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Research article

Fixed point theorems for relaxed asymptotic contractions via two quasi-metrics

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Abstract: We introduced a new class of asymptotic contractions that employed two quasi-metrics defined directly in terms of the underlying mapping. The contraction condition compared these two quantities via a sequence of bounding functions that converged locally uniformly to a Boyd-Wong function. This framework relaxed the hypotheses of Kirk's asymptotic fixed point theorem and theoretically contained it as a special case. Assuming only the continuity of the map and the boundedness of some orbit in a complete metric space, we proved both the existence and uniqueness of a fixed point, along with the convergence of all iterates to that point.

Keywords: fixed point; relaxed asymptotic contractions; lower quasi-metric; upper quasi-metric; local uniform convergence; Boyd-Wong condition
Mathematics Subject Classification: 47H10,47H09

1. Introduction

The evolution of metric fixed point theory has been propelled by a persistent weakening of contractive hypotheses. In 1922, Banach [1] formulated the classical contraction mapping principle: if a self-map T of a complete metric space (X, d) satisfies

$$d(Tx, Ty) \leq \alpha\, d(x, y) \quad (\alpha < 1),$$

then T possesses a unique fixed point, to which the iterates of any starting point converge. This theorem remains a cornerstone of nonlinear analysis. Over the following decades, numerous generalizations appeared: Brouwer's topological fixed point theorem for compact convex sets in Euclidean spaces [2], Schauder's extension to infinite-dimensional Banach spaces via compactness [3], nonlinear contractive conditions by Rakotch [4] and Edelstein [5], and the Browder–Göhde–Kirk fixed point theorems for nonexpansive maps in uniformly convex Banach spaces [6, 7].

A major unification of nonlinear contractions was achieved by Boyd and Wong [8], who considered

$$d(Tx,Ty) \leqslant \psi\left(d(x,y)\right),$$

where $\psi\colon [0,\infty) \to [0,\infty)$ is upper semicontinuous and $\psi(t) < t$ for all $t > 0$. This condition encompasses many earlier contractive definitions. A particularly influential generalization was introduced by Ćirić [9], who proposed the quasi-contraction condition

$$d(Tx,Ty) \leq \alpha \max\Big\{d(x,y),\, d(x,Tx),\, d(Ty,y),\, d(Tx,y),\, d(Ty,x)\Big\}$$

with $0 < \alpha < 1$. The insight of using a maximum of several strategic distances rather than a single pair inspired a large body of subsequent research. For instance, Gu [10] studied parallel iterative algorithms for Ćirić quasi-contractive operators. More recently, Lakzian, Kocev, and Rakočević [11] introduced Ćirić-generalized contractions via *wt*-distance and applied their results to nonlinear fractional differential equations. Păcurar [12] established Maia-type fixed point theorems for nonexpansive Ćirić–Prešić operators in ordered spaces. Bekri, Fabiano, and Alsharidi [13] developed iterated ρ-Ćirić contractions, extending Singh's framework to higher iterates.

A decisive step toward asymptotic formulations was taken by Kirk [14], who introduced asymptotic contractions: there exist functions $\psi_n\colon [0,\infty) \to [0,\infty)$ such that for all x, y, and n,

$$d(T^n x, T^n y) \leq \psi_n\left(d(x,y)\right),$$

where $\{\psi_n\}$ converges uniformly on $[0,\infty)$ to a Boyd-Wong function ψ. Kirk proved that a continuous map possessing a bounded orbit in a complete metric space admits a unique fixed point, to which all iterates converge. This result unified many nonlinear contraction principles under an asymptotic framework.

Since Kirk's seminal work, several generalizations have appeared. Suzuki [15] introduced asymptotic contractions of Meir-Keeler type, proving a fixed point theorem that generalizes both Meir-Keeler's and Kirk's results. Lindström and Ross [16] provided a nonstandard analysis proof of Kirk's theorem. In the context of bipolar metric spaces, Paul, Sajid, Chandra, and Gairola [17] established common fixed point theorems with applications. Singh, Saxena, Gairola, and Sajid [18] explored the geometry of fixed points and their application in economics. Gangwar, Tomar, Sajid, and Dimri [19] studied α-Krasnosel'skii mappings and obtained common fixed point and convergence results.

Despite these advances, Kirk's theorem imposes two significant limitations: (i) The convergence of $\{\psi_n\}$ is required to be uniform on the entire interval $[0,\infty)$, which is often difficult to verify; (ii) the estimate involves only $d(x,y)$, ignoring the possibility that intermediate terms like $d(Tx,x)$ or cross terms like $d(Tx,Ty)$ might provide better control.

Our contribution. In this paper we develop a new type of asymptotic contraction that overcomes these limitations. Our main idea is to introduce two quasi-metrics defined directly on the mapping T: A lower quasi-metric L_T that serves as the estimated quantity, and an upper quasi-metric U_T that serves as the comparison argument. The resulting condition is

$$L_T(T^n x, T^n y) \leq \psi_n\left(U_T(x,y)\right), \qquad \forall x,y,\ \forall n \in \mathbb{N}. \tag{1.1}$$

We prove that under this condition, any continuous self-map of a complete metric space that has a bounded orbit possesses a unique fixed point, and all iterates converge to it. The proof adapts Kirk's original argument with careful case distinctions to handle the adaptive switching in L_T. Because $L_T(x,y) \leqslant d(Tx,Ty)$ holds by construction, our definition theoretically contains Kirk's asymptotic contractions.

2. Preliminaries

Throughout, (X,d) denotes a metric space. For $T: X \to X$ and $x \in X$, the orbit is

$$O(x) = \{T^n x \mid n \geq 0\}.$$

A set $B \subset X$ is bounded if

$$\operatorname{diam}(B) = \sup\{d(u,v) \mid u,v \in B\} < \infty.$$

Definition 2.1 ([8] Upper semicontinuity)**.** *A function $\psi\colon [0,\infty) \to [0,\infty)$ is upper semicontinuous at t_0 if $\limsup_{t\to t_0} \psi(t) \leq \psi(t_0)$. It is upper semicontinuous on $[0,\infty)$ if this holds at every $t_0 \geq 0$.*

Definition 2.2 ([8] Boyd-Wong condition)**.** *A function $\psi\colon [0,\infty) \to [0,\infty)$ satisfies the classical Boyd-Wong condition if it is upper semicontinuous, $\psi(0) = 0$, and $\psi(t) < t$ for every $t > 0$.*

2.1. Lower and upper quasi-metrics

The following definitions form the core of our new contraction. They are inspired by Ćirić's five-distance maximum, but we introduce a lower quasi-metric that adaptively selects a minimal distance, and an upper quasi-metric that simply takes the maximum of four essential distances.

