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Research article

Square-free numbers of the form $\lfloor m^c \rfloor$ with smooth indices

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Abstract: The aim of this article was to study the square-free numbers of the form $\lfloor m^c \rfloor$ with smooth indices, which can be seen as an extension of a result of Cao and Zhai. As an extension to the work of Cao and Zhai, the proof of Theorem 1.1 employed the division trick, which transformed an arithmetic problem into a problem concerning exponential sum estimates. The division trick proved to be more direct and concise in comparison with alternative approaches. For the sake of completeness of our results, Theorem 1.2, serving as a complement to Theorem 1.1, can be proved by invoking results from the work of Cao and Zhai. Combining Theorems 1.1 and 1.2, we obtained our full conclusion, namely, Theorem 1.3.

Keywords: square-free numbers; floor functions; division trick; smooth numbers
Mathematics Subject Classification: 11N37, 11L07

1. Introduction

The arithmetic properties of Piatetski-Shapiro sequences has been studied by many scholars. For various interesting results concerning the Piatetski-Shapiro sequence, one may refer to the following works: Baker et al. [1] studied the basic properties of the Piatetski-Shapiro sequence; Chapman et al. [2] studied additive Ramsey theory over Piatetski-Shapiro numbers; Fan [3] studied small multiplicative functions in Piatetski-Shapiro sequences; Zhang [4] studied (k, r)-integers in Piatetski-Shapiro sequences. The application of the arithmetic properties of Piatetski-Shapiro sequences has also been studied, such as Yang, Shi, Zhang, and Zhang [5] who gave an analysis of a new time-space two-grid method for the two-dimensional nonlinear nonlocal mobile/immobile transport model, and Liu, Zhang, and Wang [6] who gave a new high-order compact Crank-Nicolson alternating direction implicit (CN-ADI) difference scheme on graded meshes for three-dimensional nonlinear partial integro-differential equations (PIDEs) with multiple weakly singular kernels. Naturally, whether certain properties also hold on some subsequences is an interesting question. Cao and Zhai [7] studied

the square-free numbers of the form $\lfloor n^c \rfloor$ with n in special sets. Recall that a positive integer $n \in \mathbb{Z}$ is called square-free if it is free of prime square factors. Let Q denote the set of a square-free number. Cao and Zhai [7] proved that if a set of positive integers $\mathfrak{A}$ satisfies the following two conditions:
(1) For every sufficient small positive number ε, we have

$$\mathfrak{A}(x) := \mathfrak{A} \cap \{n \mid n \leq x\} \gg x^{1-\varepsilon}. \tag{1.1}$$

(2) Let $\varepsilon > 0$ be a sufficient small positive number, let $x > 1$ be a real number, and let a, q be two positive integers with $(a, q) = 1$ and $2 < q < x^{\varepsilon}$. Then, there exists a real number $\varepsilon < \delta < 1/2$ that may depend on c such that

$$\sum_{n \leq x,\, n \in \mathfrak{A}} e\left(\frac{a\lfloor n^c \rfloor}{q}\right) \ll x^{1-\delta}, \tag{1.2}$$

then

$$\mathfrak{A}^c(x) \cap Q = \frac{6\mathfrak{A}(x)}{\pi^2} + O(x^{1-\varepsilon})$$

holds, where $1 < c < 149/87$ and $\mathfrak{A}^c(x)$ denotes the set $\{\lfloor n^c \rfloor \mid n \in \mathfrak{A},\ n \leq x\}$

In this paper, we study the square-free values of the forms $\lfloor m^c \rfloor$ where $m \leq x$ is a y-smooth number. For an integer n, let $P^+(n)$ be the largest prime factor of n. Given a real $y \geq 2$, if $P^+(n) \leq y$, we call n y-smooth. We remark that smooth numbers play a critical role in number theory. For more results about smooth numbers, readers can consult the literature for a variety of notable results on smooth number: Akbal [8] studied smooth number in Piatetski-Shapiro sequences; Baker [9] studied smooth numbers in Beatty sequences; Jain [10] give a short intervals results about smooth numbers; Song [11] studied the smooth number of the forms

$$\left\lfloor \frac{x}{n} \right\rfloor \quad \text{and} \quad \left\lfloor \frac{x}{n^c} \right\rfloor.$$

There are also some surveys about smooth numbers, for instances, Granville [12] presented a survey from the perspective of computational number theory, while Moree [13] provided a survey with respect to analytic number theory.

Let $2 \leq y \leq x$, and recall the notation

$$S(x, y) := \{n \leq x \mid P^+(n) \leq y\}, \qquad \Psi(x, y) := |S(x, y)|.$$

When $y = (\log x)^{\kappa}$ for some fixed $\kappa > 1$, by Eq (1.14) in paper [12], we have

$$x^{1-1/\kappa-\varepsilon} \leq \Psi(x, (\log x)^{\kappa}) \leq x^{1-1/\kappa+\varepsilon}, \qquad x \to \infty. \tag{1.3}$$

So by Eq (1.3), in the range $\log^a x < y < \log^b x$, the Condition (1) above is no longer valid, so the result of Cao and Zhai [7] is not applicable. Thus, the following results improve Cao and Zhai [7].

