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Research article

Analytical extensions of Chen-Lee-Liu equations with cubic-quintic nonlinearity and resonant effects

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Abstract: Extended and high-order complex-valued evolution equations are found ideally to model complex nonlinear scenarios, having perfectly featured the underlined physical assumptions. Thus, with the identified applications of the classical Chen-Lee-Liu equation, in particular, in different realms of nonlinear sciences, this study proposes several extensions to the Chen-Lee-Liu equation and its perturbed form by incorporating the cubic-quintic nonlinearity law and a resonant term with the view to extending the applicability spectrum of the classical model. Method-wise, the modified extended tanh method, in favor of its mathematical flexibility and effectiveness in handing both the real- and complex-valued evolution equations, is utilized as the primary tool for deriving exact solutions for the new models, in addition to being complemented by the modified Kudryashov method to extract additional solitonic expressions. The study also performs a linearization of the generalized model to conduct dispersion relation and modulation instability analyses. Several periodic and solitonic solutions are obtained analytically, and the characteristic equation governing the overall wave dynamics is derived. A numerical investigation of the involved parameters reveals that the inclusion of the resonant term enhances wave dispersion in the medium, while the added perturbation terms act to suppress the dispersion of harmonic waves in the complex system. This way, one notes that these new models provide an avenue to controlling transmission of field in the governing media, including fluid medium, optical fibers, or even in optical communication processes where pulsification trend is of paramount importance. Overall, the findings of this research are expected to have significant implications in fluid dynamics, nonlinear optics, quantum physics, and optoelectronics, among other fields in nonlinear science.

Keywords: nonlinear evolution equation; Chen-Lee-Liu equations; cubic-quintic nonlinearity; resonant force; extensional analysis; analytical study; modulation instability analysis
Mathematics Subject Classification: 35A09, 35A24, 35C07, 35C08

1. Introduction

Nonlinear evolution equations (NLEEs) play a pivotal role in modeling a wide range of real-world phenomena across various branches of nonlinear science. These equations, often expressed as partial differential equations (PDEs), have been extensively studied due to their ability to describe complex behaviors in fields such as fluid dynamics, nonlinear optics, optoelectronics, and quantum physics [1–3]. Over recent years, significant efforts have been made to enhance the modeling capacity of classical evolution equations by introducing various modifications. These include raising the order of the equations, increasing dimensionality, and incorporating perturbation terms [4–6]. In addition, external effects such as resonant terms have been embedded into nonlinear evolution equations to capture resonance-induced wave interactions and enrich the resulting soliton dynamics [7].

One of the most remarkable features of NLEEs is their support for soliton solutions–localized wave structures that retain their shape and energy over time and distance [4, 5]. However, due to the nonlinear and often intricate nature of these equations, obtaining exact analytical solutions is not always feasible. This challenge has led to the development and refinement of several analytical techniques designed to uncover closed-form solutions. Methods such as the exponential function expansion method [8], ansatz method [9, 10], sub-equation method [11], the effective analytical methods [12], Kudryashov method [13], enhanced version of the Kudryashov method [14], Liu's complete discriminant method [15], the bilinear Hirota technique [16, 17], trial function method [18], and the hyperbolic tangent expansion method [14] have been widely employed in this context. Additionally, Wang et al. [19] and Triki et al. [20] investigated resonant nonlinear Schrödinger equations with fractional temporal evolution and time-dependent coefficients, respectively. Mirzazadeh et al. [21], Ekici et al. [22], and Mirzazadeh et al. [23] reported analytical results for various resonant nonlinear Schrödinger models. Ali and Mehanna [24] studied the Gilson-Pickering equation, while Eslami et al. [25] derived soliton solutions for the resonant nonlinear Schrödinger equation.

In recent developments, Samir et al. [26] introduced a class of NLEEs governed by higher-order power-law nonlinearities, particularly the cubic-quintic-septic-onic (CQSN) and quadrupled power-law forms. This advancement broadened the applicability of the nonlinear Schrödinger equation and laid the foundation for subsequent research in this direction. Following this trajectory, several studies have adopted or extended the CQSN law to model various physical systems. Notable examples include the work by AlQahtani and Alngar [27], which modeled polarization-preserving fibers, Hussain et al. [28], who explored unique classes of evolution equations governed by CQSN-type laws, and analyses by Rehman et al. [29] and Chen and Li [30], on the nonlinear cubic-quartic Schrödinger equation. More recently, Ali and Nuruddeen [31] contributed further by incorporating both CQSN and quadrupled power-law nonlinearities into nonlinear Schrödinger-type models.

Motivated by these recent developments [26–31], the present study aims to advance this line of inquiry by extending the core ideas to the Chen-Lee-Liu class of equations [32–34], which are well known for modeling various wave phenomena, including the propagation of sub-picosecond solitons in optical fiber communications. After thorough analysis, it was determined that only the cubic-quintic form of the power-law nonlinearity is compatible with the Chen-Lee-Liu framework. Additionally, a resonant term has been incorporated to further enrich the model's dynamical behavior. To construct analytical solutions for the proposed models, the study employs the modified extended tanh method, known for its efficiency in generating solitonic solutions. Furthermore, an additional analytical technique is introduced to explore more solution structures. Undoubtedly, this study is paramount in contemporary nonlinear sciences, where highly nonlinear equations describe complex physical phenomena; see the recent development in [35], where the three known derivative nonlinear Schrödinger models were unified, in the same direction with the proposed new models, with potential applications in the propagation dynamics of Alfven and Langmuir waves, and in addition to its relevance in cold-plasma arenas. Moreover, as the models of the papers are new, a lot of applications are expected to follow suit upon applying them to real-life scenarios that are modeled using derivative nonlinear Schrödinger equations, including the classical Chen-Lee-Lie equation, the Kaup-Newell equation, and the Gerjikov-Ivanov equation.

The remainder of this study is organized as follows: Section 2 presents the governing models of interest, and Section 3 outlines the analytical procedures employed. Section 4 applies these procedures to derive explicit solutions while Section 5 provides further solutions for the generalized model. Section 6 discusses the linearized model, deriving the approximate dispersion relation and the corresponding modulation instability spectrum. Section 7 presents numerical results and analysis, and Section 8 concludes the study with final remarks and potential implications.

2. Governing equations

2.1. Classical equations

The primary version of the generalized Chen-Lee-Liu equation is expressed in dimensionless form as follows [32, 33]:

$$\iota v_t + a v_{xx} + \iota b \left(|v|^{2n}\right) v_x = 0, \tag{2.1}$$

where $v = v(x, t)$ is the complex-valued wave field in the space and temporal variables x and t, while ι is the imaginary complex number, $\iota = \sqrt{-1}$. Additionally, the first term in the equation stands for linear temporal evolution of the propagating wave, a is a non-zero parameter serving as the group-velocity dispersion and the coefficient of dispersion term v_{xx}, while b is the coefficient of self-phase modulation in favor of dual power law nonlinearity, with n as a number in $0 < n < 2$, such that $n \neq 2$ for soliton stability [32]. Moreover, as the classical Chen-Lee-Liu equation underwent several modifications and improvements, one of the best versions of Chen-Lee-Liu equations is the perturbed Chen-Lee-Liu equation with full nonlinearity (n) that reads [34]

$$\iota v_t + a v_{xx} + \iota b \left(|v|^{2n}\right) v_x = \iota \left(\alpha v_x + \beta(|v|^{2n} v)_x + \gamma(|v|^{2n})_x v\right), \tag{2.2}$$

where "α is the inter-modal dispersion, β represents self-steepening for short pulses and γ is the higher-order dispersion effect" [32]. Besides, the parameters in the latter equations a, b, α, β, and γ are all non-zero real-valued constants [6].

2.2. Proposed extensions

Let us begin by recalling that the inclusion of the resonant term [7] to the traditional settings of the nonlinear Schrödinger equation is very vital, as it widens the spectrum of the latter equation to model "high-intensity wave enveloped in dispersive media where nonlinear interaction (like the Kerr effects) are coupled with resonant or non-local terms" [36]; in addition, the resonant term paves the way for resonant interactions with physical medium, causing self-steepening and inter-model dispersion effects [37]. Moreover, the inter-play between the resonance and other dispersion coefficients in the governing model highlights complexities and the extent to which nonlinear evolution equations are able to perfectly capture physical scenarios.