Definition 2.3 (Four-point minimum P_T)**.** *For $x,y \in X$, define*

$$P_T(x,y) := \min\{\, d(x,y),\ d(x,Ty),\ d(Tx,y),\ d(Tx,Ty)\,\}. \tag{2.1}$$

The quantity $P_T(x,y)$ is a nonnegative symmetric function. This creates opportunities for a sharper contraction estimate.

Definition 2.4 (Lower quasi-metric L_T)**.** *For $x,y \in X$, define*

$$L_T(x,y) := \begin{cases} P_T(x,y), & \text{if } P_T(x,y) > 0, \\ d(Tx,Ty), & \text{if } P_T(x,y) = 0. \end{cases} \tag{2.2}$$

The lower quasi-metric adaptively chooses the best possible left-hand side for the contraction inequality. By construction, $L_T(x,y) \leqslant d(Tx,Ty)$ always holds. Consequently, any mapping satisfying Kirk's asymptotic contraction condition automatically satisfies our condition (3.1) with the same sequence $\{\psi_n\}$, because $L_T(T^n x, T^n y) \leqslant d(T^{n+1}x, T^{n+1}y)$.

Definition 2.5 (Upper quasi-metric U_T)**.** *For $x,y \in X$, define*

$$U_T(x,y) := \max\{d(x,y),\ d(Tx,x),\ d(Ty,y),\ d(Tx,Ty)\}. \tag{2.3}$$

The upper quasi-metric provides a robust comparison argument.

2.2. Basic properties

The following proposition follows naturally from the previous definitions.

Proposition 2.6 (Elementary properties). *For all $x, y \in X$:*

(1)

$$L_T(x, y) \geqslant 0, \quad U_T(x, y) \geqslant 0.$$

(2)

$$L_T(x, y) = L_T(y, x), \quad U_T(x, y) = U_T(y, x).$$

(3)

$$d(x, y) \leqslant U_T(x, y).$$

(4)

$$L_T(x, y) \leqslant d(Tx, Ty).$$

(5)

$$d(Tx, Ty) \leqslant U_T(x, y).$$

The following lemma is crucial: it shows that when $P_T(u, v)$ is positive, the distance $d(Tu, Tv)$ can be estimated by $P_T(u, v)$ plus the sum of the adjacent distances.

Lemma 2.7 (Safe estimate). *Let $u, v \in X$ be such that $P_T(u, v) > 0$. Then,*

$$d(Tu, Tv) \leqslant P_T(u, v) + d(Tu, u) + d(Tv, v). \tag{2.4}$$

Proof. Since $P_T(u, v)$ is the minimum of four distances, it must be one of $d(u, v)$, $d(u, Tv)$, $d(Tu, v)$, or $d(Tu, Tv)$. In each case, the triangle inequality yields the desired bound:

- If $P_T(u, v) = d(u, v)$, then

$$d(Tu, Tv) \leqslant d(u, v) + d(Tu, u) + d(Tv, v) = P_T + d(Tu, u) + d(Tv, v).$$

- If $P_T(u, v) = d(u, Tv)$, then

$$d(Tu, Tv) \leqslant d(u, Tv) + d(Tu, u) = P_T + d(Tu, u) \leqslant P_T + d(Tu, u) + d(Tv, v).$$

- If $P_T(u, v) = d(Tu, v)$, then

$$d(Tu, Tv) \leqslant d(Tu, v) + d(Tv, v) = P_T + d(Tv, v) \leqslant P_T + d(Tu, u) + d(Tv, v).$$

- If $P_T(u, v) = d(Tu, Tv)$, then

$$d(Tu, Tv) = P_T \leqslant P_T + d(Tu, u) + d(Tv, v).$$

In all cases, $d(Tu, Tv) \leqslant P_T + d(Tu, u) + d(Tv, v)$. □

2.3. Monotone majorant

The limit function ψ (Boyd-Wong) may not be monotone, which is inconvenient for taking limsups. We replace it by its monotone majorant.

Definition 2.8. *Let ψ satisfy the Boyd-Wong condition. Define $g\colon [0,\infty) \to [0,\infty)$ by*

$$g(t) := \sup_{0\leq s\leq t} \psi(s). \tag{2.5}$$

Explanation. This construction is standard in fixed point theory (see [8]). It yields a nondecreasing, right continuous function that still satisfies $g(t) < t$ for $t > 0$, which is essential for the limsup argument.

Lemma 2.9 (Properties of g)**.** *g is nondecreasing, right continuous, $g(0) = 0$, $g(t) \leq t$ for all t, and $g(t) < t$ for every $t > 0$.*

Proof. Recall that $g(t) = \sup\{\psi(s) \mid 0 \leq s \leq t\}$.

If $0 \leqslant t_1 \leqslant t_2$, then $\{\psi(s) : 0 \leqslant s \leqslant t_1\} \subset \{\psi(s) : 0 \leqslant s \leqslant t_2\}$, so taking suprema yields $g(t_1) \leqslant g(t_2)$. Hence, g is nondecreasing.

Since $\psi(0) = 0$, we have $g(0) = \psi(0) = 0$.

For any $t > 0$ and any $s \in [0, t]$, we have $\psi(s) < s \leqslant t$. Hence, $\psi(s) < t$, and taking the supremum gives $g(t) \leqslant t$. If equality holds, i.e., $g(t) = t$, then there would exist a sequence $\{s_n\} \subset [0, t]$ with $\psi(s_n) \to t$. Since $\psi(s_n) < s_n \leqslant t$, it follows that $s_n \to t$. By upper semicontinuity of ψ at t,

$$\limsup_{n\to\infty} \psi(s_n) \leqslant \psi(t).$$

The left-hand side equals t, so $t \leqslant \psi(t)$, contradicting $\psi(t) < t$. Therefore, $g(t) < t$ for every $t > 0$.

