2, the proposed inequality is applied to the error system of the FOCGNNs with time-varying delays and fuzzy logics. By estimating the time-varying delayed terms through a weighted envelope function method, a sufficient finite-time stability criterion is obtained.

3.1. Gronwall inequality derivation

Theorem 1. For $\alpha > 0$ and $0 < T < \infty$, assume that $u(t)$, $g(t)$, $h(t)$, $b(t)$, $l(t)$, $m(t)$, and $k(t)$ are continuous and non-negative on $[t_0, T]$, defined on

$$u(t) \leq g(t) + b(t) \int_{t_0}^{t} \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \left[h(\zeta)u(\zeta) + u(\zeta)m(\zeta)l(\zeta) + m(\zeta)k(\zeta)\right] d\zeta. \tag{19}$$

We then have,

$$\begin{aligned} u(t) \leq g(t) + B(t) \Bigg\{ \int_{t_0}^{t} & 2^{\wp-1} \left[h(s)g(s) + m(s)k(s) + g(s)m(s)l(s)\right]^{\wp} \\ & \times \exp\left(\int_{s}^{t} 2^{\wp-1} \left(h(r) + m(r)l(r)\right)^{\wp} B^{\wp}(r)\, dr \right) ds \Bigg\}^{\frac{1}{\wp}}, \end{aligned} \tag{20}$$

where $p, \wp > 0$ such that $\alpha > 1/\wp$ and $1/p + 1/\wp = 1$, and

$$B(t) = \frac{b(t)(t-t_0)^{\alpha-\frac{1}{\wp}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}, \quad t \in [t_0, T].$$

Moreover, if $g(t)$, $b(t)$ are nondecreasing, $g(t) > 0$, then

$$u(t) \leq g(t) + g(t) \left\{ \exp\left(\int_{t_0}^{t} 2^{\wp-1} B^{\wp}(t) \left[h(s) + m(s)\left(l(s) + \frac{k(s)}{g(s)} \right) \right]^{\wp} ds \right) - 1 \right\}^{\frac{1}{\wp}}. \tag{21}$$

Proof. Take $p, \wp > 0$ with $\alpha > 1/\wp$ and $1/p + 1/\wp = 1$. For $t \in [t_0, T]$, from Hölder inequality, it follows that

$$\begin{aligned} u(t) &\leq g(t) + b(t) \int_{t_0}^{t} \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \left(h(\zeta)u(\zeta) + u(\zeta)m(\zeta)l(\zeta) + m(\zeta)k(\zeta)\right) d\zeta \\ &\leq g(t) + \frac{b(t)}{\Gamma(\alpha)} \left(\int_{t_0}^{t} (t-\zeta)^{p(\alpha-1)} d\zeta \right)^{\frac{1}{p}} \left(\int_{t_0}^{t} \left(h(\zeta)u(\zeta) + u(\zeta)m(\zeta)l(\zeta) + m(\zeta)k(\zeta)\right)^{\wp} d\zeta \right)^{\frac{1}{\wp}} \\ &\leq g(t) + \frac{b(t)(t-t_0)^{\alpha-\frac{1}{\wp}}}{\Gamma(\alpha)\left[p(\alpha-1)+1\right]^{\frac{1}{p}}} \left(\int_{t_0}^{t} \left(h(\zeta)u(\zeta) + u(\zeta)m(\zeta)l(\zeta) + m(\zeta)k(\zeta)\right)^{\wp} d\zeta \right)^{\frac{1}{\wp}}. \end{aligned}$$

Consider G(t) = g(t), B(t) = $\frac{b(t)(t-t_0)^{\alpha-\frac{1}{\wp}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}$. We then obtain

$$u(t) \leq G(t) + B(t) \left(\int_{t_0}^{t} \left(h(\zeta)u(\zeta) + u(\zeta)m(\zeta)l(\zeta) + m(\zeta)k(\zeta)\right)^{\wp} d\zeta \right)^{\frac{1}{\wp}}.$$

By applying Lemma 2, it follows that

$$u(t) \leq g(t) + g(t) \left\{ \exp\left(\int_{t_0}^{t} 2^{\wp-1} B^{\wp}(t) \left[h(s) + m(s)\left(l(s) + \frac{k(s)}{g(s)} \right) \right]^{\wp} ds \right) - 1 \right\}^{\frac{1}{\wp}}.$$

To ensure that system (10) achieves finite-time stability, the method developed in [21] is used. □

Remark 6. Theorem 1 gives a Gronwall-type inequality suitable for the fractional integral inequality appearing in this paper. Different from some classical Gronwall inequalities, the proposed inequality contains the mixed term $u(t)m(t)l(t)$ and the nonhomogeneous term $m(t)k(t)$, which are generated by the coupling between the time-varying delayed states and the fuzzy logic terms. In addition, the parameters p and $\wp$ satisfying $1/p + 1/\wp = 1$ provide a certain flexibility in estimating the upper bound. This flexibility may help reduce conservativeness when suitable parameters are selected.

3.2. Sufficient conditions for FTS

Theorem 2. Given Assumptions 1–5, and $\max\{\|\omega\|, \|\varepsilon'(0)\|\} \leq \delta$, the FOCGNNs with time-varying delays and fuzzy logics (10) possess FTS with respect to $\{\delta, \eta, T\}$ if

$$(1+t) + (1+t)\left\{\exp\left(2^{\wp-1}\left[\frac{t^{\alpha-\frac{1}{\wp}}}{\Gamma(\alpha)\left[p(\alpha-1)+1\right]^{\frac{1}{p}}}\right]^{\wp} \times \int_{t_0}^{t}\left[\|\Omega\| + \|M(s)\|\left(\|\Xi\| + \frac{\|\Psi\|}{\|\omega\| + \|\varepsilon'(0)\|s}\right)\right]^{\wp} ds\right) - 1\right\}^{\frac{1}{\wp}} \leq \frac{\eta}{\delta}, \quad t \in [t_0, T], \tag{22}$$

where $p, \wp > 0$ satisfy $\frac{1}{p} + \frac{1}{\wp} = 1$ and $\|\omega\| > 0$. Moreover, we have

$$\begin{aligned}
\Omega &= \operatorname{diag}\left[\theta + F^*|I| + F^*(|A|P + |B|Q)\right] + \hat{F}|A|P^*,\\
\Xi &= F^*\mu(|E| + |K|)L,\\
\Psi &= \hat{F}\mu(|B|Q^* + |E|L + |K|L).
\end{aligned}$$

Here, the vectors and matrices are defined as follows:

$$\theta, P, Q \in \mathbb{R}_+^n, \quad I \in \mathbb{R}^n$$

and

$$A = (a_{ij})_{n\times n}, \quad B = (b_{ij})_{n\times n}, \quad E = (e_{ij})_{n\times n}, \quad K = (k_{ij})_{n\times n} \in \mathbb{R}^{n\times n}.$$

Moreover,

$$F^*, \widehat{F}, P^*, Q^*, L \in \mathbb{R}_+^{n\times n}$$

are diagonal matrices with positive entries, where

$$\begin{aligned}
F^* &= \operatorname{diag}(f_1^*, f_2^*, \ldots, f_n^*), \qquad \widehat{F} = \operatorname{diag}\left(\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n\right),\\
P^* &= \operatorname{diag}(p_1^*, p_2^*, \ldots, p_n^*), \qquad Q^* = \operatorname{diag}(q_1^*, q_2^*, \ldots, q_n^*),\\
L &= \operatorname{diag}(L_1, L_2, \ldots, L_n).
\end{aligned}$$

Furthermore,

$$|I| \in \mathbb{R}_0^{+\,n}, \quad |A|, |B|, |E|, |K| \in \mathbb{R}_0^{+\,n\times n}$$

denote the entry-wise absolute values of the corresponding vector and matrices.