Theorem 1.1. *Let X, y, b, T, and c be reals with $1 < c < \frac{13}{12}$ and $(\log X)^T < y < X^b$. Suppose b, T, c satisfied*

$$8c + \frac{12}{T} + b < 9,$$

$$17c + \frac{24}{T} + b < 18,$$

$$12c + \frac{18}{T} + b < 13,$$

$$5c + \frac{10}{T} < 6.$$

Let

$$S(X) := \sum_{\substack{m \le X \\ m \text{ is } y\text{-smooth}}} \mu^2(\lfloor m^c \rfloor),$$

where $\mu(n)$ is the Möbius function. Then, we have

$$S(X) = \frac{6\Psi(X,y)}{\pi^2} + O\left(\frac{\Psi(X,y)}{X^{\epsilon}}\right).$$

Theorem 1.2. *Let X, y, b, and c be reals with $1 < c < \frac{4-b}{3}$ and $y = X^b$ with $0 < b < 1$. Let*

$$S(X) := \sum_{\substack{m \le X \\ m \text{ is } y\text{-smooth}}} \mu^2(\lfloor m^c \rfloor),$$

where $\mu(n)$ is the Möbius function. Then, we have

$$S(X) = \frac{6\Psi(X,y)}{\pi^2} + O\left(\frac{\Psi(X,y)}{X^{\epsilon}}\right).$$

By Theorems 1.1 and 1.2, we have

Theorem 1.3. *Let X, y, and c be reals with $1 < c < \frac{13}{12}$ and $(\log X)^T < y < X^{1-\eta}$, where $\eta > 0$ is sufficiently small. Let T, c satisfy*

$$73c + \frac{120}{T} < 90 - 9\varepsilon,$$

$$12c + \frac{18}{T} < 13 - \varepsilon,$$

$$5c + \frac{10}{T} < 6.$$

Let m be a y-smooth number, and let

$$S(X) := \sum_{\substack{m \le X \\ m \text{ is } y\text{-smooth}}} \mu^2(\lfloor m^c \rfloor),$$

where $\mu(n)$ is the Möbius function. Then, we have

$$S(X) = \frac{6\Psi(X,y)}{\pi^2} + O\left(\frac{\Psi(X,y)}{X^{\epsilon}}\right).$$

The Proof Strategies of Theorems 1.1 and 1.2.

In this subsection, we summarize the proof strategies of Theorems 1.1 and 1.2 as follows. For Theorem 1.1, when $y \asymp \log^a x$, by Eq (1.3), the Condition (1) of the Theorem 1 in [7] dose not hold. Hence, we cannot use the consequence of Theorem 1 in [7]. Furthermore, the proof method for the main result in [7] is not applicable to Theorem 1.1. We use the division trick to detect the condition $d \mid \lfloor m^c \rfloor$, then we can prove our theorem by estimating exponential sums. Compared with other methods, the division trick is more straightforward and concise. For Theorem 1.2, we can use the consequence of Theorem 1 in [7] by verifying the Conditions (1) and (2) in Theorem 1 of [7]. Verifying the conditions for Theorem 1 in [7] is not quite obvious; specially, we need to use the partition of unity and the estimates of exponential sums.

Remark 1.4. We remark that the reason why we can extend the work of Cao and Zhai [7] is mainly based on the different way to detect the condition $q \mid \lfloor m^c \rfloor$. In Cao and Zhai [7], they used additive character to detect this condition, and in this paper, we use the division trick to detect this condition. The merit of our method is to avoid the Condition (1) in Cao and Zhai [7]. Our results dose dependent on the special multiplicative structure of the smooth number (see Lemma 3.3) because of the estimations of error terms.

2. Some notation

Let H and $c > 1$ be real numbers. For two integers m and M, $m \sim M$ means $M \leq m < 2M$. We define $P^+(n)$ to be the largest prime factor of n and $P^-(n)$ to be the smallest prime factor of n. ε always means a small positive number, but possibly a different one each time. Given a real number x, we write $e(x) = \mathrm{e}^{2\pi i x}$, $\{x\}$ for the fractional part of x, and $\lfloor x \rfloor$ for the greatest integer not exceeding x. We also write $\mathcal{L} = \log H$. For a function F and a real nonnegative function G, the notations $F \ll G$ and $F = O(G)$ are equivalent to mean that $|F| \leq CG$ for some constant $C > 0$. We also write $F \asymp G$ to indicate both $F \ll G$ and $G \ll F$. Finally, $\mathfrak{A}$ denotes a sequence consisting of natural numbers, and $\mathfrak{A}^c$ denotes a sequence $\{n^c \mid n \in \mathfrak{A}\}$. Also, Q denotes the set of square-free numbers.

3. Preliminaries

Before proving theorems, we list some preliminary lemmas.

Lemma 3.1. *Let x be a real number, and let $\psi(x) = \{x\} - \frac{1}{2}$. Then, there exist numbers a_h, b_h such that*

$$\left|\psi(x) - \sum_{0<|h|\leq H} a_h e(xh)\right| \leq \sum_{|h|\leq H} b_h e(xh),$$

where $a_h \ll \frac{1}{|h|}$, $b_h \ll \frac{1}{H}$.

Proof. The proof of this lemma can be found in [14]. □

To estimate exponential sums, the following lemma is also needed.

Lemma 3.2. *Let α_1, β_1, and γ_1 be real numbers with $\alpha_1(\alpha_1-1)\beta_1\gamma_1 \neq 0$; let H_1, K_1, L_1, and X_1 be four real numbers; and let*

$$S := \sum_{h\sim H_1}\sum_{n\sim K_1} a(h,n) \sum_{m\sim L_1} b(m)\, e\left(X_1 \frac{h^{\beta_1} n^{\gamma_1} m^{\alpha_1}}{H_1^{\beta_1} K_1^{\gamma_1} L_1^{\alpha_1}}\right),$$

where $a(h,n)$ and $b(m)$ are two functions. Suppose $b(m) \ll (2L_1)^{\varepsilon}$ for all $m \in (L_1, 2L_1]$, then we have

$$S \ll (H_1K_1L_1)^{1+\varepsilon}\left\{\left(\frac{X_1}{H_1K_1L_1^2}\right)^{1/4} + \frac{1}{(H_1K_1)^{1/4}} + \frac{1}{L_1^{1/2}} + \frac{1}{X_1^{1/2}}\right\}.$$

Proof. The proof of this lemma can be found in [15]. □

Lemma 3.3 ([16] Lemma 10.1)**.** *Let y, R, and x be three reals, and let n be an integer. If*

$$2 \le y \le R \le n \le x$$

with n is y-smooth, then there exists a unique integers triple (p,u,v) such that
(i) *$n = puv$,*
(ii) *$p \le y$,*
(iii) *$R/p < v \le R$ with $P^-(v) \ge p$ and $P^+(v) \le y$,*
(iv) *$u \le x/pv$ with $P^+(u) \le p$.*

To treat the floor functions in the exponential sums, we need the so-called partition of unity.