Fix $t_0 \geqslant 0$. Because g is nondecreasing, the right limit $g(t_0^+) := \inf_{t>t_0} g(t)$ exists and satisfies $g(t_0) \leqslant g(t_0^+)$. We prove the reverse inequality.

Let $\varepsilon > 0$. By upper semicontinuity of ψ at t_0, there exists $\delta > 0$ such that

$$\psi(s) \leqslant \psi(t_0) + \varepsilon \leqslant g(t_0) + \varepsilon \quad \text{for all } s \in [t_0, t_0 + \delta].$$

For any $t \in (t_0, t_0 + \delta)$,

$$g(t) = \max\Big\{\sup_{0\leqslant s\leqslant t_0} \psi(s),\ \sup_{t_0<s\leqslant t} \psi(s)\Big\} \leqslant \max\{g(t_0),\ g(t_0) + \varepsilon\} = g(t_0) + \varepsilon.$$

Taking the infimum over $t \in (t_0, t_0 + \delta)$ yields $g(t_0^+) \leqslant g(t_0) + \varepsilon$, and letting $\varepsilon \to 0^+$ gives $g(t_0^+) \leqslant g(t_0)$. Consequently $g(t_0^+) = g(t_0)$, so g is right-continuous at t_0. As t_0 was arbitrary, g is right-continuous on $[0, \infty)$. $\square$

Proposition 2.10 (Limsup exchange)**.** *If $\{a_n\}$ is a bounded nonnegative sequence and g is nondecreasing and right continuous, then*

$$\limsup_{n\to\infty} g(a_n) \leqslant g\Big(\limsup_{n\to\infty} a_n\Big). \tag{2.6}$$

Proof. Let

$$L = \limsup_{n\to\infty} a_n = \inf_{n\geqslant 1} \sup_{k\geqslant n} a_k.$$

Since $\{a_n\}$ is bounded, L is a finite nonnegative number.

Fix $\varepsilon > 0$. By definition of lim sup, there exists an integer N such that

$$\sup_{k\geqslant N} a_k < L + \varepsilon.$$

Consequently, for every $n \geqslant N$ we have $a_n \leqslant L + \varepsilon$.

Because g is nondecreasing, $a_n \leqslant L + \varepsilon$ implies

$$g(a_n) \leqslant g(L+\varepsilon) \quad \text{for all } n \geq N.$$

Now take the limsup as $n \to \infty$ on both sides. The inequality persists:

$$\limsup_{n\to\infty} g(a_n) \leqslant \limsup_{n\to\infty} g(L+\varepsilon) = g(L+\varepsilon),$$

where the last equality holds because $g(L+\varepsilon)$ is a constant independent of n.

Thus, we have shown that for every $\varepsilon > 0$,

$$\limsup_{n\to\infty} g(a_n) \leqslant g(L+\varepsilon).$$

Now let $\varepsilon \to 0^+$. Since g is right-continuous, $\lim_{\varepsilon\to 0^+} g(L+\varepsilon) = g(L)$. Hence,

$$\limsup_{n\to\infty} g(a_n) \leqslant g(L) = g\Big(\limsup_{n\to\infty} a_n\Big).$$

□

3. Main results

In this section, we present a novel asymptotic contraction framework using L_T and U_T, where the adaptive design of L_T and uniform convergence of ψ_n on bounded sets (replacing Kirk's pointwise convergence) produce a stronger fixed-point theorem.

Definition 3.1 (Relaxed asymptotic contraction with lower and upper quasi-metrics)**.** *A continuous mapping $T\colon X \to X$ is called an* adaptive relaxed asymptotic contraction *if there exists a sequence of functions $\psi_n\colon [0,\infty) \to [0,\infty)$ such that:*

(1) *For every $x, y \in X$ and every $n \in \mathbb{N}$,*

$$L_T(T^n x, T^n y) \leqslant \psi_n\left(U_T(x,y)\right), \tag{3.1}$$

where L_T and U_T are defined as above.

(2) *The sequence $\{\psi_n\}$ converges uniformly on every bounded subset of $[0,\infty)$ to a function ψ that satisfies the classical Boyd-Wong condition.*

Comparison with existing definitions.

- **Kirk asymptotic contractions.** Since $L_T(T^n x, T^n y) \leqslant d(T^{n+1}x, T^{n+1}y)$ by Proposition 2.6 (4) and $d(x, y) \leqslant U_T(x, y)$, any mapping satisfying Kirk's condition with functions ψ_n also satisfies (3.1) with the same ψ_n. Moreover, the local uniform convergence we require is weaker than Kirk's global uniform convergence. Hence, our definition theoretically contains Kirk's.
- **Boyd-Wong contractions.** If $d(Tx, Ty) \leqslant \psi(d(x, y))$ with $\psi(t) < t$, taking $\psi_n = \psi$ and noting $L_T(T^n x, T^n y) \leqslant d(T^{n+1}x, T^{n+1}y)$ and $U_T(x, y) \geqslant d(x, y)$ yields our condition. Thus, Boyd-Wong contractions are also included.
- The use of local uniform convergence is a relaxation; many sequences that do not converge uniformly on $[0, \infty)$ still converge uniformly on bounded sets.

The proof of the following theorem adapts the key technical ideas from Kirk's original argument [15], while introducing additional case distinctions to handle the adaptive quasi-metrics.

Theorem 3.2 (Main theorem)**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be a relaxed asymptotic contraction (Definition 3.1). If there exists a point $x_0 \in X$ whose orbit $\mathcal{O}(x_0)$ is bounded, then T has a unique fixed point z, and for every $x \in X$ the iterates $T^n x$ converge to z.*

Proof. Let $x_0 \in X$ have a bounded orbit. Set $x_n := T^n x_0$ and denote $d_n := d(x_n, x_{n+1})$.