Moreover, motivated by a recent study that incorporates quadrupled power-law nonlinearities into nonlinear Schrödinger equations (see [26]) and by Althobaiti and Nuruddeen's work [31] and the resonant term of the form $\left(\frac{|v|_{xx}}{|v|}\right)v$, the present study thus extends the classical models in Eqs (2.1) and (2.2) accordingly by the incorporation of generalized cubic-quintic and resonant terms as follows:

(1) Extended Chen-Lee-Liu equation with resonant term

$$\iota v_t + a v_{xx} + \iota\left(b_1|v|^{2n} + b_2|v|^{4n}\right)v_x + c\left(\frac{|v|_{xx}}{|v|}\right)v = 0. \tag{2.3}$$

(2) Perturbed extended Chen-Lee-Liu equation with resonant term

$$\iota v_t + a v_{xx} + \iota\left(b_1|v|^{2n} + b_2|v|^{4n}\right)v_x + c\left(\frac{|v|_{xx}}{|v|}\right)v = \iota\left(\alpha v_x + \beta(|v|^{2n}v)_x + \gamma(|v|^{2n})_x v\right). \tag{2.4}$$

(3) Extended perturbed extended Chen-Lee-Liu equation with resonant term

$$\begin{aligned}&\iota v_t + a v_{xx} + \iota\left(b_1|v|^{2n} + b_2|v|^{4n}\right)v_x + c\left(\frac{|v|_{xx}}{|v|}\right)v\\ &= \iota\left(\alpha v_x + (\left(\beta_1|v|^{2n} + \beta_2|v|^{4n}\right)v)_x + \left(\gamma_1|v|^{2n} + \gamma_2|v|^{4n}\right)_x v\right).\end{aligned} \tag{2.5}$$

Notably, Eq (2.3) extends the classical Chen-Lee-Liu equation (2.1) by incorporating a power-law term $|v|^{4n}v_x$ and the resonant term $\left(\frac{|v|_{xx}}{|v|}\right)v$, where b_1, b_2, and c are non-zero real parameters. Setting $b_1 = b$, $b_2 = 0$, and $c = 0$ in (2.4) recovers Eq (2.1). Furthermore, incorporating these terms extends the spectrum of the classical Chen-Lee-Liu equation to model more sophisticated wave dynamics in realm of optoelectronic communication [32–34]. In addition, Eq (2.4) proposes an extension to the perturbed Chen-Lee-Liu equation (2.2) by incorporating the power-law term $|v|^{4n}v_x$ and the resonant term $\left(\frac{|v|_{xx}}{|v|}\right)v$, giving room for the application of perturbation terms, as well as higher-order dispersion terms amidst the presence of the resonant form; β, γ, and c are non-zero real parameters. Lastly, Eq (2.5) is the generalized version of (2.3)–(2.5) by enjoying full nonlinearity terms in both the equation's main part, as well as, in the perturbation term, together with the presence of the resonant force, where β_1, β_2, γ_1, and γ_2 are non-zero real parameters, representing the perturbation effects via generalized cubic-quintic law of refractive index, while c is equally a non-zero real parameter serving as the coefficient of the resonant term. Hence, the present study revolves around the above-introduced complex-valued evolution equations (2.3)–(2.5) with regards to the construction of their valid exact analytical solitonic expressions. In addition, some existing constraint conditions, related to the aforementioned equations, will be unearthed. Moreover, the study equally makes sense of the modulation instability analysis of the last equation, having generalized the two frontiers of the proposed equations.

3. Model analysis and methodology

3.1. Model analysis

To begin with, let us consider the following nonlinear partial differential equation (NPDE) [31]:

$$E(v, v_t, v_x, v_{tt}, v_{tx}, v_{xx}, v_{ttt}, v_{ttx}, v_{txx}, v_{xxx}, ...) = 0. \tag{3.1}$$

To reduce the above NPDE to its corresponding ordinary differential equation (ODE), the following wave transformation is employed:

$$v(x,t) = e^{\iota(-kx+\omega t+\vartheta)}P(\xi), \qquad \xi = \zeta(x - \epsilon t), \tag{3.2}$$

where ι is the imaginary number, k, ω, ζ, and ϵ are constants, and ϑ is a phase constant. Accordingly, the reduced ODE becomes

$$F(P, P', P'', P''', ...) = 0, \tag{3.3}$$

where $P = P(\xi)$ and the primes represent continuous derivation with respective to ξ. By considering the cubic-quintic perturbed cubic-quintic Chen-Lee-Liu equation with the resonant term expressed in (2.5), one thus gets from the resulting real and imaginary parts of the revealing equation as follows:

$$\left(ak^2 + \alpha k + \omega\right) P(\xi) - k\,(b_1 - \beta_1)\,P(\xi)^{2n+1} - k\,(b_2 - \beta_2)\,P(\xi)^{4n+1} - \zeta^2(a+c)P''(\xi) = 0, \tag{3.4}$$

and

$$\begin{aligned}&(2ak + \alpha + \epsilon)P'(\xi) + (-b_1 + \beta_1 + 2\beta_1 n + 2\gamma_1 n)\,P(\xi)^{2n}P'(\xi)\\ &+ (-b_2 + \beta_2 + 4\beta_2 n + 4\gamma_2 n)\,P(\xi)^{4n}P'(\xi) = 0,\end{aligned} \tag{3.5}$$

correspondingly. In addition, one gets the soliton's velocity from the latter equation as follows:

$$\epsilon = -(\alpha + 2ak). \tag{3.6}$$

3.2. Methodology

The analytical methodology of the present study is based on the employment of the modified extended tanh expansion method [14]. Indeed, this method is a promising analytical method that constructs various function solutions for the governing evolution equation. Thus, the steps that follow outline the procedure to fully actualize the adopted method [14].

Step I. The first step assumes that (3.3) admits of a series solution given by

$$P(\xi) = \sum_{j=0}^{M} A_j \psi(\xi)^j + \sum_{j=1}^{M} B_j \psi^{-j}(\xi), \tag{3.7}$$

where A_j $(j \geq 0)$ and B_j $(j \geq 1)$ are constants that are not all equal to zero, while M is a positive integer.

Step II. The function $\psi(\xi)$ in (3.7) is said to satisfy the following nonlinear ODE:

$$\psi'(\xi) = \psi(\xi)^2 + w, \tag{3.8}$$

where w is a non-zero real number. In addition, the latter nonlinear ODE satisfies several exact solutions for w as follows:

For $w < 0$,

$$\psi(\xi) = \begin{cases} -\sqrt{-w}\coth(-\sqrt{-w}\xi), \\ -\sqrt{-w}\tanh(-\sqrt{-w}\xi). \end{cases} \tag{3.9}$$

For $w > 0$,

$$\psi(\xi) = \begin{cases} -\sqrt{w}\cot(\sqrt{w}\xi), \\ \sqrt{w}\tan(\sqrt{w}\xi). \end{cases} \tag{3.10}$$

Step III. The positive integer M in (3.7) is determined by balancing the highest linear and nonlinear terms in (3.3). In particular, the following relation helps in the determination of the explicit values of M [14, 31]:

$$D\{P^a\} = aM, \quad D\left\{\frac{\mathrm{d}^a P}{\mathrm{d}\xi^a}\right\} = M + a, \quad D\left\{P^a\left(\frac{\mathrm{d}^b P}{\mathrm{d}\xi^b}\right)^c\right\} = aM + c(M + b), \tag{3.11}$$

where D in the above expressions denotes the degree of the term(s) in the bracket, while a, b, and c are positive integers.

Step IV. Substituting (3.7) into (3.3) with the use of (3.8), we get a polynomial equation in $\psi(\xi)$. Equating the respective coefficients of the polynomial $\psi^j(\xi)$ for $j = 0, 1, 2, ..., M$ to zero yields an algebraic system of equations, which will be solved for A_j ($j = 0, 1, 2, ..., M$) and B_j ($j = 1, 2, ..., M$) in addition to the revelation of some of the soliton's parameters, and those parameters arising from the model itself. Thus, on tracing the above procedure backward, one gets various solutions for the governing model in (3.1) while tracing the given procedure backward. This procedure, the numerical simulations, and the graphical illustration of the study are carried out using Wolfram Mathematica 9.0 software.

4. Solitonic analysis

4.1. Extended Chen-Lee-Liu equation with resonant term

Let us consider the proposed extended Chen-Lee-Liu equation with the resonant term in (2.3) as follows:

$$\iota v_t + a v_{xx} + \iota\left(b_1|v|^{2n} + b_2|v|^{4n}\right)v_x + c\left(\frac{|v|_{xx}}{|v|}\right)v = 0.$$

Accordingly, based on the model analysis, the wave transformation in (3.2) and its subsequent explanation lead to the following real and imaginary parts of the reduced ODE:

$$\left(ak^2 + \omega\right)P(\xi) - b_1 k P(\xi)^{2n+1} - b_2 k P(\xi)^{4n+1} - \zeta^2(a + c)P''(\xi) = 0, \tag{4.1}$$

and

$$(2ak + \epsilon)P'(\xi) - (b_1 P(\xi)^{2n} + b_2 P(\xi)^{4n})P'(\xi) = 0, \tag{4.2}$$

such that the soliton's velocity is noted from the last equation as follows:

$$\epsilon = -2ak. \tag{4.3}$$

Furthermore, applying the homogeneous balance in Step III requires balancing the terms $P(\xi)^{4n+1}$, and $P''(\xi)$ should be balanced through (3.11), which yields $M = \frac{1}{2n}$ and leads to a new transformation as

$$Q(\xi) = P(\xi)^{\frac{1}{2n}}. \tag{4.4}$$

Consequently, substituting (4.4) into (4.1) gives the following:

$$\begin{aligned} &4n^2\left(ak^2+\omega\right)Q(\xi)^2 - 4b_1kn^2Q(\xi)^3 - 4b_2kn^2Q(\xi)^4 \\ &+\zeta^2(2n-1)(a+c)Q'(\xi)^2 - 2\zeta^2n(a+c)Q(\xi)Q''(\xi) = 0. \end{aligned} \tag{4.5}$$

In the same way, balancing the higher terms $Q(\xi)^4$ and $Q(\xi)Q''(\xi)$ in the last equation gives $M = 1$ such that (4.5) satisfies a predicted series solution from (3.7) as follows:

$$Q(\xi) = \sum_{j=0}^{1} A_j\psi^j(\xi) + \sum_{j=1}^{1} B_j\psi^{-j}(\xi), \tag{4.6}$$

where A_j $(j = 0,1)$ and B_j $(j = 1)$ are constants to be obtained. Accordingly, substituting the solution (4.6) together with (3.8) into (4.5) yields a system of algebraic equations. The resulting solutions can be classified into two cases, depending on the sign of w.