Lemma 3.4 ([17] Lemma 2)**.** *For any $Z \in \mathbb{N}$ and $x \in \mathbb{R}_+$, there exist periodic functions $g_z(t)$ with period 1 $(0 \le z \le 2Z-1)$, such that*

$$0 < g_z(t) \le 1, \quad \text{if } \left|t - \frac{z}{2Z}\right| < \frac{1}{2Z},$$
$$g_z(t) = 0, \quad \text{if } \frac{1}{2Z} \le \left|t - \frac{z}{2Z}\right| \le \frac{1}{2},$$
$$g_z(t) = \sum_{|n|\le Z(\log x)^4} \beta_n^{(z)} e(nt) + O(x^{-\log\log x}).$$

where

$$|\beta_n^{(z)}| \le \frac{1}{2Z}.$$

We also have

$$\sum_{z=0}^{2Z-1} g_z(t) = 1.$$

Lemma 3.5 ([18] Theorem 2.1)**.** *Let I be an interval. If f is continuously differentiable on I, f' is monotonic on I, and $\|f'\| \ge \lambda > 0$ on I, then*

$$\sum_{n\in I} e(f(n)) \ll \lambda^{-1}.$$

Lemma 3.6. *Let Z be a large enough positive real, then*

$$\int_{-1/2}^{1/2}\left|\sum_{r\in I} e(\alpha r)\right| d\alpha \ll \log Z,$$

where I is a subinterval of $[Z, 2Z]$.

Proof. We have

$$\begin{aligned}
&\int_{-1/2}^{1/2}\left|\sum_{r\in I} e(\alpha r)\right| d\alpha \\
&= \int_{-1/Z}^{1/Z}\left|\sum_{r\in I} e(\alpha r)\right| d\alpha + \int_{1/Z}^{1/2}\left|\sum_{r\in I} e(\alpha r)\right| d\alpha + \int_{-1/2}^{-1/Z}\left|\sum_{r\in I} e(\alpha r)\right| d\alpha \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

By trivial estimation, we have

$$\sum_{r\in I} e(\alpha r) \ll \sum_{r\in I} 1 \leq Z.$$

Hence,

$$I_1 \leq \int_{-1/Z}^{1/Z} Z\, d\alpha \ll 1. \tag{3.1}$$

By Lemma 3.5, we have

$$\sum_{r\in I} e(\alpha r) \ll \frac{1}{\|\alpha\|}.$$

Hence,

$$I_2 \leq \int_{1/Z}^{1/2} \frac{1}{\|\alpha\|}\, d\alpha = \int_{1/Z}^{1/2} \alpha^{-1}\, d\alpha \ll \log Z. \tag{3.2}$$

By Lemma 3.5, we have

$$\sum_{r\in I} e(\alpha r) \ll \frac{1}{\|\alpha\|}.$$

Hence,

$$I_3 \ll \int_{-1/2}^{-1/Z} \frac{1}{\|\alpha\|}\, d\alpha = \int_{-1/2}^{-1/Z} -\alpha^{-1}\, d\alpha \ll \log Z. \tag{3.3}$$

Combining Eqs (3.1)–(3.3), we have

$$\int_{-1/2}^{1/2}\left|\sum_{r\in I} e(\alpha r)\right| d\alpha \ll \log Z.$$

This finishes the proof of the lemma. □

4. The division trick

The following lemma, called the division trick, is also required to detect the condition $d \mid \lfloor m^c \rfloor$.

Lemma 4.1. *Let d and m be positive integers, and let $c > 1$ be a real. Then, $d \mid \lfloor m^c \rfloor$ if, and only if,*

$$E_d(m) := \left\lfloor \frac{m^c}{d} \right\rfloor - \left\lfloor \frac{m^c - 1}{d} \right\rfloor = 1.$$

Namely, the characteristic function of the condition $d \mid \lfloor m^c \rfloor$ is the function $E_d(m)$.

Proof. First, we suppose $d \mid \lfloor m^c \rfloor$, so $\lfloor m^c \rfloor = dk$ with some integer k, then we get

$$\begin{aligned} m^c - 1 < dk \leq m^c \Rightarrow \frac{m^c - 1}{d} < k \leq \frac{m^c}{d} \\ \Rightarrow \left\lfloor \frac{m^c - 1}{d} \right\rfloor < k \leq \left\lfloor \frac{m^c}{d} \right\rfloor. \end{aligned}$$

Since

$$0 \leq \left\lfloor \frac{m^c}{d} \right\rfloor - \left\lfloor \frac{m^c - 1}{d} \right\rfloor \leq 1,$$

if $d \mid \lfloor m^c \rfloor$, then we have

$$\left\lfloor \frac{m^c}{d} \right\rfloor - \left\lfloor \frac{m^c - 1}{d} \right\rfloor = 1.$$

Next, we suppose

$$\left\lfloor \frac{m^c}{d} \right\rfloor - \left\lfloor \frac{m^c - 1}{d} \right\rfloor = 1.$$

Let $\lfloor m^c \rfloor = dk + l, 0 \leq l \leq d - 1$, and we derive

$$\left\lfloor \frac{m^c}{d} \right\rfloor - \left\lfloor \frac{m^c - 1}{d} \right\rfloor = \left\lfloor \frac{l + \{m^c\}}{d} \right\rfloor - \left\lfloor \frac{l + \{m^c\} - 1}{d} \right\rfloor.$$

