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Research article

Symmetries of blob algebras: Derivations, outer classes and cellular transfer

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Abstract: We study derivations of the blob algebra $\mathsf{B}_n(q, \delta, \kappa)$ over a field of characteristic zero and their interaction with its Graham–Lehrer cellular structure. From the defining presentation, we obtain a necessary and sufficient extension criterion that translates the Leibniz rule into explicit compatibility conditions on the generators. When the algebra is semisimple, every derivation is inner, and hence $\mathrm{Der}(\mathsf{B}_n) = \mathrm{Inn}(\mathsf{B}_n) \cong \mathsf{B}_n/Z(\mathsf{B}_n)$. For cellular derivations satisfying a fixed-right-index compatibility condition, we construct D-connections on cell modules and give criteria for radical stability and descent to simple heads. We then study the non-semisimple regime $\delta = \kappa = 0$, with q not a root of unity. A general vector-space reduction of the outer derivation space is obtained, and in the fully degenerate rank-two case $\mathsf{B}_2(q, 0, 0)$, with $q + q^{-1} \neq 0$, the outer derivation Lie algebra is computed explicitly: it is two-dimensional and non-Abelian. Finally, using the known combinatorial bijection between the canonical bases of the relevant blob and Deguchi–Martin cell modules, we separate basis-level transport from algebra-level transport. We prove descent of Hecke-algebra derivations under stability of the defining quotient ideal; under an algebra isomorphism intertwining the module actions, D-connections transport by conjugation; and, under proportionality of the corresponding cellular Gram forms, radical stability is preserved. Low-rank examples illustrate the scope and limitations of these transfer statements.

Keywords: blob algebra; derivation; cellular algebra; Temperley–Lieb algebra; Hochschild cohomology; outer derivation; Deguchi–Martin quotient; mathematical operators
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1. Introduction

The blob algebra B_n was introduced in [1, 2] as the algebraic structure encoding the transfer matrix of the two-color Potts model with particular boundary conditions. It is a finite-dimensional associative algebra over a field $\mathbb{F}$ of characteristic zero, depending on an integer $n \geq 2$, a nonzero scalar $q \in \mathbb{F}^\times$ with $q^2 \neq 1$, and two additional parameters $\delta, \kappa \in \mathbb{F}$. The algebra is generated by a family of Temperley–Lieb operators together with a distinguished boundary generator representing the "blob"; in the normalization used here this generator satisfies $e^2 = \delta e$, so it is generally a quasi-idempotent rather than an idempotent. Representation-theoretically, the blob algebra is cellular in the sense of Graham and Lehrer [3], a property that permits a detailed analysis of its simple modules, standard modules, and the associated invariant bilinear forms [4, 5].

The Lie algebra of derivations of an associative $\mathbb{F}$-algebra A, namely the $\mathbb{F}$-linear maps $d : A \to A$ obeying $d(xy) = d(x)y + x\,d(y)$, controls the first Hochschild cohomology group with coefficients in the regular bimodule through $HH^1(A, A) \cong \mathrm{Der}_{\mathbb{F}}(A)/\mathrm{Inn}(A)$. Throughout this article, the word "derivation" always means a $\mathbb{F}$-linear derivation. For a full matrix algebra over a field of characteristic zero, every derivation is inner [6, 7]. Diagram algebras also support several other homological and cohomological theories; see, for example, [8–11]. More recently, Fisher and Graves studied cohomology for families of diagram algebras that include blob algebras in an augmented-algebra setting [12]. The present paper addresses a different, derivation-theoretic problem: explicit description of $\mathrm{Der}(\mathsf{B}_n)$, the quotient $HH^1(\mathsf{B}_n, \mathsf{B}_n)$, and their interaction with cellular modules and quotient constructions.

In this context, derivations are mathematical operators that capture infinitesimal symmetries of the underlying algebraic structure.

The present work provides a comprehensive study of derivations of the blob algebra B_n. Our main results are the following.

(i) **Extension and innerness.** We give a precise criterion for a linear map defined on the generators to extend to a derivation of the whole algebra (Theorem 3.1). Whenever the blob algebra is semisimple, we prove that every derivation is inner (Theorem 3.7). Hence $\mathrm{Der}(\mathsf{B}_n) = \mathrm{Inn}(\mathsf{B}_n)$.

(ii) **Action on standard modules.** For a cellular derivation satisfying the fixed-right-index compatibility in Proposition 4.4, we construct a D-connection on the cell module $\Delta(\lambda)$. We establish the corresponding module-theoretic Leibniz rule and determine necessary and sufficient conditions for radical stability (Theorem 4.6). When the cellular form ϕ_λ is nonzero, this yields a criterion for the connection to descend to the irreducible head $L(\lambda)$ (Theorem 4.10).

(iii) **Outer derivations in the fully degenerate case.** We investigate the non-semisimple situation $\delta = \kappa = 0$ (with q not being a root of unity). A general reduction of the outer derivation space to a relative subspace is given (Theorem 5.2). For the fully degenerate rank-two case $\mathsf{B}_2(q, 0, 0)$ with $q + q^{-1} \neq 0$, the space of outer derivations $\mathrm{Out}(\mathsf{B}_2^0)$ is computed completely: It is two-dimensional and non-Abelian (Theorems 5.6 and 5.7).