If $w < 0$, this yields the following solution types:

Case 1.

$$A_0 = -\frac{2b_1n+b_1}{4b_2n+4b_2},\ A_1 = A_1,\ B_1 = 0,\ \ w = -\frac{(2b_1n+b_1)^2}{16A_1^2b_2^2(n+1)^2},\ \ k = -\frac{\zeta^2(2n+1)(a+c)}{4A_1^2b_2n^2}. \tag{4.7}$$

Case 2.

$$\begin{aligned} &A_0 = -\frac{2b_1n+b_1}{4b_2n+4b_2},\ A_1 = A_1,\ B_1 = \frac{(2b_1n+b_1)^2}{64A_1b_2^2(n+1)^2}, \\ &w = -\frac{(2b_1n+b_1)^2}{64A_1^2b_2^2(n+1)^2},\ k = -\frac{\zeta^2(2n+1)(a+c)}{4A_1^2b_2n^2}, \end{aligned} \tag{4.8}$$

where the expression for the related $\omega = \omega_1$ in both Cases 1 and 2 is given as follows:

$$\omega_1 = \frac{(a+c)(\zeta+2\zeta n)^2\left(A_1^2b_1^2n^2 - a\zeta^2(n+1)^2(a+c)\right)}{16A_1^4b_2^2n^4(n+1)^2}.$$

Consequently, Case 1 gives the following singular and dark soliton solutions for (2.3), respectively:

$$v_1(x,t) = e^{\iota(-kx+\omega_1t+\vartheta)}\sqrt[2n]{\frac{b_1(2n+1)}{4b_2(n+1)}\left[\coth\left(\frac{b_1(2n+1)\zeta(x-\epsilon t)}{A_1(4b_2n+4b_2)}\right)-1\right]}, \tag{4.9}$$

and

$$v_2(x,t) = e^{\iota(-kx+\omega_1t+\vartheta)}\sqrt[2n]{\frac{b_1(2n+1)}{4b_2(n+1)}\left[\tanh\left(\frac{b_1(2n+1)\zeta(x-\epsilon t)}{A_1(4b_2n+4b_2)}\right)-1\right]}, \tag{4.10}$$

for $\frac{b_1}{b_2} > 0$, while Case 2 equally for $\frac{b_1}{b_2} > 0$ gives the following singular-dark combo soliton solution for (2.3):

$$v_3(x,t) = e^{\iota(-kx+\omega_1 t+\vartheta)} \sqrt[2n]{\frac{b_1(2n+1)}{4b_2(n+1)} \tanh\left(\frac{b_1(2n+1)\zeta(x-\epsilon t)}{8A_1 b_2(n+1)}\right)\left[\coth\left(\frac{b_1(2n+1)\zeta(x-\epsilon t)}{A_1(8b_2 n+8b_2)}\right)-1\right]^2}. \tag{4.11}$$

If $w > 0$, one obtains the following solution case:

Case 3.

$$A_0 = -\frac{2b_1 n + b_1}{4b_2 n + 4b_2},\quad A_1 = 0,\quad B_1 = \pm\frac{\iota b_1(2n+1)\sqrt{w}}{4\sqrt{b_2^2(n+1)^2}},\quad w = w,\quad k = \frac{4b_2\zeta^2(n+1)^2 w(a+c)}{b_1^2 n^2(2n+1)}, \tag{4.12}$$

where the expression for the related $\omega = \omega_2$ both Cases 1 and 2 is given as follows:

$$\omega_2 = -\frac{\zeta^2 w(a+c)}{n^4}\left(\frac{16ab_2^2\zeta^2(n+1)^4 w(a+c)}{b_1^4(2n+1)^2} + n^2\right).$$

Thus, with the development in (3.10), the complex-valued cotangent and tangent periodic solutions are respectively admitted by (2.3) as follows:

$$v_{4_\pm}(x,t) = e^{\iota(-kx+\omega_1 t+\vartheta)} \sqrt[2n]{\frac{\iota b_1(2n+1)}{4b_2(n+1)}(\iota \pm \cot\left(\sqrt{w}\zeta(x-\epsilon t)\right))}, \tag{4.13}$$

and

$$v_{5_\pm}(x,t) = e^{\iota(-kx+\omega_1 t+\vartheta)} \sqrt[2n]{\frac{\iota b_1(2n+1)}{4b_2(n+1)}(\iota \pm \tan\left(\sqrt{w}\zeta(x-\epsilon t)\right))}. \tag{4.14}$$

4.2. Perturbed extended Chen-Lee-Liu equation with resonant term

Let us consider the proposed perturbed extended Chen-Lee-Liu equation with the resonant term in (2.4) as follows:

$$\iota v_t + a v_{xx} + \iota\left(b_1|v|^{2n} + b_2|v|^{4n}\right)v_x + c\left(\frac{|v|_{xx}}{|v|}\right)v = \iota\left(\alpha v_x + \beta(|v|^{2n}v)_x + \gamma(|v|^{2n})_x v\right).$$

As explained, the resulting transformed ODE gives from both the real and imaginary parts as follows:

$$\left(ak^2 + \alpha k + \omega\right)P(\xi) + k\left(\beta - b_1\right)P(\xi)^{2n+1} - b_2 k P(\xi)^{4n+1} - \zeta^2(a+c)P''(\xi) = 0, \tag{4.15}$$

and

$$(2ak + \alpha + \epsilon)P'(\xi) + \left(\beta - b_1 + 2\beta n + 2\gamma n\right)P(\xi)^{2n}P'(\xi) - b_2 P(\xi)^{4n}P'(\xi) = 0, \tag{4.16}$$

with the related velocity of the soliton as follows:

$$\epsilon = -(2ak + \alpha). \tag{4.17}$$

In the same way, balancing the higher terms in the latter equation gives $M = \frac{1}{2n}$, necessitating yet a new transformation as in (4.4). Accordingly, with this new suggestion, one thus obtains the following transformed equation:

$$\begin{aligned} &4n^2\left(ak^2+\alpha k+\omega\right)Q(\xi)^2-4kn^2\left(b_1-\beta\right)Q(\xi)^3-4b_2kn^2Q(\xi)^4\\ &-\zeta^2(2n-1)(a+c)Q'(\xi)^2-2\zeta^2n(a+c)Q(\xi)Q''(\xi)=0, \end{aligned} \tag{4.18}$$

where the positive integer M is obtained from the last equation as $M = 1$. Thus, one maintains the same predicted equation in (4.6). Moreover, substituting (4.6) into (4.18) yields a system of algebraic equations, which for $w < 0$, further gives the following solution cases:

Case 1.

$$A_0=\frac{(2n+1)\left(\beta-b_1\right)}{4b_2(n+1)},\ A_1=A_1,\ B_1=0,\ w=-\frac{(2n+1)^2\left(\beta-b_1\right)^2}{16A_1^2b_2^2(n+1)^2},\ k=-\frac{\zeta^2(2n+1)(a+c)}{4A_1^2b_2n^2}. \tag{4.19}$$

Case 2.

$$\begin{aligned} &A_0=\frac{(2n+1)\left(\beta-b_1\right)}{4b_2(n+1)},\ A_1=A_1,\ B_1=\frac{(2n+1)^2\left(\beta-b_1\right)^2}{64A_1b_2^2(n+1)^2},\\ &w=-\frac{(2n+1)^2\left(\beta-b_1\right)^2}{64A_1^2b_2^2(n+1)^2},\ k=-\frac{\zeta^2(2n+1)(a+c)}{4A_1^2b_2n^2}, \end{aligned} \tag{4.20}$$

where the expression for the related $\omega = \omega_3$ in both Cases 1 and 2 is given as follows:

$$\omega_3=-\frac{\zeta^2(2n+1)(a+c)\left(a\zeta^2(n+1)^2(2n+1)(a+c)+A_1^2n^2\left(-4\alpha b_2(n+1)^2-(2n+1)\left(\beta-b_1\right)^2\right)\right)}{16A_1^4b_2^2n^4(n+1)^2}.$$