Proof. According to the definitions of $x_i(t)$ and $y_i(t)$ presented in Section 2, it follows from (10) that, for $t \in [0, T]$, we have

$$\begin{aligned}
{}^{C}D^{\alpha}\left(x_i(t)-y_i(t)\right) = &-f_i(x_i(t))\Bigg(g_i(x_i(t)) - \sum_{j=1}^{n} a_{ij}p_j(x_j(t)) - \sum_{j=1}^{n} b_{ij}q_j\left(x_j(t-\varphi_j(t))\right)\\
&- \bigwedge_{j=1}^{n} e_{ij}G_j\left(x_j(t-\varphi_j(t))\right) - \bigvee_{j=1}^{n} k_{ij}G_j\left(x_j(t-\varphi_j(t))\right) - I_i\Bigg)\\
&+ f_i(y_i(t))\Bigg(g_i(y_i(t)) - \sum_{j=1}^{n} a_{ij}p_j(y_j(t)) - \sum_{j=1}^{n} b_{ij}q_j\left(y_j(t-\varphi_j(t))\right)\\
&- \bigwedge_{j=1}^{n} e_{ij}G_j\left(y_j(t-\varphi_j(t))\right) - \bigvee_{j=1}^{n} k_{ij}G_j\left(y_j(t-\varphi_j(t))\right) - I_i\Bigg).
\end{aligned} \tag{23}$$

We apply fractional-order integration to it and derive the corresponding equivalent form

$$\begin{aligned}
x_i(t)-y_i(t) = {}& x_i(0)-y_i(0)+x_i'(0)t-y_i'(0)t\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\left(f_i(y_i(\zeta))g_i(y_i(\zeta))-f_i(x_i(\zeta))g_i(x_i(\zeta))\right)\mathrm{d}\zeta\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\left(f_i(x_i(\zeta))\sum_{j=1}^{n} a_{ij}p_j(x_j(\zeta)) - f_i(y_i(\zeta))\sum_{j=1}^{n} a_{ij}p_j(y_j(\zeta))\right)\mathrm{d}\zeta\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\left(f_i(x_i(\zeta))\sum_{j=1}^{n} b_{ij}q_j(x_j(\zeta-\varphi_j(\zeta))) - f_i(y_i(\zeta))\sum_{j=1}^{n} b_{ij}q_j(y_j(\zeta-\varphi_j(\zeta)))\right)\mathrm{d}\zeta\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\left(f_i(x_i(\zeta))\bigwedge_{j=1}^{n} e_{ij}G_j(x_j(\zeta-\varphi_j(\zeta))) - f_i(y_i(\zeta))\bigwedge_{j=1}^{n} e_{ij}G_j(y_j(\zeta-\varphi_j(\zeta)))\right)\mathrm{d}\zeta\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\left(f_i(x_i(\zeta))\bigvee_{j=1}^{n} k_{ij}G_j(x_j(\zeta-\varphi_j(\zeta))) - f_i(y_i(\zeta))\bigvee_{j=1}^{n} k_{ij}G_j(y_j(\zeta-\varphi_j(\zeta)))\right)\mathrm{d}\zeta\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} I_i\left(f_i(x_i(\zeta))-f_i(y_i(\zeta))\right)\mathrm{d}\zeta.
\end{aligned} \tag{24}$$

According to Assumptions 1 and 4,

$$\begin{aligned}
|x_i(t)-y_i(t)| \le {}& |x_i(0)-y_i(0)| + \left|x_i'(0)-y_i'(0)\right| t\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\left|f_i(y_i(\zeta))g_i(y_i(\zeta))-f_i(x_i(\zeta))g_i(x_i(\zeta))\right|\mathrm{d}\zeta\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\left|f_i(x_i(\zeta))\sum_{j=1}^{n} a_{ij}p_j(x_j(\zeta)) - f_i(y_i(\zeta))\sum_{j=1}^{n} a_{ij}p_j(y_j(\zeta))\right|\mathrm{d}\zeta\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\left|f_i(x_i(\zeta))\sum_{j=1}^{n} b_{ij}q_j(x_j(\zeta-\varphi_j(\zeta))) - f_i(y_i(\zeta))\sum_{j=1}^{n} b_{ij}q_j(y_j(\zeta-\varphi_j(\zeta)))\right|\mathrm{d}\zeta
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \left| f_i(x_i(\zeta)) \bigwedge_{j=1}^{n} e_{ij} G_j(x_j(\zeta-\varphi_j(\zeta))) - f_i(y_i(\zeta)) \bigwedge_{j=1}^{n} e_{ij} G_j(y_j(\zeta-\varphi_j(\zeta))) \right| \mathrm{d}\zeta \\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \left| f_i(x_i(\zeta)) \bigvee_{j=1}^{n} k_{ij} G_j(x_j(\zeta-\varphi_j(\zeta))) - f_i(y_i(\zeta)) \bigvee_{j=1}^{n} k_{ij} G_j(y_j(\zeta-\varphi_j(\zeta))) \right| \mathrm{d}\zeta \\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} |I_i| \left| f_i(x_i(\zeta)) - f_i(y_i(\zeta)) \right| \mathrm{d}\zeta \\
\le{}& |x_i(0)-y_i(0)| + \left| x_i'(0) - y_i'(0) \right| t \\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \vartheta_i \left| (x_i(\zeta)) - (y_i(\zeta)) \right| \mathrm{d}\zeta \\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \left| f_i(x_i(\zeta)) \sum_{j=1}^{n} a_{ij} p_j(x_j(\zeta)) - f_i(y_i(\zeta)) \sum_{j=1}^{n} a_{ij} p_j(y_j(\zeta)) \right| \mathrm{d}\zeta \\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \left| f_i(x_i(\zeta)) \sum_{j=1}^{n} b_{ij} q_j(x_j(\zeta-\varphi_j(\zeta))) - f_i(y_i(\zeta)) \sum_{j=1}^{n} b_{ij} q_j(y_j(\zeta-\varphi_j(\zeta))) \right| \mathrm{d}\zeta \\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \left| f_i(x_i(\zeta)) \bigwedge_{j=1}^{n} e_{ij} G_j(x_j(\zeta-\varphi_j(\zeta))) - f_i(y_i(\zeta)) \bigwedge_{j=1}^{n} e_{ij} G_j(y_j(\zeta-\varphi_j(\zeta))) \right| \mathrm{d}\zeta \\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \left| f_i(x_i(\zeta)) \bigvee_{j=1}^{n} k_{ij} G_j(x_j(\zeta-\varphi_j(\zeta))) - f_i(y_i(\zeta)) \bigvee_{j=1}^{n} k_{ij} G_j(y_j(\zeta-\varphi_j(\zeta))) \right| \mathrm{d}\zeta \\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} |I_i| f_i^* \left| (x_i(\zeta)) - (y_i(\zeta)) \right| \mathrm{d}\zeta \\
\le{}& |x_i(0)-y_i(0)| + \left| x_i'(0) - y_i'(0) \right| t \\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} (\vartheta_i + |I_i| f_i^*) \left| (x_i(\zeta)) - (y_i(\zeta)) \right| \mathrm{d}\zeta \\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \left| f_i(x_i(\zeta)) \sum_{j=1}^{n} a_{ij} p_j(x_j(\zeta)) - f_i(y_i(\zeta)) \sum_{j=1}^{n} a_{ij} p_j(x_j(\zeta)) \right| \mathrm{d}\zeta \\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \left| f_i(y_i(\zeta)) \sum_{j=1}^{n} a_{ij} p_j(x_j(\zeta)) - f_i(y_i(\zeta)) \sum_{j=1}^{n} a_{ij} p_j(y_j(\zeta)) \right| \mathrm{d}\zeta \\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \left| f_i(x_i(\zeta)) \sum_{j=1}^{n} b_{ij} q_j(x_j(\zeta-\varphi_j(\zeta))) - f_i(y_i(\zeta)) \sum_{j=1}^{n} b_{ij} q_j(x_j(\zeta-\varphi_j(\zeta))) \right| \mathrm{d}\zeta \\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \left| f_i(y_i(\zeta)) \sum_{j=1}^{n} b_{ij} q_j(x_j(\zeta-\varphi_j(\zeta))) - f_i(y_i(\zeta)) \sum_{j=1}^{n} b_{ij} q_j(y_j(\zeta-\varphi_j(\zeta))) \right| \mathrm{d}\zeta \\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \left| f_i(x_i(\zeta)) \bigwedge_{j=1}^{n} e_{ij} G_j(x_j(\zeta-\varphi_j(\zeta))) - f_i(y_i(\zeta)) \bigwedge_{j=1}^{n} e_{ij} G_j(x_j(\zeta-\varphi_j(\zeta))) \right| \mathrm{d}\zeta
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\left|f_i(y_i(\zeta))\bigwedge_{j=1}^{n} e_{ij}G_j(x_j(\zeta-\varphi_j(\zeta)))-f_i(y_i(\zeta))\bigwedge_{j=1}^{n} e_{ij}G_j(y_j(\zeta-\varphi_j(\zeta)))\right|\mathrm{d}\zeta\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\left|f_i(x_i(\zeta))\bigvee_{j=1}^{n} k_{ij}G_j(x_j(\zeta-\varphi_j(\zeta)))-f_i(y_i(\zeta))\bigvee_{j=1}^{n} k_{ij}G_j(x_j(\zeta-\varphi_j(\zeta)))\right|\mathrm{d}\zeta\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\left|f_i(y_i(\zeta))\bigvee_{j=1}^{n} k_{ij}G_j(x_j(\zeta-\varphi_j(\zeta)))-f_i(y_i(\zeta))\bigvee_{j=1}^{n} k_{ij}G_j(y_j(\zeta-\varphi_j(\zeta)))\right|\mathrm{d}\zeta\\
\le{}& |x_i(0)-y_i(0)|+\left|x_i'(0)-y_i'(0)\right|t\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\left(\vartheta_i+|I_i|\,f_i^*\right)|x_i(\zeta)-y_i(\zeta)|\,d\zeta\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}f_i^*\,|x_i(\zeta)-y_i(\zeta)|\sum_{j=1}^{n}\left|a_{ij}\right|P_j\,d\zeta\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\hat{f}_i\sum_{j=1}^{n}\left|a_{ij}\right|p_j^*\left|x_j(\zeta)-y_j(\zeta)\right|d\zeta\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}f_i^*\,|x_i(\zeta)-y_i(\zeta)|\sum_{j=1}^{n}\left|b_{ij}\right|\left|q_j\left(x_j(\zeta-\varphi_j(\zeta))\right)\right|d\zeta\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\hat{f}_i\sum_{j=1}^{n}\left|b_{ij}\right|q_j^*\left|x_i(\zeta-\varphi_j(\zeta))-y_i(\zeta-\varphi_j(\zeta))\right|d\zeta\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}f_i^*\,|x_i(\zeta)-y_i(\zeta)|\left|\bigwedge_{j=1}^{n} e_{ij}G_j\left(x_j(\zeta-\varphi_j(\zeta))\right)\right|d\zeta\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\hat{f}_i\left|\bigwedge_{j=1}^{n} e_{ij}G_j\left(x_i(\zeta-\varphi_j(\zeta))-y_i(\zeta-\varphi_j(\zeta))\right)\right|,d\zeta\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}f_i^*\,|x_i(\zeta)-y_i(\zeta)|\left|\bigvee_{j=1}^{n} k_{ij}G_j\left(x_j(\zeta-\varphi_j(\zeta))\right)\right|d\zeta\\
&+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\hat{f}_i\left|\bigvee_{j=1}^{n} k_{ij}G_j\left(x_i(\zeta-\varphi_j(\zeta))-y_i(\zeta-\varphi_j(\zeta))\right)\right|d\zeta.
\end{aligned}
\tag{25}
$$