Step 1. The adjacent distances tend to zero.

For any $n \geqslant 0$, compute $P_T(x_n, x_{n+1})$:

$$\begin{aligned} P_T(x_n, x_{n+1}) &= \min\{d(x_n, x_{n+1}),\ d(x_n, Tx_{n+1}),\ d(Tx_n, x_{n+1}),\ d(Tx_n, Tx_{n+1})\} \\ &= \min\{d_n,\ d(x_n, x_{n+2}),\ d(x_{n+1}, x_{n+1}),\ d_{n+1}\} \\ &= \min\{d_n,\ d(x_n, x_{n+2}),\ 0,\ d_{n+1}\} = 0. \end{aligned} \tag{3.2}$$

Thus, $P_T(x_n, x_{n+1}) = 0$. Consequently, the switching rule for L_T falls into the $P_T = 0$ branch. For any $m \in \mathbb{N}$, we have

$$L_T(T^m x_n, T^m x_{n+1}) = L_T(x_{n+m}, x_{n+m+1}) = d(Tx_{n+m}, Tx_{n+m+1}) = d_{n+m+1}, \tag{3.3}$$

$$U_T(x_n, x_{n+1}) = \max\{d_n,\ d_n,\ d_{n+1},\ d_{n+1}\} = \max\{d_n,\ d_{n+1}\}. \tag{3.4}$$

In particular, taking $m = n$ gives

$$d_{2n+1} = L_T(T^n x_n, T^n x_{n+1}) \leqslant \psi_n(U_T(x_n, x_{n+1})) = \psi_n(\max\{d_n, d_{n+1}\}). \tag{3.5}$$

Similarly, by taking $m = n - 1$ and considering d_{2n}, we obtain

$$d_{2n} = L_T(T^{n-1}x_n, T^{n-1}x_{n+1}) \leqslant \psi_{n-1}(\max\{d_n, d_{n+1}\}). \tag{3.6}$$

Since the orbit $\{x_n\}$ is bounded, the numbers $\max\{d_n, d_{n+1}\}$ are bounded by some constant $D > 0$. By local uniform convergence of $\{\psi_n\}$ to ψ, for any $\varepsilon > 0$ there exists M_1 such that for all $m \geq M_1$ and all $t \in [0, D]$,

$$\psi_m(t) \leqslant \psi(t) + \varepsilon \leqslant g(t) + \varepsilon, \tag{3.7}$$

where g is the monotone majorant of ψ (Lemma 2.9). Hence, for all $n > M_1$,

$$\max\{d_{2n}, d_{2n+1}\} \leqslant g(\max\{d_n, d_{n+1}\}) + \varepsilon \tag{3.8}$$

Let $r := \limsup_{n\to\infty} d_n$. Then, $\limsup_{n\to\infty} \max\{d_{2n}, d_{2n+1}\} = r$ as well (since the whole sequence's limsup equals the maximum of the limsups of its even and odd subsequences). Taking limsup as $n \to \infty$ in the inequality of (3.8) (for odd indices) and using Proposition 2.10, we obtain

$$\begin{aligned} r &= \limsup_{n\to\infty} \max\{d_{2n}, d_{2n+1}\} \\ &\leqslant \limsup_{n\to\infty} g(\max\{d_n, d_{n+1}\}) + \varepsilon \\ &\leqslant g\Big(\limsup_{n\to\infty} \max\{d_n, d_{n+1}\}\Big) + \varepsilon \\ &= g(r) + \varepsilon. \end{aligned}$$

The same estimate holds for the even subsequence. Since $\varepsilon > 0$ is arbitrary, we have $r \leqslant g(r)$. If $r > 0$, Lemma 2.9 gives $g(r) < r$, a contradiction. Therefore, $r = 0$, i.e.,

$$\lim_{n\to\infty} d_n = 0. \tag{3.9}$$

Step 2. The sequence $\{x_n\}$ is Cauchy.

For each $n \geqslant 0$ define

$$a_n := \sup_{p\geqslant 1} d(x_n, x_{n+p}). \tag{3.10}$$

We shall prove $a_n \to 0$. From Step 1 we already have $d_n \to 0$. Given $\varepsilon > 0$, choose N_1 such that $d_k < \varepsilon$ for all $k \geqslant N_1$. Because $\{x_n\}$ is bounded, the set

$$\mathcal{S} := \{U_T(x_n, x_{n+p}) : n \geqslant 0,\ p \geqslant 1\} \tag{3.11}$$

is contained in some bounded interval $[0, D']$. By local uniform convergence of $\{\psi_n\}$, there exists an integer $M \geqslant M_1$ such that for all $m \geqslant M$ and all $t \in [0, D']$,

$$\psi_m(t) \leqslant \psi(t) + \varepsilon \leqslant g(t) + \varepsilon. \tag{3.12}$$

Fix arbitrary indices $n \geqslant N_1$ and $p \geqslant 1$. For any $m \geqslant M$ set,

$$u := T^m x_n = x_{n+m}, \qquad v := T^m x_{n+p} = x_{n+m+p}. \tag{3.13}$$

Our goal is to estimate $d(Tu, Tv) = d(x_{n+m+1}, x_{n+m+p+1})$.

We consider two cases based on the value of $P_T(u, v)$.

Case A: $P_T(u, v) > 0$.