Consequently, Case 1 gives the following singular and dark soliton solutions for (2.4), respectively:

$$v_1(x,t)=e^{\iota(-kx+\omega_1t+\vartheta)}\sqrt[2n]{\frac{(2n+1)\left(\beta-b_1\right)}{4b_2(n+1)}\left[\coth\left(\frac{(2n+1)\zeta(x-\epsilon t)\left(\beta-b_1\right)}{A_1b_2(4n+4)}\right)+1\right]}, \tag{4.21}$$

and

$$v_2(x,t)=e^{\iota(-kx+\omega_1t+\vartheta)}\sqrt[2n]{\frac{(2n+1)\left(\beta-b_1\right)}{4b_2(n+1)}\left[\tanh\left(\frac{(2n+1)\zeta(x-\epsilon t)\left(\beta-b_1\right)}{A_1b_2(4n+4)}\right)+1\right]}, \tag{4.22}$$

for $\frac{\beta-b_1}{b_2} > 0$, while Case 2 equally for $\frac{\beta-b_1}{b_2} > 0$ gives the following singular-dark combo soliton solution for (2.4):

$$v_3(x,t)=e^{\iota(-kx+\omega_1t+\vartheta)}\sqrt[2n]{\chi\tanh\left(\frac{\chi}{A_1}\zeta(x-\epsilon t)\right)\left[\coth\left(\frac{\chi}{A_1}\zeta(x-\epsilon t)\right)+1\right]^2}, \tag{4.23}$$

where

$$\chi=\frac{(2n+1)\left(\beta-b_1\right)}{8b_2(n+1)}.$$

Case 3.

$$
\begin{aligned}
&A_0 = \frac{(2n+1)(\beta - b_1)}{4b_2(n+1)},\ A_1 = 0,\ B_1 = B_1,\\
&w = -\frac{16b_2^2B_1^2(n+1)^2}{(2n+1)^2(\beta - b_1)^2},\ k = -\frac{64b_2^3B_1^2\zeta^2(n+1)^4(a+c)}{n^2(2n+1)^3(\beta - b_1)^4},
\end{aligned}
\tag{4.24}
$$

where the expression for the related $\omega = \omega_4$ in Case 3 is given as follows:

$$
\omega_4 = \frac{8b_2^2B_1^2\zeta^2(n+1)(a+c)}{n^2(2n+1)^6(\beta - b_1)^8}(S_1(n) + S_2(n)),
$$

where $S_1(n)$ and $S_2(n)$ are expressed as follows:

$$
S_1(n) = \frac{2\beta^4n^2(n+1)(2n+1)^3\left(4\alpha b_2(n+1)^2 + \beta^2(2n+1)\right) - 512ab_2^4B_1^2\zeta^2(n+1)^7(a+c)}{n^2},
$$

and

$$
\begin{aligned}
S_2(n) = &-12\beta b_1^5(n+1)(2n+1)^4 + 2b_1^6(n+1)(2n+1)^4\\
&+ 4b_1(n+1)(\beta + 2\beta n)^3\left(-8\alpha b_2(n+1)^2 - 3\beta^2(2n+1)\right)\\
&+ 8\beta b_1^3(n+1)(2n+1)^3\left(-4\alpha b_2(n+1)^2 - 5\beta^2(2n+1)\right)\\
&+ 2b_1^4(n+1)(2n+1)^3\left(4\alpha b_2(n+1)^2 + 15\beta^2(2n+1)\right)\\
&+ 6\beta^2b_1^2(n+1)(2n+1)^3\left(8\alpha b_2(n+1)^2 + 5\beta^2(2n+1)\right).
\end{aligned}
$$

In the same passion, Case 3, for $\frac{\beta - b_1}{b_2} > 0$, yields the following singular and dark soliton solutions, respectively:

$$
v_4(x,t) = e^{\iota(-kx+\omega_1 t+\vartheta)}\sqrt[2n]{\frac{(2n+1)(\beta - b_1)}{4b_2(n+1)}\left[\coth\left(\frac{4b_2B_1(n+1)\zeta(x-\epsilon t)}{(2n+1)(\beta - b_1)}\right) + 1\right]},
\tag{4.25}
$$

and

$$
v_5(x,t) = e^{\iota(-kx+\omega_1 t+\vartheta)}\sqrt[2n]{\frac{(2n+1)(\beta - b_1)}{4b_2(n+1)}\left[\tanh\left(\frac{4b_2B_1(n+1)\zeta(x-\epsilon t)}{(2n+1)(\beta - b_1)}\right) + 1\right]}.
\tag{4.26}
$$

4.3. Extended perturbed extended Chen-Lee-Liu equation with resonant term

In the same way, we consider the proposed extended perturbed extended Chen-Lee-Liu equation with the resonant term in (2.5) as follows:

$$
\begin{aligned}
&\iota v_t + av_{xx} + \iota\left(b_1|v|^{2n} + b_2|v|^{4n}\right)v_x + c\left(\frac{|v|_{xx}}{|v|}\right)v\\
&= \iota\left(\alpha v_x + (\left(\beta_1|v|^{2n} + \beta_2|v|^{4n}\right)v)_x + (\left(\gamma_1|v|^{2n} + \gamma_2|v|^{4n}\right))_x v\right).
\end{aligned}
$$

Accordingly, the resulting ODEs are obtained after the implementation of the aforementioned procedure, with the resultant equations from the real and imaginary parts obtained from (3.4) and (3.5), respectively, as follows:

$$\left(ak^2+\alpha k+\omega\right)P(\xi)-k\left(b_1-\beta_1\right)P(\xi)^{2n+1}-k\left(b_2-\beta_2\right)P(\xi)^{4n+1}-\zeta^2(a+c)P''(\xi)=0,$$

and

$$\begin{aligned}&(2ak+\alpha+\epsilon)P'(\xi)+\left(-b_1+\beta_1+2\beta_1 n+2\gamma_1 n\right)P(\xi)^{2n}P'(\xi)\\&+\left(-b_2+\beta_2+4\beta_2 n+4\gamma_2 n\right)P(\xi)^{4n}P'(\xi)=0,\end{aligned}$$

while the corresponding soliton's velocity is obtained from (3.6) as follows:

$$\epsilon=-(\alpha+2ak).$$

Accordingly, balancing the higher terms in the latter equation yields $M=\frac{1}{2n}$, which further necessitates additional transformation with $P(\xi)=Q(\xi)^{1/2n}$ to be subsequently obtained as follows:

$$\begin{aligned}&4n^2Q(\xi)^2\left(ak^2+\alpha k+\omega\right)-4kn^2\left(b_2-\beta_2\right)Q(\xi)^4-4kn^2\left(b_1-\beta_1\right)Q(\xi)^3\\&+\zeta^2(2n-1)(a+c)Q'(\xi)^2-2\zeta^2n(a+c)Q(\xi)Q''(\xi)=0.\end{aligned}\tag{4.27}$$

Next, balancing the latter equation gives $M=1$ such that (4.6) holds. Moreover, substituting (4.6) into (4.27) yields a system of algebraic equations, which when solved, gives three solution sets only for $w<0$ as follows:

Case 1.

$$\begin{aligned}&A_0=\frac{(2n+1)\left(b_1-\beta_1\right)}{4(n+1)\left(\beta_2-b_2\right)},\;\; A_0=A_1,\;\; B_1=0,\\&w=-\frac{(2n+1)^2\left(b_1-\beta_1\right)^2}{16A_1^2(n+1)^2\left(b_2-\beta_2\right)^2},\;\; k=\frac{\zeta^2(2n+1)(a+c)}{4A_1^2n^2\left(\beta_2-b_2\right)}.\end{aligned}\tag{4.28}$$

Case 2.

$$\begin{aligned}&A_0=\frac{(2n+1)\left(b_1-\beta_1\right)}{4(n+1)\left(\beta_2-b_2\right)},\;\; A_1=A_1,\;\; B_1=\frac{(2n+1)^2\left(b_1-\beta_1\right)^2}{64A_1(n+1)^2\left(b_2-\beta_2\right)^2},\\&w=-\frac{(2n+1)^2\left(b_1-\beta_1\right)^2}{64A_1^2(n+1)^2\left(b_2-\beta_2\right)^2},\;\; k=\frac{\zeta^2(2n+1)(a+c)}{4A_1^2n^2\left(\beta_2-b_2\right)},\end{aligned}\tag{4.29}$$

where the expression for the related $\omega=\omega_5$ in both Cases 1 and 2 is given as follows:

$$\omega_5=-\frac{\zeta^2(2n+1)(a+c)}{16A_1^4n^4(n+1)^2\left(b_2-\beta_2\right)^2}S_3(n),$$

where

$$\begin{aligned}S_3(n)=&\,a\zeta^2(n+1)^2(2n+1)(a+c)\\&+A_1^2\left(4\alpha\beta_2\left(n^2+n\right)^2-n^2\left(4\alpha b_2(n+1)^2-2b_1\beta_1(2n+1)+b_1^2(2n+1)+\beta_1^2(2n+1)\right)\right).\end{aligned}$$