If $1 \leq l \leq d - 1$, we have

$$0 \leq \frac{l + \{m^c\} - 1}{d} < \frac{l + \{m^c\}}{d} < 1,$$

so

$$\left\lfloor \frac{l + \{m^c\}}{d} \right\rfloor = \left\lfloor \frac{l + \{m^c\} - 1}{d} \right\rfloor = 0.$$

This implies

$$\left\lfloor \frac{m^c}{d} \right\rfloor - \left\lfloor \frac{m^c - 1}{d} \right\rfloor = \left\lfloor \frac{l + \{m^c\}}{d} \right\rfloor - \left\lfloor \frac{l + \{m^c\} - 1}{d} \right\rfloor = 0,$$

hence, $l = 0$. This finishes the proof of the lemma. □

5. Proof of Theorem 1.1

Before proving Theorem 1.1, we list some ranges and dependencies of all parameters.

Suppose X, b, c, T, and y as in the statement of Theorem 1.1. Consider the following parameters:

$$1 \le q \le X^{2c-2+2/T+2\varepsilon}, \quad H = qX^{\varepsilon}, \qquad \frac{\Psi(x,y)}{qx^{\varepsilon}} \le M \le X,$$

$$M_1 = M^{9-b-12/T-8c-20\varepsilon}, \quad 2 \le P \le y, \qquad \frac{M_1}{2P} \le L \le M_1, \quad \frac{M}{4PL} \le K \le \frac{2M}{PL}.$$

Now, we give the proof of Theorem 1.1.

First, let

$$C(X,q) := \sum_{\substack{m \le X \\ m \in S(X,y) \\ q | \lfloor m^c \rfloor}} 1,$$

where $1 \le q \le X^{2c-2+2/T+2\varepsilon}$. By Lemma 4.1, we get

$$\begin{aligned}
C(X,q) &= \sum_{\substack{m \le X \\ m \in S(X,y)}} 1 \\
&= \sum_{\substack{m \le X \\ m \in S(X,y)}} \left(\left\lfloor \frac{m^c}{q} \right\rfloor - \left\lfloor \frac{m^c - 1}{q} \right\rfloor \right) \\
&= \sum_{\substack{m \le X \\ m \in S(X,y)}} \frac{1}{q} + \sum_{\substack{m \le X \\ m \in S(x,y)}} \left(\psi\left(\frac{m^c}{q}\right) - \psi\left(\frac{m^c - 1}{q}\right) \right) \\
&:= \frac{\Psi(X,y)}{q} + R, \qquad (5.1)
\end{aligned}$$

where

$$\begin{aligned}
R &= \sum_{\substack{m \le X \\ m \in S(X,y)}} \left(\psi\left(\frac{m^c}{q}\right) - \psi\left(\frac{m^c - 1}{q}\right) \right) \\
&\ll \left| \sum_{\substack{m \le X \\ m \in S(X,y)}} \psi\left(\frac{m^c}{q}\right) \right| + \left| \sum_{\substack{m \le X \\ m \in S(X,y)}} \psi\left(\frac{m^c - 1}{q}\right) \right| \\
&= R_1 + R_2.
\end{aligned}$$

By Lemma 3.1 with $H = qX^{\epsilon}$, we have

$$\begin{aligned}
R_1 &\ll \left| \sum_{\substack{m \le X \\ m \in S(X,y)}} \sum_{0 < h \le H} a_h e\left(\frac{hm^c}{q}\right) \right| + \left| \sum_{\substack{m \le X \\ m \in S(X,y)}} \sum_{0 \le h \le H} b_h e\left(\frac{hm^c}{q}\right) \right| \\
&\ll \frac{\Psi(X,y)X^{-\varepsilon}}{q} + \max_{1 \le h \le H} X^{\varepsilon} \left| \sum_{\substack{\Psi(X,y)/(qX^{\varepsilon}) \le m \le X \\ m \in S(X,y)}} e\left(\frac{hm^c}{q}\right) \right|. \qquad (5.2)
\end{aligned}$$

We split the range of m in the sum above into $\log N$ dyadic intervals. It is remain to estimates the following exponential sum,

$$\Sigma := X^{\varepsilon} \sum_{\substack{M\le m\le 2M \\ m\in S(X,y)}} e\left(\frac{hm^c}{q}\right), \tag{5.3}$$

uniformly for $\Psi(X,y)/(qX^{\epsilon}) \le M \le X$.

By Lemma 3.3 with $M_1 = M^{9-b-12/T-8c-20\varepsilon}$, we have

$$\sum_{\substack{m\sim M \\ P^+(m)\le y}} e\left(\frac{hm^c}{q}\right) = \sum_{p\le y} \sum_{\substack{M_1/p<u\le M_1 \\ P^-(u)\ge p \\ P^+(u)\le y}} \sum_{\substack{v\sim M/(up) \\ P^+(v)\le p}} e\left(\frac{h(puv)^c}{q}\right), \tag{5.4}$$

where $a \sim A$ means $A < a \le 2A$.

