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Research article

Characterizations and constructions of tight K-frames in quaternionic Hilbert spaces

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Abstract: This paper introduces the notion of tight K-frames in quaternionic Hilbert spaces and investigates their characterizations and constructions. By employing analysis and frame operators, we provide an operator-theoretic characterization and demonstrate that a tight K-frame with bound C_1 reduces to an ordinary tight frame with bound C_2 if and only if $C_1KK^* = C_2I_{\mathcal{H}}$. Several construction methods are presented, including operator actions and combinations of Bessel sequences, with refined results for Parseval K-frames. Moreover, we establish a bijection between tight K-frames and classical tight frames on the subspace $\mathrm{Ran}(K^*)$, showing that each can be obtained from the other via the pseudo-inverse of K.

Keywords: quaternionic Hilbert space; tight frame; K-frame; tight K-frame; Parseval K-frame
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1. Introduction

Frame theory originated in 1952 from the study of nonharmonic Fourier series by Duffin and Schaeffer [1], with its core idea being the stable reconstruction of elements in a Hilbert space using possibly overcomplete families of vectors. Although initially overlooked, the theory gained significant momentum following the seminal work of Daubechies, Grossmann, and Meyer [2] in 1986 and has since become an important branch of functional and harmonic analysis. As a natural generalization of orthonormal bases, frames have found extensive applications in signal processing, quantum mechanics, compressed sensing, and other fields [3, 4].

Classical frame theory is primarily concerned with the reconstruction of elements throughout the entire space. However, many practical problems require reconstruction only from the range of a bounded linear operator. To address this more general setting, Găvruţa [5] introduced the notion of K-frames in 2012. Unlike classical frames, the frame operator of a K-frame is generally not invertible, its synthesis operator need not be surjective, and reconstruction typically depends on a specific dual frame, whose choice is generally not unique [6]. Since then, a substantial body of results on the characterization, construction, and duality theory of K-frames has emerged [7–9]. However, the aforementioned work is confined to real or complex Hilbert spaces, and relatively few studies address K-frame theory in the quaternionic setting.

On the other hand, Hilbert spaces can also be constructed over the quaternion skew field [10]. In 1936, Birkhoff and von Neumann [11] proposed that quaternionic Hilbert spaces could be used to study quantum mechanics in place of complex Hilbert spaces, a perspective that has since been validated (see [12, 13]). Nevertheless, the noncommutativity of quaternionic multiplication has long hindered the development of functional and harmonic analysis in this setting [14, 15]. It was not until 2017 that Khokulan et al. [16] first introduced discrete frames into finite-dimensional quaternionic Hilbert spaces, though their work was restricted to the finite-dimensional case and did not involve K-frames. Subsequently, Sharma and Goel [17] established frame theory on general quaternionic Hilbert spaces, and notions such as Riesz bases, dual frames, weaving frames, and fusion frames have been successively extended to the quaternionic setting [18–20]. In particular, Charfi and Ellouz [21] generalized K-frames to separable quaternionic Hilbert spaces, providing a formal definition and fundamental characterizations, thereby enabling element reconstruction from the range of bounded linear operators in this context. In general, the reconstruction associated with a K-frame depends on a specific dual frame, whose choice is not unique; a tightness structure naturally eliminates this dependence.

A tight K-frame, whose frame operator has the form CKK^*, retains the ability to reconstruct elements from the range of the operator while achieving a reconstruction formula as simple as that of a classical tight frame, without resorting to dual frames. Theorem 3.8 below shows that tight K-frames are in one-to-one correspondence with classical tight frames on the subspace $\mathrm{Ran}(K^*)$; hence, the former can be viewed as an extension of the latter to the range of an operator.

Based on the above considerations, this paper combines the notion of tight frames with K-frames and introduces tight K-frames in quaternionic Hilbert spaces. The main results are as follows. Theorem 3.1 gives an operator-theoretic characterization of tight K-frames, which serves as a foundational tool for the subsequent analysis. Theorem 3.3 establishes a necessary and sufficient condition for a tight K-frame to reduce to a classical tight frame, which extends the classical tight-frame criterion to the K-frame setting. Theorems 3.5 and 3.6 provide two construction methods via operator actions and operator compositions, respectively; Corollary 3.4 treats the Parseval case separately. Theorem 3.8 establishes a one-to-one correspondence between tight K-frames and classical tight frames on the subspace $\mathrm{Ran}(K^*)$.

The remainder of this paper is organized as follows. Section 2 recalls the necessary preliminaries on quaternionic Hilbert spaces and frames. Section 3 presents the main results, including operator characterizations, construction methods for tight K-frames, and their connection with classical tight frames on subspaces.

2. Preliminaries

This section recalls the foundational elements of quaternionic algebra and the theory of right quaternionic Hilbert spaces. It also introduces the definitions and notations for frames within this setting that will be used throughout the paper.

For convenience, the following notation is adopted throughout: $\mathbb{H}$ denotes the noncommutative quaternion skew field, $\mathcal{J}$ is a countable index set, $\mathcal{H}_r(\mathbb{H})$ represents a separable right quaternionic Hilbert space, and $I_{\mathcal{H}}$ is the identity operator on it. Given two right quaternionic Hilbert spaces $\mathcal{H}_r^1(\mathbb{H})$ and $\mathcal{H}_r^2(\mathbb{H})$, the set of all bounded right-linear operators from $\mathcal{H}_r^1(\mathbb{H})$ to $\mathcal{H}_r^2(\mathbb{H})$ is denoted by $\mathscr{B}(\mathcal{H}_r^1(\mathbb{H}), \mathcal{H}_r^2(\mathbb{H}))$. For simplicity, we write $\mathscr{B}(\mathcal{H}_r(\mathbb{H})) = \mathscr{B}(\mathcal{H}_r(\mathbb{H}), \mathcal{H}_r(\mathbb{H}))$.

2.1. Quaternion algebras and right quaternionic Hilbert spaces

Due to the noncommutative nature of quaternionic multiplication, quaternionic Hilbert spaces are typically classified into right and left types. This paper focuses on the former, and we therefore concentrate here on the basic properties of quaternionic algebra, right quaternionic Hilbert spaces, and their linear operators. For a more comprehensive treatment, we refer the reader to [14, 15].

The quaternion skew field $\mathbb{H}$ is a four-dimensional associative algebra over the real numbers $\mathbb{R}$, with zero element denoted by 0 and multiplicative identity by 1. An element of $\mathbb{H}$ can be written as

$$q = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k},\ \ x_0, x_1, x_2, x_3 \in \mathbb{R},$$

where x_0 is called the real part, $x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ the imaginary part, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the three imaginary units satisfying the multiplication rules

$$\mathbf{i^2} = \mathbf{j^2} = \mathbf{k^2} = \mathbf{-1},\ \mathbf{ij} = \mathbf{-ji} = \mathbf{k},\ \mathbf{jk} = \mathbf{-kj} = \mathbf{i},\ \text{ and } \mathbf{ki} = \mathbf{-ik} = \mathbf{j}.$$

The conjugate of q is $\bar{q} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}$, and its modulus is $|q| = (q\bar{q})^{1/2} = (\bar{q}q)^{1/2} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$.