Consequently, Case 1 gives the following singular and dark soliton solutions, respectively:

$$v_1(x,t) = e^{\iota(-kx+\omega_1 t+\vartheta)} \sqrt[2n]{\frac{(2n+1)(b_1-\beta_1)}{4(n+1)(b_2-\beta_2)}\left[\coth\left(\frac{(2n+1)\zeta(x-\epsilon t)(b_1-\beta_1)}{A_1(4n+4)(b_2-\beta_2)}\right)-1\right]}, \tag{4.30}$$

and

$$v_2(x,t) = e^{\iota(-kx+\omega_1 t+\vartheta)} \sqrt[2n]{\frac{(2n+1)(b_1-\beta_1)}{4(n+1)(b_2-\beta_2)}\left[\tanh\left(\frac{(2n+1)\zeta(x-\epsilon t)(b_1-\beta_1)}{A_1(4n+4)(b_2-\beta_2)}\right)-1\right]}, \tag{4.31}$$

where $\frac{b_1-\beta_1}{b_2-\beta_2} > 0$ and $b_2 \neq \beta_2$. In addition, for the same mentioned constraints conditions, Case 2 gives the following singular-dark combo soliton solution:

$$v_3(x,t) = e^{\iota(-kx+\omega_1 t+\vartheta)} \sqrt[2n]{\frac{\delta}{8}A_1\tanh\left(\frac{\delta}{8}\zeta(x-\epsilon t)\right)\left[\coth\left(\frac{\delta}{8}\zeta(x-\epsilon t)\right)-1\right]^2}, \tag{4.32}$$

where

$$\delta = \frac{(2n+1)(b_1-\beta_1)}{(n+1)(b_2-\beta_2)A_1}.$$

Case 3.

$$\begin{aligned} A_0 &= \frac{(2n+1)(b_1-\beta_1)}{4(n+1)(\beta_2-b_2)},\quad A_1=0,\quad B_1=B_1,\\ w &= -\frac{16B_1^2(n+1)^2(b_2-\beta_2)^2}{(2n+1)^2(b_1-\beta_1)^2},\quad k=-\frac{64B_1^2\zeta^2(n+1)^4(a+c)(b_2-\beta_2)^3}{n^2(2n+1)^3(b_1-\beta_1)^4}, \end{aligned} \tag{4.33}$$

where the expression for the related $\omega = \omega_6$ in Case 3 is given as follows:

$$\omega_6 = -\frac{16B_1^2\zeta^2(n+1)^2(a+c)(b_2-\beta_2)^2}{n^4(2n+1)^6(b_1-\beta_1)^8}S_4(n),$$

with

$$\begin{aligned} S_4(n) &= 256ab_2^4B_1^2\zeta^2(n+1)^6(a+c) + 6b_1^5\beta_1 n^2(2n+1)^4 - b_1^6 n^2(2n+1)^4\\ &\quad - \beta_1^6 n^2(2n+1)^4 - 1024ab_2^3\beta_2 B_1^2\zeta^2(n+1)^6(a+c)\\ &\quad + 1536ab_2^2\beta_2^2B_1^2\zeta^2(n+1)^6(a+c) + 4\alpha\beta_1^4\beta_2(2n+1)^3\left(n^2+n\right)^2\\ &\quad + 256a\beta_2^4B_1^2\zeta^2(n+1)^6(a+c)\\ &\quad - 3b_1^2\beta_1^2n^2(2n+1)^3\left(8\alpha b_2(n+1)^2 - 8\alpha\beta_2(n+1)^2 + 5\beta_1^2(2n+1)\right)\\ &\quad + 2b_1\beta_1^3n^2(2n+1)^3\left(8\alpha b_2(n+1)^2 - 8\alpha\beta_2(n+1)^2 + \beta_1^2(6n+3)\right)\\ &\quad + 4b_1^3\beta_1 n^2(2n+1)^3\left(4\alpha b_2(n+1)^2 - 4\alpha\beta_2(n+1)^2 + 5\beta_1^2(2n+1)\right)\\ &\quad - b_1^4n^2(2n+1)^3\left(4\alpha b_2(n+1)^2 - 4\alpha\beta_2(n+1)^2 + 15\beta_1^2(2n+1)\right)\\ &\quad - 4b_2(n+1)^2\left(256a\beta_2^3B_1^2\zeta^2(n+1)^4(a+c) + \alpha\beta_1^4n^2(2n+1)^3\right). \end{aligned}$$

In the same passion, upon imposition of the constraint conditions that $\frac{b_1-\beta_1}{b_2-\beta_2} > 0$ and $b_2 \neq \beta_2$, Case 3 yields the following singular and dark soliton solutions, respectively:

$$v_4(x,t) = e^{\iota(-kx+\omega_1 t+\vartheta)} \sqrt[2n]{\frac{(2n+1)(b_1-\beta_1)}{4(n+1)(b_2-\beta_2)}\left[\coth\left(\frac{4B_1(n+1)\zeta(x-\epsilon t)(b_2-\beta_2)}{(2n+1)(b_1-\beta_1)}\right)-1\right]}, \tag{4.34}$$

and

$$v_5(x,t) = e^{\iota(-kx+\omega_1 t+\vartheta)} \sqrt[2n]{\frac{(2n+1)(b_1-\beta_1)}{4(n+1)(b_2-\beta_2)}\left[\tanh\left(\frac{4B_1(n+1)\zeta(x-\epsilon t)(b_2-\beta_2)}{(2n+1)(b_1-\beta_1)}\right)-1\right]}. \tag{4.35}$$

5. More solitons to the generalized perturbed equation

To deepen the solitonic investigation with regards to the proposed models, the generalized perturbed equation in (2.5) is further analyzed in this section, checking the possibilities of the complex-valued nonlinear evolution equation to satisfy other solitonic expressions, including singular and bright solitonic expressions. In this regard, the promising enhanced version of the famous Kudryashov's method [13] is adopted for the investigation, the so-called modified Kudryashov's method [38, 39]. Besides, this method [13] and its modifications [38, 39], including [40–42], are very effective in the construction of exponential function solutions, which can be recast to various forms of hyperbolic solitonic expressions. Thus, to present the method, the outlined procedure for the adopted modified extended tanh expansion method in (3.7) through (3.11) is equally maintained here. However, the slight difference is associated with the predicted solution (3.7), where the enhanced Kudryashov's method takes the following form:

$$P(\xi) = \sum_{j=0}^{M} \psi(\xi)^j, \tag{5.1}$$

where the function $\psi(\xi)$ in (5.1) is said to satisfy the following nonlinear ODE:

$$\psi'(\xi) = \psi(\xi)\sqrt{1 - r\psi(\xi)^2}, \tag{5.2}$$

where r is a non-zero integer; M in (5.1) is equally a positive integer to be determined from the relations expressed in (3.11). In addition, the latter nonlinear ODE satisfies several exact solutions for w as follows:

$$\psi(\xi) = \frac{4f}{4f^2e^{\xi} + re^{-\xi}} = \begin{cases} \frac{1}{2f}\operatorname{csch}(\xi), & r = -4f^2, \\ \frac{1}{2f}\operatorname{sech}(\xi), & r = 4f^2, \end{cases} \tag{5.3}$$

where f is a non-zero integer.

Hence, with the deployment of the enhanced Kudryashov's method, the proposed extended perturbed extended Chen-Lee-liu equation with the resonant form, as expressed in (2.5) via the said method, reveals the following solution cases:

Case 1.

$$A_0 = \frac{\beta_1 - b_1}{3(b_2-\beta_2)}, \;\; A_1 = \frac{\sqrt{2}\sqrt{r}(b_1-\beta_1)}{3\sqrt{(b_2-\beta_2)^2}}, \;\; n = \frac{1}{2}, \;\; k = \frac{9\zeta^2(a+c)(b_2-\beta_2)}{(b_1-\beta_1)^2}, \tag{5.4}$$

which results in the attainment of the following singular and bright solitonic expressions when $b_2 \neq \beta_2$ as

$$v_{6_\pm}(x,t) = e^{\iota(-kx+\omega_1 t+\vartheta)} \sqrt[2n]{\frac{b_1-\beta_1}{3\,(b_2-\beta_2)}\iota\left(\pm\sqrt{2}\mathrm{csch}(\zeta(x-\epsilon t))+\iota\right)}, \tag{5.5}$$

and

$$v_{7_\pm}(x,t) = e^{\iota(-kx+\omega_1 t+\vartheta)} \sqrt[2n]{\frac{b_1-\beta_1}{3\,(b_2-\beta_2)}\left(\pm\sqrt{2}\mathrm{sech}(\zeta(x-\epsilon t))-1\right)}, \tag{5.6}$$

where the expression for the related $\omega = \omega_7$ in Case 1 is given as follows:

$$\omega_7 = \frac{\zeta^2(a+c)}{(b_1-\beta_1)^4} S_5(n),$$

where

$$\begin{aligned} S_5(n) = {} & 9b_2\left(18a\beta_2\zeta^2(a+c)-\alpha\beta_1^2\right)-81ab_2^2\zeta^2(a+c) \\ & + 9\alpha\beta_1^2\beta_2 - 81a\beta_2^2\zeta^2(a+c) - 3b_1^2\left(-3\alpha\beta_2+3\alpha b_2+4\beta_1^2\right) \\ & + 2\beta_1 b_1\left(-9\alpha\beta_2+9\alpha b_2+4\beta_1^2\right)+8\beta_1 b_1^3-2b_1^4-2\beta_1^4. \end{aligned}$$

