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Research article

Finite-time and predefined-time neurodynamic approach with time-varying coefficients for solving inverse quasi-variational inequality problems

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Abstract: This paper proposes finite-time (FT) and predefined-time (PdT) neurodynamic models with time-varying coefficients for solving inverse quasi-variational inequality (IQVI) problems. Two projection-based neurodynamic models are developed, in which time-varying gains are designed to regulate the convergence speed and improve transient performance. A nominal projected model is introduced to establish the equivalence between its equilibrium points and the solutions of IQVI problems. Under Lipschitz continuity and strong monotonicity assumptions, the existence, uniqueness, and global convergence of the proposed models are established. By employing Lyapunov theory, FT and PdT convergence are rigorously proved together with explicit settling-time estimates. In particular, the PdT model guarantees convergence within a user-prescribed time bound independent of the initial conditions. Furthermore, a forward Euler discretization is developed to obtain implementable numerical algorithms, and robustness of the PdT model under bounded disturbances is established. Numerical experiments, including an additional higher-dimensional IQVI problem example and a sparse signal recovery application, validate the theoretical results and demonstrate the effectiveness, robustness, and computational efficiency of the proposed framework.

Keywords: finite-time stability; predefined-time stability; neurodynamic approach; time-varying coefficients; inverse quasi-variational inequality
Mathematics Subject Classification: 47J30, 49J53, 49M37

1. Introduction

The k-dimensional Euclidean space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$ is denoted by $\mathbb{R}^k$. Let $\mathcal{J}$: $\mathbb{R}^k \to \mathbb{R}^k$ be a continuous mapping and let $\Omega \subset \mathbb{R}^k$ be a nonempty, closed, and convex set. The classical variational inequality problems (VIPs) consist of finding a point $z^* \in \Omega$ such that

$$\langle \mathcal{J}(z^*), v - z^* \rangle \geq 0, \quad \forall v \in \Omega. \tag{1.1}$$

Equation (1.1) characterizes the classical variational inequality problem, where the objective is to determine a point $z^* \in \Omega$ such that the operator $\mathcal{J}$ satisfies the variational inequality over the feasible set Ω. Geometrically, the inequality states that the vector $\mathcal{J}(z^*)$ forms a non-acute angle with every feasible direction $v - z^*$, thereby characterizing an equilibrium point. VIPs have been extensively studied in both theory and applications, including optimization, complementarity systems, saddle-point problems, Nash equilibrium models, and fixed-point theory. Recently, time-varying dynamical systems [1], second-order approaches for inverse quasi-variational inequalities [2], inertial splitting methods for variational inclusion problems [3], and iterative schemes for fixed-point and split variational inclusion problems [4] have received considerable attention.

Numerous computational techniques have also been developed for variational inequality problems. Fundamental theoretical foundations can be found in [5, 6]. Extragradient-type methods were studied in [7, 8], whereas projection-based algorithms were investigated in [9]. More recently, inertial splitting techniques have been developed for inclusion and fixed-point problems [10]. By means of the projection theorem [5], a point $z^* \in \Omega$ solves VIPs defined in (1.1) $\iff$ it satisfies the following projection equation for some constant $\eta > 0$:

$$z^* = \mathcal{P}_\Omega(z^* - \eta \mathcal{J}(z^*)). \tag{1.2}$$

A natural extension of the VIPs is its inverse formulation. Let $\Gamma : \mathbb{R}^k \to \mathbb{R}^k$ be a single-valued operator. If the inverse relation

$$\mathtt{w}^* = \Gamma^{-1}(z^*) = \mathcal{J}(z^*)$$

exists, then the VIPs can be reformulated as an inverse variational inequality (IVI) problems, which aims to find $\mathtt{w}^* \in \mathbb{R}^k$ such that

$$\Gamma(\mathtt{w}^*) \in \Omega, ; \quad \langle v - \Gamma(\mathtt{w}^*), \mathtt{w}^* \rangle \geq 0, \quad \forall v \in \Omega. \tag{1.3}$$

IVI problems arise in several application areas, such as in economic equilibrium modeling [11] and traffic network analysis [12]. Nevertheless, compared with classical VIPs, relatively fewer theoretical and numerical methods are available for solving the IVI problem defined in (1.3). This problem is equivalent to solving the projection equation

$$\Gamma(\mathtt{w}^*) = \mathcal{P}_\Omega[\Gamma(\mathtt{w}^*) - \eta \mathtt{w}^*], \tag{1.4}$$

where $\eta > 0$ is a given constant and $\mathcal{P}_\Omega$: $\mathbb{R}^k \to \Omega$ denotes the metric projection operator defined by

$$\mathcal{P}_\Omega(w) = \arg\min_{v \in \Omega} \|w - v\|, \quad w \in \mathbb{R}^k. \tag{1.5}$$

In 2016, Zou et al. [13] proposed a dynamical model, interpreted as a neural network model, for solving the IVI problem in (1.3), given by

$$\dot{w} = \sigma\Big[\mathcal{P}_\Omega(\Gamma(w) - \eta w) - \Gamma(w)\Big], \tag{1.6}$$

where $\dot{w} = \frac{dw}{dt}$, $\mathcal{P}_\Omega$ is defined in (1.5), and $\sigma > 0$ is a constant parameter.

He et al. [14] further investigated the IVI problem and proposed a projection-based iterative method, which has been effectively applied to various equilibrium problems. Such models have important applications in engineering sciences, particularly in transportation and telecommunication networks [15].

Motivated by these developments, several generalizations of IVI problems have been introduced, leading to inverse quasi-variational inequality (IQVI) problems [16, 17]. In this setting, the constraint set depends on the unknown variable itself. The problem is to find a point $\mathtt{w}^* \in \mathbb{R}^k$ such that

$$\Gamma(\mathtt{w}^*) \in \Phi(\mathtt{w}^*); \quad \langle \mathtt{w}^*, v - \Gamma(\mathtt{w}^*)\rangle \geq 0 \quad \forall v \in \Phi(\mathtt{w}^*), \tag{1.7}$$

where $\Gamma\colon \mathbb{R}^k \to \mathbb{R}^k$ is a single-valued mapping and $\Phi\colon \mathbb{R}^k \to 2^{\mathbb{R}^k}$ is a set-valued mapping. It is evident that when $\Phi(w) = \Omega$ for all $w \in \mathbb{R}^k$, the IQVI problem described above simplifies to the IVI problem. First-order neurodynamic models have since been widely applied to variational problems due to their stability and parallel computational structure [13, 18]. Their convergence and stability properties have also been extensively investigated in the literature [19, 20]. Projection-based iterative methods for IQVI problems were also developed by He et al. [14], with applications in equilibrium modeling and engineering models [15, 21]. Beyond economics and transportation, IQVI problems can model trust evaluation, intrusion detection, and behavior-based access control in cyber-physical models.

The concept of finite-time (FT) convergence, introduced in [22], has motivated the development of fast dynamical models for consensus, control, and equilibrium problems [23–25]. Related FT control approaches have also been investigated for nonlinear systems [26]. In FT models, the settling time depends on the initial conditions and may increase as the initial state moves away from equilibrium, which limits their applicability in scenarios where the initial state is unknown. To overcome this drawback, predefined-time (PdT) convergence has been introduced, which guarantees that the settling time can be assigned in advance as a user-defined constant, independent of initial conditions (see [27–29]). This stronger notion provides improved predictability and controllability, making it more suitable for real-time applications.

Both FT and PdT convergence frameworks have been successfully applied in optimization, control, sparse recovery, and equilibrium problems [25, 27, 30]. Similar ideas have also been explored in dynamical sparse recovery [31]. In particular, neurodynamic models with time-varying coefficients have shown enhanced convergence performance in applications such as signal recovery and equilibrium computation [29, 32, 33]. However, robustness under bounded disturbances and controllable convergence behavior remain important challenges.

Recent inertial and adaptive algorithms have shown promising convergence behavior for equilibrium and monotone inclusion problems [34–36] and related applications [37]. A comparison with representative existing dynamical approaches is presented in Table 1, highlighting the distinctive features and advantages of the proposed framework.

Motivated by these observations, this paper proposes a unified framework for FT and PdT neurodynamic models with time-varying coefficients (TVCNDMs) to solve IQVI problems. Two first-order dynamical models are developed, where time-dependent gains are employed to regulate the convergence speed and improve transient performance.

A nominal projected model is first introduced to establish the equivalence between its equilibrium points and the solutions of IQVI problems. Under Lipschitz continuity and strong monotonicity assumptions, the existence, uniqueness, and global convergence of the proposed models are established. By means of Lyapunov stability theory, FT convergence and PdT stability are rigorously derived together with explicit settling-time estimates. In particular, the PdT model guarantees convergence within a user-prescribed time bound independent of the initial conditions. Furthermore, a forward Euler discretization is developed to obtain implementable numerical algorithms, and robustness of the PdT-TVCNDM under bounded disturbances is analyzed. Numerical experiments, including an additional higher-dimensional IQVI problem example and a sparse signal recovery application, verify the theoretical results and demonstrate the effectiveness and applicability of the proposed framework.

Main contributions

(1) A unified FT and PdT neurodynamic framework with time-varying coefficients is proposed to solve the IQVI problem, together with a comparative analysis against existing representative dynamical approaches to highlight its distinctive features.

(2) Two first-order TVCNDMs are proposed, providing distinct FT and PdT convergence mechanisms for inverse quasi-variational inequality problems.

(3) Establishment of existence, uniqueness, and Lyapunov-based convergence analysis under Lipschitz continuity and strong monotonicity assumptions.

(4) Derivation of explicit settling-time estimates, highlighting the user-prescribed convergence property of the PdT model.

(5) Development of forward Euler discretization schemes for both TVCNDMs together with numerical validation.

(6) Robustness analysis of the PdT-TVCNDM under bounded disturbances.

(7) Extensive numerical experiments, including an additional higher-dimensional IQVI problem example and a sparse signal recovery application, demonstrating the effectiveness and general applicability of the proposed framework.

The remainder of this paper is organized as follows. Section 2 presents the preliminary definitions and auxiliary results. A comparative study with existing representative dynamical approaches is presented to highlight the distinctive features of the proposed framework. Section 3 develops the FT neurodynamic model together with its stability analysis. Section 4 introduces the PdT neurodynamic model and establishes its convergence properties. Section 5 investigates robustness under bounded disturbances. Sections 6 and 7 provide numerical experiments and an application to sparse signal recovery. Finally, Section 8 concludes the paper and outlines future research directions.

2. Preliminaries

In this part, we recall several fundamental definitions that will be utilized in the subsequent analysis.

Definition 2.1. *[38] The operator Γ: $\mathbb{R}^k \to \mathbb{R}^k$ is said to be:*
(a) Monotone on $\mathbb{R}^k$, if

$$\langle \Gamma(w) - \Gamma(v), w - v \rangle \geq 0, \quad \forall w, v \in \mathbb{R}^k.$$

(b) Strongly monotone on $\mathbb{R}^k$ with modulus $\zeta > 0$, if

$$\langle \Gamma(w) - \Gamma(v), w - v \rangle \geq \zeta \|w - v\|^2, \quad \forall w, v \in \mathbb{R}^k.$$

(c) L-Lipschitz continuous on $\mathbb{R}^k$ if $\exists$ a $L \geq 0$, such that

$$\|\Gamma(w) - \Gamma(v)\| \leq L\|w - v\|, \quad \forall w, v \in \mathbb{R}^k.$$

It is worth noting that when Ω is nonempty, closed, and convex, $\mathcal{P}_\Omega(\cdot)$ exists and is unique. Some useful properties of the projection operator are listed below.