Let $\mathbf{H}_r(\mathbb{H})$ be a right linear space over $\mathbb{H}$. If there exists a mapping $\langle \cdot \mid \cdot \rangle : \mathbf{H}_r(\mathbb{H}) \times \mathbf{H}_r(\mathbb{H}) \to \mathbb{H}$ satisfying, for all $x, y, z \in \mathbf{H}_r(\mathbb{H})$ and $q \in \mathbb{H}$,

(i) $\langle x \mid y \rangle = \overline{\langle y \mid x \rangle}$;

(ii) $\langle x \mid x \rangle \geq 0$ with equality if and only if $x = 0$;

(iii) $\langle x \mid y + z \rangle = \langle x \mid y \rangle + \langle x \mid z \rangle$;

(iv) $\langle x \mid yq \rangle = \langle x \mid y \rangle q$ and $\langle xq \mid y \rangle = \bar{q}\langle x \mid y \rangle$,

then $\mathbf{H}_r(\mathbb{H})$ is called a right quaternionic inner product space (or right quaternionic pre-Hilbert space). Furthermore, if $\mathbf{H}_r(\mathbb{H})$ is complete with respect to the norm $\|\cdot\| = \sqrt{\langle \cdot \mid \cdot \rangle}$ induced by the inner product, it is called a right quaternionic Hilbert space, denoted as $\mathcal{H}_r(\mathbb{H})$.

Right quaternionic Hilbert spaces retain many analytical properties of classical Hilbert spaces, as shown in the following proposition.

Proposition 2.1. [15] Let $\mathcal{H}_r(\mathbb{H})$ be a right quaternionic Hilbert space. Then for any $x, y \in \mathcal{H}_r(\mathbb{H})$, we have

(i) $|\langle x \mid y \rangle|^2 \leq |\langle x \mid x \rangle||\langle y \mid y \rangle|$ (Cauchy-Schwarz inequality);

(ii) $\|x\| = \sup_{\|y\|=1} |\langle x \mid y \rangle|$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$;

(iv) The polarization identity holds

$$4\langle x \mid y\rangle = \|x+y\|^2 - \|x-y\|^2 + (\|x\mathbf{i}+y\|^2 - \|x\mathbf{i}-y\|^2)\mathbf{i} + (\|x\mathbf{j}+y\|^2 - \|x\mathbf{j}-y\|^2)\mathbf{j} + (\|x\mathbf{k}+y\|^2 - \|x\mathbf{k}-y\|^2)\mathbf{k}.$$

As in the classical case, orthonormal bases play a central role in quaternionic Hilbert spaces.

Proposition 2.2. [15] Let $\mathcal{N} \subset \mathcal{H}_r(\mathbb{H})$ be an orthonormal subset, i.e., for any $z, z' \in \mathcal{N}$, $\langle z \mid z'\rangle = 0$ if $z \neq z'$ and $\langle z \mid z\rangle = 1$. Then the following statements are equivalent:

(i) For all $x, y \in \mathcal{H}_r(\mathbb{H})$, the series $\sum_{z\in\mathcal{N}}\langle x \mid z\rangle\langle z \mid y\rangle$ converges absolutely, and

$$\langle x \mid y\rangle = \sum_{z\in\mathcal{N}}\langle x \mid z\rangle\langle z \mid y\rangle;$$

(ii) For all $x \in \mathcal{H}_r(\mathbb{H})$, $\|x\|^2 = \sum_{z\in\mathcal{N}}|\langle z \mid x\rangle|^2$;

(iii) $\mathcal{N}^{\perp} = \{x \in \mathcal{H}_r(\mathbb{H}) : \langle x \mid z\rangle = 0, \forall z \in \mathcal{N}\} = \{0\}$;

(iv) Span $\mathcal{N}$ is dense in $\mathcal{H}_r(\mathbb{H})$.

An orthonormal subset $\mathcal{N}$ satisfying these conditions is called a Hilbert basis (or orthonormal basis) for $\mathcal{H}_r(\mathbb{H})$. The existence of such a basis is guaranteed by the following result.

Proposition 2.3. [15] Every right quaternionic Hilbert space $\mathcal{H}_r(\mathbb{H})$ possesses a Hilbert basis $\mathcal{N}$. Moreover, if $\mathcal{N}$ is a Hilbert basis for $\mathcal{H}_r(\mathbb{H})$, then every $x \in \mathcal{H}_r(\mathbb{H})$ has a unique representation

$$x = \sum_{z\in\mathcal{N}} z\langle z \mid x\rangle,$$

and this series converges absolutely in $\mathcal{H}_r(\mathbb{H})$.

2.2. *Bounded linear operators*

A map $T : \mathcal{H}_r(\mathbb{H}) \to \mathcal{H}_r(\mathbb{H})$ is called a right-linear operator if for all $x, y \in \mathcal{H}_r(\mathbb{H})$ and $p, q \in \mathbb{H}$,

$$T(xp + yq) = T(x)p + T(y)q.$$

If there exists a constant $M \geq 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in \mathcal{H}_r(\mathbb{H})$, then T is called bounded. The set of all bounded right-linear operators on $\mathcal{H}_r(\mathbb{H})$ is denoted by $\mathscr{B}(\mathcal{H}_r(\mathbb{H}))$.

For $T \in \mathscr{B}(\mathcal{H}_r(\mathbb{H}))$, its adjoint operator T^* is the unique operator satisfying

$$\langle x \mid Ty\rangle = \langle T^*x \mid y\rangle, \ \forall x, y \in \mathcal{H}_r(\mathbb{H}).$$

We denote the kernel and range of T by $\ker T$ and $\operatorname{Ran} T$, respectively. If $\operatorname{Ran} T$ is closed, the Moore-Penrose pseudo-inverse $T^\dagger$ of T satisfies

$$T^\dagger T = P_{\operatorname{Ran}(T^*)}, \ \ TT^\dagger = P_{\operatorname{Ran}(T)},$$

where $P_{\operatorname{Ran}(T)}$ denotes the orthogonal projection onto the subspace $\operatorname{Ran} T$. An operator T is called self-adjoint if $T = T^*$. The norm of a self-adjoint operator T can be expressed as

$$\|T\| = \sup_{\|x\|=1} |\langle Tx \mid x\rangle|.$$

2.3. Frames in right quaternionic Hilbert spaces

We first define the sequence space $\ell^2(\mathbb{H})$ as follows:

$$\ell^2(\mathbb{H}) = \left\{ \{q_j\}_{j\in\mathcal{J}} \subset \mathbb{H} : \sum_{j\in\mathcal{J}} |q_j|^2 < \infty \right\}.$$

Its inner product is defined by

$$\langle p \mid q\rangle = \sum_{j\in\mathcal{J}} \overline{p_j} q_j, \quad p = \{p_j\}_{j\in\mathcal{J}} \text{ and } q = \{q_j\}_{j\in\mathcal{J}} \in \ell^2(\mathbb{H}).$$

Then $\ell^2(\mathbb{H})$ forms a right quaternionic Hilbert space.