Case 2.

$$A_0 = \frac{3\,(b_1-\beta_1)}{8\,(\beta_2-b_2)},\;\; A_1 = \pm\frac{3\sqrt{r}\,(b_1-\beta_1)}{8\sqrt{(b_2-\beta_2)^2}},\;\; n=1,\;\; k = \frac{16\zeta^2(a+c)\,(b_2-\beta_2)}{3\,(b_1-\beta_1)^2}, \tag{5.7}$$

which results in the attainment of the following singular and bright solitonic expressions when $b_2 \neq \beta_2$ as

$$v_{8_\pm}(x,t) = e^{\iota(-kx+\omega_1 t+\vartheta)} \sqrt[2n]{\frac{3(b_1-\beta_1)}{8\,(b_2-\beta_2)}\iota\,(\pm\mathrm{csch}(\zeta(x-\epsilon t))+\iota)}, \tag{5.8}$$

and

$$v_{9_\pm}(x,t) = e^{\iota(-kx+\omega_1 t+\vartheta)} \sqrt[2n]{\frac{3(b_1-\beta_1)}{8\,(b_2-\beta_2)}\,(\pm\mathrm{sech}(\zeta(x-\epsilon t))-1)}, \tag{5.9}$$

where the expression for the related $\omega = \omega_8$ in Cases 2 is given as follows:

$$\omega_8 = \frac{\zeta^2(a+c)}{36\,(b_1-\beta_1)^4} S_6(n),$$

with

$$\begin{aligned} S_6(n) = {} & 64b_2\left(32a\beta_2\zeta^2(a+c)-3\alpha\beta_1^2\right)-1024ab_2^2\zeta^2(a+c) \\ & + 192\alpha\beta_1^2\beta_2 - 1024a\beta_2^2\zeta^2(a+c) - 6b_1^2\left(32\alpha\,(b_2-\beta_2)+45\beta_1^2\right) \\ & + 12\beta_1 b_1\left(32\alpha\,(b_2-\beta_2)+15\beta_1^2\right)+180\beta_1 b_1^3-45b_1^4-45\beta_1^4. \end{aligned}$$

Notably, the derived singular and bright solutions in (5.5)-(5.6) and (5.8)-(5.9) are acquired for specific cases of n, that is, when $n = 1/2$ and $n = 1$. Indeed, one may note that the deployed enhanced Kudryashov's method is inadequate in revealing generalized solitonic expressions when n assumes the full nonlinearity, as given by the earlier adopted modified extended tanh expansion method. Moreover, the acquired solutions in this section serve as supplementary solutions, especially whenever bright solitonic solutions are the target of the pulse propagation in the governing optical medium.

6. Dispersion relation and modulation instability

To analyze the resulting dispersion relation and the subsequent modulation instability of the proposed models, this section thus considers the most general model proposed in (2.5), in which all the equations have been generalized. In this regard, the model is first linearized for the deployment of the analytical approach. Moreover, considering the following perturbed solution [43, 44]:

$$v(x,t) = e^{\iota m x}\left(V(x,t) + \sqrt{m}\right), \tag{6.1}$$

where m is the normalized incident power, $\iota = \sqrt{-1}$, and $V(x,t)$ is the perturbed field, the governing model (2.5) is, therefore, linearized as follows:

$$\chi_1(V + V^*) + \iota\chi_2\left(V_x + V_x^*\right) + \iota\chi_3 V_t + \chi_4 V_{xx} = 0, \tag{6.2}$$

where V^* is the conjugate of $V = V(x,t)$, while χ_j for $j = 1,2,3,4$ are expressed as follows:

$$\begin{aligned} \chi_1 &= 2m^2\left(-am + \alpha + 3\beta_2 m^2 + 2\beta_1 m\right), \qquad \chi_4 = am + c\sqrt{m}, \\ \chi_2 &= m\left(2am - \alpha - 5\beta_2 m^2 - 4\gamma_2 m^2 - 3\beta_1 m - 2\gamma_1 m\right), \ \chi_3 = m. \end{aligned} \tag{6.3}$$

Next, the following harmonic-type solution is adopted to analyze the latter linearized complex-valued wave equation:

$$V(x,t) = c_1 e^{\iota(x\varpi - t\Psi)} + c_2 e^{-\iota(x\varpi - t\Psi)}, \ V^*(x,t) = c_1 e^{-\iota(x\varpi - t\Psi)} + c_2 e^{\iota(x\varpi - t\Psi)}, \tag{6.4}$$

where c_1 and c_2 are constants to be determined from the corresponding boundary conditions, $\iota = \sqrt{-1}$, ϖ is the wavenumber, and Ψ is the frequency. However, since a general scenario is considered, the corresponding boundary conditions are not imposed. Substituting (6.4) into (6.2) yields the following dispersion matrix:

$$\Gamma = \begin{pmatrix} \chi_1 + \varpi\chi_2 & -\chi_4\varpi^2 + \chi_2\varpi + \chi_1 - \Psi\chi_3 \\ -\chi_4\varpi^2 - \chi_2\varpi + \chi_1 + \Psi\chi_3 & \chi_1 - \varpi\chi_2 \end{pmatrix}. \tag{6.5}$$

By setting the determinant of the above dispersion matrix equal to zero, we get

$$2\chi_2\chi_3\Psi\varpi - \chi_3^2\Psi^2 + \chi_4^2\varpi^4 - 2\chi_1\chi_4\varpi^2 = 0. \tag{6.6}$$

Remarkably, the dispersion relation (6.6) will deeply be examined graphically to assess the impact of the involving model's parameters on the dispersion of waves in the medium.

Furthermore, for the modulation instability, the dispersion relation (6.6) is solved for the frequency Ψ as follows:

$$\Psi = \frac{\varpi\left(\chi_2 \pm \sqrt{\chi_2^2 - 2\chi_1\chi_4 + \chi_4^2\varpi^2}\right)}{\chi_3}. \tag{6.7}$$

Accordingly, one derives the resting gain function $G(\varpi)$ from the latter expression as follows:

$$G(\varpi) = 2\Im\left(\frac{\varpi\left(\chi_2 \pm \sqrt{\chi_2^2 - 2\chi_1\chi_4 + \chi_4^2\varpi^2}\right)}{\chi_3}\right), \tag{6.8}$$

twice the imaginary part of the latter frequency, such that the modulation instability occurs when

$$\chi_2^2 - 2\chi_1\chi_4 + \chi_4^2\varpi^2 < 0. \tag{6.9}$$

Moreover, in what follows, the determined approximate dispersion in (6.6) will be extensively examined, assessing the impact of the involved wave's parameters on the dispersion of waves in the medium. Indeed, this analysis is in favor of the generalized model in (2.5), generalizing all the models in (2.1)–(2.4).

7. Graphical analysis and discussion

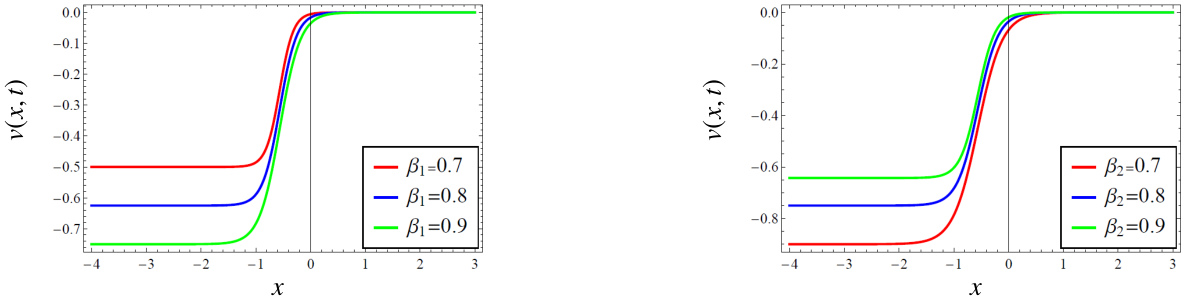
This current section graphically examines some of the constructed exact solitonic solutions in the previous sections, and the consequential approximate dispersion relation for the governing extended perturbed extended Chen-Lee-Liu equation embedded in cubic-quintic nonlinearity and the resonant force (2.5). Moreover, Eq (2.5) demonstrates the model's capability to generalize the previously proposed models represented by Eqs (2.3) and (2.4). Thus, some interesting fixed values for the involving parameters are considered for the simulation exercise; these results are portrayed using two-dimensional (2D) plots, three-dimensional (3D) plots, and some interesting contour plots for the dependencies of frequency Ψ over the wavenumber ϖ, respectively.