Then, $L_T(u, v) = P_T(u, v)$. The contraction condition (3.1) applied to the pair (x_n, x_{n+p}) with iteration index m yields

$$P_T(u, v) = L_T(T^m x_n, T^m x_{n+p}) \leqslant \psi_m(U_T(x_n, x_{n+p})) \leqslant g(U_T(x_n, x_{n+p})) + \varepsilon. \tag{3.14}$$

By Lemma 2.7, we have

$$d(Tu, Tv) \leqslant P_T(u, v) + d(Tu, u) + d(Tv, v) = P_T(u, v) + d_{n+m} + d_{n+m+p}.$$

Since $n \geqslant N_1$ and $m \geqslant M \geqslant M_1$, we have $d_{n+m} < \varepsilon$ and $d_{n+m+p} < \varepsilon$. Hence,

$$d(Tu, Tv) \leqslant g\big(U_T(x_n, x_{n+p})\big) + 3\varepsilon.$$

Case B: $P_T(u, v) = 0$**.**

Then, $L_T(u, v) = d(Tu, Tv)$. The contraction condition directly provides

$$d(Tu, Tv) = L_T(T^m x_n, T^m x_{n+p}) \leqslant \psi_m(U_T(x_n, x_{n+p})) \leqslant g\big(U_T(x_n, x_{n+p})\big) + \varepsilon. \tag{3.15}$$

Thus, in all cases we have established the uniform estimate

$$d(x_{n+m+1}, x_{n+m+p+1}) \leqslant g\big(U_T(x_n, x_{n+p})\big) + 3\varepsilon. \tag{3.16}$$

We now show that $U_T(x_n, x_{n+p}) \leqslant a_n + 2\varepsilon$. By definition,

$$U_T(x_n, x_{n+p}) = \max\{d(x_n, x_{n+p}),\ d_n,\ d_{n+p},\ d(x_{n+1}, x_{n+p+1})\}. \tag{3.17}$$

Using the triangle inequality,

$$d(x_{n+1}, x_{n+p+1}) \leqslant d(x_n, x_{n+p}) + d_n + d_{n+p} \leqslant a_n + 2\varepsilon,$$

and $d(x_n, x_{n+p}) \leqslant a_n,\ d_n,\ d_{n+p} < \varepsilon$. Hence, $U_T(x_n, x_{n+p}) \leqslant \max\{a_n,\ \varepsilon,\ a_n + 2\varepsilon\} \leqslant a_n + 2\varepsilon$.

Since g is nondecreasing, (3.16) yields

$$d(x_{n+m+1}, x_{n+m+p+1}) \leqslant g(a_n + 2\varepsilon) + 3\varepsilon. \tag{3.18}$$

The righthand side does not depend on p. Taking supremum over $p \geqslant 1$ gives

$$a_{n+m+1} \leqslant g(a_n + 2\varepsilon) + 3\varepsilon. \tag{3.19}$$

Let $L := \limsup_{n\to\infty} a_n$. Because shifting the index by $m + 1$ does not affect the limsup, we let $n \to \infty$ in (3.19) and apply Proposition 2.10 together with the right continuity of g:

$$L \leqslant g(L + 2\varepsilon) + 3\varepsilon. \tag{3.20}$$

Letting $\varepsilon \to 0^+$, we obtain $L \leqslant g(L)$. If $L > 0$, Lemma 2.9 forces $g(L) < L$, a contradiction. Hence, $L = 0$, i.e., $a_n \to 0$. This means that $\{x_n\}$ is a Cauchy sequence.

Step 3. Existence and uniqueness of the fixed point.

Since X is complete, $\{x_n\}$ converges to some $z \in X$. Continuity of T implies

$$Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z, \tag{3.21}$$

so z is a fixed point.

To prove uniqueness, let z_1, z_2 be two fixed points. Compute $P_T(z_1, z_2)$:

$$P_T(z_1, z_2) = \min\{d(z_1, z_2), d(z_1, z_2), d(z_1, z_2), d(z_1, z_2)\} = d(z_1, z_2). \tag{3.22}$$

If $d(z_1, z_2) > 0$, then $P_T(z_1, z_2) > 0$ and $L_T(z_1, z_2) = P_T(z_1, z_2) = d(z_1, z_2)$. If $d(z_1, z_2) = 0$, then $P_T(z_1, z_2) = 0$ and $L_T(z_1, z_2) = d(Tz_1, Tz_2) = d(z_1, z_2) = 0$. In either case, for any $n \in \mathbb{N}$,

$$L_T(T^n z_1, T^n z_2) = L_T(z_1, z_2) = d(z_1, z_2), \tag{3.23}$$

and

$$U_T(z_1, z_2) = \max\{d(z_1, z_2),\ 0,\ 0,\ d(z_1, z_2)\} = d(z_1, z_2). \tag{3.24}$$

The contraction condition gives $d(z_1, z_2) \leqslant \psi_n(d(z_1, z_2))$ for every n. Letting $n \to \infty$ and using pointwise convergence of ψ_n to ψ, we obtain $d(z_1, z_2) \leqslant \psi(d(z_1, z_2))$. If $d(z_1, z_2) > 0$, the Boyd-Wong condition $\psi(t) < t$ yields a contradiction. Hence, $d(z_1, z_2) = 0$ and $z_1 = z_2$.

Step 4. Convergence from any starting point.

Let z be the unique fixed point obtained from the bounded orbit of x_0. Take an arbitrary $y \in X$ and set $y_n := T^n y$, $b_n := d(y_n, z)$.

We first prove that the orbit $\{y_n\}$ is bounded. Applying the contraction condition to the pair (y, z) yields, for every $n \in \mathbb{N}$,

$$L_T(y_n, z) \leqslant \psi_n(U_T(y, z)). \tag{3.25}$$

Because $\{\psi_n\}$ converges uniformly on bounded subsets of $[0, \infty)$ and $U_T(y, z)$ is a fixed number, the sequence $\psi_n(U_T(y, z))$ is bounded above by some constant $C > 0$.

Now compute the four-point minimum for the pair (y_n, z):

$$\begin{aligned} P_T(y_n, z) &= \min\{d(y_n, z),\ d(y_n, Tz),\ d(Ty_n, z),\ d(Ty_n, Tz)\} \\ &= \min\{b_n,\ b_n,\ b_{n+1},\ b_{n+1}\} \\ &= \min\{b_n, b_{n+1}\}. \end{aligned} \tag{3.26}$$

Two cases appear:

- If $P_T(y_n, z) > 0$, then
$$L_T(y_n, z) = P_T(y_n, z) = \min\{b_n, b_{n+1}\} \leqslant C;$$
- If $P_T(y_n, z) = 0$, then
$$L_T(y_n, z) = d(Ty_n, Tz) = b_{n+1} \leqslant C$$
and also $\min\{b_n, b_{n+1}\} = 0 \leqslant C$.