Definition 2.1. [17] Let $\{u_j\}_{j\in\mathcal{J}} \subset \mathcal{H}_r(\mathbb{H})$. If there exist constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{j\in\mathcal{J}} |\langle u_j \mid x\rangle|^2 \leq B\|x\|^2, \quad \forall x \in \mathcal{H}_r(\mathbb{H}), \tag{2.1}$$

then $\{u_j\}_{j\in\mathcal{J}}$ is called a frame for $\mathcal{H}_r(\mathbb{H})$, with lower frame bound A and upper frame bound B. If only the right-hand inequality in (2.1) holds, it is called a Bessel sequence with Bessel bound B. If $A = B$, it is called a tight frame. If $A = B = 1$, it is called a Parseval frame.

For a Bessel sequence $\{u_j\}_{j\in\mathcal{J}}$, we can define its synthesis operator $\Theta : \ell^2(\mathbb{H}) \to \mathcal{H}_r(\mathbb{H})$ by

$$\Theta(\{q_j\}_{j\in\mathcal{J}}) = \sum_{j\in\mathcal{J}} u_j q_j, \quad \forall \{q_j\}_{j\in\mathcal{J}} \in \ell^2(\mathbb{H}),$$

which is a bounded linear operator. Its adjoint operator $\Theta^* : \mathcal{H}_r(\mathbb{H}) \to \ell^2(\mathbb{H})$, called the analysis operator, satisfies

$$\Theta^* x = \{\langle u_j \mid x\rangle\}_{j\in\mathcal{J}}, \ \forall x \in \mathcal{H}_r(\mathbb{H}).$$

Furthermore, the frame operator $S = \Theta\Theta^* : \mathcal{H}_r(\mathbb{H}) \to \mathcal{H}_r(\mathbb{H})$ is defined by

$$Sx = \sum_{j\in\mathcal{J}} u_j \langle u_j \mid x\rangle, \ \forall x \in \mathcal{H}_r(\mathbb{H}).$$

Definition 2.2. [18] Let $K \in \mathscr{B}(\mathcal{H}_r(\mathbb{H}))$ and $\{u_j\}_{j\in\mathcal{J}} \subset \mathcal{H}_r(\mathbb{H})$. If there exist constants $C, D > 0$ such that

$$C\|K^* x\|^2 \leq \sum_{j\in\mathcal{J}} |\langle u_j \mid x\rangle|^2 \leq D\|x\|^2, \quad \forall x \in \mathcal{H}_r(\mathbb{H}), \tag{2.2}$$

then $\{u_j\}_{j\in\mathcal{J}}$ is called a K-frame for $\mathcal{H}_r(\mathbb{H})$.

Remark 2.1. By definition, a K-frame is always a Bessel sequence. When $K = I_{\mathcal{H}}$, the K-frame reduces to a classical frame.

We now extend the concept of tight frames to K-frames.

Definition 2.3. Let $K \in \mathscr{B}(\mathcal{H}_r(\mathbb{H}))$ and $\{u_j\}_{j\in\mathcal{J}} \subset \mathcal{H}_r(\mathbb{H})$. If there exists a constant $C > 0$ such that

$$C\|K^*x\|^2 = \sum_{j\in\mathcal{J}} |\langle u_j \mid x\rangle|^2, \quad \forall x \in \mathcal{H}_r(\mathbb{H}), \tag{2.3}$$

then $\{u_j\}_{j\in\mathcal{J}}$ is called a tight K-frame for $\mathcal{H}_r(\mathbb{H})$ with frame bound C. In particular, if $C = 1$, it is called a Parseval K-frame.

Remark 2.2. When $K = I_{\mathcal{H}}$, a tight K-frame is exactly a classical tight frame, and a Parseval K-frame is a classical Parseval frame. Thus, tight K-frames generalize classical tight frames by replacing the identity operator with an arbitrary bounded operator K.

3. Characterizations and constructions of tight K-frames

This section characterizes and constructs tight K-frames in quaternionic Hilbert spaces. We first provide an operator characterization using synthesis and frame operators and establish conditions under which such frames reduce to classical tight frames. Several construction methods are then developed, including operator actions and combinations of Bessel sequences, with refined results for Parseval K-frames. Finally, we establish a bijection between tight K-frames and classical tight frames on the subspace $\mathrm{Ran}(K^*)$.

To facilitate the discussion, we first recall several auxiliary results that will be used throughout this section.

3.1. Auxiliary lemmas

The first lemma is a quaternionic analogue of the classical Douglas theorem, characterizing the inclusion of ranges of bounded operators.

Lemma 3.1. [18] Let $L_1 \in \mathscr{B}(\mathcal{H}_r^1(\mathbb{H}), \mathcal{H}_r(\mathbb{H}))$ and $L_2 \in \mathscr{B}(\mathcal{H}_r^2(\mathbb{H}), \mathcal{H}_r(\mathbb{H}))$. Then the following statements are equivalent:

(a) $\mathrm{Ran}(L_1) \subseteq \mathrm{Ran}(L_2)$;

(b) There exists $\lambda > 0$ such that $L_1L_1^* \leq \lambda^2 L_2L_2^*$;

(c) There exists a bounded operator $M \in \mathscr{B}(\mathcal{H}_r^1(\mathbb{H}), \mathcal{H}_r^2(\mathbb{H}))$ such that $L_1 = L_2M$.

The following lemma provides several equivalent characterizations of K-frames.

Lemma 3.2. [18] Let $K \in \mathscr{B}(\mathcal{H}_r(\mathbb{H}))$ and $\{u_j\}_{j\in\mathcal{J}} \subset \mathcal{H}_r(\mathbb{H})$. Then the following statements are equivalent:

(a) $\{u_j\}_{j\in\mathcal{J}}$ is a K-frame for $\mathcal{H}_r(\mathbb{H})$;

(b) $\{u_j\}_{j\in\mathcal{J}}$ is a Bessel sequence, and there exists a Bessel sequence $\{v_j\}_{j\in\mathcal{J}} \subset \mathcal{H}_r(\mathbb{H})$ such that

$$Kx = \sum_{j\in\mathcal{J}} u_j\langle v_j|x\rangle, \ \forall x \in \mathcal{H}_r(\mathbb{H});$$

(c) $\{u_j\}_{j\in\mathcal{J}}$ is a Bessel sequence and $\mathrm{Ran}(K) \subseteq \mathrm{Ran}(\Theta)$, where Θ is the synthesis operator of $\{u_j\}_{j\in\mathcal{J}}$.