Lemma 2.1. *[39] Suppose Ω is a nonempty, closed, convex set in $\mathbb{R}^k$. Let $u, y \in \mathbb{R}^k$. Then, we have*

(1) $v = \mathcal{P}_\Omega(u) \iff \langle u - v, w - v \rangle \leq 0,\ \forall w \in \Omega$.

(2) $\mathcal{P}_\Omega$ is 1-Lipschitz continuous on Ω.

Lemma 2.2. *For any $a, b, c \in \mathbb{R}^k$, the following inequalities are satisfied:*

(1) $\langle a, b \rangle \leq \|a\|\|b\|$.

(2) (Pythagoras identity), it holds that $2\langle a - b, c - b \rangle = \|a - b\|^2 + \|c - b\|^2 - \|a - c\|^2$.

2.1. Lyapunov stability conditions

Consider the time-varying model

$$\dot{w} = \chi(t, w(t)), \quad w(t_0) = w_0, \tag{2.1}$$

where w: $[t_0, +\infty) \to \mathbb{R}^k$, and χ: $[t_0, +\infty) \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous.

Definition 2.2. *[22] We say that $w^* \in \mathbb{R}^k$ is an equilibrium point of (2.1) if*

$$\chi(t, w^*) = 0, \quad \forall\, t > t_0. \tag{2.2}$$

*1) **Lyapunov stability:** For any $\mathcal{E} > 0$, there exists $\Sigma = \Sigma(\mathcal{E}, t_0)$ such that*

$$\|w_0 - w^*\| < \Sigma \ \Rightarrow\ \|w(t) - w^*\| < \mathcal{E}, \quad \forall\, t > t_0. \tag{2.3}$$

*2) **Uniform stability:** For any $\mathcal{E} > 0$, there exists $\Sigma = \Sigma(\mathcal{E}) > 0$, independent of t_0, such that (2.3) holds.*

Definition 2.3. *[40] From model* (2.1)*, we obtain an equilibrium point* $\mathtt{w}^*$ *of* (2.1)*, said to be FT convergent if there exists a FT,* $\mathcal{T}\colon \mathbb{R}^k \setminus \{0\} \to (0,\infty)$ *(i.e.,* $\mathcal{T} < \infty$*), such that for all* $w(0) \in N \setminus \{0\}$,

$$\lim_{t\to\mathcal{T}} w(t) = 0 \quad \textit{and} \quad \mathcal{T} = \mathcal{T}(w(0)) < \infty.$$

Definition 2.4. *[41] The equilibrium point of* (2.1) *is said to be fixed-time stable if it is FT stable and there exists an upper bound* $\mathcal{T}_{\max} < \infty$ *such that*

$$\sup_{w_0\in\mathbb{R}^k} \mathcal{T}(w_0) \le \mathcal{T}_{\max}.$$

Definition 2.5. *(Uniform PdT stability) If system* (2.1) *is uniformly stable and there exists a PdT parameter* $\mathcal{T}_p > 0$ *s.t. the convergence time* $\mathcal{T}(\mathtt{w}^*)$ *satisfies*

$$\mathcal{T}(\mathtt{w}^*) < \mathcal{T}_p, \tag{2.4}$$

then system (2.1) *is said to be uniformly PdT stable.*

We now establish Lyapunov stability conditions for PdT stability of the time-varying dynamical system (2.1). To this end, the following assumptions are introduced.

Assumption 2.1. *Let* $\phi_2\colon [t_0,+\infty) \to \mathbb{R}_{++}$ *be continuous and satisfy*

$$\int_{t_0}^{+\infty} \phi_2(z)\,dz > \frac{\mathcal{T}_p}{\mathcal{G}_p m_2(\Theta_2 - 1)}, \tag{2.5}$$

where $m_2 \in (0,1)$*,* $\Theta_2 > 1$*,* $\mathcal{T}_p > 0$*, and* $\mathcal{G}_p > 0$.

Assumption 2.2. *Let* $\phi_1\colon [t_0,+\infty) \to \mathbb{R}_{++}$ *be continuous and satisfy*

$$\int_{t_0}^{+\infty} \phi_1(z)\,dz > \mathcal{F}_1 \circ \mathcal{F}_2^{-1}\left(\frac{\mathcal{T}_p}{\mathcal{G}_p m_2(\Theta_2 - 1)}\right) + \frac{\mathcal{T}_p}{\mathcal{G}_p m_1(1 - \Theta_1)}, \tag{2.6}$$

where $m_1 \in (0,1)$*,* $\Theta_1 \in (0,1)$*, and*

$$\mathcal{F}_1(t) = \int_{t_0}^{t} \phi_1(z)\,dz, \quad \mathcal{F}_2(t) = \int_{t_0}^{t} \phi_2(z)\,dz, \tag{2.7}$$

are strictly increasing functions.

Lemma 2.3. *[33] Consider the time-varying dynamical model* (2.1)*. Suppose there exists a continuously differentiable, positive definite function* $\mathcal{L}\colon \mathfrak{H} \to \mathbb{R}_+$ *and a PdT parameter* $\mathcal{T}_p > 0$ *s.t.:*

(1)

$$\mathcal{L}(t, w(t)) = 0 \iff \dot{w}(t) = 0. \tag{2.8}$$

(2) For all $t \ge t_0$ *and any solution* $w(t)$ *of* (2.1),

$$\dot{\mathcal{L}}(t, w(t)) \le -\frac{\mathcal{G}_p}{\mathcal{T}_p}\Big(m_1\phi_1(t)\mathcal{L}^{\Theta_1}(t, w(t)) + m_2\phi_2(t)\mathcal{L}^{\Theta_2}(t, w(t))\Big), \tag{2.9}$$

where $\mathcal{G}_p > 0$*,* $m_1, m_2, \Theta_1 \in (0,1)$*,* $\Theta_2 > 1$*, and*

$$\dot{\mathcal{L}}(t, w(t)) = \frac{d\mathcal{L}}{dt}(t, w) + \frac{\partial\mathcal{L}}{\partial w}(t, w)\dot{w}(t). \tag{2.10}$$

If Assumptions 2.1 and 2.2 hold and the parameters $\mathcal{T}_p$ and $\mathcal{G}_p$ satisfy

$$\mathcal{F}_1^{-1}\left(\mathcal{F}_1 \circ \mathcal{F}_2^{-1}\left(\frac{\mathcal{T}_p}{\mathcal{G}_p m_2(\Theta_2 - 1)}\right) + \frac{\mathcal{T}_p}{\mathcal{G}_p m_1(1 - \Theta_1)}\right) = \mathcal{T}_p, \tag{2.11}$$

then the system (2.1) *is PdT stable.*

2.2. A nominal time-varying neurodynamic model for IQVI problems

In this section, we first recall the nominal neurodynamic model proposed in [17] for IQVI problems (1.7) and a result on the existence and uniqueness of solutions of IQVI problems (1.7). We then present several auxiliary results that will be useful in the subsequent analysis. In particular, we establish key inequalities that play a crucial role in studying FT and PdT stability.

To solve the IQVI problem (1.7), we consider the following nominal time-varying neurodynamic model (TVNDM):

$$\dot{w}(t) = -\phi(t)\Big(\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\Big), \tag{2.12}$$

where $\phi(t) > 0$ is a continuous time-dependent gain for all $t > 0$. For any nonempty, closed, and convex set $\Phi(w) \subseteq \mathbb{R}^k$, let $\mathcal{P}_{\Phi(w)}$: $\mathbb{R}^k \to \Phi(w)$ denote the metric projection operator onto the feasible set $\Phi(w)$. The residual $\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)$ measures the violation of the IQVI problem and vanishes only at its solution. Therefore, the equilibrium points of the TVNDM are precisely the solutions of the IQVI problems.

Lemma 2.4. *[17] An equilibrium of TVNDM* (2.12) *corresponds exactly to a solution of IQVI problem* (1.7)*. Then, solving IQVI problem* (1.7) *is equivalent to finding $\mathtt{w}^* \in \mathbb{R}^k$ satisfying*

$$\Gamma(\mathtt{w}^*) = \mathcal{P}_{\Phi(\mathtt{w}^*)}[\Gamma(\mathtt{w}^*) - \eta \mathtt{w}^*]. \tag{2.13}$$

To analyze the FT and PdT stability, we make the following assumption.

Assumption 2.3. *Let Φ: $\mathbb{R}^k \longrightarrow 2^{\mathbb{R}^k}$ be a set-valued mapping with nonempty, closed, and convex values, and Γ: $\mathbb{R}^k \to \mathbb{R}^k$ be an L-Lipschitz continuous and ζ-strongly monotone single-valued mapping. Assume that there exists some $\mu > 0$ such that*

$$\|\mathcal{P}_{\Phi(a)}(w) - \mathcal{P}_{\Phi(b)}(w)\| \le \mu\|a - b\|, \quad \forall a, b, w \in \mathbb{R}^k \tag{2.14}$$

and

$$\theta = \sqrt{L^2 + \eta^2 - 2\eta\zeta} + \mu < \eta.$$

We recall a result on the existence and uniqueness of solutions to the IQVI problem (1.7).

Lemma 2.5. *[17] Suppose Assumption 2.3 holds. Then, $\mathtt{w}^* \in \mathbb{R}^k$ is the unique solution of the IQVI problem.*

Lemma 2.6. *[42] Suppose Assumption 2.3 holds. Assume that the solution set of the IQVI problem* (1.7) *is nonempty. Let $\mathtt{w}^* \in \mathbb{R}^k$ be a solution of the IQVI problem* (1.7)*. Then, the following statements hold:*

(1)

$$\left\|\mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w) - \mathcal{P}_{\Phi(\mathtt{w}^*)}(\Gamma(\mathtt{w}^*) - \eta \mathtt{w}^*)\right\| \le \theta\|w - \mathtt{w}^*\|, \qquad \forall w \in \mathbb{R}^k. \tag{2.15}$$

(2)

$$0 < (\zeta - \theta)\|w - \mathtt{w}^*\| \le \|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\| \le (L + \theta)\|w - \mathtt{w}^*\|, \quad \forall w \neq \mathtt{w}^*. \tag{2.16}$$

(3)

$$\langle w - \mathtt{w}^*, \Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\rangle \ge (\zeta - \theta)\|w - \mathtt{w}^*\|^2 > 0, \quad \forall w \neq \mathtt{w}^*. \tag{2.17}$$

Lemma 2.7. *[42] Suppose Assumption 2.3 holds. Define the operator*

$$\Pi(w) := \Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w), \qquad w \in \mathbb{R}^k.$$

Then, Π *is Lipschitz continuous with modulus* $(2L + \eta + \mu) > 0$*.*

Lemma 2.8. *Suppose Assumption 2.3 holds. Then, the TVNDM* (2.12) *has a unique equilibrium point, which is also the unique solution of IQVI problem* (1.7)*.*

Proof. By Lemma 2.4, a point $w^* \in \mathbb{R}^k$ is an equilibrium point of the nominal TVNDM (2.12) if and only if it satisfies the projection equation

$$\Gamma(w^*) = P_{\Phi(w^*)}(\Gamma(w^*) - \eta w^*),$$

which is equivalent to the IQVI problem (1.7). Moreover, by Lemma 2.5, under Assumption 2.3, the IQVI problem (1.7) admits a unique solution. In addition, Lemma 2.5 establishes the fundamental properties of the projection residual operator, namely

$$(\zeta - \theta)\|w - w^*\| \le \|\Gamma(w) - P_{\Phi(w)}(\Gamma(w) - \eta w)\|,$$

which implies that the projection residual vanishes only at $w = w^*$. Consequently, the equilibrium point of the nominal TVNDM is unique. Hence, the nominal TVNDM (2.12) possesses a unique equilibrium point, which coincides with the unique solution of the IQVI problem (1.7). □

Table 1 highlights the differences between the proposed method and existing dynamical approaches. Unlike the existing methods, the proposed model combines time-varying coefficients with both FT and PdT convergence for inverse quasi-variational inequality problems, thereby providing a more general computational framework.