In either case we obtain

$$\min\{b_n, b_{n+1}\} \leqslant C \qquad \forall n \in \mathbb{N}. \tag{3.27}$$

We now prove that $\{b_n\}$ is bounded. Applying the contraction condition to the pair (y, Ty) and noting, as in Step 1, that $P_T(y_n, y_{n+1}) = 0$ for all n, we obtain

$$d(y_{n+1}, y_{n+2}) \leqslant \psi_n(U_T(y, Ty)) \leqslant C_1 \qquad (\forall n \in \mathbb{N}), \tag{3.28}$$

where C_1 is an upper bound for $\{\psi_n(U_T(y, Ty))\}$ (such a bound exists by local uniform convergence).

Set $M = \max\{C + d(y, Ty),\ C + C_1\}$. We claim $b_n \leqslant \max\{b_0, M\}$ for all n. The statement holds for $n = 0$. Suppose it holds for some $k \geq 0$ and assume, for contradiction, that $b_{k+1} > \max\{b_0, M\}$. Then, (3.27) forces $b_k \leqslant C$. Now:

- If $k = 0$, then

$$b_1 \leqslant b_0 + d(y, Ty) \leqslant C + d(y, Ty) \leqslant M,$$

a contradiction.
- If $k \geqslant 1$, then by (3.28) we have $d(y_k, y_{k+1}) \leqslant C_1$, hence

$$b_{k+1} \leqslant b_k + C_1 \leqslant C + C_1 \leqslant M,$$

a contradiction.

Thus, $b_{k+1} \leqslant \max\{b_0, M\}$, completing the induction. Hence, $\{b_n\}$ is bounded.

Having established boundedness of the orbit $\{y_n\}$, we can now apply Steps 1–3 to the initial point y. Consequently, $\{y_n\}$ is Cauchy and converges to some fixed point. By uniqueness, the limit must be z. Therefore, $T^n y \to z$ for every $y \in X$.

This completes the proof of Theorem 3.2. □

Corollaries. We now present several direct consequences of Theorem 3.2 that are useful in applications.

Corollary 3.3 (Bounded invariant set)**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be a relaxed asymptotic contraction. If there exists a nonempty bounded closed subset $K \subset X$ such that $T(K) \subset K$, then T has a unique fixed point in K, and for every $x \in K$ the iterates $T^n x$ converge to this fixed point.*

Proof. Pick any $x_0 \in K$. Since $T(K) \subset K$, the orbit $O(x_0)$ remains in K and is therefore bounded. By Theorem 3.2, T possesses a unique fixed point $z \in X$ and $T^n x_0 \to z$. Because K is closed and $O(x_0) \subset K$, the limit z belongs to K. The convergence of iterates from any starting point in K follows again from Theorem 3.2. □

Corollary 3.4 (Existence of a fixed point implies global convergence)**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be a relaxed asymptotic contraction. If T has a fixed point $z \in X$, then z is the unique fixed point of T, and for every $x \in X$ the iterates $T^n x$ converge to z.*

Proof. Since z is a fixed point, the orbit of z is bounded (it is the singleton $\{z\}$). By Theorem 3.2, T possesses a unique fixed point (which must be z) and all iterates converge to it. Note that the theorem requires the existence of some point with a bounded orbit, and z itself serves this purpose. □

Corollary 3.5 (Kirk's theorem is a special case)**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be a continuous mapping for which there exist functions $\psi_n : [0, \infty) \to [0, \infty)$ converging uniformly on $[0, \infty)$ to a Boyd-Wong function ψ, such that*

$$d(T^n x, T^n y) \leqslant \psi_n(d(x, y)) \qquad \forall x, y \in X, \ \forall n \in \mathbb{N}.$$

If some orbit of T is bounded, then T has a unique fixed point and all iterates converge to it. In particular, the hypotheses of Kirk's asymptotic contraction theorem [14] imply that T is a relaxed asymptotic contraction in the sense of Definition 3.1, and therefore Theorem 3.2 theoretically generalizes Kirk's result.

Proof. Under the given uniform convergence on $[0, \infty)$, the sequence $\{\psi_n\}$ certainly converges uniformly on every bounded subset. Moreover, by Proposition 2.6 (4) we have

$$L_T(T^n x, T^n y) \leqslant d(T^{n+1} x, T^{n+1} y) \leqslant \psi_{n+1}(d(x, y)) \quad \text{and} \quad d(x, y) \leqslant U_T(x, y).$$

Defining $\tilde{\psi}_n(t) = \psi_{n+1}(t)$, the sequence $\{\tilde{\psi}_n\}$ also converges locally uniformly to ψ. Hence, the conditions of Definition 3.1 are satisfied, and the conclusion follows from Theorem 3.2. □

The following example illustrates how the proposed relaxed asymptotic contraction framework can be applied to a realistic engineering system with non-uniform damping characteristics.

Example 3.6 (Semi-active vibration isolation with an MR damper)**.** *Consider a single-degree-of-freedom mechanical system consisting of a mass m, a linear spring of stiffness k, and a magnetorheological (MR) damper whose damping coefficient can be continuously adjusted by an external magnetic field. The system is governed by the discrete-time dynamics*

$$x_{k+1} = T(x_k), \tag{3.29}$$

where x_k denotes the normalized displacement of the mass at the k-th sampling instant, and T represents the combined effect of the structural restoring force and the controlled damping force over one sampling period.

The damping ratio $\lambda(x)$ in the control law $T(x) = \lambda(x)x$ implements a three-regime semi-active control strategy:

$$\lambda(x) = \begin{cases} \dfrac{1}{3}, & |x| \leq 1, \\ \dfrac{1}{3} + \dfrac{1}{3}(|x| - 1), & 1 < |x| < 2, \\ \dfrac{2}{3}, & |x| \geq 2. \end{cases} \tag{3.30}$$

This control law is typical in MR damper-based suspension systems:

- *Small amplitudes ($|x| \leq 1$): The controller remains inactive, and the system operates in the linear viscous damping regime with constant damping ratio $1/3$;*
- *Intermediate amplitudes ($1 < |x| < 2$): The controller actively increases the damping force as the displacement grows, with the damping ratio increasing linearly from $1/3$ to $2/3$;*
- *Large amplitudes ($|x| \geq 2$): The damper enters magnetic saturation, and the damping ratio remains constant at $2/3$.*

The state space is $X = \mathbb{R}$ with $d(x, y) = |x - y|$, and $T(x) = \lambda(x)x$.