Lemma 3.3. [18] Let $T : \mathcal{H}_r^1(\mathbb{H}) \to \mathcal{H}_r^2(\mathbb{H})$ be a bounded right-linear operator with closed range $\operatorname{Ran} T$. Then there exists a unique bounded right-linear operator $T^\dagger : \mathcal{H}_r^2(\mathbb{H}) \to \mathcal{H}_r^1(\mathbb{H})$, called the Moore-Penrose pseudo-inverse of T, satisfying

$$\ker(T^\dagger) = \operatorname{Ran}(T)^\perp, \ \operatorname{Ran}(T^\dagger) = \ker(T)^\perp, \ \text{ and } TT^\dagger x = x, \ \forall x \in \operatorname{Ran}(T).$$

Moreover, $TT^\dagger$ and $T^\dagger T$ are the orthogonal projections of $\mathcal{H}_r^2(\mathbb{H})$ onto $\operatorname{Ran} T$ and of $\mathcal{H}_r^1(\mathbb{H})$ onto $\operatorname{Ran}(T^*)$, respectively.

When constructing new frames, it is often necessary to combine Bessel sequences linearly. The following lemma guarantees the Bessel property of such combinations.

Lemma 3.4. Let $T_1, T_2 \in \mathscr{B}(\mathcal{H}_r^1(\mathbb{H}), \mathcal{H}_r^2(\mathbb{H}))$. Suppose $\{u_j\}_{j\in\mathcal{J}}$ and $\{v_j\}_{j\in\mathcal{J}}$ are Bessel sequences in $\mathcal{H}_r^1(\mathbb{H})$ with Bessel bounds D_1 and D_2, respectively. Then $\{T_1 u_j + T_2 v_j\}_{j\in\mathcal{J}}$ is a Bessel sequence in $\mathcal{H}_r^2(\mathbb{H})$ with bound $(\sqrt{D_1}\|T_1\| + \sqrt{D_2}\|T_2\|)^2$.

Proof. Since $\{u_j\}_{j\in\mathcal{J}}$ and $\{v_j\}_{j\in\mathcal{J}}$ are Bessel sequences in $\mathcal{H}_r^1(\mathbb{H})$ with bounds D_1 and D_2, for any $x \in \mathcal{H}_r^1(\mathbb{H})$, we have

$$\sum_{j\in\mathcal{J}} |\langle u_j|x\rangle|^2 \le D_1\|x\|^2, \quad \sum_{j\in\mathcal{J}} |\langle v_j|x\rangle|^2 \le D_2\|x\|^2.$$

For any $y \in \mathcal{H}_r^2(\mathbb{H})$, applying the triangle inequality and the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\left(\sum_{j\in\mathcal{J}} \left|\langle T_1 u_j + T_2 v_j \mid y\rangle\right|^2\right)^{\frac{1}{2}} &= \left(\sum_{j\in\mathcal{J}} |\langle u_j \mid T_1^* y\rangle + \langle v_j \mid T_2^* y\rangle|^2\right)^{\frac{1}{2}} \\
&\le \left(\sum_{j\in\mathcal{J}} |\langle u_j \mid T_1^* y\rangle|^2\right)^{\frac{1}{2}} + \left(\sum_{j\in\mathcal{J}} |\langle v_j \mid T_2^* y\rangle|^2\right)^{\frac{1}{2}} \\
&\le \sqrt{D_1}\|T_1^* y\| + \sqrt{D_2}\|T_2^* y\| \\
&\le \left(\sqrt{D_1}\|T_1\| + \sqrt{D_2}\|T_2\|\right)\|y\|.
\end{aligned}$$

Therefore, $\{T_1 u_j + T_2 v_j\}_{j\in\mathcal{J}}$ is a Bessel sequence in $\mathcal{H}_r^2(\mathbb{H})$ with the stated bound. □

3.2. Operator characterizations of tight K-frames

The following theorem characterizes a Bessel sequence as a tight K-frame in terms of K and its synthesis operator Θ.

Theorem 3.1. Let $K \in \mathscr{B}(\mathcal{H}_r(\mathbb{H}))$ and let $\{u_j\}_{j\in\mathcal{J}} \subset \mathcal{H}_r(\mathbb{H})$ be a Bessel sequence for $\mathcal{H}_r(\mathbb{H})$ with synthesis operator Θ. Then $\{u_j\}_{j\in\mathcal{J}}$ is a tight K-frame for $\mathcal{H}_r(\mathbb{H})$ with bound C if and only if

$$CKK^* = \Theta\Theta^*.$$

Proof. **Necessity:** If $\{u_j\}_{j\in\mathcal{J}}$ is a tight K-frame for $\mathcal{H}_r(\mathbb{H})$ with bound C, then for all $x \in \mathcal{H}_r(\mathbb{H})$,

$$C\|K^* x\|^2 = \sum_{j\in\mathcal{J}} |\langle u_j \mid x\rangle|^2 = \|\Theta^* x\|^2.$$

Since $||K^*x||^2 = \langle KK^*x \mid x\rangle$ and $||\Theta^*x||^2 = \langle \Theta\Theta^*x \mid x\rangle$, we have

$$C\langle KK^*x \mid x\rangle = \langle \Theta\Theta^*x \mid x\rangle, \quad \forall x \in \mathcal{H}_r(\mathbb{H}).$$

Both CKK^* and $\Theta\Theta^*$ are self-adjoint operators, so $CKK^* = \Theta\Theta^*$.

Sufficiency: If $CKK^* = \Theta\Theta^*$, then for any $x \in \mathcal{H}_r(\mathbb{H})$,

$$C||K^*x||^2 = C\langle KK^*x \mid x\rangle = \langle \Theta\Theta^*x \mid x\rangle = ||\Theta^*x||^2 = \sum_{j\in\mathcal{J}} |\langle u_j \mid x\rangle|^2,$$

which means $\{u_j\}_{j\in\mathcal{J}}$ is a tight K-frame for $\mathcal{H}_r(\mathbb{H})$ with bound C. □

For the special case $C = 1$, we obtain a characterization for Parseval K-frames.