(iv) **Transfer to the Deguchi–Martin quotient.** We recall the canonical basis bijection between the relevant blob and Deguchi–Martin cell modules established by Alotaibi [13]. We distinguish this vector-space correspondence from algebra isomorphism. We prove the ideal-stability criterion for Hecke-algebra derivations to descend to $H_n^{1,1}$ (Theorem 6.3); we record transport through an actual algebra isomorphism (Theorem 6.4); and, under explicit compatibility hypotheses on the module actions and cellular Gram forms, we transport D-connections and radical stability (Theorems 6.6 and 6.8).

Throughout the paper, $\mathbb{F}$ is a field of characteristic zero. The parameter q is assumed to be not a root of unity unless stated otherwise. We use the notation $[2]_q = q + q^{-1}$.

2. Preliminaries

2.1. The blob algebra

Throughout this paper, let $\mathbb{F}$ be a field of characteristic zero and let $q \in \mathbb{F}^{\times}$. We write $[2]_q = q + q^{-1}$. For an integer $n \geq 2$ and the scalars $\delta, \kappa \in \mathbb{F}$, the blob algebra $\mathsf{B}_n = \mathsf{B}_n(q, \delta, \kappa)$ is the unital associative $\mathbb{F}$-algebra generated by $U_1, \ldots, U_{n-1}, e$ subject to the following defining relations:

$$U_i^2 = [2]_q\, U_i, \qquad 1 \leq i \leq n-1, \tag{2.1}$$
$$U_i U_{i+1} U_i = U_i,\ U_{i+1} U_i U_{i+1} = U_{i+1}, \qquad 1 \leq i \leq n-2, \tag{2.2}$$
$$U_i U_j = U_j U_i, \qquad |i - j| > 1, \tag{2.3}$$
$$e^2 = \delta e, \tag{2.4}$$
$$U_1 e U_1 = \kappa U_1, \tag{2.5}$$
$$e U_i = U_i e, \qquad 2 \leq i \leq n-1. \tag{2.6}$$

The subalgebra generated by $U_1, \ldots, U_{n-1}$ is the Temperley–Lieb algebra TL_n with the loop parameter $[2]_q$. The extra generator e is called the blob generator. Algebraically, it records the interaction of diagrams with the distinguished boundary.

There is a natural $\mathbb{F}$-linear anti-involution on B_n, denoted by ι, which is determined by $\iota(U_i) = U_i, \iota(e) = e$, and extended anti-multiplicatively as follows:

$\iota(xy) = \iota(y)\iota(x)$, for all $x, y \in \mathsf{B}_n$. This is the anti-involution used in the cellular structure of the blob algebra.

2.2. Cellular algebras and cell modules

Let A be a finite-dimensional $\mathbb{F}$-algebra. Following Graham and Lehrer [3], a cellular datum for A consists of $(\Lambda, \mathcal{T}, C, \iota)$, where Λ is a finite partially ordered set, $\mathcal{T}(\lambda)$ is a finite set for each $\lambda \in \Lambda$, $\iota : A \to A$ is an $\mathbb{F}$-linear anti-involution, and $C : \coprod_{\lambda \in \Lambda} \mathcal{T}(\lambda) \times \mathcal{T}(\lambda) \longrightarrow A,\ (S, T) \longmapsto c^{\lambda}_{S,T}$, is an injective map whose image is an $\mathbb{F}$-basis of A. These data are required to satisfy $\iota(c^{\lambda}_{S,T}) = c^{\lambda}_{T,S}$ and, for every $a \in A$, every $\lambda \in \Lambda$, and all $S, T \in \mathcal{T}(\lambda)$, we have $a\, c^{\lambda}_{S,T} \equiv \sum_{S' \in \mathcal{T}(\lambda)} r_a(S', S) c^{\lambda}_{S',T} \pmod{A^{<\lambda}}$, where $A^{<\lambda}$ denotes the $\mathbb{F}$-span of all basis elements $c^{\mu}_{U,V}$ with $\mu < \lambda$. The coefficients $r_a(S', S) \in \mathbb{F}$ are independent of T.

For each $\lambda \in \Lambda$, the corresponding cell module, denoted $\Delta(\lambda)$, is the $\mathbb{F}$-vector space with basis $\{c^{\lambda}_S \mid S \in \mathcal{T}(\lambda)\}$. The A-module structure is defined by $a\, c^{\lambda}_S = \sum_{S' \in \mathcal{T}(\lambda)} r_a(S', S) c^{\lambda}_{S'}$. The cellular datum also gives a bilinear form $\phi_\lambda : \Delta(\lambda) \times \Delta(\lambda) \longrightarrow \mathbb{F}$ defined by the congruence $c^{\lambda}_{S,T} c^{\lambda}_{U,V} \equiv \phi_\lambda(c^{\lambda}_T, c^{\lambda}_U) c^{\lambda}_{S,V}$ (mod $A^{<\lambda}$). The radical of $\Delta(\lambda)$ is $\operatorname{rad}\Delta(\lambda) = \{x \in \Delta(\lambda) \mid \phi_\lambda(x, y) = 0 \text{ for all } y \in \Delta(\lambda)\}$. If $\phi_\lambda \neq 0$, then $L(\lambda) = \Delta(\lambda)/\operatorname{rad}\Delta(\lambda)$ is a simple A-module. Moreover, the nonzero modules $L(\lambda)$, as λ ranges over the labels for which $\phi_\lambda \neq 0$, form a complete set of pairwise non-isomorphic simple A-modules.

A derivation $D \in \mathrm{Der}(A)$ is called *cellular* (or *compatible with the cellular filtration*) if $D(A^{\leq\lambda}) \subseteq A^{\leq\lambda}$ and $D(A^{<\lambda}) \subseteq A^{<\lambda}$ for all $\lambda \in \Lambda$. This notion will be used systematically in later sections.