In the following, the inequalities shown in the figure are derived term by term, and we denote the error as

$$\varepsilon_i(t)=x_i(t)-y_i(t),\qquad \varepsilon_i'(0)=x_i'(0)-y_i'(0).$$

By Lemma 4, the time-varying delayed terms satisfy

$$|\varepsilon_i(t-\varphi_j(\zeta))|\le M_\varepsilon(\zeta)$$

and

$$|x_j(t-\varphi_j(\zeta))|\le M_x(t).$$

According to Assumption 3, by utilizing Lemma 3, we have

$$
\begin{aligned}
|x_i(t)-y_i(t)| \le{}& |x_i(0)-y_i(0)| + \left|x_i'(0)-y_i'(0)\right| t \\
&+ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \left(\vartheta_i + |I_i| f_i^*\right) |x_i(\zeta)-y_i(\zeta)|\, d\zeta \\
&+ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} f_i^* |x_i(\zeta)-y_i(\zeta)| \sum_{j=1}^n \left|a_{ij}\right| P_j\, d\zeta \\
&+ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \hat{f}_i \sum_{j=1}^n \left|a_{ij}\right| p_j^* \left|x_j(\zeta)-y_j(\zeta)\right| d\zeta \\
&+ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} f_i^* |x_i(\zeta)-y_i(\zeta)| \sum_{j=1}^n \left|b_{ij}\right| Q_j\, d\zeta \\
&+ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \hat{f}_i \mu \mathcal{M}_{\varepsilon\rho}(\zeta) \sum_{j=1}^n \left|b_{ij}\right| q_j^*\, d\zeta \\
&+ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} f_i^* |x_i(\zeta)-y_i(\zeta)| \sum_{j=1}^n \left|e_{ij}\right| L_j \left|x_j(\zeta-\varphi_j(\zeta))\right| d\zeta \\
&+ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \hat{f}_i \sum_{j=1}^n \left|e_{ij}\right| L_j \mu \mathcal{M}_{\varepsilon\rho}(\zeta) d\zeta \\
&+ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} f_i^* |x_i(\zeta)-y_i(\zeta)| \sum_{j=1}^n \left|k_{ij}\right| L_j \left|x_j(\zeta-\varphi_j(\zeta))\right| d\zeta \\
&+ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \hat{f}_i \sum_{j=1}^n \left|k_{ij}\right| L_j \mu \mathcal{M}_{\varepsilon\rho}(\zeta) d\zeta. \\
={}& |x_i(0)-y_i(0)| + \left|x_i'(0)-y_i'(0)\right| t + \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \Bigg(\vartheta_i + |I_i| f_i^* + f_i^* \sum_{j=1}^n \left(\left|a_{ij}\right| P_j + \left|b_{ij}\right| Q_j\right) \\
&+ f_i^* \mu \mathcal{M}_{x\rho}(\zeta) \sum_{j=1}^n L_j \left(\left|e_{ij}\right| + \left|k_{ij}\right|\right)\Bigg) |x_i(\zeta)-y_i(\zeta)|\, d\zeta \\
&+ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \hat{f}_i \sum_{j=1}^n \left|a_{ij}\right| p_j^* \left|x_j(\zeta)-y_j(\zeta)\right| d\zeta + \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \hat{f}_i \mu \mathcal{M}_{\varepsilon\rho}(\zeta) \sum_{j=1}^n \left|b_{ij}\right| q_j^*\, d\zeta \\
&+ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \hat{f}_i \sum_{j=1}^n \left|e_{ij}\right| L_j \mu \mathcal{M}_{\varepsilon\rho}(\zeta) d\zeta + \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \hat{f}_i \sum_{j=1}^n \left|k_{ij}\right| L_j \mu \mathcal{M}_{\varepsilon\rho}(\zeta) d\zeta \\
={}& |x_i(0)-y_i(0)| + \left|x_i'(0)-y_i'(0)\right| t + \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \Bigg(\vartheta_i + |I_i| f_i^* + f_i^* \sum_{j=1}^n \left(\left|a_{ij}\right| P_j + \left|b_{ij}\right| Q_j\right) \\
&+ f_i^* \mu \mathcal{M}_{x\rho}(\zeta) \sum_{j=1}^n \left(\left|e_{ij}\right| + \left|k_{ij}\right|\right) L_j\Bigg) |x_i(\zeta)-y_i(\zeta)|\, d\zeta \\
&+ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \hat{f}_i \sum_{j=1}^n \left|a_{ij}\right| p_j^* \left|x_j(\zeta)-y_j(\zeta)\right| d\zeta
\end{aligned}
$$

$$+\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\hat{f}_i\mu\mathcal{M}_{\varepsilon\rho}(\zeta)\left(\sum_{j=1}^n \left|b_{ij}\right| q_j^* + \sum_{j=1}^n \left|e_{ij}\right| L_j + \sum_{j=1}^n \left|k_{ij}\right| L_j\right)d\zeta. \tag{26}$$