Table 1. Comparison of the proposed method with representative dynamical approaches for inverse variational inequality problems.

Reference	Problem type	Second-order	Time-varying coefficient	FT	PdT
[43]	VIP	×	×	×	×
[29]	Mixed VIP	×	×	×	✓
[14]	IVI problem	×	×	×	×
[23]	Equilibrium problem	×	×	✓	×
[17]	IQVI problem	×	×	×	×
[42]	IQVI problem	×	×	✓	Fixed-time
Proposed method	IQVI problem	×	✓	✓	✓

3. FT stability analysis for a time-varying model

Before presenting the main results, we now establish the Lyapunov stability conditions for FT convergence of the time-varying neurodynamic model.

Lemma 3.1. *Consider the dynamical model* (2.1)*, which is locally Lipschitz in* w *and let* $\mathtt{w}^* \in \mathbb{R}^k$ *denote one of its equilibrium points. Suppose that* $\exists$ *a neighborhood* $\mathcal{V}$ *of* $\mathtt{w}^*$ *and a continuously differentiable function* $\mathcal{L}$*:* $[t_0, \infty) \times \mathbb{R}^k \to \mathbb{R}_+$ *such that* $\mathcal{L}(t, 0) = 0$*,* $\mathcal{L}(t, w) > 0\ \forall w \neq 0$*. Suppose that along the trajectories of the model*

$$\dot{\mathcal{L}}(t, w(t)) \leq -K\phi(t)(\mathcal{L}(t, w(t)))^{\alpha}, \tag{3.1}$$

for all $t \geq t_0$ *and* $w \in \mathfrak{U} \setminus \{\mathtt{w}^*\}$*, for some open neighborhood* $\mathfrak{U} \subseteq \mathcal{V}$ *of* $\mathtt{w}^*$*, where* $K > 0$*,* $0 < \alpha < 1$*, and* $\phi(t)$*:* $[t_0, \infty) \to \mathbb{R}_+$ *is continuous. Further, suppose that*

$$\int_{t_0}^{+\infty} \phi(s)\, ds > \frac{\mathcal{L}(w(0))^{1-\alpha}}{K(1-\alpha)}, \tag{3.2}$$

where $H(t) = \int_{t_0}^{t} \phi(s)\, ds$*. Then,* $\mathtt{w}^*$ *is an FT stable equilibrium point of* (2.1)*. Furthermore, the settling time* $\mathcal{T}$ *satisfies*

$$\mathcal{T}(w(0)) \leq H^{-1} \frac{\mathcal{L}(w(0))^{1-\alpha}}{K(1-\alpha)} \tag{3.3}$$

for all $w(0) \in \mathfrak{U}$*.*

Proof. Since $\phi(t)$ is positive and continuous, the function $H(t) = \int_{t_0}^{t} \phi(s)\, ds$ is strictly increasing. Let w: $[t_0, +\infty) \to \mathbb{R}^k$ be a solution of (2.1). Let $\mathcal{L}$: $[t_0, +\infty) \to \mathbb{R}_+$ be given by $\mathcal{L}(t) = \mathcal{L}(t, w(t))$. It follows from (3.1) that, for all $t \geq t_0$,

$$\dot{\mathcal{L}}(t) \leq -K\phi(t)\mathcal{L}(t)^{\alpha}. \tag{3.4}$$

Now, we define $\mathcal{T} = \sup\{t_1 \geq t_0 : \mathcal{L}(t) > 0 \text{ for all } t \in [t_0, t_1]\}$. By the definition of $\mathcal{T}$, the Lyapunov function satisfies $\mathcal{L}(t) > 0$ for every $t \in [t_0, \mathcal{T})$. Hence, $\mathcal{L}(t)^{-\alpha}$ is well defined in this interval, allowing the variables in (3.4) to be separated. Integrating the resulting differential inequality over $[t_0, t_1]$, where $t_1 < \mathcal{T}$, yields an estimate for the decrease of the Lyapunov function in terms of the accumulated time-varying gain $\int_{t_0}^{t_1} \phi(s)\, ds$, which forms the basis for deriving the explicit settling-time estimate. By Eq (3.4), for all $t \in [t_0, \mathcal{T})$, $\mathcal{L}(t)^{-\alpha}\dot{\mathcal{L}}(t) \leq -K\phi(t)$. For any $t_1 \in [t_0, \mathcal{T})$, integrating this inequality on $[t_0, t_1]$ yields

$$\frac{\mathcal{L}(t_1)^{1-\alpha}}{K(1-\alpha)} - \frac{\mathcal{L}(t_0)^{1-\alpha}}{K(1-\alpha)} \leq -\int_{t_0}^{t_1} \phi(s)\, ds = -H(t_1).$$

By combining with $\alpha < 1$, $\mathcal{L}(t_1) \leq \mathcal{L}(t_0)$, and $w \in \mathfrak{U} \setminus \{\mathtt{w}^*\}$, for some open neighborhood $\mathfrak{U} \subseteq \mathcal{V}$ of $\mathtt{w}^*$, and (3.2), we obtain

$$H(t_1) \leq \frac{\mathcal{L}(t_0)^{1-\alpha}}{K(1-\alpha)} - \frac{\mathcal{L}(t_1)^{1-\alpha}}{K(1-\alpha)} < \frac{\mathcal{L}(w(0))^{1-\alpha}}{K(1-\alpha)} < \int_{t_0}^{+\infty} \phi(s)\, ds,$$

which holds for all $w(0) \in \mathfrak{U}$. Moreover,

$$\lim_{t\to\infty} H(t) = \int_{t_0}^{+\infty} \phi(s)\, ds,$$

hence

$$H(t_1) < \frac{\mathcal{L}(w(0))^{1-\alpha}}{K(1-\alpha)} < \lim_{t\to\infty} H(t).$$

As H is strictly increasing,

$$t_1 < H^{-1}\left(\frac{\mathcal{L}(w(0))^{1-\alpha}}{K(1-\alpha)}\right) < +\infty,$$

which implies that

$$\mathcal{T} \leq H^{-1}\left(\frac{\mathcal{L}(w(0))^{1-\alpha}}{K(1-\alpha)}\right) < +\infty. \tag{3.5}$$

Now, the definition of $\mathcal{T}$ and the continuity of $\mathcal{L}$ imply that $\mathcal{L}(\mathcal{T}) = 0$. Since $\alpha > 0$, and $\dot{\mathcal{L}}(t) \leq -K\phi(t)\mathcal{L}(t)^\alpha \leq 0$, we have that $\mathcal{L}$ is nonincreasing. Combining with the assumption that $\mathcal{L}$ is nonnegative, we obtain that $\mathcal{L}(t) = 0$ for all $t \geq \mathcal{T}$, and so, for all $t \geq \mathcal{T}$, $\mathcal{L}(t, w(t)) = \mathcal{L}(t) = 0$. □

To study the FT stability of the IQVI problem (1.7), a first-order FT-TVCNDM is constructed, whose equilibrium point corresponds to the solution of (1.7). Under appropriate parameter conditions, the proposed neurodynamic model is shown to be FT stable. Accordingly, the FT-TVCNDM associated with (1.7) is defined as follows:

$$\dot{w} = -\phi(t)\frac{\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)}{\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\|^{\frac{\varsigma-2}{\varsigma-1}}}, \tag{3.6}$$

where $\varsigma > 2$ is a parameter and $\phi(t)$: $[t_0, +\infty) \to \mathbb{R}_{++}$ is a continuous function.

It should be noted that for any $w \in \mathbb{R}^k$ satisfying $\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w) = 0$, the right-hand side of (3.6) remains well defined for all $\varsigma > 2$. In this case, the expression evaluates to zero. Indeed, we have

$$\left\| \frac{\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)}{\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\|^{\frac{\varsigma-2}{\varsigma-1}}} \right\| = \|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\|^{\frac{1}{\varsigma-1}} = 0. \tag{3.7}$$

The following result shows that the equilibrium points of (3.6) coincide with TVNDM (2.12).

Proposition 3.1. *If* $\mathtt{w}^* \in \mathbb{R}^k$ *is an equilibrium point of* (3.6) $\iff$ *it is also an equilibrium point of TVNDM* (2.12).

Proof. A necessary and sufficient condition for $\mathtt{w}^* \in \mathbb{R}^k$ to be an equilibrium point of FT-TVCNDM (3.6) is that

$$\begin{aligned}\phi(t)\frac{\Gamma(\mathtt{w}^*) - \mathcal{P}_{\Phi(\mathtt{w}^*)}(\Gamma(\mathtt{w}^*) - \eta\mathtt{w}^*)}{\|\Gamma(\mathtt{w}^*) - \mathcal{P}_{\Phi(\mathtt{w}^*)}(\Gamma(\mathtt{w}^*) - \eta\mathtt{w}^*)\|^{\frac{\varsigma-2}{\varsigma-1}}} = 0 &\iff \frac{\|\Gamma(\mathtt{w}^*) - \mathcal{P}_{\Phi(\mathtt{w}^*)}(\Gamma(\mathtt{w}^*) - \eta\mathtt{w}^*)\|}{\|\Gamma(\mathtt{w}^*) - \mathcal{P}_{\Phi(\mathtt{w}^*)}(\Gamma(\mathtt{w}^*) - \eta\mathtt{w}^*)\|^{\frac{\varsigma-2}{\varsigma-1}}} = 0 \\ &\iff (\Gamma(\mathtt{w}^*) - \mathcal{P}_{\Phi(\mathtt{w}^*)}(\Gamma(\mathtt{w}^*) - \eta\mathtt{w}^*))^{1-\frac{\varsigma-2}{\varsigma-1}} = 0.\end{aligned} \tag{3.8}$$

From $1 - \frac{\varsigma-2}{\varsigma-1} = \frac{1}{\varsigma-1} > 0$, $\forall\varsigma > 2$, we conclude that Eq (3.8) holds $\iff$

$$\Gamma(\mathtt{w}^*) - \mathcal{P}_{\Phi(\mathtt{w}^*)}(\Gamma(\mathtt{w}^*) - \eta\mathtt{w}^*) = 0, \tag{3.9}$$

or, equivalently,

$$\Gamma(\mathtt{w}^*) = \mathcal{P}_{\Phi(\mathtt{w}^*)}(\Gamma(\mathtt{w}^*) - \eta\mathtt{w}^*). \tag{3.10}$$

The last equality follows directly from the projection characterization of IQVI problem established in Lemma 2.4. Therefore, the projection operator establishes the equivalence between the equilibrium point of the proposed FT-TVCNDM and the solution of the IQVI problem. Hence, $\mathtt{w}^*$ is an equilibrium point of TVNDM (2.12). □

The result below is an immediate consequence of Lemma 2.8 together with Proposition 3.1.

Corollary 3.1. *Suppose $\mathtt{w}^*$ be a solution of IQVI problem* (1.7) $\iff$ *it is an equilibrium point of FT-TVCNDM* (3.6).

Proof. It is a direct consequence of Lemma 2.8 together with Proposition 3.1. □

The following result establishes the FT stability of the FT-TVCNDM (3.6).