Now, we divide the ranges of p, v, and u into $\log^3 N$ dyadic intervals to get

$$\begin{aligned}
&\sum_{p\le y} \sum_{\substack{M_1/p<u\le M_1 \\ P^-(u)\ge p \\ P^+(u)\le y}} \sum_{\substack{v\sim M/(up) \\ P^+(v)\le p}} e\left(\frac{h(puv)^c}{q}\right) \\
&\ll \sum_{p\le y} \sum_{\substack{M_1/p<u\le M_1 \\ P^-(u)\ge p \\ P^+(u)\le y}} \left| \sum_{\substack{v\sim M/(up) \\ P^+(v)\le p}} e\left(\frac{h(puv)^c}{q}\right)\right| \\
&\le \sum_{p\le y} \sum_{M_1/p<u\le M_1} \left| \sum_{\substack{v\sim M/(up) \\ P^+(v)\le p}} e\left(\frac{h(puv)^c}{q}\right)\right| \\
&= \sum_{p\le y} \sum_{M_1/p<u\le M_1} \sum_{\substack{v\sim M/(up) \\ P^+(v)\le p}} b(u,p) e\left(\frac{h(puv)^c}{q}\right) \\
&\le \sum_{\substack{2\leqslant P\leqslant y \\ P=2^k}} \sum_{\substack{M_1/2P\leqslant L\leqslant M_1 \\ L=2^r}} \sum_{\substack{M/4PL\le K\le 2M/PL \\ K=2^j}} |S'(P,K,L)|,
\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}
S' &:= S'(P,K,L) && (5.6)\\
&= \sum_{v\sim K} \sum_{\substack{p\sim P \\ P^+(v)\leqslant p}} \sum_{\substack{u\sim L \\ uvp\sim M}} b(u,p) e\left(\frac{hp^c u^c v^c}{q}\right), && (5.7)
\end{aligned}$$

and $b(u,p) \ll 1$ is a complex number.

We also note that

$$
\begin{aligned}
S' &= \int_{-1/2}^{1/2} \sum_{v\sim K} \sum_{\substack{p\sim P\\ P^+(v)\le p}} \sum_{u\sim L} e(\alpha up) b(u,p) e\left(\frac{hp^c u^c v^c}{q}\right) \sum_{d\sim M/v} e(-\alpha d)\, d\alpha \\
&= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \sum_{v\sim K}\sum_{d\sim M/v} e(-\alpha d) \sum_{p\sim P}\sum_{u\sim L} e(\alpha up-\beta p) b(u,p) e\left(\frac{hp^c u^c v^c}{q}\right) \sum_{\substack{g\sim P\\ g\ge P^+(v)}} e(\beta g)\, d\alpha\, d\beta \\
&= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \sum_{v\sim K}\sum_{d\sim M/v} e(-\alpha d) \sum_{\substack{g\sim P\\ g\ge P^+(v)}} e(\beta g) \sum_{p\sim P}\sum_{u\sim L} e(\alpha up-\beta p) b(u,p) e\left(\frac{hp^c u^c v^c}{q}\right) d\alpha\, d\beta.
\end{aligned}
$$

Therefore,

$$
|S'| = \left| \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \sum_{v\sim K} a(v;\alpha,\beta) \sum_{u\sim L}\sum_{p\sim P} b(u,p;\alpha,\beta) e\left(\frac{hp^c u^c v^c}{q}\right) d\alpha d\beta \right|,
$$

where

$$
a(v;\alpha,\beta) = \sum_{d\sim M/v} e\left(-\alpha d\right) \sum_{\substack{g\sim P\\ g\ge P^+(v)}} e\left(\beta g\right),
$$

and

$$
b(u,p;\alpha,\beta) = e\left(\alpha up - \beta p\right) b(u,p).
$$

Let $r = up$. Then, we have

$$
S' \ll \left| \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \sum_{v\sim K} a(v;\alpha,\beta) \sum_{r\sim PL} c(r;\alpha,\beta) e\left(\frac{hr^c v^c}{q}\right) d\alpha d\beta \right|,
$$

where

$$
c(r) = \tau(r)\, b(u,p;\alpha,\beta).
$$

If $(\alpha,\beta)\in\Omega$, $\max_{v\sim K}|a(v;\alpha,\beta)| \ne 0$, and $\max_{r\sim PL}|c(r;\alpha,\beta)|\ne 0$, let

$$
T_0(\alpha,\beta) = \sum_{v\sim K} \frac{a(v;\alpha,\beta)}{\max_{v\sim K}|a(v;\alpha,\beta)|} \sum_{r\sim PL} \frac{c(r;\alpha,\beta)}{\max_{r\sim PL}|c(r;\alpha,\beta)|} e\left(\frac{hr^c v^c}{q}\right).
$$

Consequently,

$$
\begin{aligned}
S' &\ll \left| \int\int_{(\alpha,\beta)\in\Omega} \max_{v\sim K}|a(v;\alpha,\beta)| \max_{r\sim PL}|c(r;\alpha,\beta)||T_0(\alpha,\beta)| d\alpha d\beta \right| \\
&= \int\int_{(\alpha,\beta)\in\Omega} \max_{v\sim K}|a(v;\alpha,\beta)| \max_{r\sim PL}|c(r;\alpha,\beta)| f(h) T_0(\alpha,\beta) d\alpha d\beta, \qquad (5.8)
\end{aligned}
$$

where $|f(h)| = 1$.

Now we estimate $T_0(\alpha,\beta)$. By Lemma 3.2 with $H_1 = 1$, $K_1 = K$, $L_1 = PL$, $X_1 = \frac{h(PLK)^c}{q}$, $\beta_1 = 1$, and $\alpha_1 = \gamma_1 = c$, we have

$$
\begin{aligned}
T_0(\alpha,\beta) &\ll M^{1+\varepsilon}\left\{\left(\frac{hM^c}{qK(PL)^2}\right)^{1/4} + \frac{1}{K^{1/4}} + \frac{1}{(PL)^{1/2}} + \frac{q^{1/2}}{h^{1/2}M^{c/2}}\right\} \\
&=: V_1 + V_2 + V_3 + V_4,
\end{aligned}
$$

where

$$
V_1 = M^{1+\varepsilon}\left(\frac{hM^c}{qK(PL)^2}\right)^{1/4}, \quad V_2 = \frac{M^{1+\varepsilon}}{K^{1/4}}, \quad V_3 = \frac{M^{1+\varepsilon}}{(PL)^{1/2}}, \quad V_4 = M^{1+\varepsilon}\frac{q^{1/2}}{h^{1/2}M^{c/2}}.
$$

