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Research article

Equivariant degree approach to random periodic solutions in second-order stochastic systems with symmetry

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Abstract: We study second-order stochastic systems with finite symmetry through an associated deterministic mean-field equation. This reduction provides a symmetry-preserving setting in which equivariant degree methods can be applied to the periodic problem. After reformulating the mean-field equation as a compact perturbation of the identity, we define a mean-field equivariant degree and introduce a degree-jump invariant for detecting bifurcation from the trivial branch. A nonzero degree jump yields nontrivial 2π-periodic solutions, while its Burnside-ring components provide information on possible symmetry types of the bifurcating branches. We then formulate conditional lifting principles connecting mean-field periodic solutions with random periodic solutions of the original stochastic system through measurable complete random trajectories with prescribed mean motion. A sufficient framework based on a compact invariant random set and exponential contraction is also established. Examples with dihedral symmetry illustrate the computation of the invariant and the detection of maximal orbit types.

Keywords: random periodic solutions; equivariant degree; stochastic systems; symmetry; bifurcation
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1. Introduction

Random periodic solutions describe recurrent behavior in systems affected by noise. They occur in many models, including those arising in neuroscience, mechanics, and oscillatory networks. The general framework of random dynamical systems and the stability theory of stochastic differential equations provide the mathematical foundation for studying such recurrent behavior [1,2]. More recent developments formulate random periodicity through cocycles and shifts of the underlying noise and establish existence results for stochastic systems [3, 4]. Thus, in the stochastic setting, periodicity differs from ordinary deterministic periodicity because the period is measured relative to the shift of the driving noise.

Symmetry also plays an important role in nonlinear problems. When an equation is equivariant, its solutions can be organized according to their isotropy subgroups, and bifurcating branches may possess symmetry different from that of the trivial solution. General equivariant degree theories and their applications to symmetric bifurcation problems provide the main topological framework for this analysis [5, 6]. Related methods have been applied to reversible equations and periodic solutions in systems with spatial or temporal symmetry [7, 8]. Further developments establish multiplicity and existence results for symmetric second-order differential systems [9, 10]. These studies show that equivariant degree is an effective tool for detecting periodic solutions and symmetry-breaking bifurcation.

These two viewpoints usually lead to different methods. Random periodic solutions are studied through random dynamical systems, while symmetry-breaking bifurcation is often treated by equivariant degree. In this paper, we connect them through a deterministic mean-field equation associated with the stochastic system. The degree theory is applied to this mean-field problem, which still preserves the symmetry of the original equation.

Alternative approaches to stochastic non-autonomous dynamics have also been developed. In particular, Lyapunov-type methods provide useful tools for investigating qualitative and long-term behavior in stochastic systems with time-dependent coefficients; see, for example, [11].

Recent studies continue to demonstrate broad interest in the qualitative and computational analysis of complex differential systems. For second-order and inertial models with delayed interactions, the current work has addressed periodic behavior and stability through characteristic and related analytical methods [12, 13]. In parallel, nonlinear wave dynamics have been extensively investigated through soliton structures, interaction mechanisms, and higher-dimensional coherent patterns in nonlocal and coupled nonlinear Schrödinger-type equations [14]. Data-driven techniques, including deep learning and physics-informed neural networks, have also been increasingly applied to forward and inverse problems, soliton reconstruction, and fractional dissipative wave models [15].

In this paper, we consider second-order stochastic systems of the form

$$x''(t) = f(\alpha, x(t)) + \sigma(x(t))\,\xi(t), \tag{1.1}$$

where $\xi(t)$ denotes formal white noise.

Our approach is to replace the stochastic equation by a mean-field problem. Under appropriate averaging or assumptions, this leads to a deterministic operator equation that keeps the symmetry of the original system. This mean-field equation is not intended to be pathwise equivalent to the stochastic system. Rather, it provides the deterministic setting in which equivariant degree can be used.

For this mean-field operator, we define an equivariant degree. When this degree is nonzero, the mean-field equation has nontrivial periodic solutions. We also introduce a degree-jump invariant at critical parameters. This invariant detects bifurcation from the trivial solution and gives information about the possible symmetry types of the bifurcating branches.

The contribution of this work lies in combining a symmetry-preserving mean-field reduction with equivariant degree theory and in formulating explicit conditions under which deterministic periodic bifurcation information can be lifted to random periodic dynamics of the original stochastic system.

We now introduce the mathematical setting of the problem. Let Γ be a finite group acting on $\mathbb{R}^n$ by permuting its coordinates so that $\mathbb{R}^n$ becomes an orthogonal representation of Γ. Let x: $\mathbb{R} \to \mathbb{R}^n$, σ: $\mathbb{R}^n \to \mathbb{R}^n$, and let $f(\alpha, x)$: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a C^2 function.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is the canonical Wiener space, $\mathcal{F}$ is a σ-algebra, and $\mathbb{P}$ is a probability measure. Let $W(t)$, $t \geq 0$, be an n-dimensional standard Wiener process with $W(0) = 0$, independent increments, and covariance

$$\mathbb{E}[W(t)W(s)^\top] = \min(t, s)\, I_n, \quad t, s \geq 0.$$

Interpreting $\xi(t)$ as the formal derivative of $W(t)$, i.e.,

$$\xi(t) = \frac{dW(t)}{dt},$$

system (1.1) can be written in Itô form as

$$\begin{cases} dx = v\, dt, \\ dv = f(\alpha, x(t))\, dt + \sigma(x(t))\, dW(t). \end{cases} \tag{1.2}$$

We impose the following assumptions:

(A1) $\mathbb{E}[x(t + 2\pi, \omega)] = \mathbb{E}[x(t, \omega)]$ and $\mathbb{E}[v(t + 2\pi, \omega)] = \mathbb{E}[v(t, \omega)]$.
(A2) f is Γ-equivariant.
(A3) For every $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$f(\alpha, -x) = -f(\alpha, x).$$

Thus, f is odd with respect to the state variable x.
(A4) The stationary point $(\alpha, 0)$ is nondegenerate, i.e.,

$$\det(D_x f(\alpha, 0)) \neq 0.$$

We next recall the notion of random periodic solutions. Let $(\theta_t)_{t\in\mathbb{R}}$ be the metric dynamical system on Ω defined by

$$(\theta_t\omega)(s) = \omega(s + t) - \omega(t).$$

Definition 1.1. *(Random periodic solution) Let Φ be the cocycle generated by system* (1.2). *A measurable mapping*

$$z^* : \mathbb{R} \times \Omega \to \mathbb{R}^n \times \mathbb{R}^n$$

is called a random T-periodic solution if

$$z^*(t + s, \omega) = \Phi\big(t, \theta_s\omega, z^*(s, \omega)\big)$$

for all $t \geq 0$ and $s \in \mathbb{R}$, and

$$z^*(t + T, \omega) = z^*(t, \theta_T\omega).$$

Remark 1.1. *In this paper, random periodic solutions are detected through the mean-field equation. Under suitable averaging assumptions, zeros of the mean-field operator provide candidates for random periodic solutions of the original stochastic system [3, 4].*

2. Function space setting and symmetry group structure

Since we seek 2π-periodic solutions, we work in the Sobolev space

$$X := H^2_{\text{per}}([0, 2\pi], V),$$

where $V = \mathbb{R}^n$ is equipped with its standard finite-dimensional Hilbert space structure.

The space X is equipped with the standard Sobolev inner product

$$\langle x, y\rangle_X = \int_0^{2\pi} \left(x''(t) \cdot y''(t) + x'(t) \cdot y'(t) + x(t) \cdot y(t)\right) dt, \quad x, y \in X,$$

which induces the norm

$$\|x\|_X = \left(\int_0^{2\pi} \left(|x''(t)|^2 + |x'(t)|^2 + |x(t)|^2\right) dt\right)^{1/2}.$$

Next, we describe the symmetry structure. The orthogonal group in dimension two, denoted by $O(2)$, admits the decomposition

$$O(2) = SO(2) \cup SO(2)\zeta, \quad \zeta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

For each $\tau \in \mathbb{R}$, we define the rotation matrix

$$R(\tau) = \begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix} \in SO(2).$$

The reflection and rotation satisfy the relation

$$\zeta R(\tau) = R(-\tau)\zeta,$$

which, in complex notation, can be written as

$$\zeta e^{i\tau} = e^{-i\tau}\zeta.$$

We introduce the symmetry group

$$G := \Gamma \times \mathbb{Z}_2 \times O(2).$$

Then, X becomes an orthogonal Hilbert G-representation with the action defined by

$$(\gamma, \pm 1, e^{i\tau})x(t) = \pm\gamma\, x(t + \tau),$$
$$(\gamma, \pm 1, e^{i\tau}\zeta)x(t) = \pm\gamma\, x(-t + \tau),$$

for all $x \in X$, $\gamma \in \Gamma$, and $e^{i\tau} \in SO(2)$.

Proposition 2.1. *The above action defines a well-defined orthogonal representation of G on X.*

Proof. First, translations and reflections of the time variable preserve 2π-periodicity and H^2-regularity. Hence, for every $g \in G$ and $x \in X$, $gx \in X$, the action is well defined.

Moreover, for $g_1, g_2 \in G$ and $x \in X$, the definition of the action gives

$$g_1(g_2x) = (g_1g_2)x,$$

and the identity element $e \in G$ satisfies

$$ex = x.$$

Thus, the action defines a linear representation of G on X.

It remains to verify orthogonality. Since the action of Γ on V is orthogonal and $\varepsilon^2 = 1$, we have

$$\langle \varepsilon\gamma u, \varepsilon\gamma v\rangle_V = \langle u, v\rangle_V$$

for all $u, v \in V$.

If ρ is a rotation, then the action has the form

$$(\rho x)(t) = x(t + \theta)$$

for some $\theta \in \mathbb{R}$. By periodicity and the change of variables

$$s = t + \theta,$$

we obtain

$$\langle gx, gy\rangle_X = \langle x, y\rangle_X.$$

If ρ is a reflection, then the action has the form

$$(\rho x)(t) = x(-t + \theta).$$

In this case,

$$\frac{d}{dt}x(-t + \theta) = -x'(-t + \theta)$$

and

$$\frac{d^2}{dt^2}x(-t + \theta) = x''(-t + \theta).$$

Therefore, the sign appearing in the first derivative cancels inside the inner product, and a change of variables again yields

$$\langle gx, gy\rangle_X = \langle x, y\rangle_X.$$

Consequently,

$$\langle gx, gy\rangle_X = \langle x, y\rangle_X$$

for all $g \in G$ and $x, y \in X$. Hence, G acts orthogonally on X. □

This orthogonal representation is fundamental for the subsequent symmetry analysis. In particular, for every subgroup $H \leq G$, the fixed-point space

$$X^H = \{x \in X : hx = x \text{ for all } h \in H\}$$

is a closed linear subspace of X. The isotropy subgroups and corresponding fixed-point spaces encode the symmetry types of periodic solutions and provide the representation-theoretic framework for the isotypical decomposition and the computation of the G-equivariant degree.

We denote by f_{mf} the deterministic drift obtained from the prescribed mean-field reduction of system (1.2). Throughout the paper, f_{mf} is assumed to be C^2, Γ-equivariant, and odd in the state variable. We consider the mean-field equation

$$x'' = f_{\mathrm{mf}}(\alpha, x).$$

The Γ-equivariance of f_{mf} gives the spatial symmetry Γ, while oddness yields the additional $\mathbb{Z}_2$-symmetry $x \mapsto -x$. Moreover, since the mean-field equation is autonomous and second order with 2π-periodic boundary conditions, it is invariant under the natural $O(2)$-action generated by time shifts and time reversal. Therefore, the full symmetry group of the mean-field periodic problem is

$$G = \Gamma \times \mathbb{Z}_2 \times O(2).$$

Hence, the associated mean-field periodic problem is G-equivariant.