Theorem 3.2. *Suppose Assumption 2.3 holds. Then, $\mathtt{w}^* \in \mathbb{R}^k$, the solution of* (1.7)*, is a FT stable equilibrium point of FT-TVCNDM* (3.6) *with a settling time*

$$\mathcal{T}(w(0)) \leq \mathcal{T}_{\max} = H^{-1}\frac{(w(0) - \mathtt{w}^*)^{2(1-\alpha)}}{2^{1-\alpha}K(1-\alpha)} \tag{3.11}$$

for constants $K, \mathcal{T} > 0$ and $\alpha \in (0.5, 1)$, where $H(t) = \int_{t_0}^{t} \phi(s)\, ds$.

Proof. We take the Lyapunov function $\mathcal{L}$: $\mathbb{R}^k \to \mathbb{R}$ as:

$$\mathcal{L}(w) := \frac{1}{2}\|w - \mathtt{w}^*\|^2. \tag{3.12}$$

Now, differentiating (3.12), we obtain

$$\dot{\mathcal{L}}(w) = \langle w - \mathtt{w}^*, \dot{w}\rangle.$$

Thus, from any $w(0) \in \mathbb{R}^k \setminus \{\mathtt{w}^*\}$, where $\mathtt{w}^* \in Zer(\Pi)$ is unique with $\Pi(\cdot) = \Gamma(\cdot) - \mathcal{P}_{\Phi(\cdot)}(\Gamma(\cdot) - \eta(\cdot))$, we obtain

$$\begin{aligned}\dot{\mathcal{L}} &= -\left\langle w - \mathtt{w}^*, \phi(t)\frac{\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)}{\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\|^{\frac{\varsigma-2}{\varsigma-1}}}\right\rangle \\ &= -\phi(t)\frac{\langle w - \mathtt{w}^*,\, \Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\rangle}{\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\|^{\frac{\varsigma-2}{\varsigma-1}}}.\end{aligned} \tag{3.13}$$

By part (3) of Lemma 2.6, it follows that

$$\dot{\mathcal{L}} \leq -\phi(t)(\zeta - \theta)\frac{\|w - \mathtt{w}^*\|^2}{\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\|^{\frac{\varsigma-2}{\varsigma-1}}}. \tag{3.14}$$

By part (2) of Lemma 2.6, we obtain

$$\begin{aligned}\dot{\mathcal{L}} &\leq -\phi(t)(\zeta - \theta)(\zeta - \theta)^{\frac{\varsigma-2}{\varsigma-1}}\frac{\|w - \mathtt{w}^*\|^2}{\|w - \mathtt{w}^*\|^{\frac{\varsigma-2}{\varsigma-1}}} \\ &= -\phi(t)(\zeta - \theta)^{\frac{1}{\varsigma-1}}\|w - \mathtt{w}^*\|^{\frac{\varsigma}{\varsigma-1}}.\end{aligned} \tag{3.15}$$

Set

$$N := (\zeta - \theta)^{\frac{1}{\varsigma-1}},$$

where

$$\theta = \sqrt{(L^2 + \eta^2 - 2\eta\zeta)} + \mu.$$

Then, condition $\theta < \zeta$ implies that $N > 0$, and Eqs (3.12) and (3.15) can be written as

$$\begin{aligned} \dot{\mathcal{L}} &\leq -N\phi(t)\|w - \mathtt{w}^*\|^{\frac{\varsigma}{\varsigma-1}} \\ &\leq -N\phi(t)\, 2^{\frac{\varsigma}{2(\varsigma-1)}} \left(\frac{1}{2}\|w - \mathtt{w}^*\|^2\right)^{\frac{\varsigma}{2}\frac{1}{\varsigma-1}} && (3.16) \\ &= -N\phi(t)\, 2^{\frac{\varsigma}{2(\varsigma-1)}} \mathcal{L}^{\frac{\varsigma}{2}\frac{1}{\varsigma-1}}. && (3.17) \end{aligned}$$

Applying Lemma 3.1 and observing that

$$0.5 < \alpha := \frac{\varsigma}{2}\frac{1}{\varsigma - 1} < 1 \quad \text{for all } \varsigma > 2, \text{ with } K := N \cdot 2^{\alpha} > 0,$$

the conclusion follows. The proof is complete. □

Remark 3.1. *By discretizing the continuous-time model* (3.6) *via the forward Euler method, we arrive at the following discrete scheme:*

$$w_{k+1} = w_k - h_k\, \Theta(w_k, t) \left(\Gamma(w_k) - \mathcal{P}_{\Phi(w)}(\Gamma(w_k) - \eta w_k)\right), \tag{3.18}$$

where $h_k > 0$ denotes the iteration-dependent step size, i.e., $h_k = h(k) > 0$, and initialize $w_0 \in \mathbb{R}^k$. The scaling factor $\Theta(w_k, t)$ is defined as

$$\Theta(w_k, t) = \frac{\phi(t)}{\left\|\Gamma(w_k) - \mathcal{P}_{\Phi(w)}(\Gamma(w_k) - \eta w_k)\right\|^{\frac{\varsigma-2}{\varsigma-1}}},$$

where $\varsigma > 2$ and $\phi(t)$: $[t_0, +\infty) \to \mathbb{R}_{++}$ is continuous. For $k = 0, 1, 2, \ldots$. This discrete scheme provides an implementable iterative form of the proposed continuous-time model with a variable step size.

4. PdT stability analysis

Here, we are analyzing the PdT stability of solutions of IQVI problem (1.7). We will introduce another projection TVCNDM for the problem (1.7). Next, we describe the solutions of IQVI problems in terms of the equilibrium states of the proposed dynamical system. In addition, we prove the PdT convergence of the suggested TVCNDM.

4.1. The PdT-TVCNDM for IQVI problems

Inspired by research on continuous-time neurodynamic optimization methods for PdT convergence. We propose the following PdT-TVCNDM:

$$\dot{w} = -\aleph(w, t)\frac{\mathcal{G}_p}{\mathcal{T}_p}(\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)), \tag{4.1}$$

where

$$\aleph(w,t) := \begin{cases} \dfrac{\phi_1(t)}{\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\|^{1-\beta_1}} \\ +\dfrac{\phi_2(t)}{\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\|^{1-\beta_2}}, \text{ if } w \in \mathbb{R}^k \setminus \mathrm{Zer}(\Pi), \\ 0, \quad \text{otherwise}, \end{cases} \tag{4.2}$$

with $0 < \beta_1 < 1, \beta_2 > 1, \mathcal{T}_p > 0$, $\mathcal{G}_p > 0, \Pi(\cdot) = \mathcal{P}_{\Phi(w)}(\Gamma(\cdot) - \mu(\cdot)) - \Gamma(\cdot)$, and $\phi_1(t), \phi_2(t)$: $[t_0, +\infty) \to \mathbb{R}_{++}$ are continuous functions. The PdT-TVCNDM (4.1) allows greater flexibility through its time-varying coefficients. Moreover, note that PdT-TVCNDM (4.1) can be written in an equivalent form as follows:

$$\dot{w} = -\frac{\mathcal{G}_p}{\mathcal{T}_p}\phi_1(t)\kappa(w)\Big\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\Big\|^{\beta_1} \times \left(1 + \frac{\phi_2(t)}{\phi_1(t)}\Big\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\Big\|^{\beta_2-\beta_1}\right), \tag{4.3}$$

where

$$\kappa(w) := \frac{\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)}{\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\|}.$$

Remark 4.1. *(1) In the PdT-TVCNDM* (4.1)*, the term* $-\kappa(w)$ *can be interpreted as the search direction, while*

$$\eta_t := -\frac{\mathcal{G}_p}{\mathcal{T}_p}\phi_1(t)\Big\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\Big\|^{\beta_1} \times \left(1 + \frac{\phi_2(t)}{\phi_1(t)}\Big\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\Big\|^{\beta_2-\beta_1}\right)$$

represents the stepsize at the current iterate $w = w(t)$*. Once the functions* $\phi_1(t), \phi_2(t)$ *and* β_1, β_2 *are specified, the stepsize* η_t *is automatically updated with respect to* t*. Therefore, PdT-TVCNDM* (4.1) *may also be viewed as a PdT-TVCNDM with a self-adaptive dynamical step size.*

(2) By taking $\phi_2(t) = 1$*, PdT-TVCNDM* (4.1) *is reduces to*

$$\dot{w} = -\frac{\mathcal{G}_p}{\mathcal{T}_p}\phi_1(t)\kappa(w)\Big\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\Big\|^{\beta_1} \times \left(1 + \frac{1}{\phi_1(t)}\Big\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\Big\|^{\beta_2-\beta_1}\right).$$

(3) If $\phi_2(t) = 0$, $\mathcal{G}_p = \mathcal{T}_P = 1$*, and* $\phi_1(t) = \sigma$ *is a constant function, then the model* (4.1) *reduces to the nominal neurodynamic model studied in [17], namely*

$$\dot{w}(t) = -\sigma(\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)), \tag{4.4}$$

where $\sigma > 0$ *is a constant. This reduced model was analyzed in [17] in the context of stability for the IQVI problem* (1.7)

The next result establishes the equivalence between the equilibrium points of the proposed PdT neurodynamic model and those of the nominal TVNDM (2.12).

Proposition 4.1. *A point* $\mathtt{w}^* \in \mathbb{R}^k$ *is an equilibrium point of model* (4.1) *if and only if it is an equilibrium point of TVNDM* (2.12)*.*

Proof. The conclusion follows from the definition of the gain function $\theta(\cdot, t)$ in (4.2). Indeed,

$$\aleph(\mathtt{w}^*, t) = 0 \quad \Longleftrightarrow \quad \Gamma(\mathtt{w}^*) = \mathcal{P}_{\Phi(\mathtt{w}^*)}(\Gamma(\mathtt{w}^*) - \eta\mathtt{w}^*),$$

which coincides with the equilibrium condition of TVNDM (2.12). □

Combining Proposition 4.1 with Lemma 2.4, we obtain the following characterization of solutions of the IQVI problem (1.7).

Corollary 4.1. *Suppose Assumption 2.3 holds. Then,* $\mathtt{w}^*$ *solves the IQVI problem* (1.7) *if and only if it is an equilibrium point of the PdT-TVCNDM* (4.1).

Proof. The statement follows immediately from Proposition 4.1 and Lemma 2.4. □

The following lemma gives conditions so that solutions of the given neurodynamic model exist and are uniquely determined in the classical sense.

Lemma 4.1. *[29] Let* Π*:* $\mathbb{R}^k \to \mathbb{R}^k$ *be a locally Lipschitz continuous vector field if*

$$\Pi(\tilde{w}) = 0\ ;\ \langle w - \tilde{w}, \Pi(w)\rangle > 0$$

for all $w \in \mathbb{R}^k \setminus \{\tilde{w}\}$*, where* $\tilde{w} \in \mathbb{R}^k$ *is a solution (like* $\tilde{w} = \mathtt{w}^*$*).*

We now show that solutions to the PdT-TVCNDM (4.1) exist and are uniquely determined.

The following result guarantees well-posedness of the proposed PdT-TVCNDM.