2.3. Cellular structure of the blob algebra

The blob algebra is cellular; see Martin–Woodcock [5] and Ryom-Hansen [14]. For the ordinary blob algebra, one may take the cellular labels to be $\Lambda_n = \{n, n-2, n-4, \ldots, -n\}$. We use the standard cellular partial order $t \geq s \iff |t| < |s|$ or $t = s$. (Equivalently, one may refine this to the usual total order used for the graded cellular basis; none of the arguments below depends on the choice of such a refinement.) For $t \in \Lambda_n$, we denote the corresponding cell module by $\Delta_n(t)$. Diagrammatically, $\Delta_n(t)$ has a basis indexed by upper half-blob diagrams with $|t|$ propagating lines, with the sign of t recording the boundary decoration convention. Equivalently, under the parametrization used in the graded cellular basis of Plaza–Ryom–Hansen [15], these basis elements may be described by standard one-line bitableaux. Thus, in the ordinary blob case, one should not speak of arbitrary bipartitions of n; rather, the relevant combinatorial objects are one-line bipartitions or, equivalently, blob half-diagrams.

The full cellular basis of B_n is obtained by gluing an upper and a lower half-diagram with the same cellular label. Accordingly, the cellular basis elements may be written in the form $c^t_{S,T}$, $t \in \Lambda_n$, $S, T \in \mathcal{T}(t)$, where $\mathcal{T}(t)$ denotes the indexing set of half-diagrams, or, equivalently, the corresponding set of standard one-line bitableaux.

2.4. The Deguchi–Martin quotient

Let $\mathcal{H}_n(q)$ be the Hecke algebra of type A_{n-1} over $\mathbb{F}$. It is generated by $T_1, \ldots, T_{n-1}$, subject to

$$\begin{aligned} T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, && 1 \leq i \leq n-2, \\ T_iT_j &= T_jT_i, && |i-j| > 1, \\ (T_i - q)(T_i + q^{-1}) &= 0, && 1 \leq i \leq n-1. \end{aligned}$$

Let $P, M \geq 0$, and let $\chi = ((M+1)^{P+1})$ be the rectangular partition with $P+1$ rows, each of length $M+1$. Let E_χ and F_χ denote, respectively, the corresponding q-symmetrizer and q-antisymmetrizer in the appropriate parabolic subalgebra of $\mathcal{H}_n(q)$. The Deguchi–Martin quotient [16] is the quotient $H_n^{P,M} = \mathcal{H}_n(q)/\mathcal{J}_n^{P,M}$, where $\mathcal{J}_n^{P,M}$ is the two-sided ideal generated by $F_\chi\, \mathcal{H}_{(P+1)(M+1)}(q)\, E_\chi$. In particular, in the $(1|1)$-case, one has $P = M = 1$, and hence $\chi = (2,2)$. For $n \geq 4$, the defining ideal is understood via the standard parabolic embedding of $\mathcal{H}_4(q)$ into $\mathcal{H}_n(q)$. For $n < 4$, we adopt the natural convention $\mathcal{J}_n^{1,1} = 0$, since no partition of n contains the rectangular obstruction $(2,2)$; thus $H_n^{1,1} = \mathcal{H}_n(q)$ in these ranks. Hence $H_n^{1,1}$ is obtained from $\mathcal{H}_n(q)$ by eliminating the cellular part corresponding to $(2,2)$, and the cell modules which survive are indexed by the hook partitions $\lambda = (r, 1^{n-r})$, $1 \leq r \leq n$. We denote the corresponding cell module of $H_n^{1,1}$ as R_n^λ, and we denote its standard word basis as $Q_n(\lambda)$.

The precise comparison with the blob algebra is as follows. For $\lambda = (r, 1^{n-r})$, put $t = t(\lambda) = n+1-2r$. Then $t \in \Lambda_{n-1} = \{n-1, n-3, \ldots, 3-n, 1-n\}$. Alotaibi's theorem is stated over $\mathbb{C}$. Accordingly, whenever we invoke that theorem literally we take $\mathbb{F} = \mathbb{C}$. The same purely combinatorial basis correspondence, and the linear statements derived from it, remain valid over any characteristic-zero field for which the corresponding cellular bases and their base change have been fixed. By Alotaibi's basis correspondence [13, Theorem 3.4.1], there is a natural bijection $B^{\mathrm{top}}_{n-1}(t) \longleftrightarrow Q_n(\lambda)$, where $B^{\mathrm{top}}_{n-1}(t)$ is the canonical half-diagram basis of the blob cell module $\Delta_{n-1}(t)$, and $Q_n(\lambda)$ is the word basis of the

Deguchi–Martin cell module R_n^λ. Equivalently, we have $\lambda = (r, 1^{n-r}) \iff t = n + 1 - 2r$. This gives a bijection between the standard cellular bases of $\Delta_{n-1}(t)$ and R_n^λ.

It is important to stress that this is a bijection of indexing sets and cell-module bases. It is not, by itself, a proof that the algebras B_{n-1} and $H_n^{1,1}$ are isomorphic. Any statement transferring algebraic structures, such as derivations, from one algebra to the other must therefore be made either through an explicitly constructed algebra homomorphism or through the cell-based correspondence.