Corollary 3.1. Let $K \in \mathscr{B}(\mathcal{H}_r(\mathbb{H}))$, and let $\{u_j\}_{j\in\mathcal{J}} \subset \mathcal{H}_r(\mathbb{H})$ be a Bessel sequence with synthesis operator Θ. Then $\{u_j\}_{j\in\mathcal{J}}$ is a Parseval K-frame for $\mathcal{H}_r(\mathbb{H})$ if and only if

$$KK^* = \Theta\Theta^*.$$

Theorem 3.1 directly yields several fundamental properties of tight K-frames, as summarized in the following result.

Theorem 3.2. Let $K \in \mathscr{B}(\mathcal{H}_r(\mathbb{H}))$, and let $\{u_j\}_{j\in\mathcal{J}} \subset \mathcal{H}_r(\mathbb{H})$ be a tight K-frame for $\mathcal{H}_r(\mathbb{H})$ with bound C. Let S and Θ be its frame operator and synthesis operator, respectively. Then

(1) $\mathrm{Ran}(K) = \mathrm{Ran}(\Theta)$ and $S = CKK^*$;

(2) $||\Theta||^2 = C||K||^2$.

Proof. (1) From Theorem 3.1, we have $CKK^* = \Theta\Theta^*$. Applying Lemma 3.1, we obtain $\mathrm{Ran}(K) = \mathrm{Ran}(\Theta)$. Since the frame operator satisfies $S = \Theta\Theta^*$, it follows that $S = CKK^*$.

(2) As $\{u_j\}_{j\in\mathcal{J}}$ is a tight K-frame with bound C, Theorem 3.1 implies $C||K^*x||^2 = ||\Theta^*x||^2$ for all $x \in \mathcal{H}_r(\mathbb{H})$. Therefore,

$$||\Theta||^2 = ||\Theta^*||^2 = \sup_{||x||=1} ||\Theta^*x||^2 = C \sup_{||x||=1} ||K^*x||^2 = C||K^*||^2 = C||K||^2.$$

This completes the proof. □

Remark 3.1. When $K = I_{\mathcal{H}}$, a tight K-frame is a classical tight frame. However, the converse is not necessarily true; a tight K-frame may also be a classical tight frame without K being the identity, or even invertible. This is illustrated in the following example.

Example 3.1. Let $\{e_j\}_{j\in\mathcal{J}}$ be an orthonormal basis for $\mathcal{H}_r(\mathbb{H})$. Define $K : \mathcal{H}_r(\mathbb{H}) \to \mathcal{H}_r(\mathbb{H})$ by

$$Ke_1 = 0,\ Ke_j = e_{j-1} \text{ for } j \geq 2.$$

It is easy to see that K is bounded, and its adjoint satisfies $K^*e_j = e_{j+1}$ for all $j \geq 1$. It follows that $||K^*x|| = ||x||$ for all $x \in \mathcal{H}_r(\mathbb{H})$. Setting $u_j = e_j$ for each $j \in \mathcal{J}$, we have

$$\sum_{j\in\mathcal{J}} |\langle u_j \mid x\rangle|^2 = ||x||^2 = ||K^*x||^2,\ \forall x \in \mathcal{H}_r(\mathbb{H}).$$

So, $\{u_j\}_{j\in\mathcal{J}}$ is simultaneously a Parseval K-frame and a Parseval frame for $\mathcal{H}_r(\mathbb{H})$. However, K is not injective because $Ke_1 = 0$ while $e_1 \neq 0$. K is surjective but not injective, and hence not invertible.

The following theorem characterizes precisely when a tight K-frame is also a classical tight frame.

Theorem 3.3. Let $K \in \mathscr{B}(\mathcal{H}_r(\mathbb{H}))$, and let $\{u_j\}_{j\in\mathcal{J}} \subset \mathcal{H}_r(\mathbb{H})$ be a tight K-frame for $\mathcal{H}_r(\mathbb{H})$ with bound C_1. Then $\{u_j\}_{j\in\mathcal{J}}$ is also a tight frame for $\mathcal{H}_r(\mathbb{H})$ with bound C_2 if and only if K is right-invertible and $\frac{C_1}{C_2}K^*$ is a right inverse of K.

Proof. **Necessity:** If $\{u_j\}_{j\in\mathcal{J}}$ is both a tight K-frame with bound C_1 and a tight frame with bound C_2, then for all $x \in \mathcal{H}_r(\mathbb{H})$,

$$C_1\|K^*x\|^2 = \sum_{j\in\mathcal{J}} |\langle u_j \mid x\rangle|^2 = C_2\|x\|^2.$$

By Theorem 3.1, this implies $C_1KK^* = C_2I_{\mathcal{H}}$, and consequently

$$K\left(\frac{C_1}{C_2}K^*\right) = I_{\mathcal{H}}.$$

Hence, K is right-invertible with $\frac{C_1}{C_2}K^*$ as a right inverse.

Sufficiency: If K is right-invertible with $\frac{C_1}{C_2}K^*$ as a right inverse, then $C_1KK^* = C_2I_{\mathcal{H}}$. From Theorem 3.1, we have

$$C_1\|K^*x\|^2 = \|\Theta^*x\|^2, \forall x \in \mathcal{H}_r(\mathbb{H}).$$

Substituting the operator equality yields

$$\|\Theta^*x\|^2 = C_2\|x\|^2,$$

proving that $\{u_j\}_{j\in\mathcal{J}}$ is a tight frame for $\mathcal{H}_r(\mathbb{H})$ with bound C_2. □

Corollary 3.2. A Parseval K-frame is a Parseval frame on $\mathcal{H}_r(\mathbb{H})$ if and only if K is right-invertible and K^* is a right inverse of K.

Example 3.2. In Example 3.1, $\{e_j\}_{j\in\mathcal{J}}$ is both a Parseval K-frame and a Parseval frame. We verify that K is right-invertible with right inverse K^*.

For any $x = \sum_{j\in\mathcal{J}} e_jq_j \in \mathcal{H}_r(\mathbb{H})$, we have

$$K^*x = \sum_{j\in\mathcal{J}} e_{j+1}q_j,$$

and thus

$$K(K^*x) = K\left(\sum_{j\in\mathcal{J}} e_{j+1}q_j\right) = \sum_{j\in\mathcal{J}} Ke_{j+1}q_j = \sum_{j\in\mathcal{J}} e_jq_j = x.$$

Hence, $KK^*x = x$ for all $x \in \mathcal{H}_r(\mathbb{H})$, i.e., $KK^* = I_{\mathcal{H}}$, confirming that K^* is a right inverse of K.

We conclude this subsection with a necessary condition for tight K-frames concerning the existence of a dual Bessel sequence with a norm-squared sum bounded below.