3. Mean-field equivariant degree

Definition 3.1. *(Mean-field operator) Define the operator*

$$F : \mathbb{R} \times X \to L^2_{\mathrm{per}}([0, 2\pi], \mathbb{R}^n)$$

by

$$F(\alpha, x)(t) := x''(t) - f_{\mathrm{mf}}(\alpha, x(t)).$$

We introduce the following operators:

$$\begin{aligned} L : X \to L^2(S^1; V), &\quad Lx := x'' - x, \\ N_{f_{\mathrm{mf}}} : \mathbb{R} \times C^1(S^1; V) \to L^2(S^1; V), &\quad (N_{f_{\mathrm{mf}}}(\alpha, x))(t) := f_{\mathrm{mf}}(\alpha, x(t)), \\ i : X \to C^1(S^1; V), &\quad (ix)(t) := x(t), \\ \pi : \mathbb{R} \times X \to X, &\quad \pi(\alpha, x) := x. \end{aligned}$$

These mappings fit into a commutative diagram, illustrated in Figure 1, where $N_{f_{\mathrm{mf}}}$ is the Nemytsky operator associated with the function f_{mf}.

$$\begin{tikzcd} \mathbb{R} \times X \arrow[rr, "L \circ \pi"] \arrow[dr, "\mathrm{Id} \times i"'] & & L^2(S^1; V) \\ & \mathbb{R} \times C^1(S^1; V) \arrow[ur, "N_{f_{mf}}"'] & \end{tikzcd}$$

Figure 1. Commutative diagram arising from the reformulation of the system.

The diagram allows us to express the mean-field equation in the equivalent form

$$L(\pi(\alpha, x)) = N_{f_{\mathrm{mf}}}(\alpha, i(x)) - i(x).$$

Lemma 3.1. *The mapping L: $X \to L^2(S^1; V)$ is an isomorphism.*

For each fixed parameter α, the mean-field equation can therefore be recast as a fixed point equation in X:

$$F_{\mathrm{mf}}(\alpha, x) := x - L^{-1}(N_{f_{\mathrm{mf}}}(\alpha, i(x)) - i(x)) = 0. \tag{3.1}$$

The fixed point equation is equivalent to the mean-field boundary value problem. Indeed, applying L to the identity

$$x = L^{-1}(N_{f_{\mathrm{mf}}}(\alpha, i(x)) - i(x))$$

gives

$$x'' - x = f_{\mathrm{mf}}(\alpha, x) - x,$$

and hence

$$x'' = f_{\mathrm{mf}}(\alpha, x).$$

Thus, zeros of F_{mf} are exactly the 2π-periodic solutions of the mean-field equation.

We use this fixed point form for the degree argument. The map

$$x \mapsto L^{-1}\big(N_{f_{\mathrm{mf}}}(\alpha, i(x)) - i(x)\big)$$

is compact on bounded subsets of X, since i: $X \hookrightarrow C^1(S^1, \mathbb{R}^n)$ is compact and L^{-1} is bounded. Hence, F_{mf} is a compact perturbation of the identity.

For $u, v \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, one has

$$L(\sin(kt)u + \cos(kt)v) = -(k^2 + 1)(\sin(kt)u + \cos(kt)v).$$

The operator L commutes with the G-action, and therefore L^{-1} is also G-equivariant. Since f_{mf} is assumed to be equivariant, the operator F_{mf} is G-equivariant.

We now define the degree used below. This is the equivariant degree of the deterministic mean-field operator F_{mf}. Thus, the stochastic system enters the argument only through its mean-field reduction.

Definition 3.2. *(Mean-field equivariant degree) Let $\Omega \subset X$ be a bounded G-invariant open set such that*

$$F_{\mathrm{mf}}(\alpha, x) \neq 0 \quad \textit{for all } x \in \partial\Omega.$$

The mean-field equivariant degree *of F on Ω is defined by*

$$G\text{-}\deg^{\mathrm{mf}}(F, \Omega) := G\text{-}\deg(F_{\mathrm{mf}}, \Omega),$$

where G- deg *denotes the classical equivariant degree taking values in the Burnside ring $A(G)$.*

The mean-field equivariant degree inherits the standard properties of the equivariant degree:

(1) **Existence.** If

$$G\text{-}\deg^{\mathrm{mf}}(F, \Omega) \neq 0 \in A(G),$$

then F_{mf} has at least one zero in Ω, hence the mean-field equation admits a solution in Ω.

(2) **G-equivariance.** The degree is invariant under the group action:

$$G\text{-}\deg^{\mathrm{mf}}(F, \Omega) = G\text{-}\deg^{\mathrm{mf}}(F, g\Omega), \quad \forall g \in G.$$

(3) **Additivity.** If $\Omega = \Omega_1 \cup \Omega_2$ with disjoint G-invariant open sets such that no solutions lie on the boundary intersections, then

$$G\text{-}\deg^{\mathrm{mf}}(F, \Omega) = G\text{-}\deg^{\mathrm{mf}}(F, \Omega_1) + G\text{-}\deg^{\mathrm{mf}}(F, \Omega_2).$$

(4) **Homotopy invariance.** If $H_{\mathrm{mf}}(x, \lambda)$ is a G-equivariant homotopy of mean-field operators such that

$$H_{\mathrm{mf}}(x, \lambda) \neq 0 \quad \text{for } x \in \partial\Omega,\ \lambda \in [0, 1],$$

then

$$G\text{-}\deg^{\mathrm{mf}}(H_{\mathrm{mf}}(\cdot, 0), \Omega) = G\text{-}\deg^{\mathrm{mf}}(H_{\mathrm{mf}}(\cdot, 1), \Omega).$$

4. Linearization and computation of the degree

Define

$$A_\alpha := D_x f_{\mathrm{mf}}(\alpha, 0).$$

The linearization of the mean-field operator at $(\alpha, 0)$ is

$$D_x F_{\mathrm{mf}}(\alpha, 0) := \mathrm{Id} - L^{-1}(N_{A_\alpha} - \mathrm{Id}). \tag{4.1}$$

Since $D_x F_{\mathrm{mf}}$ is a Fredholm operator of index zero [16], invertibility is equivalent to

$$0 \notin \sigma(D_x F_{\mathrm{mf}}).$$

Since

$$L^{-1}(u_k \cos(kt) + v_k \sin(kt)) = \frac{1}{-k^2 - 1}(u_k \cos(kt) + v_k \sin(kt)),$$

we obtain the spectral representation

$$D_x F_{\mathrm{mf}} = \mathrm{Id} + \frac{1}{1 + k^2}(A_\alpha - \mathrm{Id}).$$

Hence,

$$\sigma(D_x F_{\mathrm{mf}}) = \left\{ 1 + \frac{1}{1 + k^2}(\mu(\alpha) - 1) :\ \mu(\alpha) \in \sigma(A_\alpha),\ k = 0, 1, 2, \cdots \right\}. \tag{4.2}$$

In particular, $D_x F_{\mathrm{mf}}$ is an isomorphism provided that

$$\mu(\alpha) \neq -k^2 \quad \text{for } k = 0, 1, 2, \cdots.$$

Definition 4.1. *The set of* critical parameters *is defined by*

$$C := \{(\alpha, 0) : D_x F_{\mathrm{mf}}(\alpha, 0) \textit{ is not invertible}\}.$$

Equivalently,

$$(\alpha, 0) \in C \quad \Longleftrightarrow \quad \mu(\alpha) = -k^2 \textit{ for some } k \in \mathbb{N}_0.$$

If $(\alpha, 0)$ is a bifurcation point, then $D_x F_{\mathrm{mf}}(\alpha, 0)$ is not an isomorphism.
Let $\Omega_\epsilon = \{x \in X : \|x\|_X < \epsilon\}$ with $\epsilon > 0$ sufficiently small.

Lemma 4.1. *[16] Let $\alpha \notin C$. Assume (A1)–(A4). Then, F_{mf} defines an Ω_ϵ-G-equivariant homotopy to $D_x F_{\mathrm{mf}}$.*

Consider a real orthogonal representation V of $\Gamma \times \mathbb{Z}_2$, and let

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r,$$

where each V_j is modeled on the irreducible representation $\mathcal{V}_j$.

Let $V_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} V$ denote the complexification of V. For each $k \geq 0$, let W_k be the irreducible real representation of $O(2)$ of frequency k, where rotations act by angle multiplication $k\theta$, and reflections act by complex conjugation in the standard model $W_k \cong \mathbb{C}$ as a real vector space.

Then, $V_{\mathbb{C}}$ becomes a representation of $\Gamma \times \mathbb{Z}_2 \times O(2)$ via the natural extension of the actions, and we obtain the isotypical decomposition

$$X \cong \bigoplus_{k \geq 0} \bigoplus_{j=1}^{r} V_{k,j},$$

where the component $V_{k,j} = W_k \otimes_{\mathbb{R}} V_j$ is modeled on the irreducible G-representation $\mathcal{V}_{k,j} = W_k \otimes_{\mathbb{R}} \mathcal{V}_j$.

For a non-critical point $(\alpha, 0)$, let $\mu(\alpha) \in \sigma(A_\alpha)$ be an eigenvalue of A_α, and denote its corresponding eigenspace by $E(\mu(\alpha)) \subset V$. For each isotypical component $\mathcal{V}_j$, we define

$$m_j(\mu(\alpha)) := \dim \left(E(\mu(\alpha)) \cap \mathcal{V}_j\right),$$

which represents the dimension of the part of the eigenspace lying in the $\mathcal{V}_j$-component.

Since our interest is restricted to negative eigenvalues, for each $k \geq 0$ and $\mu(\alpha) \in \sigma(A_\alpha)$, we introduce the $\mathcal{V}_{k,j}$-multiplicities by

$$\mathfrak{m}_-^{k,j}(\mu(\alpha)) := \begin{cases} \dfrac{m_j(\mu(\alpha))}{\dim\left(\mathcal{V}_{k,j}\right)}, & \text{if } \mu(\alpha) < -k^2, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

Let $B_\rho \subset X$ be a sufficiently small G-invariant ball centered at 0. Then, the equivariant degree admits the factorization

$$\deg_G(D_x F_{\mathrm{mf}}, B_\rho) = \prod_{k \geq 0} \prod_{j=1}^{r} \prod_{\mu \in \sigma(A_\alpha)} \left(\deg_{\mathcal{V}_{k,j}}\right)^{\mathfrak{m}_-^{k,j}(\mu)}. \tag{4.4}$$

5. Bifurcation invariant

Let $(\alpha_*, 0)$ be a critical point of (3.1). We say that $(\alpha_*, 0)$ is a bifurcation point if every neighborhood of this point contains a nontrivial solution different from $(\alpha_*, 0)$. To detect such a bifurcation, assume that there exists $\eta > 0$ such that $D_x F_{\mathrm{mf}}(\alpha, 0)$ is an isomorphism for all $0 < |\alpha - \alpha_*| < \eta$. We then choose two regular parameters α_*^- and α_*^+ in $(\alpha_* - \eta, \alpha_* + \eta)$ satisfying

$$\alpha_*^- < \alpha_* < \alpha_*^+.$$

We define the G-equivariant degree jump at $(\alpha_*, 0)$ by

$$\omega[\alpha_*, 0] = G\text{-}\deg\left(F_{\mathrm{mf}}(\alpha_*^+, \cdot), B_\rho\right) - G\text{-}\deg\left(F_{\mathrm{mf}}(\alpha_*^-, \cdot), B_\rho\right).$$

This element of $A(G)$ measures the change of the equivariant degree as α passes through the critical value α_*. Consequently, by (4.4), the equivariant invariant $\omega[\alpha_*, 0]$ can be computed explicitly as

$$\omega[\alpha_*, 0] = \prod_k \prod_{j=1}^r \prod_{\mu\in\sigma(A_{\alpha_*^+})} \left(\deg_{\mathcal{V}_{k,j}}\right)^{\mathfrak{m}_-^{k,j}(\mu(\alpha_*^+))} - \prod_k \prod_{j=1}^r \prod_{\mu\in\sigma(A_{\alpha_*^-})} \left(\deg_{\mathcal{V}_{k,j}}\right)^{\mathfrak{m}_-^{k,j}(\mu(\alpha_*^-))}. \tag{5.1}$$

We now extend a classical bifurcation result of Kuratowski [17] in the spirit of Krasnosel'skii.