3. Derivations of the blob algebra

Let $B = \mathsf{B}_n = \mathsf{B}_n(q, \delta, \kappa)$ be the blob algebra defined in Section 2. A derivation of B is a $\mathbb{F}$-linear map $D : B \longrightarrow B$ satisfying the Leibniz rule $D(xy) = D(x)y + xD(y)$ $(x, y \in B)$. We use $\mathrm{Der}(B)$ to denote the $\mathbb{F}$-vector space of all derivations of B. For $a \in B$, the map $\mathrm{ad}_a : B \longrightarrow B$, $\mathrm{ad}_a(x) = ax - xa$, is a derivation. Such derivations are called inner derivations. The space of all inner derivations is denoted by $\mathrm{Inn}(B) = \{\mathrm{ad}_a \mid a \in B\}$. The quotient $\mathrm{Out}(B) = \mathrm{Der}(B)/\mathrm{Inn}(B)$ is the space of outer derivation classes of B. By Proposition 3.6 below, it also carries the induced Lie algebra structure.

3.1. Derivations determined by the generators

Let $\mathcal{F} = \mathbb{F}\langle X_1, \ldots, X_{n-1}, E\rangle$ be the free associative $\mathbb{F}$-algebra on n noncommuting generators, and let $\pi : \mathcal{F} \longrightarrow B$ be the canonical surjective algebra homomorphism defined by $\pi(X_i) = U_i$, $\pi(E) = e$. Let $\mathcal{R} = \ker \pi$. Then $B \cong \mathcal{F}/\mathcal{R}$, where $\mathcal{R}$ is the two-sided ideal generated by the defining relators of the blob algebra.

Since B is generated by $U_1, \ldots, U_{n-1}, e$, every derivation $D \in \mathrm{Der}(B)$ is uniquely determined by the elements $d_i = D(U_i)$ $(1 \leq i \leq n-1)$, $d_e = D(e)$. However, not every choice of $d_1, \ldots, d_{n-1}, d_e \in B$ defines a derivation of B. The chosen elements must be compatible with the defining relations of B. The next result gives the exact compatibility condition.

Theorem 3.1 (Extension criterion)**.** *Let $d_1, \ldots, d_{n-1}, d_e \in B$. There is a unique derivation $D \in \mathrm{Der}(B)$ satisfying $D(U_i) = d_i$ $(1 \leq i \leq n-1)$, $D(e) = d_e$, if and only if the following identities hold in B:*

$$d_iU_i + U_id_i = [2]_q\, d_i, \qquad 1 \leq i \leq n-1, \tag{3.1}$$
$$d_iU_{i+1}U_i + U_id_{i+1}U_i + U_iU_{i+1}d_i = d_i, \qquad 1 \leq i \leq n-2, \tag{3.2}$$
$$d_{i+1}U_iU_{i+1} + U_{i+1}d_iU_{i+1} + U_{i+1}U_id_{i+1} = d_{i+1}, \qquad 1 \leq i \leq n-2, \tag{3.3}$$
$$d_iU_j + U_id_j = d_jU_i + U_jd_i, \qquad |i-j| > 1, \tag{3.4}$$
$$d_ee + ed_e = \delta d_e, \tag{3.5}$$
$$d_1eU_1 + U_1d_eU_1 + U_1ed_1 = \kappa d_1, \tag{3.6}$$
$$d_eU_i + ed_i = d_ie + U_id_e, \qquad 2 \leq i \leq n-1. \tag{3.7}$$

Proof. First, suppose that $D \in \mathrm{Der}(B)$ is a derivation such that $D(U_i) = d_i$, $D(e) = d_e$. Since the scalars q, δ, κ are fixed elements of $\mathbb{F}$, we have $D(\alpha 1_B) = 0$ $(\alpha \in \mathbb{F})$. Applying D to each defining relation of the blob algebra gives the required conditions.

Indeed, from $U_i^2 = [2]_qU_i$ we obtain $D(U_i^2) = D([2]_qU_i)$. Using the Leibniz rule, this gives $D(U_i)U_i + U_iD(U_i) = [2]_qD(U_i)$, which is exactly (3.1).

Next, applying D to $U_iU_{i+1}U_i = U_i$ gives $D(U_i)U_{i+1}U_i + U_iD(U_{i+1})U_i + U_iU_{i+1}D(U_i) = D(U_i)$, which is (3.2). Similarly, applying D to $U_{i+1}U_iU_{i+1} = U_{i+1}$ gives (3.3).

For $|i-j| > 1$, the defining relation $U_iU_j = U_jU_i$ implies $D(U_i)U_j + U_iD(U_j) = D(U_j)U_i + U_jD(U_i)$, which is (3.4). Applying D to $e^2 = \delta e$ gives $D(e)e + eD(e) = \delta D(e)$, which is (3.5). Applying D to $U_1eU_1 = \kappa U_1$ gives $D(U_1)eU_1 + U_1D(e)U_1 + U_1eD(U_1) = \kappa D(U_1)$, which is (3.6). Finally, applying D to $eU_i = U_ie$ $(2 \le i \le n-1)$ gives (3.7). Hence, every derivation satisfies (3.1)–(3.7).

Conversely, assume that the elements $d_1, \ldots, d_{n-1}, d_e \in B$ satisfy (3.1)–(3.7). Since $\mathcal{F}$ is the free associative algebra on $X_1, \ldots, X_{n-1}, E$, there is a unique $\mathbb{F}$-linear derivation $\widetilde{D} : \mathcal{F} \longrightarrow B$ with respect to the $\mathcal{F}$-bimodule structure on B induced by π, such that $\widetilde{D}(X_i) = d_i$, $\widetilde{D}(E) = d_e$. This means that $\widetilde{D}(fg) = \widetilde{D}(f)\pi(g) + \pi(f)\widetilde{D}(g)$ $(f, g \in \mathcal{F})$.