7.1. Analysis of wave fields

Having obtained several exact analytical solutions for the governing extended perturbed extended Chen-Lee-Liu equation embedded in cubic-quintic nonlinearity and a resonant force (2.5), it is, therefore, pertinent to mention here that the dark (tanh[.]) and bright (sech[.]) solitonic expressions have applications in nonlinear sciences, including fluid flow modeling, optical and quantum processes, and in modern communication networking for brevity; however, readers are referred to [45], who extensively outlined various applications of these solitons in fluid dynamics, to mention a few. Besides, one equally reads about the divergent property of the singular soliton (csch[.] and coth[.]), where singularity points exist, which in physics usually means that the wavefield intensity diverges.

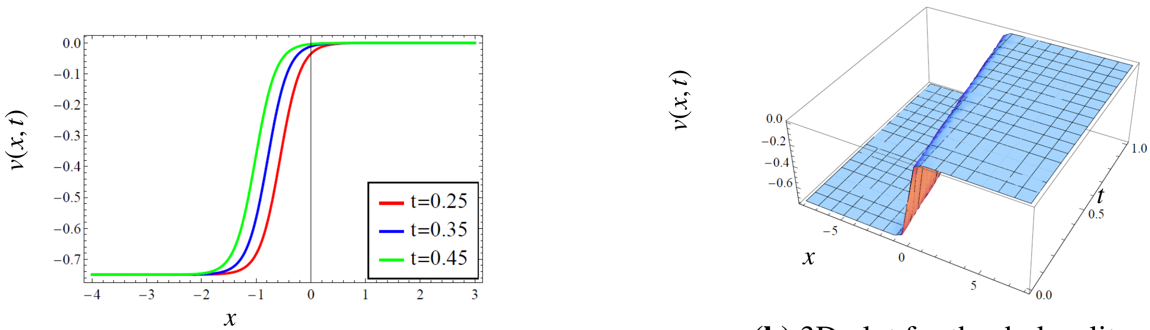
Thus, the obtained dark and bright solitonic expressions in (4.35) and (5.9), respectively, are examined. In addition, the impact of the temporal variation t and the involving perturbation term coefficients β_1 and β_2 are graphically portrayed; moreover, the impact of c, γ_1, and γ_2 on the complex-valued wave field $v(x,t)$ has been eliminated by the deployed analytical methods. Consequently, to begin with, Figures 1 and 2 graphically examine the obtained dark solitonic solution in (4.35), assessing the influence of increasing perturbation coefficients β_1 and β_2, respectively, in Figure 1(a) and Figure 1(b), sequentially. Notably, an increase in the coefficient β_1 reduces the amplitude of the propelling wave in Figure 1(a), while an opposite trend has been noted in Figure 1(b) concerning the increase of β_2. In addition, variation of short time alteration has been captured in Figure 2(a), where an increase in time t delays the wave's movement; see also Figure 2(b) for the 3D depiction of the dark soliton solution.

Furthermore, Figures 3 and 4 graphically capture the variation of the obtained bright solitonic solution in (5.9) in response to the alteration of the involved parameters. Remarkably, one equally notes the same interpretations of the dark soliton solution here. In fact, an increase in the coefficient β_1 reduces the amplitude of the wave displacement $v(x,t)$ in Figure 3(a), while the opposite trend is noted in Figure 3(b) while increasing of the parameter β_2. Additionally, short time variation has equally been captured in Figure 4(a), where an increased time t delays the wave's movement; see also Figure 4(b) for the 3D depiction of the bright soliton solution. Notably, the time variation plots in Figures 2(b) and Figure 4(b) have been noticed to be backward; indeed, this is due the consideration of the negative sign in the considering solutions in (4.35) and (5.9), correspondingly.



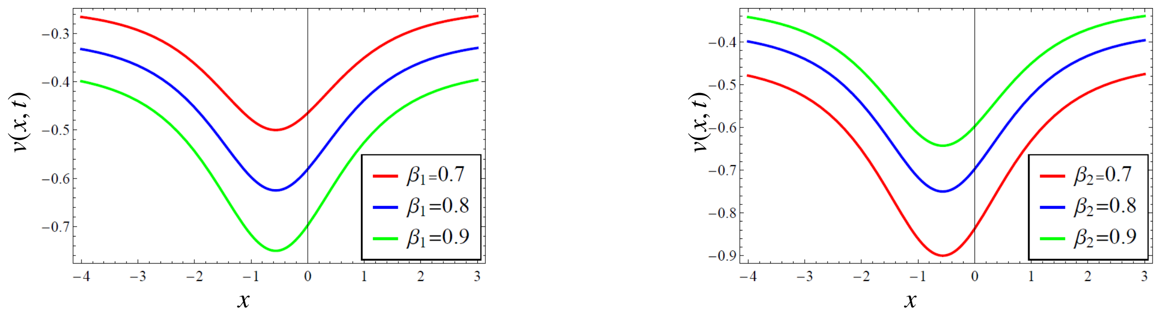
(a) Impact of β_1 on the dark soliton. **(b)** Impact of β_2 on the dark soliton.

Figure 1. Impact of β_1 and β_2 on the propagation of dark soliton (4.35) when $n = 1$, $b_1 = 0.3$, $b_2 = 0.2$, $\zeta = 1$, $\alpha = 0.25$, $k = 1$, $B_1 = 1$, $a = 1$, and $t = 0.25$ for (a) $\beta_2 = 0.8$ and (b) $\beta_1 = 0.9$.



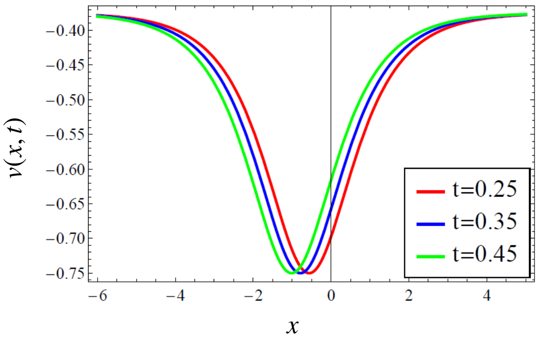
(a) Impact of t on the dark soliton. **(b)** 3D plot for the dark soliton.

Figure 2. Impact of t on the propagation of the dark soliton (4.35) when $n = 1$, $b_1 = 0.3$, $\beta_1 = 0.9$, $b_2 = 0.2$, $\beta_2 = 0.8$, $\zeta = 1$, $\alpha = 0.25$, $k = 1$, $B_1 = 1$, and $a = 1$ in (a), while (b) gives the 3D depiction.

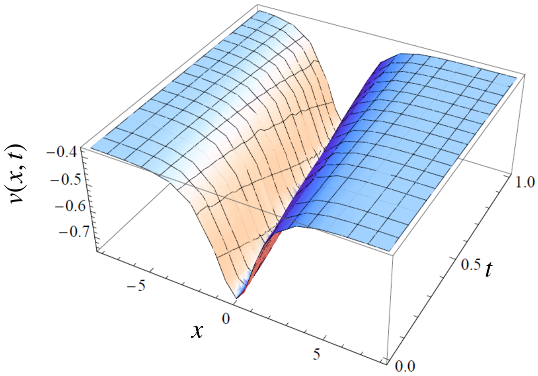


(a) Impact of β_1 on the bright soliton. **(b)** Impact of β_2 on the bright soliton.

Figure 3. Impact of β_1 and β_2 on the propagation of bright soliton (5.9) when $n = 1$, $b_1 = 0.3$, $b_2 = 0.2$, $\zeta = 1$, $\alpha = 0.25$, $k = 1$, $B_1 = 1$, $a = 1$, and $t = 0.25$, for (a) $\beta_2 = 0.8$ and (b) $\beta_1 = 0.9$.



(a) Impact of t on the bright soliton.



(b) 3D plot for the bright soliton.

Figure 4. Impact of t on the propagation of the dark soliton (4.35) when $n = 1$, $b_1 = 0.3$, $\beta_1 = 0.9$, $b_2 = 0.2$, $\beta_2 = 0.8$, $\zeta = 1$, $\alpha = 0.25$, $k = 1$, $B_1 = 1$, and $a = 1$ in (a), while (b) gives the 3D depiction.