Theorem 5.1. *Conditions* (A1)–(A4) *hold, and* $(\alpha_*, 0)$ *is an isolated critical point of* (3.1). *Suppose that* $D_xF_{\mathrm{mf}}(\alpha, 0)$ *is an isomorphism for* $0 < |\alpha - \alpha_*| < \eta$. *Let* $\alpha_*^- < \alpha_* < \alpha_*^+$ *be chosen close to* α_*. *If*

$$\omega[\alpha_*, 0] \neq 0,$$

then at least one branch of 2π*-periodic solutions of the mean-field equation*

$$x'' = f_{\mathrm{mf}}(\alpha, x)$$

bifurcates from $(\alpha_*, 0)$.

If the nonzero contribution to the degree jump is associated with a Fourier mode $k \geq 1$*, then the bifurcating branch contains nonconstant* 2π*-periodic solutions.*

Moreover, if

$$\omega[\alpha_*, 0] = n_1(H_1) + n_2(H_2) + \cdots + n_N(H_N), \quad n_j \neq 0,$$

and (H_j) *is a maximal orbit type, then the bifurcating branches contain solutions with orbit types detected by* (H_j).

Proof. Suppose, to the contrary, that $(\alpha_*, 0)$ is not a bifurcation point of the mean-field equation (3.1). Then, there exist $\delta > 0$ and $\rho > 0$ such that

$$F_{\mathrm{mf}}(\alpha, x) \neq 0$$

for every

$$\alpha \in [\alpha_* - \delta, \alpha_* + \delta]$$

and every

$$x \in \overline{B_\rho} \setminus \{0\}.$$

Since $(\alpha_*, 0)$ is an isolated critical point, we may choose regular parameters

$$\alpha_*^- < \alpha_* < \alpha_*^+$$

with

$$\alpha_*^\pm \in (\alpha_* - \delta, \alpha_* + \delta)$$

such that

$$D_xF_{\mathrm{mf}}(\alpha_*^\pm, 0)$$

are isomorphisms.

By the choice of ρ,

$$F_{\mathrm{mf}}(\alpha, x) \neq 0 \quad \text{for all } \; x \in \partial B_\rho$$

and every

$$\alpha \in [\alpha_*^-, \alpha_*^+].$$

Define

$$H : [0,1] \times \overline{B_\rho} \to X, \quad H(s,x) := F_{\mathrm{mf}}(\alpha(s), x),$$

where

$$\alpha(s) := (1-s)\alpha_*^- + s\alpha_*^+.$$

Since F_{mf} is G-equivariant and depends continuously on α, H is an admissible G-equivariant homotopy. Hence, by homotopy invariance of the equivariant degree,

$$G\text{-}\deg(F_{\mathrm{mf}}(\alpha_*^-, \cdot), B_\rho) = G\text{-}\deg(F_{\mathrm{mf}}(\alpha_*^+, \cdot), B_\rho).$$

Therefore,

$$\omega[\alpha_*, 0] = G\text{-}\deg(F_{\mathrm{mf}}(\alpha_*^+, \cdot), B_\rho) - G\text{-}\deg(F_{\mathrm{mf}}(\alpha_*^-, \cdot), B_\rho) = 0,$$

which contradicts the hypothesis

$$\omega[\alpha_*, 0] \neq 0.$$

Hence, $(\alpha_*, 0)$ is a bifurcation point of nontrivial solutions of the mean-field equation.

Finally, suppose that

$$\omega[\alpha_*, 0] = n_1(H_1) + \cdots + n_N(H_N)$$

and that $n_j \neq 0$ for a maximal orbit type (H_j). By the existence property of the equivariant degree, the nonzero coefficient of (H_j) detects nontrivial zeros whose isotropy contains a subgroup conjugate to H_j. By maximality of (H_j) among the relevant orbit types, the bifurcating set contains solutions with orbit type (H_j). If this orbit type is associated with a time-dependent Fourier component $k \geq 1$, then such solutions do not belong to the constant Fourier mode and are therefore nonconstant 2π-periodic solutions. □

6. Detection of periodic solutions

Let $\mathcal{V}_{k,j}$ be one of the isotypical components in the decomposition of the G-representation. The basic degree corresponding to $\mathcal{V}_{k,j}$ has the form

$$\deg_{\mathcal{V}_{k,j}} = (G) - n_i(H_i) + \text{other orbit-type terms},$$

where (H_i) is a maximal orbit type for this representation.

For a subgroup $H_i \subset G$, define

$$\mathcal{V}_{k,j}^{H_i} = \{x \in \mathcal{V}_{k,j} : G_x \supset H_i\}.$$

Thus, $\mathcal{V}_{k,j}^{H_i}$ is the part of $\mathcal{V}_{k,j}$ whose points have at least the symmetry (H_i). The coefficient n_i is determined by the parity of this space and by the Weyl group

$$W(H_i) = N(H_i)/H_i.$$

More precisely,

$$n_i = \begin{cases} 0, & \text{if } \dim \mathcal{V}_{k,j}^{H_i} \text{ is even,} \\ 1, & \text{if } \dim \mathcal{V}_{k,j}^{H_i} \text{ is odd and } |W(H_i)| = 2, \\ 2, & \text{if } \dim \mathcal{V}_{k,j}^{H_i} \text{ is odd and } |W(H_i)| = 1. \end{cases}$$

Therefore, an orbit type contributes to the basic degree only when the corresponding fixed-point space has odd dimension. This simple parity rule is what allows us to read symmetry information from the degree.

Since the local degree of the linearized problem is a product of basic degrees, we need to know how such orbit-type contributions behave under multiplication in $A(G)$. There are two cases.

6.1. The same orbit type appears in two factors

Assume that the maximal orbit type (H_i) appears in both $\deg_{\mathcal{V}_{k_1,j_1}}$ and $\deg_{\mathcal{V}_{k_2,j_2}}$. Then, we may write

$$\deg_{\mathcal{V}_{k_1,j_1}} = (G) - n_i(H_i) + \cdots$$

and

$$\deg_{\mathcal{V}_{k_2,j_2}} = (G) - n_l(H_i) + \cdots .$$

Multiplication in the Burnside ring gives

$$\begin{aligned} \deg_{\mathcal{V}_{k_1,j_1}} \deg_{\mathcal{V}_{k_2,j_2}} &= ((G) - n_i(H_i) + \cdots)((G) - n_l(H_i) + \cdots) \\ &= (G) - (n_i + n_l - n_i n_l |W(H_i)|)(H_i) + \cdots . \end{aligned} \tag{6.1}$$

Thus, if the same orbit type appears in more than one factor, the corresponding coefficient can cancel after multiplication. In practice, this cancellation is controlled by the parity of the fixed-point dimensions. If the two factors contribute the same parity for H_i, the coefficient may disappear. If the parities are different, then (H_i) remains with a nonzero coefficient.

6.2. The orbit type appears in only one factor

Now assume that (H_i) appears in $\deg_{\mathcal{V}_{k_1,j_1}}$, but not in $\deg_{\mathcal{V}_{k_2,j_2}}$. Then, we write

$$\deg_{\mathcal{V}_{k_1,j_1}} = (G) - n_i(H_i) + \cdots ,$$

while

$$\deg_{\mathcal{V}_{k_2,j_2}} = (G) - n_l(H_l) + \cdots , \quad H_l \neq H_i.$$

In this case,

$$\begin{aligned} \deg_{\mathcal{V}_{k_1,j_1}} \deg_{\mathcal{V}_{k_2,j_2}} &= ((G) - n_i(H_i) + \cdots)((G) - n_l(H_l) + \cdots) \\ &= (G) - n_i(H_i) - n_l(H_l) + \cdots . \end{aligned} \tag{6.2}$$

Here, the contribution of (H_i) is not canceled by the second factor, because (H_i) does not occur there. Therefore, once (H_i) appears with a nonzero coefficient in one factor, it survives in the product unless it is canceled later by another factor containing the same orbit type.

We now apply this discussion to the degree jump at a critical point:

$$\omega[\alpha_*, 0] = \deg_G(F_{\mathrm{mf}}(\alpha_*^+, \cdot), B_\rho) - \deg_G(F_{\mathrm{mf}}(\alpha_*^-, \cdot), B_\rho).$$

The two degrees in this difference are products of basic degrees corresponding to the negative directions of the linearized operator on the two sides of α_*.

For each $k \geq 0$ and $j = 1, \dots, r$, let

$$\mathcal{V}_{k,j}^{\alpha_*^-} \quad \text{and} \quad \mathcal{V}_{k,j}^{\alpha_*^+}$$

denote the isotypical components that actually occur in the degree products at α_*^- and α_*^+, respectively. For a fixed orbit type (H_i), set

$$\mathcal{V}_{k,j}^{H_i,\alpha_*^-} = \{x \in \mathcal{V}_{k,j}^{\alpha_*^-} : G_x \supset H_i\}$$

and

$$\mathcal{V}_{k,j}^{H_i,\alpha_*^+} = \{x \in \mathcal{V}_{k,j}^{\alpha_*^+} : G_x \supset H_i\}.$$

These spaces measure how much of the H_i-symmetry is present in the negative part of the linearization before and after the critical value.

The next theorem gives a direct criterion ensuring that a prescribed orbit type appears in the bifurcating set.

Theorem 6.1. *Assume that conditions (A1)–(A4) hold, and let $(\alpha_*, 0)$ be an isolated critical point of*

$$F_{\mathrm{mf}}(\alpha, x) = 0.$$

Suppose that $D_x F_{\mathrm{mf}}(\alpha, 0)$ is invertible for $0 < |\alpha - \alpha_*| < \eta$. Choose regular parameters

$$\alpha_*^- < \alpha_* < \alpha_*^+$$

sufficiently close to α_*.

Assume further that there exist an eigenvalue $\mu(\alpha_*^-) \in \sigma(A_{\alpha_*^-})$ and an integer $k \in \mathbb{N}_0$ such that

$$\mu(\alpha_*^-) < -k^2.$$

Suppose also that, for some k, j, and eigenvalue μ, at least one of the multiplicities

$$\mathfrak{m}_-^{k,j}(\mu(\alpha_*^-)), \qquad \mathfrak{m}_-^{k,j}(\mu(\alpha_*^+))$$

is odd.

Let (H_i) be a maximal orbit type appearing in one of the basic degrees $\deg_{\mathcal{V}_{k,j}}$. If exactly one of the two integers

$$\sum_{k,j} \dim \mathcal{V}_{k,j}^{H_i,\alpha_*^-}, \quad \sum_{k,j} \dim \mathcal{V}_{k,j}^{H_i,\alpha_*^+}$$

is odd, then the coefficient of (H_i) in the degree jump $\omega[\alpha_*, 0]$ is nonzero.

Consequently, a branch of nontrivial 2π-periodic solutions bifurcates from $(\alpha_*, 0)$. Moreover, the bifurcating set contains solutions whose symmetry is detected by the orbit type (H_i).

Proof. The condition

$$\mu(\alpha_*^-) < -k^2$$

shows that the linearized operator has a negative direction in the isotypical component $\mathcal{V}_{k,j}$. Moreover, the assumption that at least one of the multiplicities

$$\mathfrak{m}_-^{k,j}(\mu(\alpha_*^-)), \quad \mathfrak{m}_-^{k,j}(\mu(\alpha_*^+))$$

is odd implies that the corresponding basic degree enters the degree product with odd exponent and therefore contributes nontrivially.

Fix the maximal orbit type (H_i). By the structure of the basic degree $\deg_{\mathcal{V}_{k,j}}$, the coefficient of (H_i) is determined by the parity of the fixed-point space $\mathcal{V}_{k,j}^{H_i}$.