We claim that $\widetilde{D}(\mathcal{R}) = 0$. It is enough to check this on the defining relators generating $\mathcal{R}$. For example, $X_i^2 - [2]_qX_i$ is sent to $d_iU_i + U_id_i - [2]_qd_i$, which is zero by (3.1). Similarly, the relators $X_iX_{i+1}X_i - X_i$ and $X_{i+1}X_iX_{i+1} - X_{i+1}$ are sent to zero by (3.2) and (3.3); the commutation relators are sent to zero by (3.4); the idempotent relator $E^2 - \delta E$ is sent to zero by (3.5); the blob relator $X_1EX_1 - \kappa X_1$ is sent to zero by (3.6); and the boundary commutation relators $EX_i - X_iE$ $(2 \le i \le n-1)$ are sent to zero by (3.7).

Now let r be any defining relator and let $a, b \in \mathcal{F}$. Since $\pi(r) = 0$ and $\widetilde{D}(r) = 0$, we have $\widetilde{D}(arb) = \widetilde{D}(a)\pi(r)\pi(b) + \pi(a)\widetilde{D}(r)\pi(b) + \pi(a)\pi(r)\widetilde{D}(b) = 0$. Because $\mathcal{R}$ is the two-sided ideal generated by the defining relators, it follows that $\widetilde{D}(\mathcal{R}) = 0$.

Therefore, the rule $D(\pi(f)) = \widetilde{D}(f)$ $(f \in \mathcal{F})$ defines a well-defined $\mathbb{F}$-linear map $D : B \longrightarrow B$. Indeed, if $\pi(f) = \pi(g)$, then $f - g \in \mathcal{R}$, and hence $\widetilde{D}(f-g) = 0$. Thus $\widetilde{D}(f) = \widetilde{D}(g)$.

Finally, for $x = \pi(f)$ and $y = \pi(g)$, we have $D(xy) = D(\pi(fg)) = \widetilde{D}(fg) = \widetilde{D}(f)\pi(g) + \pi(f)\widetilde{D}(g) = D(x)y + xD(y)$. Hence, D is a derivation of B. By construction, $D(U_i) = d_i$, $D(e) = d_e$. Uniqueness follows because B is generated as an algebra by $U_1, \ldots, U_{n-1}, e$. This completes the proof. □

3.2. *Explicit constraints*

The extension criterion can be restated in a compact form as follows.

Proposition 3.2 (Quadratic constraints)**.** *Let $D \in \mathrm{Der}(B)$. Then, for every $1 \le i \le n-1$, $D(U_i)U_i + U_iD(U_i) = [2]_qD(U_i)$. Equivalently, if $d_i = D(U_i)$, then $d_iU_i + U_id_i = [2]_qd_i$.*

Proof. Apply D to the defining relation $U_i^2 = [2]_qU_i$. The Leibniz rule gives $D(U_i^2) = D(U_i)U_i + U_iD(U_i)$, while $\mathbb{F}$-linearity gives $D([2]_qU_i) = [2]_qD(U_i)$. Hence, $D(U_i)U_i + U_iD(U_i) = [2]_qD(U_i)$. □

Proposition 3.3 (Temperley–Lieb constraints)**.** *Let $D \in \mathrm{Der}(B)$. Then, for $1 \le i \le n-2$,*

$$D(U_i)U_{i+1}U_i + U_iD(U_{i+1})U_i + U_iU_{i+1}D(U_i) = D(U_i),$$
$$D(U_{i+1})U_iU_{i+1} + U_{i+1}D(U_i)U_{i+1} + U_{i+1}U_iD(U_{i+1}) = D(U_{i+1}).$$

Moreover, if $|i-j| > 1$, then $D(U_i)U_j + U_iD(U_j) = D(U_j)U_i + U_jD(U_i)$.

Proof. The first two identities follow by applying D to $U_iU_{i+1}U_i = U_i$ and $U_{i+1}U_iU_{i+1} = U_{i+1}$, respectively. The commutation identity follows by applying D to $U_iU_j = U_jU_i$ $(|i-j| > 1)$. In each case, the result follows immediately from the Leibniz rule. □

Proposition 3.4 (Blob constraints)**.** *Let $D \in \mathrm{Der}(B)$. Then $D(e)e + eD(e) = \delta D(e)$, $D(U_1)eU_1 + U_1D(e)U_1 + U_1eD(U_1) = \kappa D(U_1)$, and, for every $2 \le i \le n-1$, $D(e)U_i + eD(U_i) = D(U_i)e + U_iD(e)$.*

Proof. The first identity follows by applying D to $e^2 = \delta e$. The second follows by applying D to $U_1eU_1 = \kappa U_1$. The third follows by applying D to $eU_i = U_ie$ $(2 \le i \le n-1)$. All three identities are obtained directly by expanding with the Leibniz rule. □

Proposition 3.5 (Inner derivations)**.** *For every $a \in B$, the map $\mathrm{ad}_a : B \longrightarrow B$, $\mathrm{ad}_a(x) = ax - xa$, is a derivation of B. Moreover, $\ker(a \mapsto \mathrm{ad}_a) = Z(B)$, where $Z(B)$ is the centre of B. Consequently, $\mathrm{Inn}(B) \cong B/Z(B)$ as $\mathbb{F}$-vector spaces.*

Proof. Let $a, x, y \in B$. Then $\mathrm{ad}_a(xy) = axy - xya$. Adding and subtracting xay, we obtain $axy - xya = (ax - xa)y + x(ay - ya)$. Hence $\mathrm{ad}_a(xy) = \mathrm{ad}_a(x)y + x\,\mathrm{ad}_a(y)$. Thus ad_a is a derivation.