Theorem 3.4. Let $K \in \mathscr{B}(\mathcal{H}_r(\mathbb{H}))$ with $K \neq 0$, and let $\{u_j\}_{j\in\mathcal{J}}$ be a tight K-frame for $\mathcal{H}_r(\mathbb{H})$ with bound C. Then there exists a Bessel sequence $\{v_j\}_{j\in\mathcal{J}}$ in $\mathcal{H}_r(\mathbb{H})$ such that

$$Kx = \sum_{j\in\mathcal{J}} u_j\langle v_j \mid x\rangle, \ \forall x \in \mathcal{H}_r(\mathbb{H}), \tag{3.1}$$

and $\sum_{j\in\mathcal{J}} \|v_j\|^2 \geq 1/C$.

Proof. By Lemma 3.2, there exists a Bessel sequence $\{v_j\}_{j\in\mathcal{J}} \subset \mathcal{H}_r(\mathbb{H})$ such that

$$Kx = \sum_{j\in\mathcal{J}} u_j\langle v_j \mid x\rangle, \ \forall x \in \mathcal{H}_r(\mathbb{H}). \tag{3.2}$$

For any $x, y \in \mathcal{H}_r(\mathbb{H})$, from (3.2), we have

$$\langle Kx \mid y\rangle = \sum_{j\in\mathcal{J}}\langle u_j\langle v_j \mid x\rangle \mid y\rangle = \sum_{j\in\mathcal{J}} \overline{\langle v_j \mid x\rangle}\langle u_j \mid y\rangle.$$

Since $\langle Kx \mid y\rangle = \langle x \mid K^*y\rangle$, we obtain

$$\langle x \mid K^*y\rangle = \sum_{j\in\mathcal{J}}\langle x \mid v_j\rangle\langle u_j \mid y\rangle, \forall x, y \in \mathcal{H}_r(\mathbb{H}).$$

By the arbitrariness of x, this implies

$$K^*y = \sum_{j\in\mathcal{J}} v_j\langle u_j \mid y\rangle, \forall y \in \mathcal{H}_r(\mathbb{H}). \tag{3.3}$$

For any $x \in \mathcal{H}_r(\mathbb{H})$, the definition of a tight K-frame and (3.3) yield

$$\sum_{j\in\mathcal{J}} |\langle u_j \mid x\rangle|^2 = C\|K^*x\|^2 = C\left\|\sum_{j\in\mathcal{J}} v_j\langle u_j \mid x\rangle\right\|^2. \tag{3.4}$$

Using the supremum representation of the norm and the Cauchy-Schwarz inequality,

$$\left\|\sum_{j\in\mathcal{J}} v_j\langle u_j \mid x\rangle\right\| \leq \left(\sum_{j\in\mathcal{J}} |\langle u_j \mid x\rangle|^2\right)^{1/2} \sup_{\|y\|=1}\left(\sum_{j\in\mathcal{J}} |\langle v_j \mid y\rangle|^2\right)^{1/2}.$$

Combining this with (3.4) gives

$$\sum_{j\in\mathcal{J}} |\langle u_j \mid x\rangle|^2 \leq C\left(\sum_{j\in\mathcal{J}} |\langle u_j \mid x\rangle|^2\right) \cdot \sup_{\|y\|=1}\left(\sum_{j\in\mathcal{J}} |\langle v_j \mid y\rangle|^2\right). \tag{3.5}$$

Since $K \neq 0$, there exists $x_0 \in \mathcal{H}_r(\mathbb{H})$ such that $K^*x_0 \neq 0$, and thus

$$\sum_{j\in\mathcal{J}} |\langle u_j \mid x_0\rangle|^2 = C\|K^*x_0\| > 0.$$

Dividing by this positive quantity yields

$$1 \leq C\sup_{\|y\|=1}\left(\sum_{j\in\mathcal{J}} |\langle v_j \mid y\rangle|^2\right) \leq C\sum_{j\in\mathcal{J}} \|v_j\|^2,$$

where the last inequality follows from $|\langle v_j \mid y\rangle| \leq \|v_j\|\|y\|$. Thus, $\sum_{j\in\mathcal{J}} \|v_j\|^2 \geq 1/C$. □

Remark 3.2. It is worth noting that the noncommutativity of quaternionic multiplication plays a role in the derivation of the dual representation in the proof above. Specifically, when passing from

$$Kx = \sum_{j\in\mathcal{J}} u_j\langle v_j \mid x\rangle$$

to

$$\langle Kx \mid y\rangle = \sum_{j\in\mathcal{J}} \overline{\langle v_j \mid x\rangle}\langle u_j \mid y\rangle,$$

the order of multiplication in the quaternionic inner product must be carefully preserved, since

$$\langle u_j\langle v_j \mid x\rangle \mid y\rangle = \overline{\langle v_j \mid x\rangle}\langle u_j \mid y\rangle.$$

This is precisely a point where the noncommutativity of $\mathbb{H}$ genuinely affects the computation in contrast to the complex case, where the order of multiplication is immaterial.

3.3. Constructions of tight K-frames

This subsection presents several practical methods for constructing tight K-frames from existing ones.

Theorem 3.5. Let $\{u_j\}_{j\in\mathcal{J}} \subset \mathcal{H}_r(\mathbb{H})$ be a tight frame for $\mathcal{H}_r(\mathbb{H})$ with bound C. Then for any $K \in \mathscr{B}(\mathcal{H}_r(\mathbb{H}))$, the sequence $\{Ku_j\}_{j\in\mathcal{J}}$ is a tight K-frame for $\mathcal{H}_r(\mathbb{H})$ with the same bound C.

Proof. Since $\{u_j\}_{j\in\mathcal{J}}$ is a tight frame for $\mathcal{H}_r(\mathbb{H})$ with bound C, we have

$$\sum_{j\in\mathcal{J}} |\langle Ku_j \mid x\rangle|^2 = \sum_{j\in\mathcal{J}} |\langle u_j \mid K^*x\rangle|^2 = C\|K^*x\|^2, \forall x \in \mathcal{H}_r(\mathbb{H}),$$

which is precisely the defining condition for a tight K-frame. □

This theorem shows that any tight frame can be transformed into a tight K-frame via the operator K. Choosing an orthonormal basis yields the following immediate corollary.

Corollary 3.3. If $\{e_j\}_{j\in\mathcal{J}}$ is an orthonormal basis for $\mathcal{H}_r(\mathbb{H})$, then for any $K \in \mathscr{B}(\mathcal{H}_r(\mathbb{H}))$, $\{Ke_j\}_{j\in\mathcal{J}}$ is a Parseval K-frame for $\mathcal{H}_r(\mathbb{H})$.

The next result shows that applying an operator to an existing tight K-frame produces a tight frame for the composition of operators.

Theorem 3.6. Let $\{u_j\}_{j\in\mathcal{J}}$ be a tight K-frame for $\mathcal{H}_r(\mathbb{H})$ with bound C. Then for any $F \in \mathscr{B}(\mathcal{H}_r(\mathbb{H}))$, the sequence $\{Fu_j\}_{j\in\mathcal{J}}$ is a tight FK-frame for $\mathcal{H}_r(\mathbb{H})$ with the same bound C.