Proposition 4.2. *Suppose that Assumption 2.3 holds. Then, model* (4.1) *has a unique solution in a classical sense.*

Proof. By Lemma 2.5, the IQVI problem (1.7) has a unique solution. It follows that the set $\mathrm{Zer}(\Pi)$ is a singleton, where $\Pi(w) \equiv \Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)$. As a result, the vector field defined in (2.12) admits a unique equilibrium point given by $\tilde{w} = \mathtt{w}^*$. By Lemma 2.6, part (3), the vector field $\Pi(w) \equiv \Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)$ satisfies the following property:

$$\langle w - \tilde{w}, \Pi(w)\rangle > 0$$

for all $w \in \mathbb{R}^k \backslash \{\tilde{w}\}$, where $\{\tilde{w}\} = \mathrm{Zer}(\Pi)$. Also, by Lemma 2.7, the operator $\Pi(\cdot) = \mathcal{P}_{\Phi(w)}(\Gamma(\cdot) - \mu(\cdot)) - \Gamma(\cdot)$ is Lipschitz continuous. This implies Lipschitz continuity on $\mathbb{R}^k$ of the vector field on the right-hand side of (2.12). Finally, the conclusion follows from Lemma 4.1. □

4.2. PdT convergence

The following theorem presents the main result of this section. Suppose $\mathtt{w}^* \in \mathbb{R}^k$ denotes the unique equilibrium of PdT-TVCNDM (4.1). Then, the equilibrium $\mathtt{w}^*$, which coincides with the solution of the IQVI problem (1.7), is PdT stable.

Theorem 4.3. *Suppose Assumption 2.3 is satisfied. Furthermore, suppose that*

$$\frac{\mathcal{T}_p}{m_2 \mathcal{G}_p(\vartheta_2 - 1)} < \int_{t_0}^{+\infty} \phi_2(z)\, dz$$

and

$$\mathcal{F}_1 \circ \mathcal{F}_2^{-1}\left(\frac{\mathcal{T}_p}{m_2 \mathcal{G}_p(\vartheta_2 - 1)}\right) + \frac{\mathcal{T}_p}{m_1 \mathcal{G}_p(1 - \vartheta_1)} < \int_{t_0}^{+\infty} \phi_1(z)\, dz,$$

where $\vartheta_1 = \frac{1+\beta_1}{2}$, $\vartheta_2 = \frac{1+\beta_2}{2}$, $m_1 = 2^{\vartheta_1}\frac{(\zeta-\theta)}{(L+\theta)^{1-\beta_1}}$, *and* $m_2 = 2^{\vartheta_2}(\zeta-\theta)^{\beta_2}$ *for* $\theta < \zeta$ *with*

$$\mathcal{F}_1(t) = \int_0^t \phi_1(z)\,dz, \qquad \mathcal{F}_2(t) = \int_0^t \phi_2(z)\,dz.$$

Then, the state $w(t)$ *of* (4.1) *convergent within* $\mathcal{T}_p$*, the convergence time satisfies as follows:*

$$\mathcal{T} \le \mathcal{F}_1^{-1}\left(\mathcal{F}_1 \circ \mathcal{F}_2^{-1}\left(\frac{\mathcal{T}_p}{\mathcal{G}_p m_2(\vartheta_2-1)}\right) + \frac{\mathcal{T}_p}{\mathcal{G}_p m_1(1-\vartheta_1)}\right) = \mathcal{T}_p. \tag{4.5}$$

Proof. By Proposition 4.2, a solution of PdT-TVCNDM (4.1) admits a uniquely determined solution for all forward times. Next, we take the Lyapunov function:

$$\mathcal{L}(t,w) = \frac{1}{2}\|w-\mathtt{w}^*\|^2. \tag{4.6}$$

Then, $\mathcal{L}(w) = 0 \iff w = \mathtt{w}^*$ and $\mathcal{L}(w) \to \infty$ as $\|w\| \to \infty$. Taking the time derivative of (4.6) along the trajectories of (4.1), the Lyapunov function (4.6) is considered for any initial point $w(0) \in \mathbb{R}^k \setminus \{\mathtt{w}^*\}$. Since $\mathtt{w}^*$ is the unique element in $\mathrm{Zer}(\Gamma(\cdot) - P_{\mathfrak{D}}(\Gamma(\cdot) - \mu\,\mathrm{id}(\cdot)))$, it follows that:

$$\begin{aligned}
\dot{\mathcal{L}}(t,w) &= \langle w-\mathtt{w}^*, \dot{w}\rangle \\
&= -\frac{\mathcal{G}_p}{\mathcal{T}_p}\phi_1(t)\frac{1}{\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w)-\eta w)\|^{1-\beta_1}}\langle w-\mathtt{w}^*, \Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w)-\eta w)\rangle \\
&\quad + \frac{\mathcal{G}_p}{\mathcal{T}_p}\phi_2(t)\frac{1}{\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w)-\eta w)\|^{1-\beta_2}}\langle w-\mathtt{w}^*, \Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w)-\eta w)\rangle.
\end{aligned}$$

From Lemma 2.6, we obtain

$$\langle w-\mathtt{w}^*, \Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w)-\eta w)\rangle \ge (\zeta-\theta)\|w-\mathtt{w}^*\|^2$$

and

$$(\zeta-\theta)\|w-\mathtt{w}^*\| \le \|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w)-\eta w)\| \le (L+\theta)\|w-\mathtt{w}^*\|.$$

Thus,

$$\begin{aligned}
\dot{\mathcal{L}}(t,w(t)) &\le -\frac{\mathcal{G}_p}{\mathcal{T}_p}(\zeta-\theta)\phi_1(t)\frac{\|w-\mathtt{w}^*\|^2}{(L+\theta)^{1-\beta_1}\|w-\mathtt{w}^*\|^{1-\beta_1}} - \frac{\mathcal{G}_p}{\mathcal{T}_p}(\zeta-\theta)\phi_2(t)\frac{\|w-\mathtt{w}^*\|^2}{(\zeta-\theta)^{1-\beta_2}\|w-\mathtt{w}^*\|^{1-\beta_2}} \\
&= -\frac{\mathcal{G}_p}{\mathcal{T}_p}\frac{(\zeta-\theta)}{(L+\theta)^{1-\beta_1}}\phi_1(t)\|w-\mathtt{w}^*\|^{1+\beta_1} - \frac{\mathcal{G}_p}{\mathcal{T}_p}(\zeta-\theta)^{\beta_2}\phi_2(t)\|w-\mathtt{w}^*\|^{1+\beta_2}.
\end{aligned} \tag{4.7}$$

Therefore, from Eqs (4.6) and (4.7), we obtain

$$\dot{\mathcal{L}}(t,w(t)) \le -\frac{\mathcal{G}_p}{\mathcal{T}_p}2^{\frac{1+\beta_1}{2}}\frac{(\zeta-\theta)}{(L+\theta)^{1-\beta_1}}\phi_1(t)\mathcal{L}^{\frac{1+\beta_1}{2}} - \frac{\mathcal{G}_p}{\mathcal{T}_p}2^{\frac{1+\beta_2}{2}}(\zeta-\theta)^{\beta_2}\phi_2(t)\mathcal{L}^{\frac{1+\beta_2}{2}}. \tag{4.8}$$

Let $\vartheta_1 = \frac{1+\beta_1}{2}$, $\vartheta_2 = \frac{1+\beta_2}{2}$, $m_1 = 2^{\vartheta_1}\frac{(\zeta-\theta)}{(L+\theta)^{1-\beta_1}}$, and $m_2 = 2^{\vartheta_2}(\zeta-\theta)^{\beta_2}$. Then, inequality (4.8), becomes

$$\dot{\mathcal{L}} \le -\frac{\mathcal{G}_p}{\mathcal{T}_p}(m_1\phi_1(t)\mathcal{L}^{\vartheta_1} - m_2\phi_2(t)\mathcal{L}^{\vartheta_2}).$$

Applying Lemma 2.3, we define

$$\mathcal{F}_1(t) = \int_{t_0}^{t} \phi_1(z)\,dz, \quad \mathcal{F}_2(t) = \int_{t_0}^{t} \phi_2(z)\,dz,$$

then TVCDM (4.1) achieve PdT stability, where $\mathcal{T}_p$ and $\mathcal{G}_p$ satisfy

$$\mathcal{T} \leq \mathcal{F}_1^{-1}\left(\mathcal{F}_1 \circ \mathcal{F}_2^{-1}\left(\frac{\mathcal{T}_p}{\mathcal{G}_p m_2(\vartheta_2 - 1)}\right) + \frac{\mathcal{T}_p}{\mathcal{G}_p m_1(1 - \vartheta_1)}\right) = \mathcal{T}_p.$$

This completes the proof. □

We now present several representative cases of the model (4.1), obtained through different choices of time-varying coefficient functions. Consequently, various forms of the $\mathcal{G}_p$ are derived corresponding to these coefficient selections.

Corollary 4.2. *Let the Assumption* 2.3 *holds. Without loss of generality, we assume that* $t_0 = 0$. *When* $\varpi_0 > 0$, $\varrho_0 > 0$, $\delta > 0$, *and* $\tau > 0$, *three specific coefficients* $\mathcal{G}_p$ *can be derived, where* $\Theta_1 = \frac{1+\beta_1}{2}$, $\Theta_2 = \frac{1+\beta_2}{2}$, $m_1 = 2^{\Theta_1}\frac{(1-\Lambda)}{(1+\Lambda)^{1-\beta_1}}$, *and* $m_2 = 2^{\Theta_2}(1-\Lambda)^{\beta_2}$.

(1) If one selects $\phi_1(t) = \varpi_0$ *and* $\phi_2(t) = \varrho_0$ *in PdT-TVCNDM* (4.1), *then the state* $w(t)$ *converges to* $\mathtt{w}^*$ *in the frequency domain, with*

$$\mathcal{G}_p = \frac{1}{\varrho_0 m_2(\Lambda - 1)} + \frac{1}{\varpi_0 m_1(1 - \delta)}. \tag{4.9}$$

(2) If one selects $\phi_1(t) = \frac{\varpi_0}{t+\tau}$ *and* $\phi_2(t) = \frac{\varrho_0}{t+\tau}$ *in PdT-TVCNDM* (4.1), *then* $w(t)$ *converges to* $\mathtt{w}^*$ *in the frequency domain, with*

$$\mathcal{G}_p = \frac{\mathcal{T}_p}{\ln\left(\frac{\tau + \mathcal{T}_p}{\tau}\right)}\left(\frac{1}{\varrho_0 m_2(\Theta_2 - 1)} + \frac{1}{\varpi_0 m_1(1 - \Theta_1)}\right). \tag{4.10}$$

(3) If one selects $\phi_1(t) = \dfrac{\varpi_0}{t^2+\tau}$ *and* $\phi_2(t) = \dfrac{\varrho_0}{t^2+\tau}$ *in PdT-TVCNDM* (4.1), *then* $w(t)$ *converges to* $\mathtt{w}^*$ *in the frequency domain, with*

$$\mathcal{G}_p = \frac{\mathcal{T}_p\left(\frac{\sqrt{\tau}}{\varrho_0 m_2(\Theta_2 - 1)} + \frac{\sqrt{\tau}}{\varpi_0 m_1(1-\Theta_1)}\right)}{\arctan\left(\frac{\mathcal{T}_p}{\sqrt{\tau}}\right)}. \tag{4.11}$$

(4) If one selects $\phi_1(t) = \varpi_0$ *and* $\phi_2(t) = \frac{\varrho_0}{t+\tau}$ *in PdT-TVCNDM* (4.1), *then* $w(t)$ *converges to* $\mathtt{w}^*$ *in the frequency domain, with settling time bounded by*

$$T \leq \tau \exp\left(\frac{1}{\varrho_0 m_2(\vartheta_2 - 1)}\right) + \frac{1}{\varpi_0 m_1(1 - \vartheta_1)} - \tau. \tag{4.12}$$

As highlighted in Corollary 4.2, the derived settling-time bounds depend on the exponents β_1 and β_2, the operator parameters θ, ζ, and the Lipschitz constant L, along with the step size μ and the time-varying gain functions $\phi_1(t)$ and $\phi_2(t)$. Therefore, for fixed values of β_1, β_2, θ, ζ, L, and μ, the convergence-time bound can be flexibly controlled through an appropriate selection of the time-varying coefficients $\phi_1(t)$ and $\phi_2(t)$.