Moreover, $\mathrm{ad}_a = 0$ if and only if $ax - xa = 0$ for all $x \in B$. This is equivalent to saying that $a \in Z(B)$. Therefore, the kernel of the linear map $B \longrightarrow \mathrm{Inn}(B)$, $a \longmapsto \mathrm{ad}_a$, is exactly $Z(B)$. The first isomorphism theorem gives $\mathrm{Inn}(B) \cong B/Z(B)$. □

Proposition 3.6 (Inner derivations form a Lie ideal)**.** *The space $\mathrm{Inn}(B)$ is a Lie ideal of the Lie algebra $\mathrm{Der}(B)$. More precisely, for every $D \in \mathrm{Der}(B)$ and $a \in B$, $[D, \mathrm{ad}_a] = \mathrm{ad}_{D(a)}$. Consequently, $\mathrm{Out}(B) = \mathrm{Der}(B)/\mathrm{Inn}(B)$ inherits a well-defined Lie algebra structure from the commutator bracket on $\mathrm{Der}(B)$.*

Proof. For $x \in B$,

$$\begin{aligned}[D, \mathrm{ad}_a](x) &= D(ax - xa) - aD(x) + D(x)a \\ &= D(a)x + aD(x) - D(x)a - xD(a) - aD(x) + D(x)a \\ &= D(a)x - xD(a) \\ &= \mathrm{ad}_{D(a)}(x).\end{aligned}$$

Thus $[D, \mathrm{ad}_a] \in \mathrm{Inn}(B)$, proving that $\mathrm{Inn}(B)$ is a Lie ideal. The quotient therefore carries the induced Lie bracket. □

3.3. Cellular derivations and innerness

Let $B = \mathsf{B}_n$. Let $\{c^\lambda_{S,T} \mid \lambda \in \Lambda,\ S, T \in \mathcal{T}(\lambda)\}$ be a cellular basis of B. For $\lambda \in \Lambda$, define $B^{\le\lambda} = \mathrm{span}_{\mathbb{F}}\{c^\mu_{S,T} \mid \mu \le \lambda\}$, $B^{<\lambda} = \mathrm{span}_{\mathbb{F}}\{c^\mu_{S,T} \mid \mu < \lambda\}$. A derivation $D \in \mathrm{Der}(B)$ is called *cellular* if $D(B^{\le\lambda}) \subseteq B^{\le\lambda}$ and $D(B^{<\lambda}) \subseteq B^{<\lambda}$ $(\lambda \in \Lambda)$. This definition was presented earlier in Section 2.2; we repeat it here for convenience. Equivalently, a cellular derivation preserves the cellular filtration and hence induces a derivation on every associated cellular subquotient $B^{\le\lambda}/B^{<\lambda}$.

Theorem 3.7 (Innerness in the semisimple case)**.** *Assume that $B = \mathsf{B}_n(q, \delta, \kappa)$ is semisimple over $\mathbb{F}$. Then every derivation of B is inner. Consequently, we have $\mathrm{Der}(B) = \mathrm{Inn}(B) \cong B/Z(B)$. In particular, every cellular derivation of B is inner.*

Proof. Since $\mathbb{F}$ has characteristic zero, every finite-dimensional semisimple $\mathbb{F}$-algebra is separable. Hence, its first Hochschild cohomology vanishes as follows:

$HH^1(B, B) = 0$. But $HH^1(B, B) \cong \mathrm{Der}(B)/\mathrm{Inn}(B)$. Therefore $\mathrm{Der}(B) = \mathrm{Inn}(B)$.

For completeness, we recall the concrete argument in the split semisimple case. Suppose $B \cong \bigoplus_{\lambda\in\Lambda_0} M_{d_\lambda}(\mathbb{F})$ as a finite direct sum of full matrix algebras. Let $1 = \sum_{\lambda\in\Lambda_0} z_\lambda$ be the decomposition of the identity into primitive central idempotents, so that $B_\lambda = Bz_\lambda$ is a simple two-sided ideal.

We first show that every derivation $D \in \mathrm{Der}(B)$ fixes each z_λ. Since $z_\lambda^2 = z_\lambda$, the Leibniz rule gives $D(z_\lambda) = D(z_\lambda^2) = D(z_\lambda)z_\lambda + z_\lambda D(z_\lambda)$. Moreover, because z_λ is central, for every $x \in B$, we have $z_\lambda x = xz_\lambda$. Applying D to this equality yields $D(z_\lambda)x + z_\lambda D(x) = D(x)z_\lambda + xD(z_\lambda)$. Since z_λ is central, $z_\lambda D(x) = D(x)z_\lambda$, and hence $D(z_\lambda)x = xD(z_\lambda)$ $(x \in B)$. Thus $D(z_\lambda) \in Z(B)$. Therefore $D(z_\lambda) = D(z_\lambda^2) = 2z_\lambda D(z_\lambda)$. Multiplying this equality by z_λ, and using $\mathrm{char}\,\mathbb{F} = 0$, gives $z_\lambda D(z_\lambda) = 0$. Hence $D(z_\lambda) = 2z_\lambda D(z_\lambda) = 0$. It follows that $D(B_\lambda) = D(Bz_\lambda) \subseteq Bz_\lambda = B_\lambda$.