7.2. Analysis of dispersion relation

In the same manner, this subsection examines the obtained approximate analytical dispersion relation (6.6) for the governing extended perturbed extended Chen-Lee-Liu equation embedded in cubic-quintic nonlinearity and a resonant force (2.5) graphically, assessing the impact of the involved wave's parameters through the frequency-wavenumber dependency graphs. Moreover, throughout these plots in Figures 5, 6, 7, and 8, only the first harmonic curve has been analyzed, with regard to the impact of the embedding resonant term, presided over by the coefficient c, and the involving perturbation term coefficients β_1, β_2, γ_1, and γ_2, are graphically portrayed. In this regard, the effect of the embedding resonant term coefficient c on the wave's dispersion has been captured in Figure 5, where it is vividly noted that an increase in the resonance coefficient increases the dispersion of waves in the medium; least dependency has been noted when $c = 0$. Moreover, the first perturbation term is coefficitiated by α, where Figure 6 has it that an increase in α decreases the dependency of Ψ on ϖ. In addition, the influence of the perturbation coefficients β_1 and β_2 has been illustrated in Figures 7(a) and 7(b), respectively, where an increase in both coefficients results in the decrease of the wave's dispersion in the medium; indeed, the parameter β_1 has been noted to be more pronounced than β_2 as small values are equally effective. Lastly, the influence of the perturbation coefficients γ_1 and γ_2 is illustrated in Figures 8(a) and 8(b), respectively. The results indicate that increasing both coefficients leads to a reduction in the wave's dispersion within the medium. Moreover, the influence of γ_1 is more significant than that of γ_2. In addition, the impact of the normalized incident power m on the modulation instability is finally assessed via Figure 9 through examining the gain function established in (6.8) upon satisfaction of the constraining condition in (6.9). Indeed, the modulation behavior is rightly captured in Figure 9, setting all the parameters to unity with the exception of the perturbation parameter, the generalized quintic nonlinearity, which is assumed as $\beta_2 = -9$. Notably, the variation of the normalized power parameter m shows that increasing the normalization parameter increases the gain function, which thus increases the modulation instability.

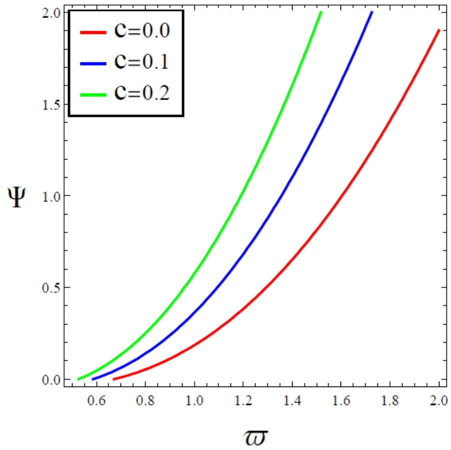


Figure 5. Impact of resonant coefficient c on the dispersion of waves (6.6) when $a = 1$, $m = 0.1$, $\alpha = 1$, $\beta_1 = 1$, $\beta_2 = 1$, $\gamma_1 = 1$, and $\gamma_2 = 1$.

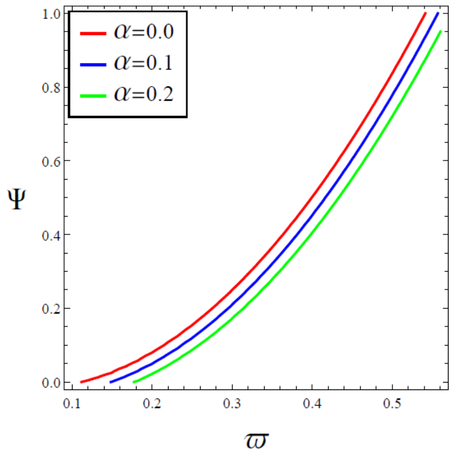
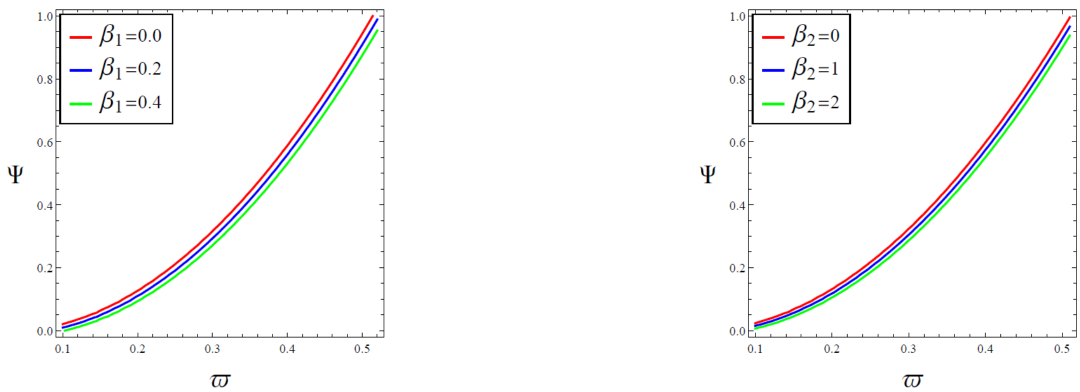
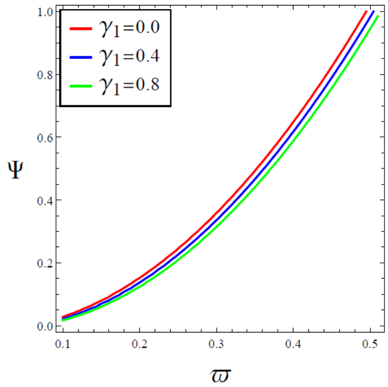


Figure 6. Impact of α on the dispersion of waves (6.6) when $a = 1$, $m = 0.1$, $\beta_1 = 1$, $\beta_2 = 1$, $\gamma_1 = 1$, $\gamma_2 = 1$, and $c = 1$.

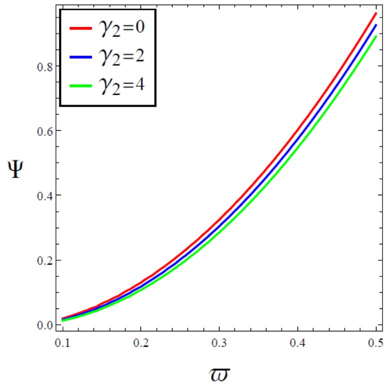


(a) Impact of β_1 on the dispersion of waves. **(b)** Impact of β_2 on the dispersion of waves.

Figure 7. Impact of β_1 and β_2 on the dispersion of waves (6.6) when $a = 1$, $m = 0.1$, $\beta_2 = 1$, $\gamma_1 = 1$, $\gamma_2 = 1$, $c = 1$, and $\alpha = 0.1$ for (a) $\beta_2 = 1$ and (b) $\beta_1 = 0.1$.



(a) Impact of γ_1 on the dispersion of waves.



(b) Impact of γ_2 on the dispersion of waves.

Figure 8. Impact of γ_1 and γ_2 on the dispersion of waves (6.6) when $a = 1, m = 0.1, \beta_1 = 0.1, \beta_2 = 1$, $c = 1$, and $\alpha = 0.1$ for (a) $\gamma_2 = 1$ and (b) $\gamma_1 = 0.8$.

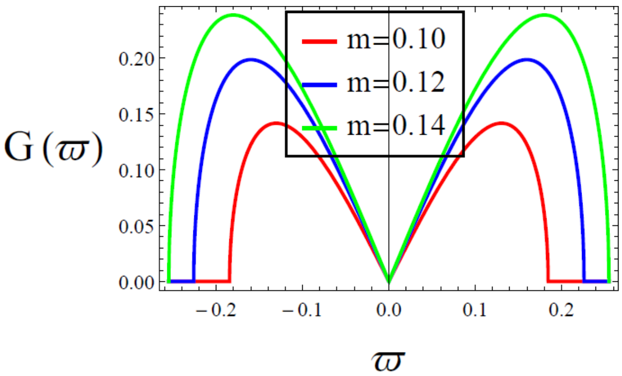


Figure 9. Impact of the normalized incident power m on the modulation instability via the gain function (6.8) when a =1, $c = 1$, $\alpha = 1$, $\beta_1 = 1$, and $\beta_2 = -9$.

8. Conclusions

The present study proposed several extensions to the classical Chen-Lee-Liu equation by embedding the cubic-quintic nonlinearity law amidst the incorporation of the resonant force, thereby enriching both the applicability and dynamicity of the traditional model. A promising analytical approach was deployed to successfully construct various periodic and solitonic solutions of the models, in addition to the employment of the relatively recent Kudryashov's modification method for solitonic constructions. In fact, several periodic solutions, bright and dark solitons, and many combos were constructed for the proposed class of equations. In addition, the study got hold of the approximate dispersion relation of the linearized generalized model, ensuring the spectrum of modulation stability and instability regions. Moreover, some numerical simulations were conducted, examining the involving parameters on the influence of the wave's dispersion in the medium. Notably, the presence of the resonant term in the model increased the dispersion of waves in the medium. In contrast, all the perturbation terms were noted to oppose the dispersion of waves in the system. Lastly, this study could be extended in different ways, including the incorporation of higher-order power laws, coupling the model for birefringent fiber applications, and the superposition of higher dimensions for more applications in real-life scenarios.

In addition, the present study has profound potential in fluid dynamics, optics and quantum physics, and optoelectronics, to mention a few. What is more, the findings of this study are limited to the proposed models in this study; however, the idea can be extended to other variants of the Chen-Lee-Liu equations, including to other forms of the derivative nonlinear Schrödinger equations, where derivative dispersion terms play vital parts in the description of pulsification processes.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

The author declares no conflict of interest to this work.

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