Remark 4.2. *By discretizing the continuous-time model* (4.1) *via the forward Euler method, we arrive at the following discrete scheme:*

$$w_{k+1} = w_k - h_k\, \aleph(w_k, t)\frac{\mathcal{G}_p}{\mathcal{T}_p}\left(\Gamma(w_k) - \mathcal{P}_{\Phi(w)}(\Gamma(w_k) - \eta w_k)\right), \tag{4.13}$$

where $h_k > 0$ *denotes the iteration-dependent step size, i.e.,* $h_k = h(k) > 0$ *and initialize* $w_0 \in \mathbb{R}^k$*. The scaling factor* $\aleph(w_k, t)$ *is defined as*

$$\aleph(w_k, t) := \begin{cases} \dfrac{\phi_1(t)}{\|\Gamma(w_k) - \mathcal{P}_{\Phi(w_k)}(\Gamma(w_k) - \eta w_k)\|^{1-\beta_1}} \\ + \dfrac{\phi_2(t)}{\|\Gamma(w_k) - \mathcal{P}_{\Phi(w_k)}(\Gamma(w_k) - \eta w_k)\|^{1-\beta_2}}, & \text{if } w_k \in \mathbb{R}^k \setminus \mathrm{Zer}(\Pi), \\ 0, & \text{otherwise}, \end{cases} \tag{4.14}$$

with $0 < \beta_1 < 1$*,* $\beta_2 > 1$ $\Pi(\cdot) = \mathcal{P}_{\Phi(w)}(\Gamma(\cdot) - \mu(\cdot)) - \Gamma(\cdot)$*.* $\phi_1(t), \phi_2(t)$*:* $[t_0, +\infty) \to \mathbb{R}_{++}$ *are continuous functions. For* $k = 0, 1, 2, \ldots$*, this discrete scheme provides an implementable iterative form of the proposed continuous-time model with a variable step size.*

5. Robust PdT analysis

In this subsection, we investigate the robustness of the proposed PdT converging PdT-TVCNDM, when it is subject to an external disturbance term $\mathcal{D}$: $\mathbb{R}^k \to \mathbb{R}^k$. Specifically, we consider the scenario where PdT convergence remains valid when PdT-TVCNDM (4.1) is exposed to interference:

$$\begin{aligned} \dot{w} = &-\frac{\mathcal{G}_p}{\mathcal{T}_p}\left(\frac{\phi_1(t)}{\left\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\right\|^{1-\beta_1}} + \frac{\phi_2(t)}{\left\|\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\right\|^{1-\beta_2}}\right) \\ &\times \left(\Gamma(w) - \mathcal{P}_{\Phi(w)}(\Gamma(w) - \eta w)\right) + \mathcal{D}(w), \end{aligned} \tag{5.1}$$

where $\mathcal{D}(w)$ represents the interference affecting PdT-TVCNDM (4.1).

Assumption 5.1. *For some constant* $q > 0$ *such that for any* $w \neq \mathtt{w}^* \in \mathbb{R}^k$*,*

$$\|\mathcal{D}(w)\| < q\|w - \mathtt{w}^*\|.$$

Assumption 5.2. *The coefficient* $\mathcal{G}_p$ *and parameter* $\mathcal{T}_p$ *satisfy*

$$\mathcal{T}_p > \mathcal{G}_p.$$

It follows from Assumption 5.1 that $\mathtt{w}^*$ is an equilibrium point of model (5.1). The following result shows that the PdT convergent model (5.1) is robust with respect to external disturbances.

Theorem 5.1. *Let Assumptions 2.3 and 5.2 hold. Further, suppose that*

$$\min\left\{\frac{(\zeta-\theta)}{(L+\theta)^{1-\beta_1}}\phi_1(t),\ (\zeta-\theta)^{\beta_2}\phi_2(t)\right\} > q,\ \frac{\mathcal{T}_p}{m_2\mathcal{G}_p(\vartheta_2-1)} < \int_{t_0}^{+\infty}\phi_2(z)\,dz$$

and

$$\mathcal{F}_1\circ\mathcal{F}_2^{-1}\left(\frac{\mathcal{T}_p}{m_2\mathcal{G}_p(\vartheta_2-1)}\right)+\frac{\mathcal{T}_p}{m_1\mathcal{G}_p(1-\vartheta_1)} < \int_{t_0}^{+\infty}\phi_1(z)\,dz,$$

where $\vartheta_i=\frac{1+\lambda_i}{2},\ i=1,2,\ m_1=2^{\vartheta_1}\frac{(\zeta-\theta)}{(L+\theta)^{1-\beta_1}}$, *and* $m_2=2^{\vartheta_2}(\zeta-\theta)^{\beta_2}$ *for* $\theta<\zeta$ *with*

$$\mathcal{F}_1(t)=\int_0^t\left(\frac{(\zeta-\theta)}{(L+\theta)^{1-\beta_1}}\phi_1(z)-q\right)dz, \mathcal{F}_2(t)=\int_0^t\left((\zeta-\theta)^{\beta_2}\phi_2(z)-q\right)dz.$$

Under Assumption 5.1, the trajectory $w(t)$ of model (5.1) *converges to* $\mathtt{w}^*$ *within* $\mathcal{T}_p$.

Proof. Consider the Lyapunov function introduced in (4.6). Taking its time derivative and using Assumption 5.1, we obtain

$$\begin{aligned}\dot{\mathcal{L}}(t,w)&=\langle w-\mathtt{w}^*,\dot{w}\rangle\\&=-\frac{\mathcal{G}_p}{\mathcal{T}_p}\phi_1(t)\frac{1}{\|\Gamma(w)-\mathcal{P}_{\Phi(w)}(\Gamma(w)-\eta w)\|^{1-\beta_1}}\langle w-\mathtt{w}^*,\Gamma(w)-\mathcal{P}_{\Phi(w)}(\Gamma(w)-\eta w)\rangle\\&\quad+\frac{\mathcal{G}_p}{\mathcal{T}_p}\phi_2(t)\frac{1}{\|\Gamma(w)-\mathcal{P}_{\Phi(w)}(\Gamma(w)-\eta w)\|^{1-\beta_2}}\langle w-\mathtt{w}^*,\Gamma(w)-\mathcal{P}_{\Phi(w)}(\Gamma(w)-\eta w)\rangle\\&\quad+q\|w-\mathtt{w}^*\|^2.\end{aligned}\tag{5.2}$$

From Lemma 2.6, we obtain

$$\langle w-\mathtt{w}^*,\Gamma(w)-\mathcal{P}_{\Phi(w)}(\Gamma(w)-\eta w)\rangle\geq(\zeta-\theta)\|w-\mathtt{w}^*\|^2$$

and

$$(\zeta-\theta)\|w-\mathtt{w}^*\|\leq\|\Gamma(w)-\mathcal{P}_{\Phi(w)}(\Gamma(w)-\eta w)\|\leq(L+\theta)\|w-\mathtt{w}^*\|.$$

Thus,

$$\begin{aligned}\dot{\mathcal{L}}(t,w(t))&\leq-\frac{\mathcal{G}_p}{\mathcal{T}_p}(\zeta-\theta)\phi_1(t)\frac{\|w-\mathtt{w}^*\|^2}{(L+\theta)^{1-\beta_1}\|w-\mathtt{w}^*\|^{1-\beta_1}}\\&\quad-\frac{\mathcal{G}_p}{\mathcal{T}_p}(\zeta-\theta)\phi_2(t)\frac{\|w-\mathtt{w}^*\|^2}{(\zeta-\theta)^{1-\beta_2}\|w-\mathtt{w}^*\|^{1-\beta_2}}+q\|w-\mathtt{w}^*\|^2\\&=-\frac{\mathcal{G}_p}{\mathcal{T}_p}\frac{(\zeta-\theta)}{(L+\theta)^{1-\beta_1}}\phi_1(t)\|w-\mathtt{w}^*\|^{1+\beta_1}-\frac{\mathcal{G}_p}{\mathcal{T}_p}(\zeta-\theta)^{\beta_2}\phi_2(t)\|w-\mathtt{w}^*\|^{1+\beta_2}+q\|w-\mathtt{w}^*\|^2.\end{aligned}\tag{5.3}$$

We define $\Xi:=\{w\mid\|w-\mathtt{w}^*\|\leq1\}$. We consider two cases.

Case 1: $w\in\Xi$. Since $0<\beta_1<1$ and $\|w-\mathtt{w}^*\|\leq1$, consequently,

$$\|w-\mathtt{w}^*\|^2-\|w-\mathtt{w}^*\|^{1+\beta_1}\leq0.$$

Thus,

$$\dot{\mathcal{L}}(t, w(t)) \leq -N_1\|w - \mathtt{w}^*\|^{1+\beta_1} - N_2\|w - \mathtt{w}^*\|^{1+\beta_2}, \tag{5.4}$$

where $N_1 = \frac{(\zeta-\theta)\phi_1(t)}{(L+\theta)^{1-\beta_1}} - q$ and $N_2 = (\zeta - \theta)^{\beta_2}\phi_2(t) - q$.

Case 2: $w \notin \Xi$. Since $\|w - \mathtt{w}^*\| > 1$ and $\beta_2 > 1$, one has

$$\|w - \mathtt{w}^*\|^2 - \|w - \mathtt{w}^*\|^{1+\beta_2} \leq 0,$$

and inequality (5.4) remains valid. Combining both cases gives

$$\dot{\mathcal{L}}(t, w(t)) \leq -\left(\frac{\mathcal{G}_p}{\mathcal{T}_p}\frac{(\zeta-\theta)}{(L+\theta)^{1-\beta_1}}\phi_1(t) - q\right)\|w - \mathtt{w}^*\|^{1+\beta_1} - \left(\frac{\mathcal{G}_p}{\mathcal{T}_p}(\zeta-\theta)^{\beta_2}\phi_2(t) - q\right)\|w - \mathtt{w}^*\|^{1+\beta_2}.$$

Rewriting in terms of $\mathcal{L}$, we obtain

$$\dot{\mathcal{L}}(t, w(t)) \leq -2^{\frac{1+\beta_1}{2}}\mathcal{F}_1(t)\mathcal{L}^{\frac{1+\beta_1}{2}} - 2^{\frac{1+\beta_2}{2}}\mathcal{F}_2(t)\mathcal{L}^{\frac{1+\beta_2}{2}},$$

where

$$\mathcal{F}_1 = \frac{(\zeta-\theta)}{(L+\theta)^{1-\beta_1}}\phi_1(t) - q$$

and

$$\mathcal{F}_2 = (\zeta - \theta)^{\beta_2}\phi_2(t) - q.$$

Therefore, by Lemma 2.3, TVCNDM (5.1) under noise interference still achieves PdT stability, where $\mathcal{T}_p$ and $\mathcal{G}_p$ satisfy (4.5). □

Remark 5.1. *The disturbance condition*

$$\|D(w)\| \leq q\|w - w^*\|, \quad w \in \mathbb{R}^k,$$

represents a state-dependent bounded disturbance that vanishes at the equilibrium point. This assumption is adopted to facilitate the Lyapunov-based robustness analysis. Extending the proposed framework to bounded nonvanishing disturbances will be considered in future work.