Thus D restricts to a derivation $D_\lambda : B_\lambda \longrightarrow B_\lambda$ for each $\lambda \in \Lambda_0$. Since $B_\lambda \cong M_{d_\lambda}(\mathbb{F})$, and every derivation of a full matrix algebra over a field is inner, there exists $a_\lambda \in B_\lambda$ such that $D_\lambda(x) = [a_\lambda, x]$ $(x \in B_\lambda)$. Set $a = \sum_{\lambda\in\Lambda_0} a_\lambda \in B$. Every $x \in B$ has a unique decomposition $x = \sum_{\lambda\in\Lambda_0} x_\lambda,\ x_\lambda \in B_\lambda$. Since distinct simple two-sided ideals multiply to zero, we obtain $D(x) = \sum_\lambda D_\lambda(x_\lambda) = \sum_\lambda [a_\lambda, x_\lambda] = [a, x]$. Therefore, $D = \mathrm{ad}_a$, and so D is inner.

Finally, the canonical map $B \longrightarrow \mathrm{Inn}(B)$, $a \longmapsto \mathrm{ad}_a$, has the kernel $Z(B)$. Hence $\mathrm{Inn}(B) \cong B/Z(B)$. This proves the result. □

Remark 3.8. *The cellularity assumption is not needed in Theorem 3.7. Once B is semisimple over a field of characteristic zero, all derivations are inner. The cellular condition only becomes relevant in the non-semisimple case, where outer derivations may occur and the cellular filtration can be used to organize them.*

3.4. The rank-two case

We now examine the case $n = 2$. Put $\theta = [2]_q = q + q^{-1}$. Throughout this subsection we assume $\theta \neq 0$. This is automatic, for example, if q is not a root of unity. Write $U = U_1$. Then $B = \mathsf{B}_2$ is generated by U and e, subject to $U^2 = \theta U,\ e^2 = \delta e,\ UeU = \kappa U.$

Lemma 3.9 (Rank-two normal form)**.** *The six elements* $\mathcal{B} = \{1,\ U,\ e,\ Ue,\ eU,\ W\}$, $W = eUe$, *form a* $\mathbb{F}$*-basis of* B_2. *In particular,* $\dim_{\mathbb{F}} \mathsf{B}_2 = 6$.

Proof. Consider the length-reducing rewriting rules $UU \longrightarrow \theta U,\ ee \longrightarrow \delta e,\ UeU \longrightarrow \kappa U.$ Every word in U, e reduces to a linear combination of words containing none of UU, ee, UeU. Such an irreducible word has length at most three: after excluding UU and ee, a word is alternating, and every alternating word of length at least four contains UeU. Hence the only irreducible words are $1,\ U,\ e,\ Ue,\ eU,\ eUe$. Thus these six elements span.

It remains to verify uniqueness of the reduction. Since every rule strictly reduces word length, termination is immediate. The only overlap ambiguities are UUU, eee, $UUeU$, $UeUU$, $UeUeU$. Both reductions of UUU give $\theta^2 U$, both reductions of eee give $\delta^2 e$, both reductions of $UUeU$ and $UeUU$ give $\theta\kappa U$, and both reductions of $UeUeU$ give $\kappa^2 U$. Thus all critical overlaps are resolvable. The terminating rewriting system is therefore confluent, so the irreducible words give unique normal forms. Consequently the six spanning elements are linearly independent and form a basis. □

The multiplication table with respect to this basis is as follows:

$\cdot$	1	U	e	Ue	eU	W
1	1	U	e	Ue	eU	W
U	U	θU	Ue	θUe	κU	κUe
e	e	eU	δe	W	δeU	δW
Ue	Ue	κU	δUe	κUe	$\delta\kappa U$	$\delta\kappa Ue$
eU	eU	θeU	W	θW	κeU	κW
W	W	κeU	δW	κW	$\delta\kappa eU$	$\delta\kappa W$

Theorem 3.10 (Derivations of B_2)**.** *Assume $\theta \neq 0$. Let $D \in \mathrm{Der}(\mathsf{B}_2)$. Then D is uniquely determined by $D(U)$ and $D(e)$, and these elements have the following form.*

First, we have $D(U) = aU + b\,Ue + c\,eU$, where $\theta a + \kappa(b + c) = 0$.

If $\delta \neq 0$, then $D(e) = r\,Ue + s\,eU + tW$, where $r + s + \delta t = 0$ and $\kappa(\theta\delta - \kappa)\left(\frac{b+c}{\theta} - t\right) = 0$.

If $\delta = 0$, then $D(e) = h\,e + r(Ue - eU) + tW$, where $\kappa\left(h + \kappa t - \frac{\kappa}{\theta}(b + c)\right) = 0$.

Conversely, every assignment $D(U), D(e)$ satisfying the above conditions extends uniquely to a derivation of B_2.

Proof. Let $D(U) = a_0\,1 + a_1U + a_2e + a_3Ue + a_4eU + a_5W$ and $D(e) = b_0\,1 + b_1U + b_2e + b_3Ue + b_4eU + b_5W$. Since D is a derivation, it must satisfy the three derived defining relations

$$D(U)U + UD(U) = \theta D(U), \tag{3.8}$$

$$D(e)e + eD(e) = \delta D(e), \tag{3.9}$$

$$D(U)eU + UD(e)U + UeD(U) = \kappa D(U). \tag{3.10}$$

Using the multiplication table, (3.8) gives $a_0 = 0,\ a_2 = 0,\ a_5 = 0$, and $\theta a_1 + \kappa(a_3 + a_4) = 0$. Thus $D(U) = a_1U + a_3Ue + a_4eU$. Writing $a = a_1,\ b = a_3,\ c = a_4$, we obtain $D(U) = aU + bUe + ceU,\ \theta a + \kappa(b + c) = 0$.