By the choices of parameters, we have

$$
\begin{aligned}
V_1 &\ll M^{1+\varepsilon}M^{(c-1)/4}(PL)^{-1/4} \ll X^{9c/4+3/T+b/4-3/2+\varepsilon} \ll \frac{\Psi(X,y)}{qX^{\epsilon}}, \\
V_2 &\ll M^{1+\varepsilon}\frac{(PL)^{1/4}}{M^{1/4}} \ll X^{3-2c-3/T-4\varepsilon} \ll \frac{\Psi(X,y)}{qX^{\epsilon}}, \\
V_3 &\ll \frac{M^{1+\varepsilon}}{(PL)^{1/2}} \ll \frac{M^{1+\varepsilon}}{M_1^{1/2}} \ll X^{4c+6/T+b/2-7/2+10\varepsilon} \ll \frac{\Psi(X,y)}{qX^{\epsilon}}, \\
V_4 &\ll M^{1+\varepsilon}\frac{q^{1/2}}{M^{c/2}} \ll X^{c/2+1/T+\varepsilon} \ll \frac{\Psi(X,y)}{qX^{\epsilon}}.
\end{aligned}
$$

Consequently,

$$
T_0(\alpha,\beta) \ll \frac{\Psi(X,y)}{qX^{\epsilon}}. \tag{5.9}
$$

By Eqs (5.7) and (5.8) and Lemma 3.6, we have

$$
S' \ll \frac{\Psi(X,y)}{qX^{\epsilon}}.
$$

By Eqs (5.3)–(5.5) and the above inequalities, we have

$$
\Sigma \ll \frac{\Psi(X,y)}{qX^{\epsilon}}.
$$

Using Eq (5.2), it follows that

$$
R_1 \ll \frac{\Psi(X,y)}{qX^{\epsilon}}.
$$

Similarly, we have

$$
R_2 \ll \frac{\Psi(X,y)}{qX^{\epsilon}}.
$$

Using Eq (5.1), it follows that

$$
C(X,q) = \frac{\Psi(X,y)}{q} + O\left(\frac{\Psi(X,y)}{qX^{\epsilon}}\right). \tag{5.10}
$$

Finally, we have

$$
\begin{aligned}
S(X) &= \sum_{\substack{m\le X\\ m\in S(X,y)}} \mu^2(\lfloor m^c\rfloor)\\
&= \sum_{\substack{m\le X\\ m\in S(X,y)}} \sum_{d^2\|\lfloor m^c\rfloor} \mu(d) = \sum_{1\le d\le X^{c/2}} \mu(d)\, C(X,d^2)\\
&= \sum_{X^{c-1+1/T+\varepsilon}<d} \mu(d)\, C(X,d^2) + \sum_{1\le d\le X^{c-1+1/T+\varepsilon}} \mu(d)\, C(X,d^2)\\
&= \sum_{1\le d\le X^{c-1+1/T+\varepsilon}} \mu(d)\, C(X,d^2) + O\left(\frac{\Psi(X,y)}{X^{\varepsilon}}\right),
\end{aligned}
\tag{5.11}
$$

where we used trivial estimates for $C(X,d^2)$.

For the remaining sum in Eq (5.10), we use Eq (5.9) to derive

$$
S(X) = \Psi(X,y) \sum_{1\le d\le X^{c-1+1/T+\varepsilon}} \frac{\mu(d)}{d^2} + O\left(\frac{\Psi(X,y)}{X^{\varepsilon}}\right) \tag{5.12}
$$

$$
= \frac{6}{\pi^2}\Psi(X,y) + O\left(\frac{\Psi(X,y)}{X^{\varepsilon}}\right). \tag{5.13}
$$

This finishes the proof of Theorem 1.1.

6. Proof of Theorem 1.2

Before proving Theorem 1.2, we list some ranges and dependencies of all parameters.

Suppose X, b, c, and y as in the statement of Theorem 1.2. Let

$$
2\le q\le X^{\eta},\quad Z=\lfloor X^{\delta}\rfloor+1,\quad x^{1-\delta}\le M\le X,\quad \frac{1}{q}\le |b_1|\le X^{\delta+\varepsilon}
$$

$$
M_1 = M^{(1-b)/3-\varepsilon},\quad 2\le P\le y,\quad \frac{M_1}{2P}\le L\le M_1,\quad \frac{M}{4PL}\le K\le \frac{2M}{PL}.
$$

We now give a proof of Theorem 1.2. To apply the results in [7], the following lemma is required.

Lemma 6.1. *Let $\eta>0$ be a sufficient small real number, let X be a sufficiently large real number, let a and q be two integers with $2\le q\le X^{\eta}$, $0<a<q$, and $(a,q)=1$, and let $1<c<\frac{4-b}{3}$ be a real. Then there exists a real number $\eta<\delta<\frac{1}{2}$ such that*

$$
S := \sum_{n\in S(x,y)} e\left(\frac{a\lfloor n^c\rfloor}{q}\right) \ll X^{1-\delta}. \tag{6.1}
$$