In particular, an odd fixed-point dimension yields a nonzero contribution of (H_i), whereas an even fixed-point dimension gives no contribution. When several basic degrees are multiplied, possible cancellations of the (H_i)-term are governed by the product formulas (6.1) and (6.2).

By assumption, exactly one of

$$\sum_{k,j} \dim \mathcal{V}_{k,j}^{H_i,\alpha_*^-} \quad \text{and} \quad \sum_{k,j} \dim \mathcal{V}_{k,j}^{H_i,\alpha_*^+}$$

is odd. Hence, the total contribution of (H_i) has different parity in the two local degree products. Therefore, the coefficients of (H_i) in

$$\deg_G(F_{\mathrm{mf}}(\alpha_*^-, \cdot), B_\rho)$$

and

$$\deg_G(F_{\mathrm{mf}}(\alpha_*^+, \cdot), B_\rho)$$

cannot coincide. Consequently, the coefficient of (H_i) in

$$\omega[\alpha_*, 0] = \deg_G(F_{\mathrm{mf}}(\alpha_*^+, \cdot), B_\rho) - \deg_G(F_{\mathrm{mf}}(\alpha_*^-, \cdot), B_\rho)$$

is nonzero. In particular,

$$\omega[\alpha_*, 0] \neq 0.$$

Theorem 5.1 therefore implies that $(\alpha_*, 0)$ is a bifurcation point of nontrivial 2π-periodic solutions of the mean-field equation. Since the coefficient of the maximal orbit type (H_i) in the degree jump is nonzero, the bifurcating set contains solutions whose symmetry is detected by (H_i). □

7. From mean-field periodic solutions to random periodic solutions

The preceding sections establish bifurcation of periodic solutions for the deterministic mean-field equation. Passing from such a periodic mean-field orbit to a random periodic solution of the original stochastic system is not automatic. The mean-field reduction retains averaged information but generally loses pathwise information about the stochastic cocycle. We therefore formulate the connection as a conditional lifting principle. The required lift consists of a measurable complete random trajectory whose expectation agrees with the prescribed mean-field periodic orbit. Its existence must be verified separately from the degree computation. One sufficient framework for the existence of complete measurable random trajectories is described later through conditions *(B1)–(B3)*.

Assume that system (1.2) generates a continuous random dynamical system cocycle

$$\Phi(t, \omega, z), \quad z \in \mathbb{R}^n \times \mathbb{R}^n,$$

over the metric dynamical system

$$(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}}),$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $(\theta_t)_{t\in\mathbb{R}}$ is a measurable family of $\mathbb{P}$-preserving transformations satisfying

$$\theta_0 = \mathrm{id}, \quad \theta_{t+s} = \theta_t \circ \theta_s$$

for all $t, s \in \mathbb{R}$.

Φ satisfies

$$\Phi(0, \omega, z) = z, \quad \Phi(t + s, \omega, z) = \Phi\big(t, \theta_s\omega, \Phi(s, \omega, z)\big)$$

for all $t, s \geq 0$, $z \in \mathbb{R}^n \times \mathbb{R}^n$, and every ω in a θ-invariant subset of Ω of full $\mathbb{P}$-measure.

A measurable map

$$z\colon \mathbb{R} \times \Omega \to \mathbb{R}^n \times \mathbb{R}^n$$

is called a complete random trajectory of Φ if there exists a θ-invariant set $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$z(t + s, \omega) = \Phi\big(t, \theta_s\omega, z(s, \omega)\big)$$

for all $t \geq 0$, $s \in \mathbb{R}$, and $\omega \in \Omega_0$.

Then, we have the following theorem.

Theorem 7.1. *Let*

$$\bar{x} \in H^2_{\mathrm{per}}([0, 2\pi], \mathbb{R}^n)$$

be a 2π-periodic solution of the mean-field equation

$$\bar{x}''(t) = f_{\mathrm{mf}}(\alpha, \bar{x}(t)).$$

Suppose that there exists a measurable complete random trajectory

$$z^*(t, \omega) = \big(x^*(t, \omega), v^*(t, \omega)\big)$$

such that $x^*(t, \cdot)$ and $v^*(t, \cdot)$ are integrable for every $t \in \mathbb{R}$, and

$$\mathbb{E}[x^*(t, \cdot)] = \bar{x}(t), \quad \mathbb{E}[v^*(t, \cdot)] = \bar{x}'(t)$$

for all $t \in \mathbb{R}$.

Assume further that any two measurable complete random trajectories satisfying these mean relations coincide for all $t \in \mathbb{R}$ on a common θ-invariant subset of Ω of full $\mathbb{P}$-measure. Then, z^* is a random 2π-periodic solution of system (1.2)*, namely,*

$$z^*(t + 2\pi, \omega) = z^*(t, \theta_{2\pi}\omega)$$

for all $t \in \mathbb{R}$ and every ω in a θ-invariant subset of Ω of full $\mathbb{P}$-measure.

If the cocycle Φ is Γ-equivariant, then for every $\gamma \in \Gamma$, the process

$$(\gamma z^*)(t, \omega) := \gamma z^*(t, \omega)$$

is also a random 2π-periodic solution, where Γ acts on the phase space diagonally by

$$\gamma(x, v) := (\gamma x, \gamma v).$$

Proof. Set

$$\widetilde{z}(t, \omega) := z^*(t + 2\pi, \theta_{-2\pi}\omega).$$

For $t \geq 0$ and $s \in \mathbb{R}$, the complete trajectory property of z^* gives

$$\begin{aligned}
\widetilde{z}(t + s, \omega) &= z^*(t + s + 2\pi, \theta_{-2\pi}\omega) \\
&= \Phi\Big(t, \theta_{s+2\pi}\theta_{-2\pi}\omega, z^*(s + 2\pi, \theta_{-2\pi}\omega)\Big) \\
&= \Phi(t, \theta_s\omega, \widetilde{z}(s, \omega)).
\end{aligned}$$

Thus, $\widetilde{z}$ is a complete random trajectory.

Since the base flow preserves $\mathbb{P}$,

$$\begin{aligned}
\mathbb{E}[\widetilde{x}(t, \cdot)] &= \mathbb{E}[x^*(t + 2\pi, \cdot)] \\
&= \bar{x}(t + 2\pi) \\
&= \bar{x}(t),
\end{aligned}$$

where the last equality follows from the 2π-periodicity of $\bar{x}$. Likewise,

$$\mathbb{E}[\widetilde{v}(t, \cdot)] = \bar{x}'(t + 2\pi) = \bar{x}'(t).$$

Hence, $\widetilde{z}$ has the same prescribed mean relations as z^*. The uniqueness assumption therefore implies

$$\widetilde{z}(t, \omega) = z^*(t, \omega)$$

for all $t \in \mathbb{R}$ on a θ-invariant subset of Ω of full $\mathbb{P}$-measure. By the definition of $\widetilde{z}$,

$$z^*(t + 2\pi, \theta_{-2\pi}\omega) = z^*(t, \omega).$$

Replacing ω with $\theta_{2\pi}\omega$ yields

$$z^*(t + 2\pi, \omega) = z^*(t, \theta_{2\pi}\omega),$$

so z^* is random 2π-periodic.

Now suppose that Φ is Γ-equivariant. For $\gamma \in \Gamma$,

$$\Phi\big(t, \theta_s\omega, \gamma z^*(s, \omega)\big) = \gamma\Phi\big(t, \theta_s\omega, z^*(s, \omega)\big) = \gamma z^*(t + s, \omega).$$

Thus, γz^* is again a complete random trajectory. Moreover,

$$\gamma z^*(t + 2\pi, \omega) = \gamma z^*(t, \theta_{2\pi}\omega),$$

and hence γz^* is also a random 2π-periodic solution. □

The next result provides a sufficient framework for establishing uniqueness and random periodicity of a prescribed complete random lift.

All the following assumptions are understood to hold on a θ-invariant subset

$$\Omega_0 \subset \Omega$$

satisfying

$$\mathbb{P}(\Omega_0) = 1.$$

(B1) There exists a measurable compact random set

$$\mathcal{A}(\omega) \subset \mathbb{R}^n \times \mathbb{R}^n$$

that is strictly invariant under the cocycle, namely,

$$\Phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t \omega), \quad t \geq 0.$$

Moreover, $\mathcal{A}$ has tempered diameter in the sense that, for every $\varepsilon > 0$,

$$e^{-\varepsilon r} \operatorname{diam} \mathcal{A}(\theta_{-r}\omega) \longrightarrow 0 \quad \text{as} \;\; r \to \infty.$$

(B2) There exists a measurable complete random trajectory

$$z(t, \omega) = \big(x(t, \omega), v(t, \omega)\big)$$

contained in the invariant random set, that is,

$$z(t, \omega) \in \mathcal{A}(\theta_t \omega), \quad t \in \mathbb{R},$$

and satisfying

$$z(t + s, \omega) = \Phi\big(t, \theta_s \omega, z(s, \omega)\big)$$

for all $t \geq 0$ and $s \in \mathbb{R}$. In addition, $x(t, \cdot)$ and $v(t, \cdot)$ are integrable for every $t \in \mathbb{R}$, and

$$\mathbb{E}[x(t, \cdot)] = \bar{x}(t), \quad \mathbb{E}[v(t, \cdot)] = \bar{x}'(t)$$

for all $t \in \mathbb{R}$.

(B3) There exist constants $C > 0$ and $\lambda > 0$ such that

$$|\Phi(t, \omega, z_1) - \Phi(t, \omega, z_2)| \leq C e^{-\lambda t} |z_1 - z_2|$$

for all $t \geq 0$, all

$$z_1, z_2 \in \mathcal{A}(\omega),$$

and every $\omega \in \Omega_0$.

Condition *(B1)* provides a compact strictly invariant random set whose pullback diameter grows at most subexponentially. Condition *(B2)* supplies the measurable complete random lift of the prescribed mean-field periodic solution required in Theorem 7.1. Condition *(B3)* imposes exponential contraction on the invariant random set. Together, conditions *(B1)* and *(B3)* imply uniqueness of complete random trajectories contained in $\mathcal{A}$.

Theorem 7.2. *Assume that system* (1.2) *generates the continuous cocycle described above, and let*

$$\bar{x} \in H^2_{\mathrm{per}}([0, 2\pi], \mathbb{R}^n)$$

be a 2π*-periodic solution of*

$$\bar{x}''(t) = f_{\mathrm{mf}}(\alpha, \bar{x}(t)).$$

Suppose that conditions (B1)–(B3) *hold. Then, there exists a unique measurable complete random trajectory*

$$z^*(t, \omega) = \big(x^*(t, \omega), v^*(t, \omega)\big)$$

contained in the invariant random set, that is,

$$z^*(t, \omega) \in \mathcal{A}(\theta_t \omega).$$

Moreover,

$$z^*(t + 2\pi, \omega) = z^*(t, \theta_{2\pi}\omega)$$

for all $t \in \mathbb{R}$ *and every* $\omega \in \Omega_0$*. Hence,* z^* *is a random* 2π*-periodic solution of system* (1.2) *whose mean phase-space trajectory is*

$$\big(\bar{x}(t), \bar{x}'(t)\big).$$

Proof. We first prove uniqueness of complete random trajectories contained in $\mathcal{A}$. Let z_1 and z_2 be two such trajectories. For $r > 0$, the complete trajectory relation gives

$$z_i(t, \omega) = \Phi\big(r, \theta_{t-r}\omega, z_i(t - r, \omega)\big), \quad i = 1, 2.$$

Since

$$z_i(t - r, \omega) \in \mathcal{A}(\theta_{t-r}\omega), \quad i = 1, 2.$$

Condition *(B3)*, applied with base point $\theta_{t-r}\omega$, yields

$$\begin{aligned} |z_1(t, \omega) - z_2(t, \omega)| &\le Ce^{-\lambda r}|z_1(t - r, \omega) - z_2(t - r, \omega)| \\ &\le Ce^{-\lambda r} \operatorname{diam} \mathcal{A}(\theta_{t-r}\omega). \end{aligned}$$

Choose ε such that

$$0 < \varepsilon < \lambda.$$

With $\theta_{t-r}\omega = \theta_{-r}(\theta_t\omega)$, condition *(B1)* then gives

$$e^{-\varepsilon r} \operatorname{diam} \mathcal{A}(\theta_{t-r}\omega) \longrightarrow 0 \quad \text{as} \quad r \to \infty.$$

Therefore,

$$\begin{aligned} Ce^{-\lambda r} \operatorname{diam} \mathcal{A}(\theta_{t-r}\omega) &= Ce^{-(\lambda-\varepsilon)r} \left(e^{-\varepsilon r} \operatorname{diam} \mathcal{A}(\theta_{t-r}\omega)\right) \\ &\longrightarrow 0 \quad \text{as } r \to \infty. \end{aligned}$$

Hence,

$$z_1(t, \omega) = z_2(t, \omega)$$

for all $t \in \mathbb{R}$ and every $\omega \in \Omega_0$.