6. Numerical experiment

In this section, we illustrate the performance of the proposed FT iterative scheme (3.18) for solving implicit IQVIs. All numerical experiments were performed using MATLAB R2024a on a Dell computer equipped with an Intel Core i5 processor. Thus, we consider the mappings

$$\Gamma(w) = 3w + 2, \quad \Phi(w) = [g(w) - 1,\ g(w) + 1],$$

where

$$g(w) = \frac{1}{3 + |w|}.$$

The projection operator $\mathcal{P}_{\Phi(w)}(\cdot)$ is defined as the Euclidean projection onto the interval $\Phi(w)$.

Case 1: The parameters are chosen as $\eta = 3$, $\varsigma = 3$, and $h = 0.005$. The time-varying coefficient is initially taken as

$$\phi(t) = c_1 + c_2(1 - e^{-t}),$$

where $c_1 = 0.5$ and $c_2 = 0.5$. This choice ensures that $\phi(t)$ is positive, bounded, and smooth. The initial values are selected uniformly from the interval $[-3, 3]$. The iterative process was terminated when the stopping criterion

$$\|w^{k+1} - w^k\| < 10^{-5}$$

was satisfied.

From Table 2, Figures 1 and 2, it is observed that the proposed FT iterative scheme converges in a finite number of iterations for all initial values. However, the number of iterations varies significantly depending on the initial condition. In particular, the maximum number of iterations is $T_{\max} = 489$, whereas the minimum is $T_{\min} = 1$, resulting in a spread of 488. Moreover, the CPU execution times reported in Table 2 indicate that the computational cost increases with the number of iterations. Nevertheless, the execution time remains very small for all tested initial values, demonstrating that the proposed FT iterative scheme is computationally efficient while maintaining the desired FT convergence property.

Table 2. Number of iterations and corresponding CPU execution time required for convergence for different initial values.

Initial value w_0	Iterations (k)	CPU time (s)
-3.00	489	0.00011448
-2.57	457	0.00008152
-2.14	423	0.00007308
-1.71	385	0.00006629
-1.29	341	0.00006075
-0.86	288	0.00005072
-0.43	215	0.00003681
0.00	1	0.00000027
0.43	215	0.00004307
0.86	288	0.00006084
1.29	341	0.00006501
1.71	385	0.00005904
2.14	423	0.00006596
2.57	457	0.00006901
3.00	489	0.00007363

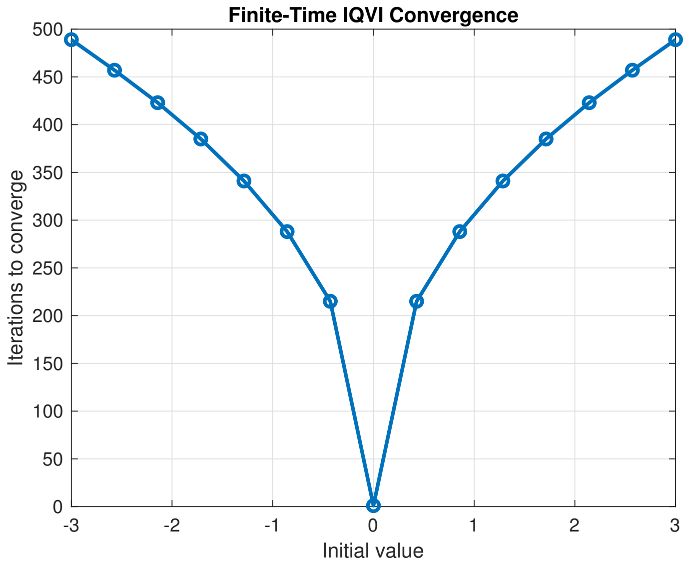


Figure 1. Number of iterations required for convergence versus initial values.

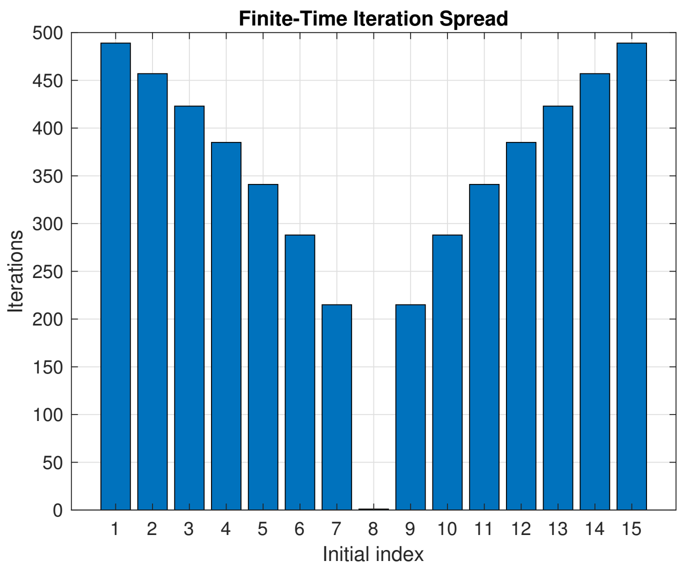


Figure 2. Bar graph of iteration counts for different initial conditions.

Case 2: We take another modified time-varying coefficient as

$$\phi(t) = c_1 + c_2 \sin(t) + t, \text{ where } c_1 = 0.5 \text{ and } c_2 = 0.5.$$

The numerical results presented in Table 3, Figures 3 and 4 demonstrate that the modified time-varying coefficient significantly improves the convergence speed while preserving the FT convergence property. In particular, the maximum number of iterations is reduced to $T_{\max} = 271$, while the minimum remains $T_{\min} = 1$, resulting in a spread of 270.

Table 3. Number of iterations and CPU time required for convergence under the modified time-varying coefficient.

Initial value w_0	Iterations (k)	CPU time (s)
-3.00	271	0.00004324
-2.57	258	0.00004049
-2.14	244	0.00004674
-1.71	227	0.00003621
-1.29	207	0.00003435
-0.86	181	0.00002744
-0.43	144	0.00002150
0.00	1	0.00000027
0.43	144	0.00002147
0.86	181	0.00002762
1.29	207	0.00003191
1.71	227	0.00003556
2.14	244	0.00003826
2.57	258	0.00004649
3.00	271	0.00006075

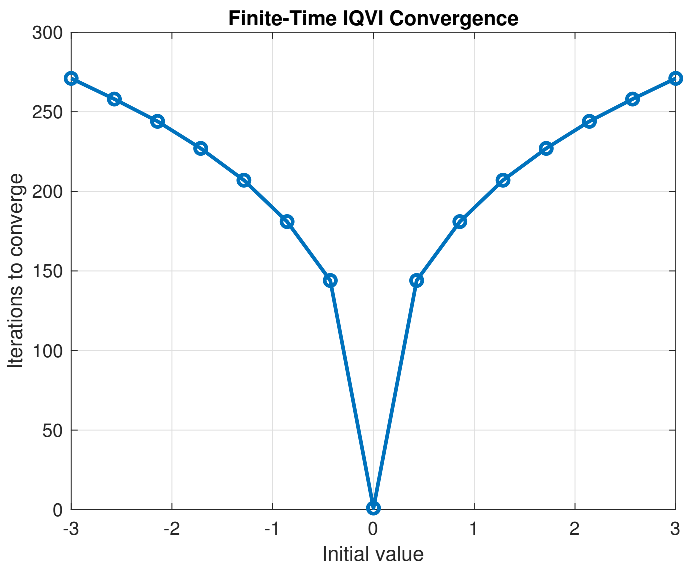


Figure 3. Iteration count versus initial values. The modified time-varying coefficient significantly reduces the convergence iterations while preserving the FT convergence property of the algorithm.

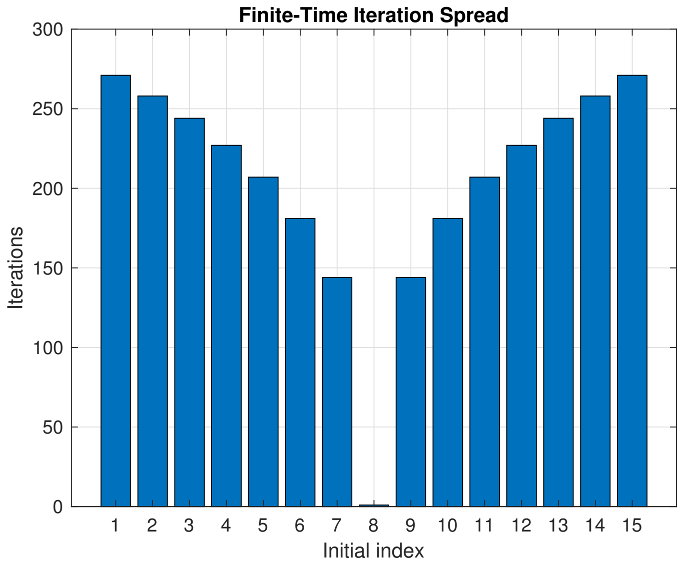


Figure 4. Bar representation of iteration counts for different initial values.

Furthermore, the CPU execution times reported in Table 3 show that the computational cost is consistent with the iteration count. Even for the largest initial values, the proposed FT iterative scheme requires only a very small execution time, indicating its computational efficiency and practical suitability for solving IQVI problems.

Table 4 presents a comparison of the computational performance of the two proposed FT iterative schemes. It can be observed that the modified time-varying coefficient employed in Case 2 significantly reduces both the maximum number of iterations and the CPU execution time compared with Case 1. These results demonstrate that the modified coefficient improves the computational efficiency of the proposed FT iterative scheme while preserving its FT convergence property.

Table 4. Comparison of the computational performance of the two proposed FT iterative schemes.

Scheme	$T_{\max}$	Average CPU time (s)	Maximum CPU time (s)
Case 1	489	0.000061	0.000114
Case 2	271	0.000034	0.000061

Therefore, the proposed algorithm exhibits FT convergence, since convergence is achieved in a finite number of iterations for all initial values, whereas the convergence time depends on the initial condition. Consequently, the algorithm does not satisfy the PdT convergence property. Overall, the numerical results and CPU execution time analysis validate the theoretical findings and further demonstrate that the modified time-varying coefficient provides superior computational performance while maintaining the effectiveness of the proposed FT iterative scheme for solving IQVI problems.