Proof. Since $\{u_j\}_{j\in\mathcal{J}}$ is a tight K-frame for $\mathcal{H}_r(\mathbb{H})$ with bound C, for any $x \in \mathcal{H}_r(\mathbb{H})$, we have

$$\begin{aligned}\sum_{j\in\mathcal{J}} |\langle Fu_j \mid x\rangle|^2 &= \sum_{j\in\mathcal{J}} |\langle u_j \mid F^*x\rangle|^2 \\ &= C\|K^*(F^*x)\|^2 \\ &= C\|(FK)^*x\|^2,\end{aligned}$$

which completes the proof. □

Corollary 3.4. Under the conditions of Theorem 3.6, $\{K^N u_j\}_{j\in\mathcal{J}}$ is a tight K^{N+1}-frame for $\mathcal{H}_r(\mathbb{H})$ with bound C for any integer $N \geq 1$.

The following technical observation gives a sufficient and necessary condition for a linear combination of two Bessel sequences to form a tight K-frame. The condition is obtained by expanding the operator identity for the frame operator of the combined sequence, and the result serves primarily as a verification tool.

Theorem 3.7. Let $\{u_j\}_{j\in\mathcal{J}}$ and $\{v_j\}_{j\in\mathcal{J}}$ be Bessel sequences in $\mathcal{H}_r^1(\mathbb{H})$ with synthesis operators Θ_u and Θ_v, respectively. Let $T_1, T_2 \in \mathscr{B}(\mathcal{H}_r^1(\mathbb{H}), \mathcal{H}_r^2(\mathbb{H}))$ and $K \in \mathscr{B}(\mathcal{H}_r^2(\mathbb{H}))$. Then $\{T_1 u_j + T_2 v_j\}_{j\in\mathcal{J}}$ is a tight K-frame for $\mathcal{H}_r^2(\mathbb{H})$ with bound C if and only if

$$CKK^* = T_1\Theta_u\Theta_u^*T_1^* + T_2\Theta_v\Theta_v^*T_2^* + T_1\Theta_u\Theta_v^*T_2^* + T_2\Theta_v\Theta_u^*T_1^*. \tag{3.6}$$

Proof. By Lemma 3.4, $\{T_1 u_j + T_2 v_j\}_{j\in\mathcal{J}}$ is a Bessel sequence in $\mathcal{H}_r^2(\mathbb{H})$. Its synthesis operator Q satisfies, for any $\{q_j\}_{j\in\mathcal{J}} \in \ell^2(\mathbb{H})$,

$$Q(\{q_j\}_{j\in\mathcal{J}}) = \sum_{j\in\mathcal{J}}(T_1u_j + T_2v_j)q_j = T_1\sum_{j\in\mathcal{J}}u_jq_j + T_2\sum_{j\in\mathcal{J}}v_jq_j = T_1\Theta_u(\{q_j\}_{j\in\mathcal{J}}) + T_2\Theta_v(\{q_j\}_{j\in\mathcal{J}}).$$

Hence, $Q = T_1\Theta_u + T_2\Theta_v$. The corresponding frame operator is then

$$\begin{aligned} S &= QQ^* = (T_1\Theta_u + T_2\Theta_v)(\Theta_u^*T_1^* + \Theta_v^*T_2^*) \\ &= T_1\Theta_u\Theta_u^*T_1^* + T_1\Theta_u\Theta_v^*T_2^* + T_2\Theta_v\Theta_u^*T_1^* + T_2\Theta_v\Theta_v^*T_2^*. \end{aligned}$$

By Theorem 3.1, $\{T_1u_j + T_2v_j\}_{j\in\mathcal{J}}$ is a tight K-frame for $\mathcal{H}_r^2(\mathbb{H})$ with bound C if and only if $S = CKK^*$, which is exactly condition (3.6). □

Remark 3.3. Theorem 3.7 provides a unified criterion for constructing tight K-frames from combinations of Bessel sequences. In particular, setting $T_1 = K$, $T_2 = 0$ and taking $\{u_j\}_{j\in\mathcal{J}}$ to be a classical tight frame for the whole space recovers Theorem 3.5; setting $T_1 = F$, $T_2 = 0$ and taking $\{u_j\}_{j\in\mathcal{J}}$ to be a tight K-frame with bound C recovers Theorem 3.6. Thus, the theorem serves as a general framework that encompasses these earlier construction results as special cases.

3.4. Connection with classical tight frames on subspaces

The following theorem establishes a fundamental one-to-one correspondence between the tight K-frame on $\mathcal{H}_r(\mathbb{H})$ and classical tight frames on the subspace $\mathrm{Ran}(K^*)$. This result can be viewed as a structural inverse to Theorem 3.5.

Theorem 3.8. Let $K \in \mathscr{B}(\mathcal{H}_r(\mathbb{H}))$ have a closed range, and let $\{u_j\}_{j\in\mathcal{J}} \subset \mathcal{H}_r(\mathbb{H})$ be a tight K-frame for $\mathcal{H}_r(\mathbb{H})$ with bound C. Then the sequence $\{v_j\}_{j\in\mathcal{J}} = \{K^\dagger u_j\}_{j\in\mathcal{J}}$ is a tight frame for the subspace $\mathrm{Ran}(K^*)$ with the same bound C. Conversely, if $\{v_j\}_{j\in\mathcal{J}} \subset \mathrm{Ran}(K^*)$ is a tight frame for $\mathrm{Ran}(K^*)$ with bound C, then the sequence $\{u_j\}_{j\in\mathcal{J}} = \{Kv_j\}_{j\in\mathcal{J}}$ is a tight K-frame for $\mathcal{H}_r(\mathbb{H})$ with bound C.

Proof. **Necessity:** Let Θ_u be the synthesis operator of $\{u_j\}_{j\in\mathcal{J}}$. By Theorem 3.1, $\Theta_u\Theta_u^* = CKK^*$. Define $\Theta_v := K^\dagger\Theta_u$. Then Θ_v is the synthesis operator for $\{v_j\}_{j\in\mathcal{J}}$. Consequently,

$$\Theta_v\Theta_v^* = K^\dagger(\Theta_u\Theta_u^*)(K^\dagger)^* = CK^\dagger KK^*(K^\dagger)^*.$$

From Lemma 3.3, $K^\dagger K = P_{\mathrm{Ran}(K^*)}$ and $K^*(K^\dagger)^* = (K^\dagger K)^* = P_{\mathrm{Ran}(K^*)}$. Hence,

$$\Theta_v \Theta_v^* = C P_{\mathrm{Ran}(K^*)}.$$

Consequently, for any $x \in \mathcal{H}_r(\mathbb{H})$,

$$\sum_{j\in\mathcal{J}} |\langle v_j \mid x\rangle|^2 = \|\Theta_v^* x\|^2 = \langle \Theta_v \Theta_v^* x \mid x\rangle = C\|P_{\mathrm{Ran}(K^*)} x\|^2.$$

If $x \in \mathrm{Ran}(K^*)$, then $P_{\mathrm{Ran}(K^*)} x = x$, and we obtain

$$\sum_{j\in\mathcal{J}} |\langle v_j \mid x\rangle|^2 = C\|x\|^2,$$

which shows that $\{v_j\}_{j\in\mathcal{J}}$ is a tight frame for $\mathrm{Ran}(K^*)$.