Thus, the complete random trajectory contained in $\mathcal{A}$ is unique. Denote it by

$$z^*(t,\omega) = \big(x^*(t,\omega), v^*(t,\omega)\big).$$

Since the trajectory supplied by condition *(B2)* is contained in $\mathcal{A}$, uniqueness implies that it coincides with z^*. Consequently,

$$\mathbb{E}[x^*(t,\cdot)] = \bar{x}(t), \quad \mathbb{E}[v^*(t,\cdot)] = \bar{x}'(t)$$

for all $t \in \mathbb{R}$.

It remains to establish the random 2π-periodicity relation. Define

$$\widetilde{z}(t,\omega) := z^*(t+2\pi, \theta_{-2\pi}\omega).$$

For $t \geq 0$ and $s \in \mathbb{R}$, the complete trajectory property of z^* gives

$$\begin{aligned}
\widetilde{z}(t+s,\omega) &= z^*(t+s+2\pi, \theta_{-2\pi}\omega) \\
&= \Phi\Big(t, \theta_{s+2\pi}\theta_{-2\pi}\omega, z^*(s+2\pi, \theta_{-2\pi}\omega)\Big) \\
&= \Phi(t, \theta_s\omega, \widetilde{z}(s,\omega)).
\end{aligned}$$

Thus, $\widetilde{z}$ is a complete random trajectory.

Moreover,

$$\begin{aligned}
\widetilde{z}(t,\omega) &= z^*(t+2\pi, \theta_{-2\pi}\omega) \\
&\in \mathcal{A}(\theta_{t+2\pi}\theta_{-2\pi}\omega) \\
&= \mathcal{A}(\theta_t\omega).
\end{aligned}$$

Hence, $\widetilde{z}$ is a complete random trajectory contained in $\mathcal{A}$. By uniqueness,

$$\widetilde{z}(t,\omega) = z^*(t,\omega).$$

Therefore,

$$z^*(t+2\pi, \theta_{-2\pi}\omega) = z^*(t,\omega).$$

Replacing ω with $\theta_{2\pi}\omega$, we obtain

$$z^*(t+2\pi, \omega) = z^*(t, \theta_{2\pi}\omega).$$

Hence, z^* is a random 2π-periodic solution of system (1.2). Finally, the identities

$$\mathbb{E}[x^*(t,\cdot)] = \bar{x}(t), \quad \mathbb{E}[v^*(t,\cdot)] = \bar{x}'(t)$$

show that its mean phase-space trajectory is

$$(\bar{x}(t), \bar{x}'(t)).$$

This completes the proof. $\square$

8. Examples

8.1. Example 1

The system

$$\begin{cases} x_1'' = \alpha x_1 - 3x_1 + x_2 + x_3 - x_1^3 + 0.1x_1\xi_1(t), \\ x_2'' = \alpha x_2 + x_1 - 3x_2 + x_3 - x_2^3 + 0.1x_2\xi_2(t), \\ x_3'' = \alpha x_3 + x_1 + x_2 - 3x_3 - x_3^3 + 0.1x_3\xi_3(t), \end{cases}$$

is $D_3 \times \mathbb{Z}_2$-equivariant, the eigenvalue of the linearization at 0 is $\mu_1 = -1$ with multiplicity 1 and $\mu_2 = -4$ with multiplicity 2. And, $\alpha_1 = 0, \alpha_2 = 3$ admit critical points of (3.1).

Let

$$D_3 = \langle r, \kappa : r^3 = \kappa^2 = 1,\ \kappa r \kappa = r^{-1} \rangle,$$

where r denotes the basic rotation and κ a reflection. We identify

$$r = (r, 1), \quad \kappa = (\kappa, 1),$$

in $D_3 \times \mathbb{Z}_2$, while

$$-1 = (1, -1), \quad -r = (r, -1), \quad -\kappa = (\kappa, -1).$$

The symbols (1), (r), (κ), (-1), $(-r)$, and $(-\kappa)$ denote the corresponding conjugacy classes. The character table of $D_3 \times \mathbb{Z}_2$ is given as shown in Table 1.

Table 1. Character table of the group $D_3 \times \mathbb{Z}_2$.

	(1)	(κ)	(r)	(-1)	$(-\kappa)$	$(-r)$
$\mathcal{V}_1$	1	1	1	1	1	1
$\mathcal{V}_2$	1	−1	−1	1	1	−1
$\mathcal{V}_3$	1	−1	1	−1	1	−1
$\mathcal{V}_4$	1	1	−1	−1	1	1
$\mathcal{V}_5$	2	−2	0	0	−1	1
$\mathcal{V}_6$	2	2	0	0	−1	−1

$V = \mathcal{V}_3 \oplus \mathcal{V}_5$ is the $D_3 \times \mathbb{Z}_2$-isotypical decomposition, and the signs of $\sigma(D_x F_{\mathrm{mf}}(\alpha_1^-, 0))$ and $\sigma(D_x F_{\mathrm{mf}}(\alpha_1^+, 0))$ are given in Tables 2 and 3.

Table 2. Sign of $\mu \in \sigma(D_x F_{\mathrm{mf}}(\alpha_1^-, 0))$.

$\mu_{k,j} \in \sigma\Big(D_x F_{\mathrm{mf}}(\alpha_1^-, 0)\big\vert_{\mathcal{V}_{k,j}}\Big)$		
$k \setminus j$	3	5
0	−	−
1	−	−
2	+	−
3	+	+
⋮	⋮	⋮

Table 3. Sign of $\mu \in \sigma(D_x F_{\mathrm{mf}}(\alpha_1^+, 0))$.

$\mu_{k,j} \in \sigma\Big(D_x F_{\mathrm{mf}}(\alpha_1^+, 0)\big\vert_{\mathcal{V}_{k,j}}\Big)$		
$k \setminus j$	3	5
0	−	−
1	+	−
2	+	+
3	+	+
⋮	⋮	⋮

Then, we have the equivariant degree at $\alpha = \alpha_1$:

$$G\text{-}\deg(D_x F_{\mathrm{mf}}(\alpha_1^-, 0), B_\rho) = \deg_{\mathcal{V}_{0,3}} \cdot \deg_{\mathcal{V}_{0,5}} \cdot \deg_{\mathcal{V}_{1,3}} \cdot \deg_{\mathcal{V}_{1,5}} \cdot \deg_{\mathcal{V}_{2,5}},$$
$$G\text{-}\deg(D_x F_{\mathrm{mf}}(\alpha_1^+, 0), B_\rho) = \deg_{\mathcal{V}_{0,3}} \cdot \deg_{\mathcal{V}_{0,5}} \cdot \deg_{\mathcal{V}_{1,5}} .$$

After computing the invariant $\omega[\alpha_1, 0]$, we omit its explicit expression because of its considerable length (see the appendix). The maximal orbit types appearing with nonzero coefficients in $\omega[\alpha_1, 0]$ are

$$(D_3^p {}^{D_3}\times^{D_1} D_2), \quad (D_1^p {}^{D_1^d}\times^{D_2} D_4), \quad (D_1^p {}^{D_1}\times^{D_2} D_4), \quad (D_3^p {}^{\mathbb{Z}_1}\times^{\mathbb{Z}_2} D_{12}).$$

For the amalgamated-subgroup notation used above, we refer the reader to [18].

This result implies that the associated mean-field equation possesses at least one nontrivial periodic solution corresponding to each of these orbit types. Under the lifting assumption in Section 7, these solutions yield random periodic solutions of the stochastic system.

For the following simulations, the stochastic system is simulated in *R* using the Euler–Maruyama method on the time interval $[0, 100]$ with uniform step size

$$\Delta t = 0.001.$$

At each time step, independent Brownian increments are generated according to

$$\Delta W_{i,k} = \sqrt{\Delta t}\, \xi_{i,k}, \quad \xi_{i,k} \sim N(0, 1).$$

We numerically simulate a periodic solution bifurcating from $(\alpha_1, 0)$. The corresponding trajectories of the components x_1–x_3 are shown in Figure 2.

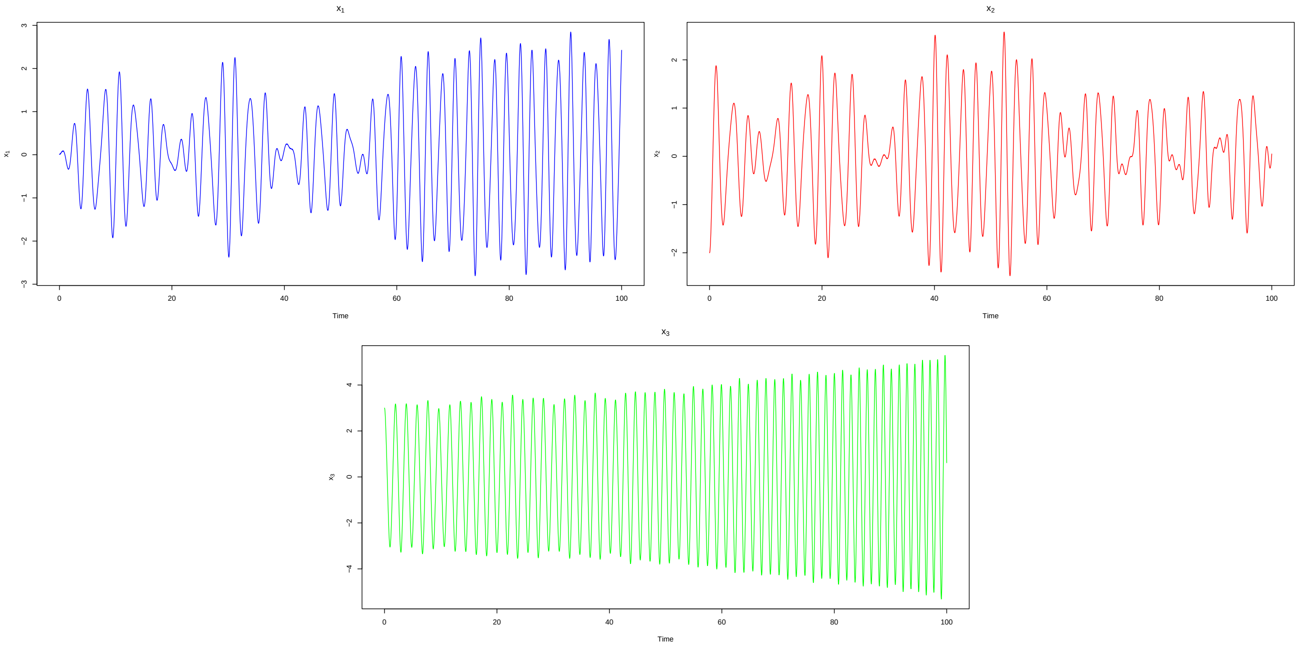

Figure 2. Periodic solution for Example 1 with $\alpha = \alpha_1$.