Additional numerical experiment for an IQVI problem in $\mathbb{R}^5$. The numerical simulation is carried out using the forward Euler discretization (4.13) of the proposed PdT-TVCNDM. We consider

IQVI problem with

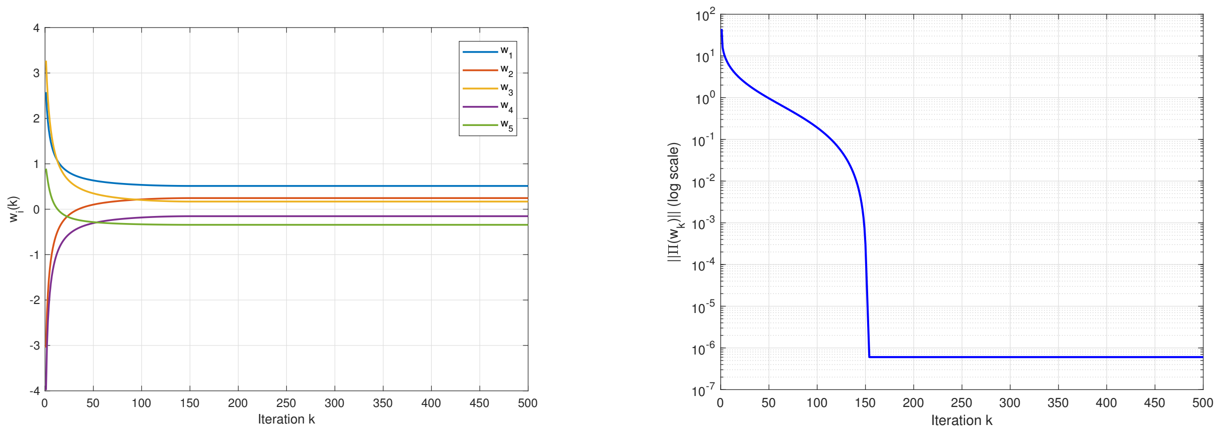
$$\Gamma(w) = Aw + c, \quad \text{where} \quad A = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}, \quad c = (-3, -2, -1, 1, 2)^T.$$

The constraint mapping is defined by

$$\Phi(w) = \left\{ v \in \mathbb{R}^5 : \|v - a(w)\| \leq \rho \right\}, \text{ where } a(w) = 0.05w.$$

The initial point is chosen as $w(0) = (8, -7, 6, -5, 9)^T$. The algorithm parameters are selected as $\beta_1 = 0.5, \beta_2 = 2, G_p = 1$, and $\mathcal{T}_p = 2$, with $\phi_1(t) = 100(1 + t), \phi_2(t) = 100\,(1 + 0.1\cos t)$. The forward Euler discretization is employed with the step size $h = 10^{-4}$, and the maximum number of iterations is set to 500.

Figure 5 presents the numerical performance of the proposed algorithm (4.13) for an additional IQVI problem in $\mathbb{R}^5$. Figure 5a shows that all five state variables converge smoothly from the prescribed initial condition to the equilibrium solution $\mathtt{w}^*$. Figure 5b illustrates that the residual $\|\Pi(w_k)\|$ decreases monotonically and reaches 6.032186×10^{-7} within 500 iterations, confirming the numerical stability and effectiveness of the proposed method for another problem instance.



(a) **(b)**

Figure 5. (a) Evolution of all state trajectories; (b) Convergence of the residual $\|\Pi(w_k)\|$ on a logarithmic scale.

Table 5 shows that the proposed algorithm (4.13) attains a highly accurate approximate solution with a residual of order 10^{-7} after only 500 iterations. These additional numerical results demonstrate that the proposed method is not restricted to the previous test problems and remains effective for another IQVI problem with a moving-center feasible set, thereby providing further evidence of its robustness and general applicability.

Table 5. Numerical performance of the proposed PdT-TVCNDM Scheme for IQVI problem.

Performance metric	Value
Problem dimension	5
Maximum iterations	500
Step size	10^{-4}
Lipschitz constant (L)	5.732051
Strong monotonicity constant (ζ)	2.267949
Radius of feasible set (ρ)	1
Parameter η	10
Parameter μ	0.1
Final residual $\|\Pi(w_k)\|$	6.032186×10^{-7}
Final equilibrium point $\mathtt{w}^*$	$(0.5126,\ 0.2453,\ 0.1690,\ -0.1536,\ -0.3436)^T$

7. Application to signal recovery

In this section, we demonstrate the performance of the proposed algorithm (4.13) from PdT-TVCNDM for sparse signal recovery in compressed sensing. Let $w \in \mathbb{R}^k$ denote an unknown sparse signal. The measurement model $\mathbf{y} = \mathbf{B}w + \boldsymbol{\varepsilon}$, where $\mathbf{B} \in \mathbb{R}^{m\times k}$ is a Gaussian sensing matrix scaled by $1/\sqrt{m}$, and $\boldsymbol{\varepsilon}$ represents additive Gaussian noise. The sparse recovery problem is formulated as $\min_{w\in\mathbb{R}^k} \frac{1}{2}\|\mathbf{B}w - \mathbf{y}\|_2^2$ subject to $\|w\|_1 \leq K$, where $K > 0$ controls the sparsity level. Define

$$\Gamma(w) = \mathbf{B}^T(\mathbf{B}w - \mathbf{y}),$$

and the projection-based operator $\mathcal{P}_\Omega(\Gamma(w) - \eta w)$, where the feasible set is

$$\Omega = \{w \in \mathbb{R}^k : \|w\|_1 \leq K\}.$$

Example 7.1. *The proposed model is given in* (4.1). *For numerical implementation, we employ an algorithm scheme* (4.13). *The sparse signal has dimension* $k = 200$ *with sparsity level* $K = 50$, *and the number of measurements is chosen as* $m = 200$. *The sensing matrix* $\mathbf{B} \in \mathbb{R}^{200\times 200}$ *is generated from a standard Gaussian distribution with entries scaled by*

$$1/\sqrt{m} = 1/\sqrt{200}.$$

The ground-truth signal w_{true} *is generated by randomly selecting* $K = 50$ *nonzero components with values in* $\{-1, 1\}$. *The measurement vector is obtained as* $\mathbf{y} = \mathbf{B}w_{\text{true}} + \boldsymbol{\varepsilon}$, *where* $\boldsymbol{\varepsilon}$ *is additive Gaussian noise with noise level* 0.01. *The algorithmic parameters are selected as* $\beta_1 = 0.5$, $\beta_2 = 2$, $\mathcal{G}_p = 1$, *and* $\mathcal{T}_p = 2$, *with time-varying coefficients* $\phi_1(t) = 100(1 + t)$, $\phi_2(t) = 100\,(1 + 0.1\cos t)$, *and projection parameter* $\eta = 0.8$. *The feasible set is*

$$\Omega = \{w \in \mathbb{R}^{200} : \|w\|_1 \leq 50\},$$

where the projection operator $P_\Omega(\cdot)$ *is computed using the exact Euclidean projection onto the* ℓ_1-*ball. The algorithm is initialized with* $w^0 = 0$, *and executed for* 2000 *iterations. The numerical* *implementation employs the step size* $h = \frac{1}{L}$, *where* $L = \|B^T B\|$ *denotes the Lipschitz constant of the gradient operator.*

The evaluation metrics are defined as

$$MSE = \frac{1}{k}\|\hat{w} - w_{true}\|_2^2, \quad RelErr = \frac{\|\hat{w} - w_{true}\|_2}{\|w_{true}\|_2}.$$

Both metrics decrease monotonically, indicating stable and efficient convergence of the proposed method. Figure 6 illustrates the performance of the proposed PdT-TVCNDM for sparse signal recovery. The left column presents the original sparse signal, the noisy measurements, and the recovered signal obtained by the proposed method. The recovered signal accurately reconstructs the support and amplitudes of the original sparse signal. The comparison plot further demonstrates the close agreement between the original and reconstructed signals. Moreover, both the mean squared error (MSE) and the relative error (RelErr) decrease rapidly and stabilize, while the MSE versus computational time indicates that the proposed method achieves accurate signal recovery with low computational cost.

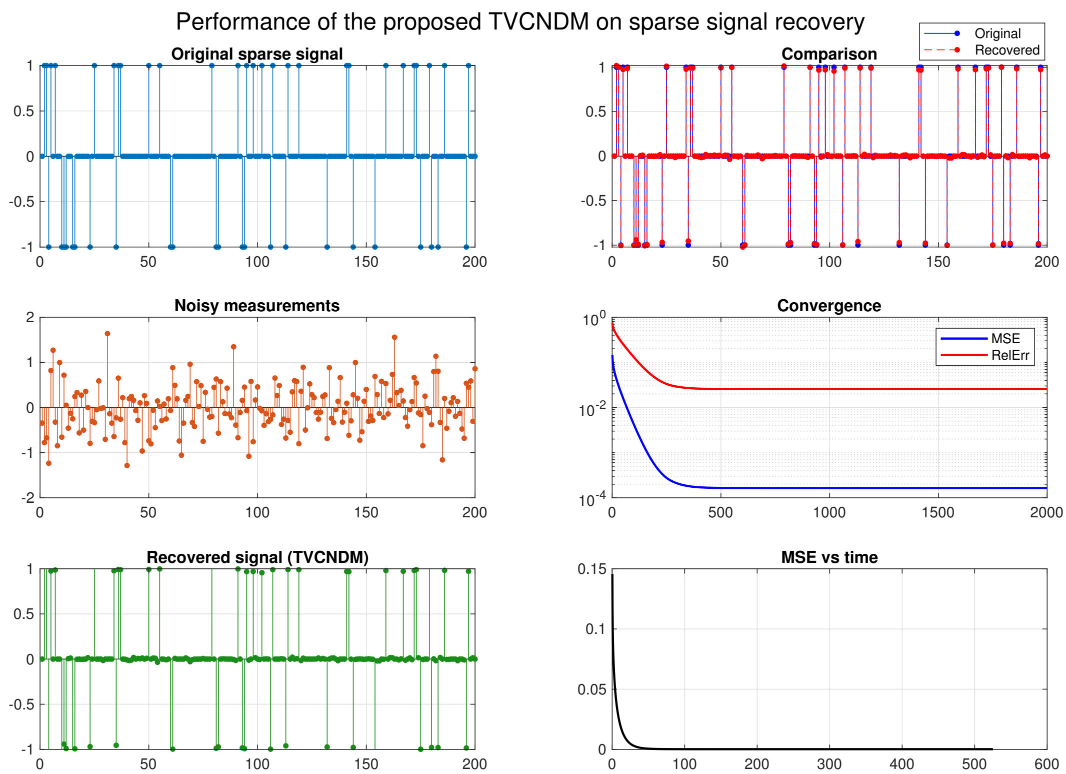


Figure 6. Performance of the proposed TVCNDM on sparse signal recovery. Left: original sparse signal, noisy measurements, and recovered signal. Right: comparison between the original and recovered signals, convergence histories of the mean squared error (MSE) and relative error (RelErr), and MSE versus computational time.

Table 6 reports the corresponding quantitative performance measures. The proposed

PdT-TVCNDM attains a final MSE of 1.646565×10^{-4} *and a final relative error of* 2.566371×10^{-2}, *confirming the high reconstruction accuracy observed in Figure 6.*

Table 6. Numerical performance of the proposed TVCNDM for sparse signal recovery.

Performance metric	Value
Final MSE	1.646565×10^{-4}
Final relative error	2.566371×10^{-2}
Recovered amplitude	$\approx \pm 1.0$
Convergence behavior	smooth and monotonic

8. Conclusions

In this paper, FT-TVCNDM and PdT-TVCNDM were proposed for solving inverse quasi-variational inequality problems. Under the assumptions of Lipschitz continuity and strong monotonicity, the existence, uniqueness, and convergence of the proposed models were established using Lyapunov theory. The FT model achieves FT convergence, whereas the PdT model guarantees convergence within a user-prescribed time independent of the initial conditions. A forward Euler discretization was developed to obtain implementable iterative algorithms, and the robustness of the PdT-TVCNDM under bounded disturbances was also established. Numerical experiments, including an additional higher-dimensional $\mathbb{R}^5$ example and a sparse signal recovery application, confirm the effectiveness, robustness, and applicability of the proposed framework.

Future work will focus on stochastic and nonconvex inverse quasi-variational inequality problems, distributed neurodynamic models, and applications to large-scale optimization and machine learning.

Author contributions

Y. Y. Xie: writing review and editing, Visualization, project administration, and funding acquisition; V. K. Khan: writing-original draft, writing-review and editing, methodology, investigation, software, conceptualization, visualization, resources; M. K. Ahmad: investigation, formal analysis, supervision; Q. B. Cai: investigation, formal analysis, supervision; All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests.

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