Let

$$\begin{aligned}\Phi_i(\zeta) =&\vartheta_i + |I_i|\, f_i^* + f_i^* \sum_{j=1}^n \left(\left|a_{ij}\right| P_j + \left|b_{ij}\right| Q_j\right)\\ &+ f_i^*\mu\mathcal{M}_{x\rho}(\zeta)\sum_{j=1}^n \left(\left|e_{ij}\right| + \left|k_{ij}\right|\right) L_j,\end{aligned}$$

then,

$$\begin{aligned}\begin{pmatrix} \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\Phi_1(\zeta)|x_1(\zeta)-y_1(\zeta)|\,d\zeta \\ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\Phi_2(\zeta)|x_2(\zeta)-y_2(\zeta)|\,d\zeta \\ \vdots \\ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\Phi_n(\zeta)|x_n(\zeta)-y_n(\zeta)|\,d\zeta \end{pmatrix} &= \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\begin{pmatrix} \Phi_1(\zeta) & 0 & \cdots & 0 \\ 0 & \Phi_2(\zeta) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Phi_n(\zeta)\end{pmatrix}\begin{pmatrix}|\varepsilon_1(\zeta)| \\ |\varepsilon_2(\zeta)| \\ \vdots \\ |\varepsilon_n(\zeta)|\end{pmatrix} d\zeta \\ &= \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{diag}\begin{pmatrix}\Phi_1(\zeta) \\ \Phi_2(\zeta) \\ \vdots \\ \Phi_n(\zeta)\end{pmatrix}\begin{pmatrix}|\varepsilon_1(\zeta)| \\ |\varepsilon_2(\zeta)| \\ \vdots \\ |\varepsilon_n(\zeta)|\end{pmatrix} d\zeta,\end{aligned} \tag{27}$$

where

$$\begin{aligned}\begin{pmatrix}\Phi_1(\zeta) \\ \Phi_2(\zeta) \\ \vdots \\ \Phi_n(\zeta)\end{pmatrix} =& \begin{pmatrix}\vartheta_1 \\ \vartheta_2 \\ \vdots \\ \vartheta_n\end{pmatrix} + \begin{pmatrix} f_1^* & 0 & \cdots & 0 \\ 0 & f_2^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_n^*\end{pmatrix}\begin{pmatrix}|I_1| \\ |I_2| \\ \vdots \\ |I_n|\end{pmatrix} + \begin{pmatrix} f_1^* & 0 & \cdots & 0 \\ 0 & f_2^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_n^*\end{pmatrix}\left[\begin{pmatrix} |a_{11}| & |a_{12}| & \cdots & |a_{1n}| \\ |a_{21}| & |a_{22}| & \cdots & |a_{2n}| \\ \vdots & \vdots & \ddots & \vdots \\ |a_{n1}| & |a_{n2}| & \cdots & |a_{nn}|\end{pmatrix}\begin{pmatrix}P_1 \\ P_2 \\ \vdots \\ P_n\end{pmatrix}\right. \\ &+ \left.\begin{pmatrix} |b_{11}| & |b_{12}| & \cdots & |b_{1n}| \\ |b_{21}| & |b_{22}| & \cdots & |b_{2n}| \\ \vdots & \vdots & \ddots & \vdots \\ |b_{n1}| & |b_{n2}| & \cdots & |b_{nn}|\end{pmatrix}\begin{pmatrix}Q_1 \\ Q_2 \\ \vdots \\ Q_n\end{pmatrix} + \mathcal{M}_{x\rho}(\zeta)\mu\begin{pmatrix} |e_{11}|+|k_{11}| & |e_{12}|+|k_{12}| & \cdots & |e_{1n}|+|k_{1n}| \\ |e_{21}|+|k_{21}| & |e_{22}|+|k_{22}| & \cdots & |e_{2n}|+|k_{2n}| \\ \vdots & \vdots & \ddots & \vdots \\ |e_{n1}|+|k_{n1}| & |e_{n2}|+|k_{n2}| & \cdots & |e_{nn}|+|k_{nn}|\end{pmatrix}\begin{pmatrix}L_1 \\ L_2 \\ \vdots \\ L_n\end{pmatrix}\right].\end{aligned}$$

Let

$$\Theta_i(\zeta) = \hat{f}_i\sum_{j=1}^n \left|a_{ij}\right| p_j^* \left|x_j(\zeta) - y_j(\zeta)\right|.$$

One then has

$$\begin{pmatrix}\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\Theta_1(\zeta)\,d\zeta\\ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\Theta_2(\zeta)\,d\zeta\\ \vdots\\ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\Theta_n(\zeta)\,d\zeta\end{pmatrix} = \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\begin{pmatrix}\hat{f}_1 & 0 & \cdots & 0\\ 0 & \hat{f}_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \hat{f}_n\end{pmatrix}\begin{pmatrix}|a_{11}| & |a_{12}| & \cdots & |a_{1n}|\\ |a_{21}| & |a_{22}| & \cdots & |a_{2n}|\\ \vdots & \vdots & \ddots & \vdots\\ |a_{n1}| & |a_{n2}| & \cdots & |a_{nn}|\end{pmatrix}$$
$$\times\begin{pmatrix}|p_1^*| & 0 & \cdots & 0\\ 0 & |p_2^*| & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & |p_n^*|\end{pmatrix}\begin{pmatrix}|\varepsilon_1(\zeta)|\\ |\varepsilon_2(\zeta)|\\ \vdots\\ |\varepsilon_n(\zeta)|\end{pmatrix} d\zeta. \tag{28}$$

Let

$$\Pi_i(\zeta) = \hat{f}_i\mu\mathcal{M}_{\varepsilon\rho}(\zeta)\left(\sum_{j=1}^n \left|b_{ij}\right|q_j^* + \sum_{j=1}^n \left|e_{ij}\right|L_j + \sum_{j=1}^n \left|k_{ij}\right|L_j\right),$$

then

$$\begin{pmatrix}\int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\Pi_1(\zeta)\,d\zeta\\ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\Pi_2(\zeta)\,d\zeta\\ \vdots\\ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\Pi_n(\zeta)\,d\zeta\end{pmatrix} = \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\mathcal{M}_{\varepsilon\rho}(\zeta)\begin{pmatrix}\mu\hat{f}_1 & 0 & \cdots & 0\\ 0 & \mu\hat{f}_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mu\hat{f}_n\end{pmatrix}\left[\begin{pmatrix}|b_{11}| & |b_{12}| & \cdots & |b_{1n}|\\ |b_{21}| & |b_{22}| & \cdots & |b_{2n}|\\ \vdots & \vdots & \ddots & \vdots\\ |b_{n1}| & |b_{n2}| & \cdots & |b_{nn}|\end{pmatrix}\begin{pmatrix}q_1^*\\ q_2^*\\ \vdots\\ q_n^*\end{pmatrix}\right.$$
$$\left.+\begin{pmatrix}|e_{11}|+|k_{11}| & |e_{12}|+|k_{12}| & \cdots & |e_{1n}|+|k_{1n}|\\ |e_{21}|+|k_{21}| & |e_{22}|+|k_{22}| & \cdots & |e_{2n}|+|k_{2n}|\\ \vdots & \vdots & \ddots & \vdots\\ |e_{n1}|+|k_{n1}| & |e_{n2}|+|k_{n2}| & \cdots & |e_{nn}|+|k_{nn}|\end{pmatrix}\begin{pmatrix}L_1\\ L_2\\ \vdots\\ L_n\end{pmatrix}\right] d\zeta. \tag{29}$$

It can be seen from (27)–(29) that

$$\begin{pmatrix}|\varepsilon_1(t)|\\ |\varepsilon_2(t)|\\ \vdots\\ |\varepsilon_n(t)|\end{pmatrix} \le \begin{pmatrix}|\varepsilon_1(0)|\\ |\varepsilon_2(0)|\\ \vdots\\ |\varepsilon_n(0)|\end{pmatrix} + \begin{pmatrix}|\varepsilon_1'(0)|t\\ |\varepsilon_2'(0)|t\\ \vdots\\ |\varepsilon_n'(0)|t\end{pmatrix} + \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{diag}\left\{\begin{pmatrix}\vartheta_1\\ \vartheta_2\\ \vdots\\ \vartheta_n\end{pmatrix} + \begin{pmatrix}f_1^* & 0 & \cdots & 0\\ 0 & f_2^* & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & f_n^*\end{pmatrix}\begin{pmatrix}|I_1|\\ |I_2|\\ \vdots\\ |I_n|\end{pmatrix}\right.$$
$$+\begin{pmatrix}f_1^* & 0 & \cdots & 0\\ 0 & f_2^* & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & f_n^*\end{pmatrix}\left[\begin{pmatrix}|a_{11}| & |a_{12}| & \cdots & |a_{1n}|\\ |a_{21}| & |a_{22}| & \cdots & |a_{2n}|\\ \vdots & \vdots & \ddots & \vdots\\ |a_{n1}| & |a_{n2}| & \cdots & |a_{nn}|\end{pmatrix}\begin{pmatrix}P_1\\ P_2\\ \vdots\\ P_n\end{pmatrix} + \begin{pmatrix}|b_{11}| & |b_{12}| & \cdots & |b_{1n}|\\ |b_{21}| & |b_{22}| & \cdots & |b_{2n}|\\ \vdots & \vdots & \ddots & \vdots\\ |b_{n1}| & |b_{n2}| & \cdots & |b_{nn}|\end{pmatrix}\begin{pmatrix}Q_1\\ Q_2\\ \vdots\\ Q_n\end{pmatrix}\right.$$