For $\alpha = \alpha_2$, the signs of $\sigma(D_x F_{\mathrm{mf}}(\alpha_2^-, 0))$ and $\sigma(D_x F_{\mathrm{mf}}(\alpha_2^+, 0))$ are given in Tables 4 and 5.

Table 4. Sign of $\mu \in \sigma(D_x F_{\mathrm{mf}}(\alpha_2^-, 0))$.

	$\mu_{k,j} \in \sigma\left(D_x F_{\mathrm{mf}}(\alpha_2^-, 0)\big\|_{\mathcal{V}_{k,j}}\right)$	
$k \setminus j$	3	5
0	+	−
1	+	−
2	+	+
3	+	+
$\vdots$	$\vdots$	$\vdots$

Table 5. Sign of $\mu \in \sigma(D_x F_{\mathrm{mf}}(\alpha_2^+, 0))$.

	$\mu_{k,j} \in \sigma\left(D_x F_{\mathrm{mf}}(\alpha_2^+, 0)\big\|_{\mathcal{V}_{k,j}}\right)$	
$k \setminus j$	3	5
0	+	−
1	+	+
2	+	+
3	+	+
$\vdots$	$\vdots$	$\vdots$

Then we have the equivalent degree at $\alpha = \alpha_2$:

$$G\text{-}\deg(D_x F_{\mathrm{mf}}(\alpha_2^-, 0), B_\rho) = \deg_{\mathcal{V}_{0,5}} \cdot \deg_{\mathcal{V}_{1,5}},$$
$$G\text{-}\deg(D_x F_{\mathrm{mf}}(\alpha_2^+, 0), B_\rho) = \deg_{\mathcal{V}_{0,5}}.$$

The invariant $\omega[\alpha_2, 0]$ admits the following maximal orbit types:

$$(D_1^{p\,D_1^d} \times^{D_1} D_2), \quad (D_1^{p\,D_1} \times^{D_1} D_2), \quad (D_3^{p\,\mathbb{Z}_1} \times^{\mathbb{Z}_1} D_6).$$

We simulate one periodic solution bifurcating from $(\alpha_2, 0)$ in Figure 3:

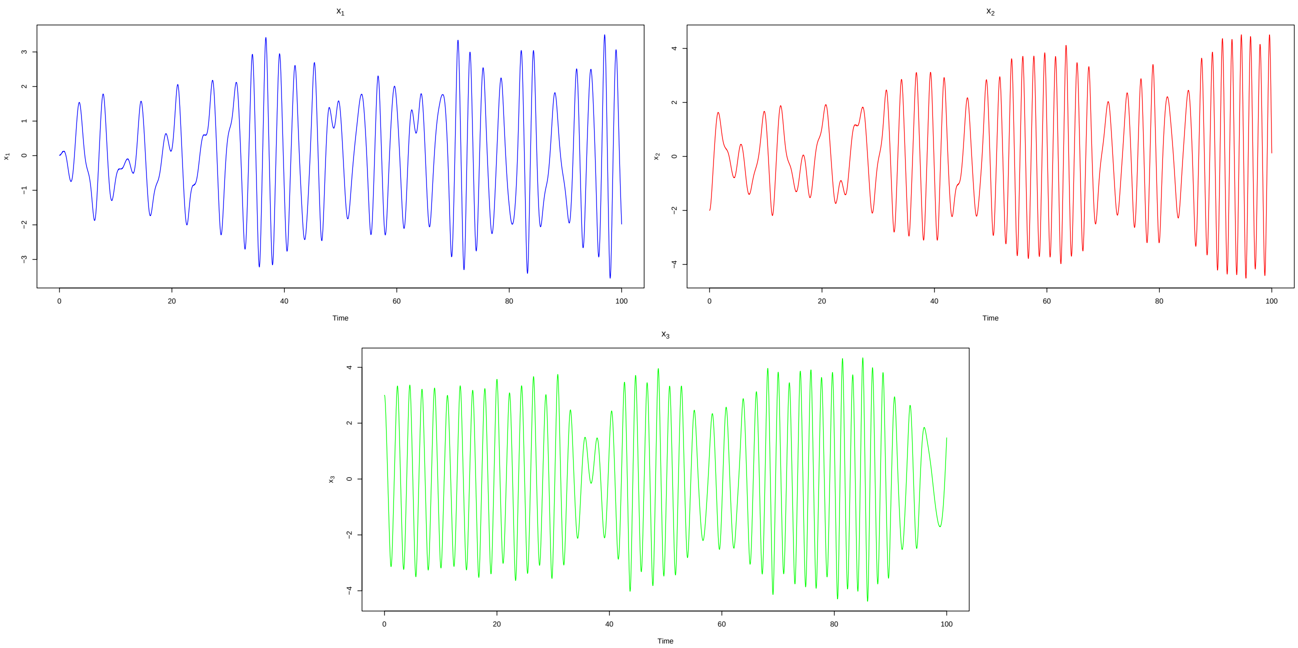

Figure 3. Periodic solution for Example 1 with $\alpha = \alpha_2$.

8.2. *Example 2*

Consider the system

$$\begin{cases} x_1'' = \alpha x_1 - 4x_1 + x_2 + x_3 + x_4 + x_5 - x_1^3 - x_1^5 + 0.1x_1\xi_1(t), \\ x_2'' = \alpha x_2 + x_1 - 4x_2 + x_3 + x_4 + x_5 - x_2^3 - x_2^5 + 0.1x_2\xi_2(t), \\ x_3'' = \alpha x_3 + x_1 + x_2 - 4x_3 + x_4 + x_5 - x_3^3 - x_3^5 + 0.1x_3\xi_3(t), \\ x_4'' = \alpha x_4 + x_1 + x_2 + x_3 - 4x_4 + x_5 - x_4^3 - x_4^5 + 0.1x_4\xi_4(t), \\ x_5'' = \alpha x_5 + x_1 + x_2 + x_3 + x_4 - 4x_5 - x_5^3 - x_5^5 + 0.1x_5\xi_5(t). \end{cases}$$

The system is $D_5 \times \mathbb{Z}_2$-equivariant. The eigenvalues of the linearization at 0 are given by $\mu_1 = -2$ with multiplicity 1,

$$\mu_2 = -4 + 2\cos\left(\frac{2\pi}{5}\right)$$

with multiplicity 2, and

$$\mu_3 = -4 + 2\cos\left(\frac{4\pi}{5}\right)$$

with multiplicity 2. Moreover, $\alpha_3 = 1$ and $\alpha_4 = -2$ yield critical points of (3.1).

Let

$$D_5 = \langle r, \kappa : r^5 = \kappa^2 = 1,\ \kappa r \kappa = r^{-1}\rangle,$$

where r denotes the basic rotation and κ a reflection. We identify

$$r = (r, 1), \quad \kappa = (\kappa, 1),$$

in $D_5 \times \mathbb{Z}_2$, and use

$$-1 = (1, -1), \quad -r = (r, -1), \quad -r^2 = (r^2, -1), \quad -\kappa = (\kappa, -1).$$

The parentheses in the table denote conjugacy classes. We also set

$$A = 2\cos\left(\frac{4\pi}{5}\right), \quad B = 2\cos\left(\frac{2\pi}{5}\right).$$

The character table of $D_5 \times \mathbb{Z}_2$ is given in the Table 6.

Table 6. Character table of $D_5 \times \mathbb{Z}_2$.

	(1)	(-1)	(κ)	$(-\kappa)$	(r)	$(-r)$	(r^2)	$(-r^2)$
$\mathcal{V}_1$	1	1	1	1	1	1	1	1
$\mathcal{V}_2$	1	−1	−1	1	1	−1	1	−1
$\mathcal{V}_3$	1	−1	1	−1	1	−1	1	−1
$\mathcal{V}_4$	1	1	−1	−1	1	1	1	1
$\mathcal{V}_5$	2	−2	0	0	A	$-A$	B	$-B$
$\mathcal{V}_6$	2	−2	0	0	B	$-B$	A	$-A$
$\mathcal{V}_7$	2	2	0	0	A	A	B	B
$\mathcal{V}_8$	2	2	0	0	B	B	A	A

V admits $D_5 \times \mathbb{Z}_2$ isotypical decomposition $V = \mathcal{V}_3 \oplus \mathcal{V}_6 \oplus \mathcal{V}_5$ and the signs of $\sigma(D_xF_{\mathrm{mf}}(\alpha_3^-, 0)), \sigma(D_xF_{\mathrm{mf}}(\alpha_3^+, 0))$ are described in Tables 7 and 8.

Table 7. Sign of $\mu \in \sigma(D_xF_{\mathrm{mf}}(\alpha_3^-, 0))$.

	$\mu_{k,j} \in \sigma(D_xF_{\mathrm{mf}}(\alpha_3^-, 0)\vert_{\mathcal{V}_{k,j}})$		
$k \setminus j$	3	6	5
0	−	−	−
1	−	−	−
2	+	+	+
3	+	+	+
⋮	⋮	⋮	⋮

Table 8. Sign of $\mu \in \sigma(D_x F_{\text{mf}}(\alpha_3^+, 0))$.

	$\mu_{k,j} \in \sigma(D_x F_{\text{mf}}(\alpha_3^+, 0)\|_{\mathcal{V}_{k,j}})$		
$k \setminus j$	3	6	5
0	−	−	−
1	+	−	−
2	+	+	+
3	+	+	+
⋮	⋮	⋮	⋮

Then, we have the equivariant degree at $\alpha = \alpha_3$:

$$G\text{-}\deg(D_x F_{\text{mf}}(\alpha_3^-, 0), B_\rho) = \deg_{\mathcal{V}_{0,3}} \cdot \deg_{\mathcal{V}_{0,6}} \cdot \deg_{\mathcal{V}_{0,5}} \cdot \deg_{\mathcal{V}_{1,3}} \cdot \deg_{\mathcal{V}_{1,6}} \cdot \deg_{\mathcal{V}_{1,5}},$$
$$G\text{-}\deg(D_x F_{\text{mf}}(\alpha_3^+, 0), B_\rho) = \deg_{\mathcal{V}_{0,3}} \cdot \deg_{\mathcal{V}_{0,6}} \cdot \deg_{\mathcal{V}_{0,5}} \cdot \deg_{\mathcal{V}_{1,6}} \cdot \deg_{\mathcal{V}_{1,5}}.$$

After computing the invariant $\omega[\alpha_3, 0]$, we omit its explicit (and rather cumbersome) expression due to space limitations. The invariant $\omega[\alpha_3, 0]$ admits one maximal orbit type $(D_2{}^{D_1}\times^{D_5} D_5^p)$, implying that the associated mean-field equation possesses at least one periodic solution corresponding to $(D_2{}^{D_1}\times^{D_5} D_5^p)$ orbit type.

We simulate one periodic solution bifurcating from $(\alpha_3, 0)$ in Figure 4.