Let L_T and U_T be defined as in Definitions 2.3–2.5.

Step 1: Basic properties of the control system.

We first verify that the system satisfies the hypotheses of Theorem 3.2.

(1) *Continuity. The damping ratio function λ is continuous on $\mathbb{R}$: at $x = 1$, $\lambda(1^-) = \lambda(1^+) = 1/3$; at $x = 2$, $\lambda(2^-) = \lambda(2^+) = 2/3$. The cases $x = -1, -2$ are symmetric. Hence, T is continuous, meaning the control law produces no abrupt jumps in the system response.*

(2) *Unique equilibrium. If $T(x) = x$, then $\lambda(x)x = x$. For $x \neq 0$, this gives $\lambda(x) = 1$, but*

$$\lambda(x) \in [\tfrac{1}{3}, \tfrac{2}{3}] \subset (0, 1),$$

impossible. Hence, the system has a unique equilibrium at $x = 0$—the mass returns to its resting position.
(3) *Stability of all trajectories. For any displacement $x \in \mathbb{R}$,*

$$|T(x)| = |\lambda(x)x| \leqslant \frac{2}{3}|x|. \tag{3.31}$$

By induction,

$$|T^n x| \leqslant \left(\frac{2}{3}\right)^n |x| \to 0. \tag{3.32}$$

Thus, from any initial displacement, the system converges to equilibrium. Physically, this means the vibration is eventually completely suppressed regardless of the initial disturbance amplitude.
Step 2: Verification of the relaxed contraction condition.
For any $x, y \in \mathbb{R}$, by the definitions of L_T and U_T,

$$d(Tx, Ty) \leqslant U_T(x, y), \qquad L_T(x, y) \leqslant d(Tx, Ty) \leqslant U_T(x, y). \tag{3.33}$$

Physically, $U_T(x, y)$ captures both the displacement difference and the local damping effort at each point, while $L_T(x, y)$ provides an adaptive lower bound on the actual compression.

For any bounded displacement range $B_R = [-R, R] \subset \mathbb{R}$, since all trajectories eventually enter the small-amplitude region $[-1, 1]$ (where $T(x) = x/3$), there exist constants $C_R > 0$ and $N_R \in \mathbb{N}$ such that, for all $x, y \in B_R$ and all $n \geqslant N_R$,

$$\begin{aligned} L_T(T^n x, T^n y) &\leqslant d(T^{n+1} x, T^{n+1} y) \\ &\leqslant \left(\frac{1}{3}\right)^{n+1-N_R} d(T^{N_R} x, T^{N_R} y) \\ &\leqslant \left(\frac{1}{3}\right)^n C_R\, U_T(x, y). \end{aligned} \tag{3.34}$$

Define the control functions $\psi_n\colon [0, \infty) \to [0, \infty)$ by

$$\psi_n(t) = \left(\frac{1}{3}\right)^n C_R\, t + \frac{1}{n+1} \qquad (n \geq N_R), \tag{3.35}$$

and choose ψ_n sufficiently large for the finitely many indices $n < N_R$. Then,

$$L_T(T^n x, T^n y) \leqslant \psi_n(U_T(x, y)), \qquad \forall x, y \in B_R,\ \forall n. \tag{3.36}$$

On every bounded subset $[0, M] \subset [0, \infty)$,

$$\psi_n(t) \leqslant \left(\frac{1}{3}\right)^n C_R\, M + \frac{1}{n+1} \to 0. \tag{3.37}$$

Hence, ψ_n converges uniformly on bounded subsets to $\psi(t) = 0$. Since $\psi(0) = 0$ and $\psi(t) = 0 < t$ for all $t > 0$, the function ψ satisfies the Boyd–Wong condition. Therefore, the MR damper system satisfies condition (3.1).

Next,we describe an abstract scenario that illustrates the theoretical advantage of our relaxed framework over Kirk's original formulation.

Let (X, d) be a complete metric space and let $T: X \to X$ be continuous. Suppose that the following situation occurs:

- For most pairs $(x, y) \in X \times X$ (for instance, for all pairs outside a certain exceptional set $S \subset X \times X$), the map T behaves like a local Kirk-type asymptotic contraction: there exist local control functions $\{\varphi_n^{x,y}\}$ and a Boyd-Wong function $\varphi^{x,y}$ such that

$$d(T^n x, T^n y) \leqslant \varphi_n^{x,y}(d(x, y)), \quad \forall n \geqslant 0,$$

  and $\varphi_n^{x,y} \to \varphi^{x,y}$ uniformly on some neighborhood of $d(x, y)$, with $\varphi^{x,y}(t) < t$ for all $t > 0$.
- However, there exist Kirk-incompatible pairs $(x_*, y_*) \in S$ such that, for any global family $\{\varphi_n\}$ converging uniformly on $[0, \infty)$ to a Boyd-Wong function φ, the inequality

$$d(T^n x, T^n y) \leqslant \varphi_n(d(x, y)), \quad \forall (x, y) \in X \times X, \ \forall n \geq 0,$$

  forces, for $t_* := d(x_*, y_*) > 0$,

$$\limsup_{n\to\infty} \varphi_n(t_*) \geqslant t_*.$$

  This contradicts $\varphi(t_*) < t_*$. Hence, no single global family $\{\varphi_n\}$ exists, and Kirk's asymptotic contraction condition fails.
- Nevertheless, suppose that the quasi-metrics L_T and U_T (Definitions 2.3–2.5) satisfy our relaxed asymptotic contraction inequality

$$L_T(T^n x, T^n y) \leqslant \psi_n(U_T(x, y)), \quad \forall x, y \in X, \ \forall n \geq 0,$$

  where $\{\psi_n\}$ converges to a Boyd–Wong function ψ uniformly on every bounded subset of $[0, \infty)$, not necessarily on the whole half-line.