$$
+ \mathcal{M}_{x\rho}(\zeta)\mu \begin{pmatrix} |e_{11}|+|k_{11}| & |e_{12}|+|k_{12}| & \cdots & |e_{1n}|+|k_{1n}| \\ |e_{21}|+|k_{21}| & |e_{22}|+|k_{22}| & \cdots & |e_{2n}|+|k_{2n}| \\ \vdots & \vdots & \ddots & \vdots \\ |e_{n1}|+|k_{n1}| & |e_{n2}|+|k_{n2}| & \cdots & |e_{nn}|+|k_{nn}| \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_n \end{pmatrix} \Bigg] \begin{pmatrix} |\varepsilon_1(\zeta)| \\ |\varepsilon_2(\zeta)| \\ \vdots \\ |\varepsilon_n(\zeta)| \end{pmatrix} \Bigg\} d\zeta
$$

$$
+ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \begin{pmatrix} \hat{f}_1 & 0 & \cdots & 0 \\ 0 & \hat{f}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{f}_n \end{pmatrix} \begin{pmatrix} |a_{11}| & |a_{12}| & \cdots & |a_{1n}| \\ |a_{21}| & |a_{22}| & \cdots & |a_{2n}| \\ \vdots & \vdots & \ddots & \vdots \\ |a_{n1}| & |a_{n2}| & \cdots & |a_{nn}| \end{pmatrix} \begin{pmatrix} |p_1^*| & 0 & \cdots & 0 \\ 0 & |p_2^*| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |p_n^*| \end{pmatrix} \begin{pmatrix} |\varepsilon_1(\zeta)| \\ |\varepsilon_2(\zeta)| \\ \vdots \\ |\varepsilon_n(\zeta)| \end{pmatrix} d\zeta
$$

$$
+ \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} M(\zeta) \begin{pmatrix} \mu\hat{f}_1 & 0 & \cdots & 0 \\ 0 & \mu\hat{f}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu\hat{f}_n \end{pmatrix} \Bigg[\begin{pmatrix} |b_{11}| & |b_{12}| & \cdots & |b_{1n}| \\ |b_{21}| & |b_{22}| & \cdots & |b_{2n}| \\ \vdots & \vdots & \ddots & \vdots \\ |b_{n1}| & |b_{n2}| & \cdots & |b_{nn}| \end{pmatrix} \begin{pmatrix} q_1^* \\ q_2^* \\ \vdots \\ q_n^* \end{pmatrix}
$$

$$
+ \begin{pmatrix} |e_{11}|+|k_{11}| & |e_{12}|+|k_{12}| & \cdots & |e_{1n}|+|k_{1n}| \\ |e_{21}|+|k_{21}| & |e_{22}|+|k_{22}| & \cdots & |e_{2n}|+|k_{2n}| \\ \vdots & \vdots & \ddots & \vdots \\ |e_{n1}|+|k_{n1}| & |e_{n2}|+|k_{n2}| & \cdots & |e_{nn}|+|k_{nn}| \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_n \end{pmatrix} \Bigg] d\zeta.
$$

Define

$$
M(\zeta) = \max\{M_\varepsilon(\zeta),\ M_x(\zeta)\}.
$$

That is,

$$
\begin{aligned}
|\varepsilon(t)| \le |\varepsilon(0)| + |\varepsilon'(0)|\, t + \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \Bigg\{ \Bigg(\operatorname{diag}\Big(\theta + F^* |I| + F^* (|A|\, P + |B|\, Q|) \\
+ F^* \mu M(\zeta) (|E| + |K|)\, L \Big) + \widehat{F}\, |A|\, P^* \Bigg) |\varepsilon(\zeta)| + \widehat{F} \mu M(\zeta) \Big(|B|\, Q^* + |E|\, L + |K|\, L \Big) \Bigg\} d\zeta. \qquad (30)
\end{aligned}
$$

Here, the vectors and matrices are defined as follows:

$$
\theta, P, Q \in \mathbb{R}_+^n, \quad I \in \mathbb{R}^n,
$$

and

$$
A = (a_{ij})_{n\times n}, \quad B = (b_{ij})_{n\times n}, \quad E = (e_{ij})_{n\times n}, \quad K = (k_{ij})_{n\times n} \in \mathbb{R}^{n\times n}.
$$

Moreover,

$$
F^*, \widehat{F}, P^*, Q^*, L \in \mathbb{R}_+^{n\times n}
$$

are diagonal matrices with positive entries, where

$$
\begin{aligned}
F^* &= \operatorname{diag}(f_1^*, f_2^*, \ldots, f_n^*), \qquad \widehat{F} = \operatorname{diag}\Big(\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n\Big), \\
P^* &= \operatorname{diag}(p_1^*, p_2^*, \ldots, p_n^*), \qquad Q^* = \operatorname{diag}(q_1^*, q_2^*, \ldots, q_n^*), \\
L &= \operatorname{diag}(L_1, L_2, \ldots, L_n).
\end{aligned}
$$

Let

$$\Omega = diag(\theta + F^* |I| + F^*(|A| P + |B| Q)) + \hat{F} |A| P^*, \ \Xi = F^* \mu (|E| + |K|) L, \ \Psi = \hat{F}\mu(|B| Q^* + |E| L + |K| L).$$

Thus, Eq (30) is transformed into

$$\|\varepsilon(t)\| \le \|\omega\| + \|\varepsilon'(0)\| t + \int_0^t \frac{(t-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \Big[\|\Omega\| \, \|\varepsilon(\zeta)\| + \|M(\zeta)\| \, \|\Xi\| \, \|\varepsilon(\zeta)\| + \|M(\zeta)\| \, \|\Psi\|\Big] d\zeta, \qquad t \in [0, T],$$

and

$$\|\varepsilon(t)\| \le \|\omega\|, \quad t \in [-\rho, 0].$$

According to Theorem 1, we have

$$\begin{aligned} u(t) &\le \|\omega\| + \|\varepsilon'(0)\| t + (\|\omega\| + \|\varepsilon'(0)\| t) \\ &\quad \times \left\{ \exp\left(\int_0^t 2^{\wp-1} B^{\wp}(t) \left[\|\Omega\| + \|M(s)\| \left(\|\Xi\| + \frac{\|\Psi\|}{\|\omega\| + \|\varepsilon'(0)\| s} \right) \right]^{\wp} ds \right) - 1 \right\}^{\frac{1}{\wp}} \\ &\le \delta(1+t) + \delta(1+t) \times \left\{ \exp\left(2^{\wp-1} \left[\frac{t^{\alpha-\frac{1}{\wp}}}{\Gamma(\alpha) \, [p(\alpha-1)+1]^{\frac{1}{p}}} \right]^{\wp} \int_0^t \Big[\|\Omega\| \right.\right. \\ &\quad \left.\left. + \|M(s)\| \left(\|\Xi\| + \frac{\|\Psi\|}{\|\omega\| + \|\varepsilon'(0)\| s} \right) \Big]^{\wp} ds \right) - 1 \right\}^{\frac{1}{\wp}}, \end{aligned} \tag{31}$$

where $t \in [0, T]$, $p, \wp > 0$ such that $\frac{1}{p} + \frac{1}{\wp} = 1$. If $\max\{\|\omega\|, \|\varepsilon'(0)\|\} \le \delta$, $\|\omega\| > 0$, and inequality (22) is satisfied, then from inequality (31), one has $\|\varepsilon(t)\| \le \eta, t \in [0, T]$. In other words, the FOCGNNs with time-varying delays and fuzzy logics in (10) possess FTS for $t \in [0, T]$. □

Remark 7. The model addressed in this study includes several existing neural network models as special cases. When $\varphi_j(t) = \tau$ is a constant, the considered system becomes a Cohen–Grossberg neural network of fractional order with constant delay. If $E = 0$ and $K = 0$, the fuzzy min and max the feedback terms disappear, and the model reduces to a non-fuzzy delayed FOCGNN. Furthermore, when $f_i(x_i(t)) = 1$ and $g_i(x_i(t))$ is chosen as a linear or constant function, the considered model can be reduced to some ordinary fractional-order delayed neural network models. Therefore, the obtained finite-time stability criterion is applicable to a broader class of fractional-order neural networks.