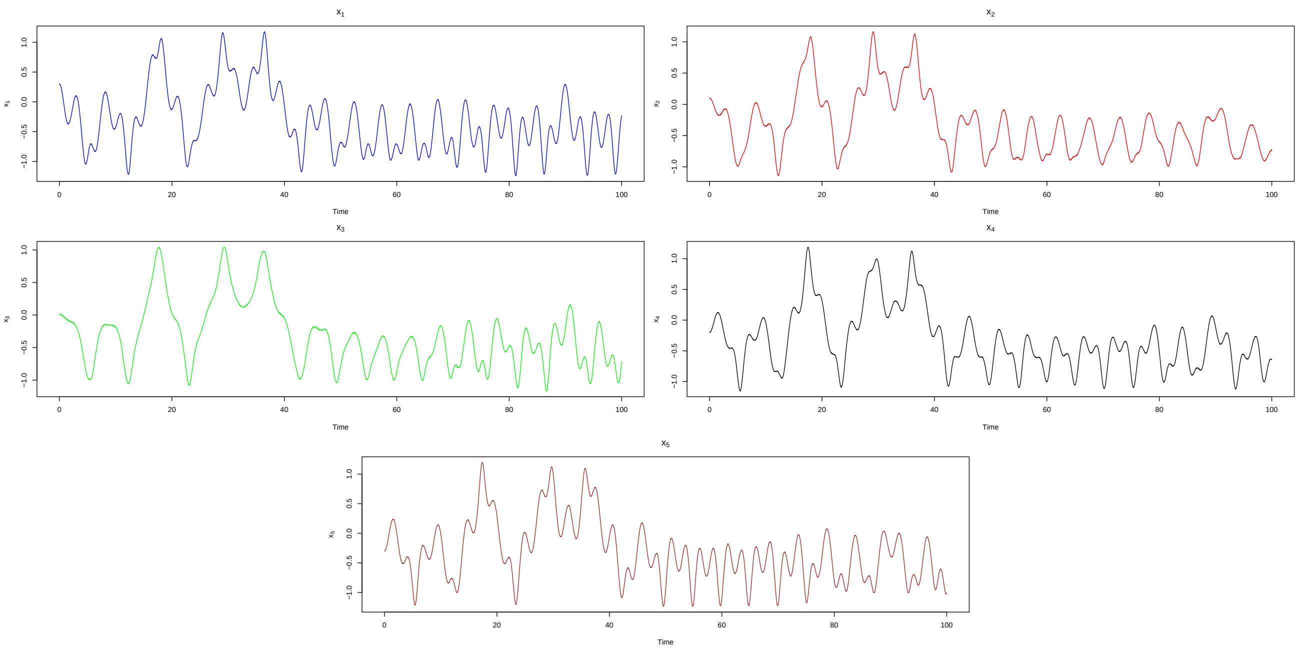

Figure 4. Periodic solution for Example 2 with $\alpha = \alpha_3$.

For $\alpha = \alpha_4$, the signs of $\sigma(D_x F_{\text{mf}}(\alpha_4^-, 0)), \sigma(D_x F_{\text{mf}}(\alpha_4^+, 0))$ are described in Tables 9 and 10.

Table 9. Sign of $\mu \in \sigma(D_x F_{\text{mf}}(\alpha_4^-, 0))$.

$\mu_{k,j} \in \sigma(D_x F_{\text{mf}}(\alpha_4^-, 0)\vert_{\mathcal{V}_{k,j}})$			
$k \setminus j$	3	6	5
0	−	−	−
1	−	−	−
2	−	−	−
3	+	+	+
⋮	⋮	⋮	⋮

Table 10. Sign of $\mu \in \sigma(D_x F_{\text{mf}}(\alpha_4^+, 0))$.

$\mu_{k,j} \in \sigma(D_x F_{\text{mf}}(\alpha_4^+, 0)\vert_{\mathcal{V}_{k,j}})$			
$k \setminus j$	3	6	5
0	−	−	−
1	−	−	−
2	+	−	−
3	+	+	+
⋮	⋮	⋮	⋮

Then, we have the equivariant degree at $\alpha = \alpha_4$:

$$G\text{-}\deg(D_x F_{\text{mf}}(\alpha_4^-, 0), B_\rho) = \deg_{\mathcal{V}_{0,3}} \cdot \deg_{\mathcal{V}_{0,6}} \cdot \deg_{\mathcal{V}_{0,5}} \cdot \deg_{\mathcal{V}_{1,3}} \cdot \deg_{\mathcal{V}_{1,6}} \cdot \deg_{\mathcal{V}_{1,5}} \cdot \deg_{\mathcal{V}_{2,3}} \cdot \deg_{\mathcal{V}_{2,6}} \cdot \deg_{\mathcal{V}_{2,5}},$$
$$G\text{-}\deg(D_x F_{\text{mf}}(\alpha_4^+, 0), B_\rho) = \deg_{\mathcal{V}_{0,3}} \cdot \deg_{\mathcal{V}_{0,6}} \cdot \deg_{\mathcal{V}_{0,5}} \cdot \deg_{\mathcal{V}_{1,3}} \cdot \deg_{\mathcal{V}_{1,6}} \cdot \deg_{\mathcal{V}_{1,5}} \cdot \deg_{\mathcal{V}_{2,6}} \cdot \deg_{\mathcal{V}_{2,5}}.$$

After computing the invariant $\omega[\alpha_4, 0]$, we omit its explicit (and rather cumbersome) expression due to space limitations. The invariant $\omega[\alpha_4, 0]$ admits one maximal orbit type $(D_4 \,{}^{D_2}\!\times^{D_5} D_5^p)$, implying that the associated mean-field equation possesses at least one periodic solution corresponding to $(D_4 \,{}^{D_2}\!\times^{D_5} D_5^p)$ orbit type.

We numerically simulate a periodic solution bifurcating from $(\alpha_4, 0)$ in Figure 5.

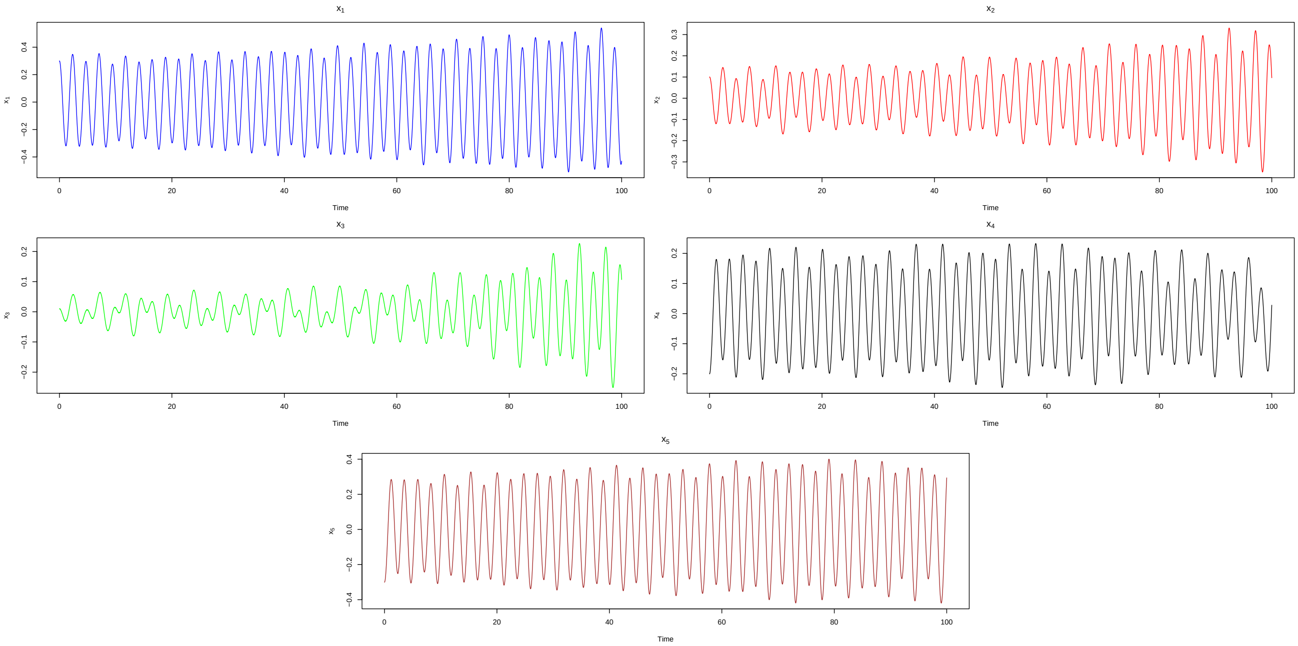

Figure 5. Periodic solution for Example 2 with $\alpha = \alpha_4$.

9. Conclusions

In this paper, we investigated second-order stochastic systems with symmetry through an associated deterministic mean-field equation. This reduction provides a setting in which equivariant degree theory can be applied to the periodic problem while retaining the relevant symmetry structure. By reformulating the mean-field equation as a compact perturbation of the identity, we introduced a mean-field equivariant degree and used it to detect nontrivial periodic solutions.

The corresponding degree jump across a critical parameter serves as a bifurcation invariant. A nonzero value yields bifurcation of nontrivial 2π-periodic solutions from the trivial branch, while the Burnside-ring components of the invariant provide information on possible orbit types of the bifurcating solutions. Thus, the approach detects both the occurrence of bifurcation and aspects of the symmetry structure of the emerging branches.

The lifting results in Theorems 7.1 and 7.2 establish a conditional connection between periodic solutions of the mean-field equation and random periodic dynamics of the original stochastic system. A measurable complete random trajectory with the prescribed mean motion provides the required lift, while uniqueness yields the random periodicity relation. In the contractive setting of Theorem 7.2, a compact invariant random set together with exponential contraction ensures uniqueness of the complete trajectory and describes a dissipative regime associated with synchronization.

The examples with dihedral symmetry illustrate how the linearized spectrum, isotypical decomposition, and Burnside-ring computations combine in concrete problems to detect bifurcation and identify possible symmetry types. These computations indicate that the mean-field equivariant-degree framework may also be useful for other stochastic or random systems in which symmetry plays a significant role.

A natural direction for future work is to extend the present framework beyond Gaussian diffusion.

In particular, Lévy-type jump perturbations may generate dynamical effects that are not captured by Brownian forcing alone; see [19]. Extending the mean-field and equivariant-degree approach to stochastic systems with jump noise would therefore be an interesting direction for further research.

Use of Generative-AI tools declaration

The authors declare he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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Appendix

Appendix A. Orbit types and fixed-point notation

We collect here the equivariant notation used throughout the paper. Let G be a compact Lie group and let $H \leq G$ be a closed subgroup. The normalizer of H is

$$N_G(H) := \left\{g \in G : gHg^{-1} = H\right\},$$

and the associated Weyl group is

$$W_G(H) := N_G(H)/H.$$

When no ambiguity is possible, we simply write $N(H)$ and $W(H)$. The conjugacy class of H is denoted by (H).

Let

$$\Phi(G) := \{(H) : H \leq G\}$$

denote the collection of conjugacy classes of closed subgroups of G. We use the standard partial order

$$(H) \leq (K)$$

whenever there exists $g \in G$ such that

$$gHg^{-1} \leq K.$$

The subcollection relevant to the Burnside ring is

$$\Phi_0(G) := \{(H) \in \Phi(G) : \dim W(H) = 0\}.$$

Suppose now that X is a G-space. For $x \in X$, its isotropy subgroup is

$$G_x := \{g \in G : gx = x\},$$

whereas its G-orbit is

$$Gx := \{gx : g \in G\}.$$

There is a natural identification

$$Gx \cong G/G_x.$$

The quotient space of all G-orbits, equipped with the quotient topology, is written as X/G.

The conjugacy class (G_x) is called the orbit type of x. Accordingly, the set of orbit types realized in X is

$$\Phi(G; X) := \{(H) \in \Phi(G) : H = G_x \text{ for some } x \in X\}.$$

For a closed subgroup $H \leq G$, we shall use

$$X_H := \{x \in X : G_x = H\}$$

and

$$X^H := \{x \in X : hx = x \text{ for every } h \in H\}.$$

Thus, X^H is the H-fixed-point set. Equivalently,

$$X^H = \{x \in X : H \leq G_x\}.$$

If X and Y are G-spaces, a map

$$f : X \to Y$$

is G-equivariant provided that

$$f(gx) = gf(x)$$

for every $g \in G$ and $x \in X$. If the action of G on Y is trivial, this condition becomes

$$f(gx) = f(x),$$

and f is called G-invariant.

For closed subgroups $L, H \leq G$, define

$$N(L, H) := \left\{g \in G : gLg^{-1} \leq H\right\}.$$

Whenever $(L), (H) \in \Phi_0(G)$, we introduce

$$n(L, H) := |N(L, H)/N(H)|.$$

This number is finite and may equivalently be interpreted as the number of subgroups conjugate to H that contain L. Further details concerning these incidence numbers and their algebraic properties can be found in [5].