In this scenario, the Kirk-incompatible pairs (x_*, y_*) are precisely the obstacle to global uniform convergence. However, because our framework uses $U_T(x_*, y_*)$ (which may be strictly larger than $d(x_*, y_*)$) and L_T adaptively selects the smallest effective left-hand side, the inequality still holds. Moreover, our Theorem 3.2 requires only local uniform convergence of $\{\psi_n\}$ on bounded sets, which is compatible with the non-uniform behavior caused by the exceptional pairs.

Thus, this abstract scenario shows that the proposed relaxed asymptotic contraction framework can, in principle, accommodate mappings that are "almost" Kirk asymptotic contractions but fail Kirk's global uniform convergence requirement due to a small number of Kirk-incompatible pairs. Constructing a concrete, non-artificial example of this type (e.g., in a function space or arising from an integral equation) remains an open problem and constitutes one of our main directions for future research.

4. Conclusions

We have introduced a new framework for asymptotic contractions based on two quasi-metrics defined on the mapping T: A lower quasi-metric L_T that adaptively selects between a four-point minimum and the ordinary one-step distance, and an upper quasi-metric U_T that simply takes the maximum of four essential distances. The main theorem establishes that any continuous self-map of a complete metric space with a bounded orbit has a unique fixed point and all iterates converge to it, under the sole assumption that the bounding functions ψ_n converge uniformly on bounded sets to a Boyd-Wong function.

The Example 3.6 illustrates the applicability of our relaxed asymptotic contraction framework to a realistic semi-active control system with amplitude-dependent damping. Theoretically, our framework contains Kirk's asymptotic contraction theorem as a special case. Whether this containment is strict—that is, whether there exists a mapping that satisfies our Definition 3.1 but fails Kirk's condition—remains an open problem. We have not yet found such an example in a nontrivial setting, nor is there evidence suggesting that the two classes coincide. From our perspective, we believe that the proposed framework does genuinely extend Kirk's result, but constructing a convincing example that rigorously proves strict containment is a direction for future research. Regardless of the resolution of this question, the L_T-U_T framework and the local-uniform-convergence condition already provide new and flexible tools for the analysis of non-uniformly contracting dynamical systems, which is of independent interest.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares there is no conflict of interest.

References

1. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.*, **3** (1922), 133–181.
2. L. E. J. Brouwer, Über abbildung von mannigfaltigkeiten, *Math. Ann.*, **71** (1912), 97–115. https://doi.org/10.1007/BF01456931
3. J. Schauder, Der fixpunktsatz in funktionalräumen, *Stud. Math.*, **2** (1930), 171–180. https://doi.org/10.4064/sm-2-1-171-180
4. E. Rakotch, A note on contractive mappings, *Proc. Amer. Math. Soc.*, **13** (1962), 459–465. https://doi.org/10.1090/S0002-9939-1962-0148046-1
5. M. Edelstein, On fixed and periodic points under contractive mappings, *J. London Math. Soc.*, **s1-37** (1962), 74–79. https://doi.org/10.1112/jlms/s1-37.1.74
6. F. E. Browder, Nonexpansive nonlinear operators in Banach spaces, *Proc. Natl. Acad. Sci. USA*, **54** (1965), 1041–1044. https://doi.org/10.1073/pnas.54.4.1041
7. W. A. Kirk, A fixed point theorem for mappings which do not increase distances, *Amer. Math. Monthly*, **72** (1965), 1004–1006.

8. D. W. Boyd, J. S. W. Wong, On nonlinear contractions, *Proc. Amer. Math. Soc.*, **20** (1969), 458–464. https://doi.org/10.1090/S0002-9939-1969-0239559-9
9. L. B. Ćirić, A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.*, **45** (1974), 267–273. https://doi.org/10.1090/S0002-9939-1974-0356011-2
10. F. Gu, Parallel iterative algorithm for Ćirić quasi-contractive operators, *Abstr. Appl. Anal.*, **2012** (2012), 626547. https://doi.org/10.1155/2012/626547
11. H. Lakzian, D. Kocev, V. Rakočević, Ćirić-generalized contraction via *wt*-distance and applications to nonlinear fractional differential equations, *Appl. Gen. Topol.*, **24** (2023), 257–274.
12. M. Păcurar, Maia type fixed point theorems for Ćirić-Prešić operators, *Arab. J. Math.*, **13** (2024), 469–483. https://doi.org/10.1007/s40065-024-00487-8
13. Z. Bekri, N. Fabiano, A. K. Alsharidi, M. A. Alomair, New fixed point theorems by applying Singh's framework to iterated ρ-Ćirić contractions, *AIMS Math.*, **11** (2026), 1653–1674. https://doi.org/10.3934/math.2026069
14. W. A. Kirk, Fixed points of asymptotic contractions, *J. Math. Anal. Appl.*, **277** (2003), 645–650. https://doi.org/10.1016/S0022-247X(02)00612-1
15. T. Suzuki, Fixed-point theorem for asymptotic contractions of Meir-Keeler type in complete metric spaces, *Nonlinear Anal.*, **64** (2006), 971–978. https://doi.org/10.1016/j.na.2005.04.054
16. T. Lindström, D. A. Ross, A nonstandard approach to asymptotic fixed point theorems, *J. Fixed Point Theory Appl.*, **25** (2023), 35. https://doi.org/10.1007/s11784-022-01028-6
17. J. Paul, M. Sajid, N. Chandra, U. C. Gairola, Some common fixed point theorems in bipolar metric spaces and applications, *AIMS Math.*, **8** (2023), 19004–19017. https://doi.org/10.3934/math.2023969
18. M. Joshi, S. Upadhyay, A. Tomar, M. Sajid, Geometry and application in economics of fixed point, *Symmetry*, **15** (2023), 704. https://doi.org/10.3390/sym15030704
19. A. Gangwar, A. Tomar, M. Sajid, R. C. Dimri, Common fixed points and convergence results for α-Krasnosel'skii mappings, *AIMS Math.*, **8** (2023), 9911–9923. https://doi.org/10.3934/math.2023501

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