4. Numerical example and simulation

This section provides three numerical simulations to evaluate the effectiveness of the proposed finite-time stability criterion. We investigate the following fractional-order Cohen–Grossberg neural network involving time-varying delays and fuzzy logic terms:

$$\begin{aligned} {}^C D_0^{\alpha} x_i(t) = -f_i(x_i(t)) \Bigg(& g_i(x_i(t)) - \sum_{j=1}^{3} a_{ij} p_j(x_j(t)) - \sum_{j=1}^{3} b_{ij} q_j(x_j(t - \varphi_j(t))) \\ & - \bigwedge_{j=1}^{3} e_{ij} G_j(x_j(t - \varphi_j(t))) - \bigvee_{j=1}^{3} k_{ij} G_j(x_j(t - \varphi_j(t))) - I_i \Bigg), \end{aligned}$$

where $i = 1, 2, 3$, $1 < \alpha < 2$, and the time-varying delay is constrained within the range

$$0 \le \varphi_j(t) \le \varphi < \rho.$$

4.1. Example 1

In this example, we take

$$\alpha = 1.8, \quad \varphi = 0.1, \quad \rho = 1.$$

We then have,

$$\mu = \left(\frac{\rho}{\rho - \varphi}\right)^{\alpha} = \left(\frac{1}{1 - 0.1}\right)^{1.8} = 1.2088.$$

The connection weight matrices are selected as follows:

$$A = \begin{pmatrix} 0.027 & 0.008 & 0.029 \\ 0.018 & 0.017 & 0.005 \\ 0.003 & 0.029 & 0.029 \end{pmatrix}, \quad B = \begin{pmatrix} 0.029 & 0.004 & 0.024 \\ 0.015 & 0.013 & 0.029 \\ 0.024 & 0.028 & 0.020 \end{pmatrix}$$

and

$$I = \begin{pmatrix} 0.0478 \\ -0.014 \\ 0.081 \end{pmatrix}.$$

For simplicity, the fuzzy connection matrices are chosen as follows:

$$E = A, \quad K = B.$$

The amplification functions, behavior functions, activation functions, and fuzzy logic functions are given by $f_i(x_i(t)) = 0.2\sin(x_i(t)) + 0.3$, $g_i(x_i(t)) = 0.8\sin(x_i(t))$, $p_j(r) = q_j(r) = 0.05(|r + 1| - |r - 1|)$, and $G_j(r) = 0.05(|r + 1| - |r - 1|)$. It is easy to obtain

$$0.1 \le f_i(x_i(t)) \le 0.5$$

and

$$f_i^* = 0.2, \qquad P_j = Q_j = p_j^* = q_j^* = 0.1.$$

Moreover, we take

$$L_f = L_l = 0.1.$$

The initial functions are chosen as

$$x(t) = \begin{pmatrix} 1.12 - 0.007t \\ 1.56 + 0.002t \\ -1.53 + 0.01t \end{pmatrix}, \quad y(t) = \begin{pmatrix} 1.125 - 0.001t \\ 1.566 + 0.006t \\ -1.525 + 0.02t \end{pmatrix},$$

where $t \in [-1, 0]$.

Let $\mathcal{E}(t) = x(t) - y(t)$, then $\|\omega\| = \sup_{t\in[-0.1,0]} \|\mathcal{E}(t)\| = 0.016$, and $\|\mathcal{E}'(0)\| = 0.02$, and thus, $\max\{\|\omega\|, \|\mathcal{E}'(0)\|\} = 0.02$.

In the following calculation, we choose $\delta = 0.02$.

According to the assumptions of the proposed theorem, let

$$\theta = \begin{pmatrix} 0.56 \\ 0.56 \\ 0.56 \end{pmatrix}, \quad F^* = 0.2I_3, \quad \widehat{F} = 0.5I_3.$$

The matrices in the proposed criterion are given by

$$\Omega = \operatorname{diag}(\theta + F^*|I| + F^*(|A|P + |B|Q)) + \widehat{F}|A|P^*,$$

$$\Xi = F^* L_f \mu(|E| + |K|)$$

and

$$\Psi = \widehat{F}\mu\left(|B|Q^* + L_f|E| + L_l|K|\right).$$

Substituting the abovementioned parameters into these matrices gives

$$\|\Omega\| = 0.579616, \quad \|\Xi\| = 0.003288, \quad \text{and} \quad \|\Psi\| = 0.012632.$$

In addition, according to the definition of the envelope function, we take $M_0 = 1.56$ such that

$$\|M(s)\| \leq M_0, \qquad s \in [0, T].$$

Then the finite-time stability condition takes the following form:

$$(1+t) + (1+t)\left\{ \exp\left[2^{p-1} \left(\frac{t^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)\left[q(\alpha-1)+1\right]^{\frac{1}{q}}} \right)^p \times \int_0^t \left[\|\Omega\| + \|M(s)\| \left(\|\Xi\| + \frac{\|\Psi\|}{\|\omega\| + \|\mathcal{E}'(0)\| s} \right) \right]^p ds \right] - 1 \right\}^{\frac{1}{p}} \leq \frac{\eta}{\delta},$$

where $p, q > 0$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

Choose

$$p = 1.3, \quad q = \frac{p}{p-1} = 4.3333.$$

According to Theorem 2, for different values of T, Table 1 summarizes the calculated estimates of the upper bound η.

Table 1. η for $\alpha = 1.8$ and $\delta = 0.02$.

T	0.4	0.8	1.2	1.6	2.0
η in Theorem 2	0.0362	0.0794	0.2169	0.8143	4.4369

It follows from Table 1 that, for $\delta = 0.02$, the estimated bound corresponding to $T = 2$ is $\eta = 4.4369$. Therefore, if the initial condition satisfies

$$\max\{\|\omega\|, \|\mathcal{E}'(0)\|\} \leq \delta,$$

then the proposed theorem guarantees that

$$\|\mathcal{E}(t)\| \leq \eta, \qquad t \in [0, 2].$$

The numerical trajectory of $100\|\mathcal{E}(t)\|$ is illustrated in Figure 1. It is evident that the error norm does not exceed the theoretical bound $\eta = 4.4369$ on the finite-time interval $[0, 2]$, which verifies the feasibility of the proposed finite-time stability criterion.

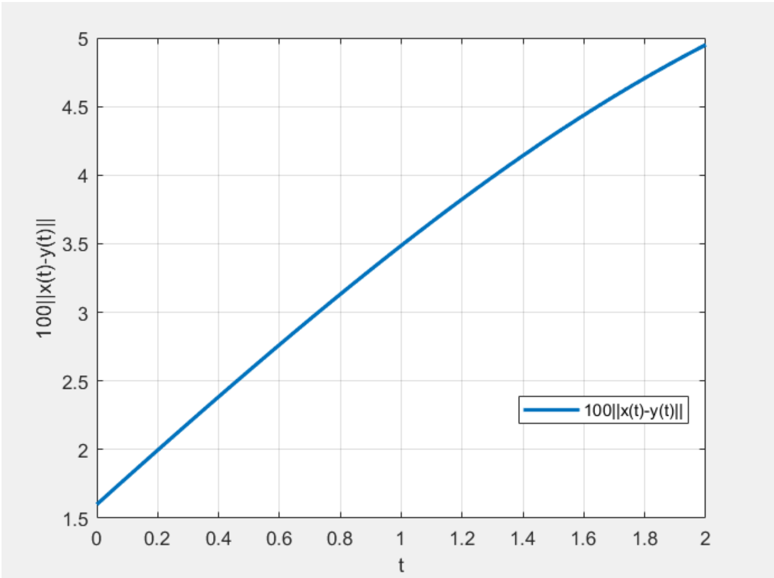

Figure 1. Amplified error norm of the FOCGNNs with time-varying delays and fuzzy logics.

Remark 8. The actual error norm $\|x(t) - y(t)\|$ is very small during the simulation process. Therefore, its variation trend is not sufficiently visible in the original scale. To make the dynamic behavior of the error norm more clearly observable, we plot the amplified quantity $100\|x(t) - y(t)\|$ in Figure 1. This amplification is only used for graphical presentation and does not affect the theoretical analysis or the validity of the finite-time stability result.

Remark 9. It should be noted that the estimated bound η is obtained from a sufficient finite-time stability condition. Hence, η is a theoretical upper bound rather than the actual maximum value of the error trajectory. The relatively large increase in η from $T = 1.6$ to $T = 2.0$ is mainly caused by the exponential term in the proposed criterion. More precisely, the bound contains the term $\exp\left\{2^{p-1}C^p(T)\int_0^T H^p(s)\,ds\right\}$. When T increases, both $C^p(T)$ and $\int_0^T H^p(s)\,ds$ increase simultaneously, and their product is further amplified by the exponential function. This explains why the estimated bound η increases from 0.8143 to 4.4369. The phenomenon reflects the conservativeness of the sufficient condition throughout a longer finite-time interval and does not indicate that the actual error trajectory experiences the same sharp growth.