Moreover, by Theorem 3.2, $\mathrm{Ran}(\Theta_u) = \mathrm{Ran}(K)$. Hence, $u_j \in \mathrm{Ran}(K)$ for all $j \in \mathcal{J}$, so $KK^\dagger u_j = u_j$ for all $j \in \mathcal{J}$. Therefore,

$$K v_j = K(K^\dagger u_j) = u_j \quad \text{for all } j \in \mathcal{J}.$$

Sufficiency: Suppose $\{v_j\}_{j\in\mathcal{J}}$ is a tight frame for $\mathrm{Ran}(K^*)$ with bound C and synthesis operator Θ_v, satisfying $\Theta_v\Theta_v^* = CP_{\mathrm{Ran}(K^*)}$. Define $\Theta_u = K\Theta_v$. Then Θ_u is the synthesis operator for $\{u_j\}_{j\in\mathcal{J}}$. Hence,

$$\Theta_u\Theta_u^* = K(\Theta_v\Theta_v^*)K^* = CKP_{\mathrm{Ran}(K^*)}K^*.$$

Since $\mathrm{Ran}(K^*) = (\ker(K))^\perp$, we have $KP_{\mathrm{Ran}(K^*)} = K$. Substituting this yields

$$\Theta_u\Theta_u^* = CKK^*.$$

By Theorem 3.1, $\{u_j\}_{j\in\mathcal{J}}$ is a tight K-frame for $\mathcal{H}_r(\mathbb{H})$ with bound C.

Since $\{v_j\}_{j\in\mathcal{J}} \subset \mathrm{Ran}(K^*)$, we have $K^\dagger K v_j = P_{\mathrm{Ran}(K^*)} v_j = v_j$ for all $j \in \mathcal{J}$. Therefore,

$$K^\dagger u_j = K^\dagger(K v_j) = v_j \quad \text{for all } j \in \mathcal{J}.$$

Thus, the two maps are inverse to each other. □

The following example illustrates the application of Theorem 3.8.

Example 3.3. Let $\mathcal{H}_r(\mathbb{H}) = \mathbb{H}^3$ with the standard inner product (i.e., $\langle x \mid y\rangle = \sum_{i=1}^3 \overline{x_i} y_i$). Define $K : \mathbb{H}^3 \to \mathbb{H}^3$ by the matrix

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then for any $x = (x_1, x_2, x_3)^T \in \mathbb{H}^3$, $Kx = (x_1, x_2, 0)^T$. A direct computation gives $K^* = K$, and hence

$$\mathrm{Ran}(K^*) = \{(x_1, x_2, 0)^T : x_1, x_2 \in \mathbb{H}\}.$$

Let $e_1 = (1, 0, 0)^T$ and $e_2 = (0, 1, 0)^T$. Then $\{e_1, e_2\}$ is an orthonormal basis for the subspace $\mathrm{Ran}(K^*)$, hence a Parseval frame on that subspace. Indeed, for any $x = (x_1, x_2, 0)^T \in \mathrm{Ran}(K^*)$,

$$\sum_{j=1}^{2} |\langle e_j \mid x\rangle|^2 = |x_1|^2 + |x_2|^2 = \|x\|^2.$$

By Theorem 3.8, the sequence $\{u_j\}_{j=1}^2$ defined by $u_j = Ke_j$ is a Parseval K-frame for $\mathbb{H}^3$. Since $Ke_1 = e_1$ and $Ke_2 = e_2$, we have $u_1 = e_1$ and $u_2 = e_2$. For any $x = (x_1, x_2, x_3)^T \in \mathbb{H}^3$, since $K^*x = (x_1, x_2, 0)^T$,

$$\sum_{j=1}^{2} |\langle u_j \mid x\rangle|^2 = |x_1|^2 + |x_2|^2 = \|K^*x\|^2.$$

Thus, $\{e_1, e_2\}$ is a Parseval K-frame for $\mathbb{H}^3$. However, it is not a frame for the whole space $\mathbb{H}^3$, as it has only two vectors and cannot cover elements of the form $(0, 0, x_3)^T$ with $x_3 \neq 0$. This example illustrates that a tight K-frame need only be a tight frame on the subspace $\mathrm{Ran}(K^*)$, not necessarily on the whole space.

Remark 3.4. Theorem 3.8 establishes a bijection between tight K-frames and classical tight frames on $\mathrm{Ran}(K^*)$. This result is, in a sense, the inverse of Theorem 3.5. Theorem 3.5 shows that a tight frame for the whole space, when acted upon by K, becomes a tight K-frame. Conversely, Theorem 3.8 demonstrates that any tight K-frame can be reduced back to a tight frame on the subspace $\mathrm{Ran}(K^*)$ via the pseudo-inverse operator $K^\dagger$.

4. Conclusions

In this paper, we have introduced tight K-frames in quaternionic Hilbert spaces. The operator characterization $\Theta\Theta^* = CKK^*$ shows that $\mathrm{Ran}(\Theta) = \mathrm{Ran}(K)$ and the frame operator equals CKK^*. Under the conditions of Theorem 3.5, a tight K-frame reduces to a classical tight frame exactly when KK^* is a scalar multiple of the identity. Three constructions are provided: applying K to a classical tight frame, taking further operator compositions, and summing two transformed Bessel sequences under a simple condition. Using the pseudo-inverse $K^\dagger$, we establish a one-to-one correspondence between tight K-frames on $\mathcal{H}_r(\mathbb{H})$ and classical tight frames on $\mathrm{Ran}(K^*)$. This correspondence shows that the theory of tight K-frames is naturally linked to classical frame theory on a subspace, allowing known results from the classical setting to be transferred to the K-frame setting.

For future work, several directions may be considered, including the characterization and construction of dual tight K-frames in the quaternionic setting, perturbation theory for ensuring stability under small perturbations, extensions to more general settings such as g-frames or fusion frames, and potential applications in quaternionic signal processing.

Author contributions

Jianping Zhang: the original draft and the formal analysis; Kepu Li: the methodology and research supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests.

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