Appendix B. The Burnside ring

Let G be a compact Lie group. The Burnside ring $A(G)$ is the free abelian group generated by the conjugacy classes in $\Phi_0(G)$:

$$A(G) := \mathbb{Z}[\Phi_0(G)].$$

Consequently, every $a \in A(G)$ has a unique finite expansion of the form

$$a = \sum_{i=1}^{m} a_i(H_i), \quad a_i \in \mathbb{Z}, \quad (H_i) \in \Phi_0(G).$$

The multiplicative structure is induced by diagonal G-actions on products of homogeneous spaces. More precisely, for

$$(H), (K) \in \Phi_0(G),$$

one writes

$$(H) \cdot (K) = \sum_{(L) \in \Phi_0(G)} c_L(L), \tag{B.1}$$

where c_L counts the G-orbits of type (L) occurring in

$$G/H \times G/K$$

under the action

$$g \cdot (aH, bK) := (gaH, gbK).$$

With this multiplication, $A(G)$ is a commutative ring whose identity element is (G).

The coefficients in (B.1) can be determined recursively with respect to the partial order on $\Phi_0(G)$. In the notation introduced above, the coefficient corresponding to (L) is given by

$$c_L = \frac{n(L,H)\,|W(H)|\,n(L,K)\,|W(K)| - \sum_{(\widetilde{L})>(L)} n(L,\widetilde{L})c_{\widetilde{L}}\left|W(\widetilde{L})\right|}{|W(L)|}. \tag{B.2}$$

Thus, once the coefficients associated with larger orbit types are known, the remaining coefficients are obtained successively from (B.2).

Appendix C. G-equivariant degree

Let V be a finite-dimensional orthogonal G-representation and let $\Omega \subset V$ be open, bounded, and G-invariant. A continuous G-equivariant map

$$f : V \to V$$

is said to be admissible on Ω when

$$f(x) \neq 0 \quad \text{for every} \quad x \in \partial\Omega.$$

We write $\mathcal{M}^G$ for the class of all such admissible pairs (f, Ω).

The equivariant Brouwer degree associates with each admissible pair an element of the Burnside ring. We recall the properties needed in the present paper.

Theorem C.1. *There is a unique assignment*

$$G\text{-}\deg : \mathcal{M}^G \to A(G)$$

such that, for every admissible G-pair (f, Ω)*,*

$$G\text{-}\deg(f, \Omega) = \sum_{(H)\in\Phi_0(G)} d_H(H), \tag{C.1}$$

with only finitely many nonzero coefficients $d_H \in \mathbb{Z}$*, and the following properties hold.*

(G1) ***Existence.*** *If the coefficient* d_H *in* (C.1) *is nonzero, then* f *has a zero* $x \in \Omega$ *whose isotropy satisfies*

$$(H) \le (G_x).$$

In particular,

$$G\text{-}\deg(f, \Omega) \neq 0$$

implies

$$f^{-1}(0) \cap \Omega \neq \emptyset.$$

(G2) ***Additivity.*** *Suppose that* $\Omega_1, \Omega_2 \subset \Omega$ *are disjoint open G-invariant sets and*

$$f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2.$$

Then,

$$G\text{-}\deg(f, \Omega) = G\text{-}\deg(f, \Omega_1) + G\text{-}\deg(f, \Omega_2).$$

(G3) ***Homotopy invariance.*** *Let*

$$h : [0, 1] \times V \to V$$

be a continuous G-equivariant homotopy satisfying

$$h(t, x) \neq 0$$

for all $t \in [0, 1]$ *and* $x \in \partial\Omega$*. Then*

$$G\text{-}\deg(h_t, \Omega)$$

does not depend on t.

(G4) ***Normalization.*** *If* Ω *is a bounded G-invariant neighborhood of the origin, then,*

$$G\text{-}\deg(\mathrm{Id}, \Omega) = (G).$$

(G5) ***Product property.*** *For admissible pairs*

$$(f_1, \Omega_1), \quad (f_2, \Omega_2),$$

one has

$$G\text{-}\deg(f_1 \times f_2, \Omega_1 \times \Omega_2) = G\text{-}\deg(f_1, \Omega_1) \cdot G\text{-}\deg(f_2, \Omega_2),$$

where the product on the right-hand side is taken in $A(G)$*.*

(G6) ***Suspension.*** *Let W be another orthogonal G-representation, and let $\mathscr{B} \subset W$ be a bounded invariant neighborhood of* 0*. Then,*

$$G\text{-}\deg(f \times \mathrm{Id}_W, \Omega \times \mathscr{B}) = G\text{-}\deg(f, \Omega).$$

(G7) ***Recurrence relation.*** *Let*

$$G\text{-}\deg(f, \Omega) = \sum_{(H)\in\Phi_0(G)} d_H (H).$$

Then, the coefficients can be recovered recursively from the ordinary Brouwer degrees of the restrictions to fixed-point spaces:

$$d_H = \frac{\deg(f^H, \Omega^H) - \sum_{(K)>(H)} d_K\, n(H,K)\, |W(K)|}{|W(H)|}, \tag{C.2}$$

where

$$f^H := f|_{V^H}$$

and

$$\Omega^H := \Omega \cap V^H.$$

The corresponding infinite-dimensional construction yields the G-equivariant Leray–Schauder degree for equivariant compact perturbations of the identity. This is the version used for the mean-field operator in the main body of the paper; see [5].

Appendix D. Basic degrees and linear degree factorization

Let $\mathcal{V}$ be an irreducible orthogonal G-representation and denote its open unit ball by

$$B(\mathcal{V}) := \{x \in \mathcal{V} : |x| < 1\}.$$

The basic degree associated with $\mathcal{V}$ is defined by

$$\deg_{\mathcal{V}} := G\text{-}\deg(-\mathrm{Id}, B(\mathcal{V})).$$

These elements form the elementary factors entering the degree of G-equivariant linear operators.

Suppose that V is an orthogonal G-representation with isotypical decomposition

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_r,$$

where V_i is the isotypical component corresponding to an irreducible representation $\mathcal{V}_i$. Let

$$T : V \to V$$

be a G-equivariant self-adjoint linear isomorphism. For $\mu \in \sigma(T)$, write

$$E_\mu := \mathrm{Ker}\,(T - \mu \mathrm{Id}).$$

For a negative eigenvalue $\mu < 0$, let $m_i(\mu)$ denote the multiplicity with which $\mathcal{V}_i$ occurs in the G-representation

$$E_\mu \cap V_i.$$

Since T preserves every isotypical component, the product property and the suspension property imply

$$G\text{-}\deg(T, B(V)) = \prod_{i=0}^{r} \prod_{\mu \in \sigma_-(T)} \left(\deg_{\mathcal{V}_i}\right)^{m_i(\mu)}, \tag{D.1}$$

where

$$\sigma_-(T) := \{\mu \in \sigma(T) : \mu < 0\}.$$

Only negative spectral directions occur in (D.1); positive directions contribute the multiplicative identity (G).

It remains to determine the basic degrees. Write

$$\deg_{\mathcal{V}_i} = \sum_{(H) \in \Phi_0(G)} b_H(H).$$

For every $H \le G$, the restriction of $-\mathrm{Id}$ to the fixed-point space $\mathcal{V}_i^H$ has ordinary Brouwer degree

$$\deg(-\mathrm{Id}, B(\mathcal{V}_i^H)) = (-1)^{\dim \mathcal{V}_i^H}.$$

Substitution into the recurrence formula (C.2) gives

$$b_H = \frac{(-1)^{\dim \mathcal{V}_i^H} - \sum_{(K)>(H)} b_K\, n(H,K)\, |W(K)|}{|W(H)|}. \tag{D.2}$$

Hence, every basic degree can be computed successively from the dimensions of the fixed-point spaces, the Weyl-group orders, and the incidence numbers $n(H,K)$. These are precisely the quantities used in the explicit degree computations in the examples of this paper.

Appendix E. Computed degree-jump invariants in the examples

The following degree-jump invariants were computed using GAP. Among the orbit types appearing with nonzero coefficients, those indicated in boldface are the maximal orbit types.

For the first critical value, we obtain

$$\begin{aligned}
\omega[\alpha_1, 0] =& 2(\mathbb{Z}_2{}^{\mathbb{Z}_1}\times^{\mathbb{Z}_1} D_1) - (\mathbb{Z}_1 \times D_1) - 2(D_1^{p}{}^{D_1^d}\times^{\mathbb{Z}_1} D_1) + (D_1^{d}{}^{\mathbb{Z}_1}\times^{D_1} D_2) + (D_1^d \times D_1) \\
&- (D_1{}^{\mathbb{Z}_1}\times^{D_1} D_2) + (D_1 \times D_1) + (\mathbb{Z}_2{}^{\mathbb{Z}_1}\times^{D_1} D_2) - 2(D_3^{z}{}^{\mathbb{Z}_1}\times^{\mathbb{Z}_1} D_3) + 2(D_3{}^{\mathbb{Z}_1}\times^{\mathbb{Z}_1} D_3) \\
&- 2(D_1^{p}{}^{D_1}\times^{D_1} D_2) - (D_3 \times D_1) + \boldsymbol{(D_3^{p}{}^{D_3}\times^{D_1} D_2)} + 2(D_1^{d}{}^{\mathbb{Z}_1}\times^{\mathbb{Z}_2} D_2) + 2(D_1{}^{\mathbb{Z}_1}\times^{\mathbb{Z}_2} D_2) \\
&+ (\mathbb{Z}_1 \times D_2) - 4(D_1^{p}{}^{\mathbb{Z}_1}\times^{\mathbb{Z}_2} D_4) - (D_1^{d}{}^{\mathbb{Z}_1}\times^{D_2} D_4) - (D_1^d \times D_2) - (\mathbb{Z}_2{}^{\mathbb{Z}_1}\times^{D_2} D_4) \\
&- 2(D_3{}^{\mathbb{Z}_1}\times^{\mathbb{Z}_2} D_6) + \boldsymbol{(D_1^{p}{}^{D_1^d}\times^{D_2} D_4)} + \boldsymbol{(D_1^{p}{}^{D_1}\times^{D_2} D_4)} + 2\boldsymbol{(D_3^{p}{}^{\mathbb{Z}_1}\times^{\mathbb{Z}_2} D_{12})}.
\end{aligned}$$

For the second critical value,

$$\begin{aligned}\omega[\alpha_2, 0] =& 2(D_1^{d\;\mathbb{Z}_1}\times^{\mathbb{Z}_1} D_1) + 2(D_1{}^{\mathbb{Z}_1}\times^{\mathbb{Z}_1} D_1) + 2(\mathbb{Z}_1 \times D_1) - 4(D_1^{p\;\mathbb{Z}_1}\times^{\mathbb{Z}_1} D_2) \\ &- (D_1^{d\;\mathbb{Z}_1}\times^{D_1} D_2) - (D_1^d \times D_1) - (D_1{}^{\mathbb{Z}_1}\times^{D_1} D_2) - (D_1 \times D_1) \\ &- (\mathbb{Z}_2{}^{\mathbb{Z}_1}\times^{D_1} D_2) + \boldsymbol{(D_1^{p^{D_1^d}}\times^{D_1} D_2)} + \boldsymbol{(D_1^{pD_1}\times^{D_1} D_2)} + 2\boldsymbol{(D_3^{p\mathbb{Z}_1}\times^{\mathbb{Z}_1} D_6)}.\end{aligned}$$

For the third critical value,

$$\omega[\alpha_3, 0] = (D_5 \times D_1) - \boldsymbol{(D_5^{pD_5}\times^{D_1} D_2)}.$$

Finally, for the fourth critical value,

$$\omega[\alpha_4, 0] = -2(D_5^{p\;D_5}\times^{\mathbb{Z}_1} D_1) + (D_5 \times D_1) - (D_5 \times D_2) + \boldsymbol{(D_5^{pD_5}\times^{D_2} D_4)}.$$

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