4.2. Example 2

In this simulation, the fractional order and delay parameters are chosen as

$$\alpha = 1.65, \quad \rho = 1, \quad \varphi = 0.09,$$

then,

$$\mu = \left(\frac{\rho}{\rho - \varphi}\right)^{\alpha} = \left(\frac{1}{1 - 0.09}\right)^{1.65} = 1.1684.$$

The connection weight matrices are selected as

$$A = \begin{pmatrix} 0.020 & 0.012 & 0.018 \\ 0.010 & 0.025 & 0.006 \\ 0.006 & 0.021 & 0.018 \end{pmatrix}, \quad B = \begin{pmatrix} 0.018 & 0.006 & 0.020 \\ 0.011 & 0.016 & 0.022 \\ 0.019 & 0.024 & 0.015 \end{pmatrix}.$$

The fuzzy feedback matrices are given by

$$E = \begin{pmatrix} 0.015 & 0.010 & 0.012 \\ 0.009 & 0.018 & 0.007 \\ 0.008 & 0.016 & 0.014 \end{pmatrix}, \quad K = \begin{pmatrix} 0.016 & 0.005 & 0.018 \\ 0.010 & 0.012 & 0.020 \\ 0.017 & 0.019 & 0.013 \end{pmatrix}.$$

The external input vector is chosen as

$$I = \begin{pmatrix} 0.035 \\ -0.020 \\ 0.060 \end{pmatrix}.$$

The amplification functions, behavior functions, activation functions, and fuzzy logic functions are defined as follows:

$$f_i(r) = 0.15\sin(r) + 0.35, \quad g_i(r) = 0.7\sin(r)$$

and

$$p_j(r) = q_j(r) = G_j(r) = 0.04(|r+1| - |r-1|), \qquad i, j = 1, 2, 3.$$

It is easy to see that

$$0.20 \le f_i(r) \le 0.50.$$

For convenience and conservativeness, we take

$$F^* = 0.2I_3, \quad \widehat{F} = 0.5I_3$$

and

$$P = Q = \begin{pmatrix} 0.1 \\ 0.1 \\ 0.1 \end{pmatrix}, \quad P^* = Q^* = 0.1I_3.$$

Moreover, the Lipschitz constants of the fuzzy logic functions are selected as follows:

$$L_f = L_l = 0.1.$$

The time-varying delays are chosen as follows:

$$\varphi_1(t) = 0.06 + 0.02\sin^2(t),$$

$$\varphi_2(t) = 0.07 + 0.02\cos^2(t)$$

and

$$\varphi_3(t) = 0.08 + 0.01\sin^2(2t).$$

Obviously, these delays satisfy

$$0 \le \varphi_j(t) \le 0.09 < 1 = \rho, \qquad j = 1, 2, 3.$$

The initial functions of the two different solutions are selected as

$$x(t) = \begin{pmatrix} 1.05 - 0.006t \\ 1.42 + 0.003t \\ -1.35 + 0.008t \end{pmatrix}, \qquad y(t) = \begin{pmatrix} 1.055 - 0.002t \\ 1.426 + 0.007t \\ -1.345 + 0.016t \end{pmatrix},$$

where $t \in [-\rho, 0]$. Let

$$\varepsilon(t) = x(t) - y(t).$$

Then

$$\|\omega\| = \sup_{t\in[-\rho,0]} \|\varepsilon(t)\| = 0.016$$

and

$$\|\varepsilon'(0)\| = 0.016.$$

Therefore, we have

$$\max\{\|\omega\|, \|\varepsilon'(0)\|\} = 0.016.$$

In the following calculation, we choose

$$\delta = 0.02,$$

which satisfies

$$\max\{\|\omega\|, \|\varepsilon'(0)\|\} \le \delta.$$

According to Theorem 2, choose

$$\theta = \begin{pmatrix} 0.56 \\ 0.56 \\ 0.56 \end{pmatrix}.$$

The matrices appearing in the finite-time stability criterion are

$$\Omega = \mathrm{diag}\,(\theta + F^*|I| + F^*(|A|P + |B|Q)) + \widehat{F}|A|P^*,$$

$$\Xi = F^* L_f \mu(|E| + |K|)$$

and

$$\Psi = \widehat{F}\mu\left(|B|Q^* + L_f|E| + L_l|K|\right).$$

Substituting the parameters above into these matrices gives

$$\|\Omega\| = 0.574306,$$

$$\|\Xi\| = 0.001963$$

and

$$\|\Psi\| = 0.008237.$$

In addition, according to the definition of the envelope function, we take

$$M_0 = 1.50$$

such that

$$\|M(s)\| \le M_0, \qquad s \in [0, T].$$

Let

$$p = 1.3, \qquad q = \frac{p}{p-1} = 4.3333.$$

For different values of T, the corresponding estimates of η are listed in Table 2.

Table 2. η for $\alpha = 1.65$ and $\delta = 0.02$.

T	0.4	0.8	1.2	1.6	2.0
η in Theorem 2	0.0357	0.0723	0.1653	0.4540	1.5608

It follows from Table 2 that, for $\delta = 0.02$, the estimated bound corresponding to $T = 2.0$ is

$$\eta = 1.5608.$$

Therefore, if the initial condition satisfies

$$\max\{\|\omega\|, \|\varepsilon'(0)\|\} \le \delta,$$

then Theorem 2 guarantees that

$$\|\varepsilon(t)\| \le \eta, \quad t \in [0, 2].$$

From Figure 2, the error norm remains bounded on the finite-time interval $[0, 2]$. Moreover, the actual value of $\|\varepsilon(t)\|$ is much smaller than the theoretical upper bound $\eta = 1.5608$. This verifies the effectiveness of the proposed finite-time stability criterion under another group of parameter settings.

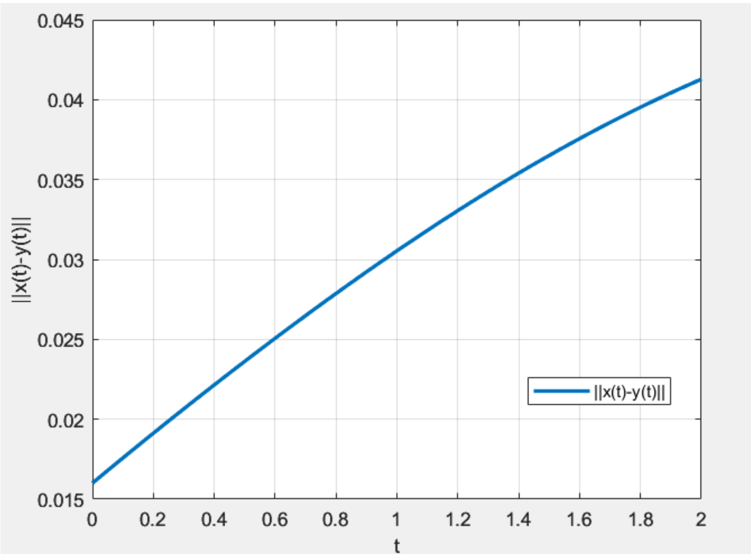

Figure 2. Error norm under an alternative parameter setting.

4.3. Example 3

In this example, we further verify the applicability of the proposed finite-time stability criterion by considering another group of parameters. Different from the previous examples, the activation functions and fuzzy logic functions are chosen as hyperbolic tangent functions. This shows that the obtained result is still effective when the nonlinear feedback functions are changed.

In this simulation, the fractional order and delay parameters are selected as

$$\alpha = 1.72, \qquad \rho = 1, \qquad \varphi = 0.085.$$

We then have,

$$\mu = \left(\frac{\rho}{\rho - \varphi}\right)^{\alpha} = \left(\frac{1}{1 - 0.085}\right)^{1.72} = 1.1651.$$

The connection matrices are chosen as

$$A = \begin{pmatrix} 0.018 & 0.010 & 0.021 \\ 0.012 & 0.022 & 0.008 \\ 0.007 & 0.019 & 0.024 \end{pmatrix}, \quad B = \begin{pmatrix} 0.020 & 0.007 & 0.017 \\ 0.013 & 0.018 & 0.019 \\ 0.016 & 0.022 & 0.014 \end{pmatrix}.$$

The fuzzy feedback matrices are given by

$$E = \begin{pmatrix} 0.012 & 0.009 & 0.015 \\ 0.010 & 0.014 & 0.006 \\ 0.007 & 0.013 & 0.016 \end{pmatrix}, \quad K = \begin{pmatrix} 0.014 & 0.006 & 0.016 \\ 0.009 & 0.011 & 0.018 \\ 0.015 & 0.017 & 0.012 \end{pmatrix}.$$