Proof. Let $\alpha=\frac{a}{q}$. We only need to show that the estimation

$$
C(\alpha) := \sum_{\substack{n\in S(M,y)\\ n\sim M}} e\left(\alpha\lfloor n^c\rfloor\right) \ll X^{1-\delta} \tag{6.2}
$$

holds uniformly for $X^{1-\delta} \leq M \leq X$. By Lemma 3.4 with $Z = \lfloor X^{\delta} \rfloor + 1$, we have

$$\begin{aligned}
C(\alpha) &= \sum_{\substack{n\in S(M,y)\\ n\sim M}} e\left(\alpha\lfloor n^c\rfloor\right) \sum_{z=0}^{2Z-1} g_z(n^c) \\
&= \sum_{z=0}^{2Z-1} \sum_{\substack{n\in S(M,y)\\ n\sim M}} \left(e\left(\alpha n^c - \frac{\alpha z}{2Z}\right) + O\left(\frac{1}{Z}\right)\right) g_z(n^c) \qquad (6.3)\\
&= \sum_{z=0}^{2Z-1} \sum_{\substack{n\in S(M,y)\\ n\sim M}} e\left(\alpha n^c - \frac{\alpha z}{2Z}\right) g_z(n^c) + O\left(\frac{1}{Z}\sum_{z=0}^{2Z-1} \sum_{\substack{n\in S(M,y)\\ n\sim M}} g_z(n^c)\right)\\
&= \sum_{z=0}^{2Z-1} \sum_{\substack{n\in S(M,y)\\ n\sim M}} e\left(\alpha n^c - \frac{\alpha z}{2Z}\right) g_z(n^c) + O(X^{1-\delta})\\
&= \sum_{z=0}^{2Z-1} \sum_{\substack{n\in S(M,y)\\ n\sim M}} \sum_{|t|\leq Z\log^4 x} \beta_n^{(z)} e\left((\alpha+t)n^c - \frac{\alpha z}{2Z}\right) + O(X^{1-\delta})\\
&\ll \sum_{z=0}^{2Z-1} \sum_{|t|\leq Z\log^4 x} \left|\beta_t^{(z)} e\left(\frac{\alpha z}{2Z}\right)\right| \left|\sum_{\substack{n\in S(M,y)\\ n\sim M}} e\left((\alpha+t)n^c\right)\right| + X^{1-\delta}\\
&\ll X^{\delta+\varepsilon} \max_{0\leq t\leq X^{\delta+\varepsilon}} \left|\sum_{\substack{n\in S(M,y)\\ n\sim M}} e\left((\alpha+t)n^c\right)\right| + X^{1-\delta}. \qquad (6.4)
\end{aligned}$$

Note that when $t < 0$, $\alpha + t \neq 0$, we have $1/q \leq |t+\alpha| \leq X^{\delta+\varepsilon}$, and when $t \geq 0$, we also have $1/q \leq |t+\alpha| \leq X^{\delta+\varepsilon}$. Let $b_1 = \alpha + t$. Therefore, it is enough to show

$$\sum_{\substack{n\in S(M,y)\\ n\sim M}} e(b_1 n^c) \ll X^{1-2\delta-\varepsilon} \qquad (6.5)$$

uniformly for $1/q \leq |b_1| \leq X^{\delta+\varepsilon}$.

By Lemma 3.3, we have

$$\sum_{\substack{n\sim M\\ P^+(n)\leq y}} e\left(b_1 n^c\right) = \sum_{p\leq y} \sum_{\substack{M_1/p<u\leq M_1\\ P^-(u)\geq p\\ P^+(u)\leq y}} \sum_{\substack{v\sim M/(up)\\ P^+(v)\leq p}} e\left(b_1 (puv)^c\right),$$

where $M_1 := M^{(1-b)/3-\varepsilon}$.

Now, we divide the ranges of p, v, and u into $\log^3 M$ dyadic intervals, and we have

$$\sum_{p\leq y} \sum_{\substack{M_1/p<u\leq M_1\\ P^-(u)\geq p\\ P^+(u)\leq y}} \sum_{\substack{v\sim M/(up)\\ P^+(v)\leq p}} e\left(b_1 (puv)^c\right) \leq \sum_{\substack{2\leqslant P\leqslant y\\ P=2^k}} \sum_{\substack{M_1/2P\leqslant L\leqslant M_1\\ L=2^r}} \sum_{\substack{M/4PL\leq K\leq 2M/PL\\ K=2^j}} \left|S'(P,K,L)\right|,$$

where

$$S' := S'(P,K,L) = \sum_{v\sim K} \sum_{\substack{p\sim P \\ P^+(v)\leqslant p}} \sum_{\substack{u\sim L \\ uvp\sim M}} b(u,p) e\left(b_1 p^c u^c v^c\right),$$

and $b(u,p) \ll 1$ is a complex number.

Here we note that

$$\begin{aligned}
S' &= \int_{-1/2}^{1/2} \sum_{v\sim K} \sum_{\substack{p\sim P \\ P^+(v)\leq p}} \sum_{u\sim L} e(\alpha up) b(u,p) e\left(b_1 p^c u^c v^c\right) \sum_{d\sim M/v} e(-\alpha d) d\alpha \\
&= \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \sum_{v\sim K} \sum_{d\sim M/v} e(-\alpha d) \sum_{p\sim P} \sum_{u\sim L} e(\alpha up - \beta p) b(u,p) e\left(b_1 p^c u^c v^c\right) \sum_{\substack{g\sim P \\ g\geq P^+(v)}} e(\beta g) d\alpha d\beta \\
&= \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \sum_{v\sim K} \sum_{d\sim M/v} e(-\alpha d) \sum_{\substack{g\sim P \\ g\geq P^+(v)}} e(\beta g) \sum_{p\sim P} \sum_{u\sim L} e(\alpha up - \beta p) b(u,p) e\left(b_1 p^c u^c v^c\right) d\alpha d\beta.
\end{aligned}$$

Therefore, we have

$$|S'| = \left| \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \sum_{v\sim K} a(v;\alpha,\beta) \sum_{u\sim L} \sum_{p\sim P} b(u,p;\alpha,\beta) e\left(b_1 p^c u^c v^c\right) d\alpha d\beta \right|,$$

where

$$a(v;\alpha,\beta) = \sum_{d\sim M/v} e\ (-\alpha d) \sum_{\substack{g\sim P \\ g\geq P^+(v)}} e\ (\beta g),$$

and $b(u,p;\alpha,\beta) = e(\alpha up - \beta p) b(u,p)$.