The external input vector is

$$I = \begin{pmatrix} 0.040 \\ -0.018 \\ 0.065 \end{pmatrix}.$$

The amplification function and behavior function are selected as

$$f_i(r) = 0.12\cos(r) + 0.38, \qquad g_i(r) = 0.65\tanh(r), \qquad i = 1, 2, 3.$$

Since $-1 \leq \cos(r) \leq 1$, one has

$$0.26 \leq f_i(r) \leq 0.50.$$

Moreover,

$$|f_i'(r)| = |-0.12\sin(r)| \leq 0.12.$$

Thus, we take

$$F^* = 0.12 I_3, \quad \widehat{F} = 0.50 I_3.$$

The activation functions and fuzzy logic functions are changed into the following tanh-type functions:

$$p_j(r) = 0.06\tanh(r),$$

$$q_j(r) = 0.05\tanh(0.8r)$$

and

$$G_j(r) = 0.045 \tanh(r), \quad j = 1, 2, 3.$$

Therefore, the following bounds can be obtained:

$$|p_j(r)| \le 0.06, \quad |p'_j(r)| \le 0.06,$$

$$|q_j(r)| \le 0.05, \quad |q'_j(r)| \le 0.04$$

and

$$|G_j(r)| \le 0.045, \qquad |G'_j(r)| \le 0.045.$$

Hence, we take

$$P = \begin{pmatrix} 0.06 \\ 0.06 \\ 0.06 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.05 \\ 0.05 \\ 0.05 \end{pmatrix},$$

$$P^* = 0.06 I_3, \quad Q^* = 0.04 I_3,$$

and

$$L_f = L_l = 0.045.$$

For the product $f_i(r)g_i(r)$, a conservative Lipschitz constant is chosen as follows:

$$\theta = \begin{pmatrix} 0.45 \\ 0.45 \\ 0.45 \end{pmatrix}.$$

The time-varying delays are defined as

$$\varphi_1(t) = 0.055 + 0.020 \sin^2(t),$$

$$\varphi_2(t) = 0.060 + 0.025 \cos^2(t)$$

and

$$\varphi_3(t) = 0.070 + 0.015 \sin^2(2t).$$

It is clear that

$$0 \le \varphi_j(t) \le 0.085 < 1 = \rho, \quad j = 1, 2, 3.$$

The initial functions of two different solutions are selected as follows:

$$x(t) = \begin{pmatrix} 1.08 - 0.005t \\ 1.38 + 0.004t \\ -1.28 + 0.007t \end{pmatrix}, \quad y(t) = \begin{pmatrix} 1.086 - 0.001t \\ 1.386 + 0.008t \\ -1.274 + 0.014t \end{pmatrix},$$

where $t \in [-\rho, 0]$. Let

$$\varepsilon(t) = x(t) - y(t).$$

Then

$$\|\omega\| = \sup_{t \in [-\rho, 0]} \|\varepsilon(t)\| = 0.018$$

and

$$\|\varepsilon'(0)\| = 0.015.$$

Thus,

$$\max\{\|\omega\|, \|\varepsilon'(0)\|\} = 0.018.$$

In this example, choose

$$\delta = 0.02.$$

We then have,

$$\max\{\|\omega\|, \|\varepsilon'(0)\|\} \leq \delta.$$

According to Theorem 2, the matrices in the finite-time stability criterion are given by

$$\Omega = \operatorname{diag}(\theta + F^*|I| + F^*(|A|P + |B|Q)) + \widehat{F}|A|P^*,$$

$$\Xi = F^* L_f \mu(|E| + |K|)$$

and

$$\Psi = \widehat{F}\mu\left(|B|Q^* + L_f|E| + L_l|K|\right).$$

Substituting the parameters above into these matrices yields

$$\|\Omega\| = 0.459420,$$
$$\|\Xi\| = 0.000522$$

and

$$\|\Psi\| = 0.003341.$$

For the envelope function, we take

$$M_0 = 1.50.$$

Choose

$$p = 1.3, \qquad q = \frac{p}{p-1} = 4.3333.$$

Then, for different values of T, the corresponding estimates of η are shown in Table 3.

Table 3. η for $\alpha = 1.72$ and $\delta = 0.02$.

T	0.4	0.8	1.2	1.6	2.0
η in Theorem 2	0.0319	0.0535	0.0943	0.1805	0.3891

From Table 3, when $T = 2.0$, the estimated upper bound is

$$\eta = 0.3891.$$

Therefore, if the initial condition satisfies

$$\max\{\|\omega\|, \|\varepsilon'(0)\|\} \leq \delta,$$

then, Theorem 2 guarantees that

$$\|\varepsilon(t)\| \leq \eta, \qquad t \in [0, 2].$$

It can be observed that the error norm remains bounded on the finite-time interval $[0, 2]$. In Figure 3, the actual maximum value of $\|\varepsilon(t)\|$ is much smaller than the theoretical upper bound $\eta = 0.3891$. This confirms that the proposed finite-time stability criterion is still valid when the activation functions are replaced by tanh-type nonlinear functions.

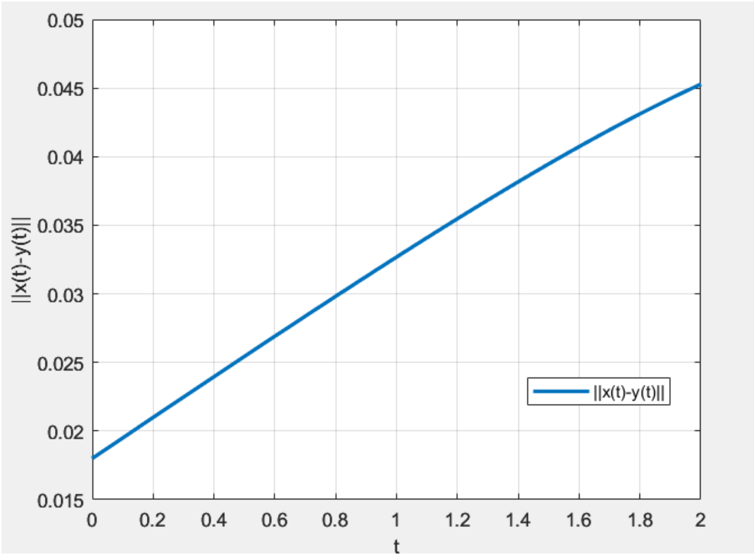

Figure 3. Error norm with tanh-type activation functions.

Remark 10. Compared with the previous examples, this example uses tanh-type activation functions instead of piecewise linear saturation functions. Since $\tanh(\cdot)$ is smooth, bounded, and globally Lipschitz continuous, the assumptions required in Theorem 2 are still satisfied. This demonstrates that the proposed finite-time stability criterion is not restricted to a specific form of activation functions.

5. Conclusions

The focus of this paper is on examining the finite-time stability properties of fractional-order CGNNs with fractional orders ranging from 1 to 2 under the influence of time-varying delays and fuzzy logic. The considered model includes Caputo fractional derivatives, Cohen–Grossberg-type amplification functions, time-varying delay terms, and fuzzy min/max feedback operators, which makes the stability analysis more complicated than that of conventional delayed neural networks. To handle the influence of time-varying delays, a weighted envelope function technique has been introduced to estimate the delayed states. Based on this treatment, a new Gronwall-type inequality has been derived. By applying the proposed inequality, a sufficient condition has been established to guarantee the finite-time stability of the addressed fractional-order CGNNs. Finally, three simulation examples are provided to verify the effectiveness of the obtained theoretical result.

From a practical viewpoint, the obtained finite-time stability criterion provides a computable tool for evaluating the transient boundedness of fractional-order neural-network-based systems with time-varying delays and fuzzy feedback. Once the system's parameters, delay bounds, and Lipschitz constants are known, the proposed criterion can be directly checked to determine whether the state error remains within a prescribed bound on a finite-time interval. Compared with some existing results, the present method can simultaneously handle Cohen–Grossberg amplification functions, time-varying delays, and fuzzy min/max feedback terms, which makes it applicable to a broader class of delayed neural network models. Future studies may consider extending the proposed method to more general fractional-order neural network models, such as systems with distributed delays, stochastic disturbances, impulsive effects, or memristive connection weights. In addition, reducing the conservativeness of the finite-time stability criterion and developing less restrictive estimation techniques for time-varying delays are also meaningful topics for further investigation.

Author contributions

Qijun Dong: draft writing, review and editing, graphics, testing, programming, material support; Ailong Wu: review and editing; Tao Li: review and editing, graphics, programming; Liguang Wan: final review, study design, financial support. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There are no conflicts of interest in this paper.

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