Let $r = up$. Then, we have

$$S' \ll \left| \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \sum_{v\sim K} a(v;\alpha,\beta) \sum_{r\sim PL} c(r;\alpha,\beta) e\left(b_1 r^c v^c\right) d\alpha d\beta \right|,$$

where $c(r) = \tau(r) b(u,p;\alpha,\beta)$.

If $(\alpha,\beta) \in \Omega$, $\max_{v\sim K} |a(v;\alpha,\beta)| \neq 0$, and $\max_{r\sim PL} |c(r;\alpha,\beta)| \neq 0$, let

$$T_0(\alpha,\beta) = \sum_{v\sim K} \frac{a(v;\alpha,\beta)}{\max_{v\sim K} |a(v;\alpha,\beta)|} \sum_{r\sim PL} \frac{c(r;\alpha,\beta)}{\max_{r\sim PL} |c(r;\alpha,\beta)|} e\left(b_1 r^c v^c\right).$$

Consequently,

$$\begin{aligned}
S' &\ll \left| \int \int_{(\alpha,\beta)\in\Omega} \max_{v\sim K} |a(v;\alpha,\beta)| \max_{r\sim PL} |c(r;\alpha,\beta)| |T_0(\alpha,\beta)| d\alpha d\beta \right| \\
&= \int \int_{(\alpha,\beta)\in\Omega} \max_{v\sim K} |a(v;\alpha,\beta)| \max_{r\sim PL} |c(r;\alpha,\beta)| f(h) T_0(\alpha,\beta) d\alpha d\beta, \qquad (6.6)
\end{aligned}$$

where $|f(h)| = 1$.

Now we estimate $T_0(\alpha,\beta)$. By Lemma 3.2 with $H_1 = 1$, $K_1 = K$, $L_1 = PL$, $X_1 = b_1(PLK)^c$, $\beta_1 = 1$, and $\alpha_1 = \gamma_1 = c$, we have

$$\begin{aligned} T_0(\alpha,\beta) &\ll M^{1+\varepsilon}\left\{\left(\frac{X_1}{K(PL)^2}\right)^{1/4} + \frac{1}{K^{1/4}} + \frac{1}{(PL)^{1/2}} + \frac{1}{X_1^{1/2}}\right\} \\ &= V_1 + V_2 + V_3 + V_4, \end{aligned}$$

where

$$V_1 = M^{1+\varepsilon}\left(\frac{X_1}{K(PL)^2}\right)^{1/4}, \quad V_2 = \frac{M^{1+\varepsilon}}{K^{1/4}}, \quad V_3 = \frac{M^{1+\varepsilon}}{(PL)^{1/2}}, \quad V_4 = \frac{M^{1+\varepsilon}}{X_1^{1/2}}.$$

By the choices of parameters, we have

$$\begin{aligned} V_1 &\ll b_1^{1/4} M^{1+\varepsilon} M^{(c-1)/4}(PL)^{-1/4} \ll X^{c/4+b/12+2/3+\delta/4+\varepsilon} \ll X^{1-2\delta-4\varepsilon}, \\ V_2 &\ll M^{1+\varepsilon}\frac{(PL)^{1/4}}{M^{1/4}} \ll X^{5/6-b/6+\varepsilon} \ll X^{1-2\delta-4\varepsilon}, \\ V_3 &\ll \frac{M^{1+\varepsilon}}{(PL)^{1/2}} \ll X^{b/6+5/6+\varepsilon} \ll X^{1-2\delta-4\varepsilon}, \\ V_4 &\ll \frac{M^{1+\varepsilon}}{X_1^{1/2}} \ll X^{1-c/2+\eta/2+\varepsilon} \ll X^{1-2\delta-4\varepsilon}. \end{aligned}$$

Therefore, we have

$$T_0(\alpha,\beta) \ll X^{1-2\delta-4\varepsilon}.$$

By Eq (6.6) and Lemma 3.6, we have

$$S' \ll X^{1-2\delta-4\varepsilon}.$$

So Eq (6.5) is held. This completes the proof of the lemma. □

When $X^{\varepsilon} \le y \le X^{\beta}$, by Eqs (1.6) and (1.8) of [12], we know

$$\Psi(x,y) \gg x^{1-\varepsilon}.$$

This implies the sequence $\lfloor m^c \rfloor$ where $m \in S(X,y)$ satisfies the Condition (1) of Theorem 1 in [7], and by Lemma 6.1, this sequence also satisfies the Condition (2) of Theorem 1 in [7]. Therefore, we can apply Theorem 1 in [7] to our sequences. This finishes the proof of the Theorem 1.2.

7. Conclusions and discussion

We remark that whether the ranges of values taken by c and y in Theorems 1.1–1.3 can be improved relies on how refined the exponential sum estimates are, but the exponential sum estimation technique adopted in this paper is among the most effective existing approaches. Therefore, the range of c and y in Theorems 1.1–1.3 is best possible by the current method.

The techniques developed in this paper cannot be directly applied to the study of k-free numbers of form $\lfloor m^c \rfloor$ with smooth indices. This remains an open problem requiring further investigation. Values satisfying other multiplicative constraints with m restricted to smooth numbers, such as almost-prime values, have been studied in another unpublished manuscript by the author (accepted for publication in the Siberian Mathematical Journal).

In summary, the main contribution of this paper is the introduction of an original technic division trick, which generalizes the results in [7]. Furthermore, this technique possesses certain potential for applications to other problems. Therefore, the results established in this work are of definite research significance.

Use of Generative-AI tools declaration

The author claims he has not used AI tools during the writing of this article.

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Conflict of interest

The author claims no conflicts of interest.

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