Next, applying (3.9), again using the multiplication table, gives $-\delta b_0 = 0, -\delta b_1 = 0, 2b_0 + \delta b_2 = 0, b_1 = 0$, and $b_3 + b_4 + \delta b_5 = 0$.

If $\delta \neq 0$, these equations imply $b_0 = b_1 = b_2 = 0$ and $b_3 + b_4 + \delta b_5 = 0$. Hence $D(e) = b_3Ue + b_4eU + b_5W$. Writing $r = b_3,\ s = b_4,\ t = b_5$, we get $D(e) = rUe + seU + tW,\ r + s + \delta t = 0$.

If $\delta = 0$, the same equations give $b_0 = 0, b_1 = 0, b_3 + b_4 = 0$. Thus $D(e) = b_2e + b_3(Ue - eU) + b_5W$. Writing $h = b_2,\ r = b_3,\ t = b_5$, we get $D(e) = he + r(Ue - eU) + tW$.

It remains to impose the blob relation (3.10). Substitution of the abovementioned forms into (3.10) gives the single additional scalar condition $\kappa\,(a + \delta(b + c) + b_2 + \theta(b_3 + b_4) + \kappa b_5) = 0$.

When $\delta \neq 0$, we have $b_2 = 0$ and $b_3 + b_4 = -\delta b_5$. Using also $a = -\frac{\kappa}{\theta}(b + c)$, the additional condition becomes $\kappa(\theta\delta - \kappa)\left(\frac{b+c}{\theta} - b_5\right) = 0$. This is exactly $\kappa(\theta\delta - \kappa)\left(\frac{b+c}{\theta} - t\right) = 0$.

When $\delta = 0$, we have $b_2 = h,\ b_3 + b_4 = 0,\ b_5 = t$. The additional condition becomes $\kappa\left(h + \kappa t - \frac{\kappa}{\theta}(b + c)\right) = 0$.

Therefore every derivation has the asserted form.

Conversely, if $D(U)$ and $D(e)$ have the stated form and satisfy the stated scalar conditions, then the three derived relations $D(U)U + UD(U) = \theta D(U)$, $D(e)e + eD(e) = \delta D(e)$, and $D(U)eU + UD(e)U + UeD(U) = \kappa D(U)$ hold. By the extension criterion, this assignment extends uniquely to a derivation of B_2. This completes the proof. □

Proposition 3.11 (Inner derivations of B_2)**.** *Assume $\theta \neq 0$. Let $x = x_0\,1 + x_1 U + x_2 e + x_3 Ue + x_4 eU + x_5 W \in \mathsf{B}_2$. Then the inner derivation* ad_x *is given on generators by*

$$\mathrm{ad}_x(U) = x_2(eU - Ue) + x_3(\kappa U - \theta Ue) + x_4(-\kappa U + \theta eU) + x_5\kappa(eU - Ue),$$
$$\mathrm{ad}_x(e) = x_1(Ue - eU) + x_3(\delta Ue - W) + x_4(W - \delta eU).$$

Moreover, we have $\dim_{\mathbb{F}} \mathrm{Inn}(\mathsf{B}_2) = 3$.

Proof. The displayed formula follows by direct multiplication: $\mathrm{ad}_x(U) = xU - Ux$, $\mathrm{ad}_x(e) = xe - ex$. Using the multiplication table gives the two asserted expressions.

Since 1 is central, the coefficient x_0 does not affect ad_x. Direct calculation also gives $\mathrm{ad}_{eU} = -\mathrm{ad}_{Ue} + \theta\,\mathrm{ad}_e + \delta\,\mathrm{ad}_U$, $\mathrm{ad}_W = \kappa\,\mathrm{ad}_e$. Hence every inner derivation lies in $\mathrm{span}_{\mathbb{F}}\{\mathrm{ad}_e, \mathrm{ad}_U, \mathrm{ad}_{Ue}\}$. These three derivations are linearly independent when $\theta \neq 0$. Indeed, if $\alpha\,\mathrm{ad}_e + \beta\,\mathrm{ad}_U + \gamma\,\mathrm{ad}_{Ue} = 0$, then evaluation at U gives $\alpha(eU - Ue) + \gamma(\kappa U - \theta Ue) = 0$. By Lemma 3.9, the coefficients of eU and Ue force $\alpha = \gamma = 0$ because $\theta \neq 0$. Evaluation at e then gives $\beta(Ue - eU) = 0$, so $\beta = 0$. Therefore $\dim_{\mathbb{F}} \mathrm{Inn}(\mathsf{B}_2) = 3$. □

Corollary 3.12 (Outer derivations in rank two)**.** *Assume $\theta \neq 0$. Then the dimensions of* $\mathrm{Der}(\mathsf{B}_2)$*,* $\mathrm{Inn}(\mathsf{B}_2)$*, and* $HH^1(\mathsf{B}_2, \mathsf{B}_2) = \mathrm{Der}(\mathsf{B}_2)/\mathrm{Inn}(\mathsf{B}_2)$ *are as follows:*

Parameters	$\dim_{\mathbb{F}} \mathrm{Der}(\mathsf{B}_2)$	$\dim_{\mathbb{F}} \mathrm{Inn}(\mathsf{B}_2)$	$\dim_{\mathbb{F}} HH^1(\mathsf{B}_2, \mathsf{B}_2)$
$\delta \neq 0,\ \kappa \neq 0,\ \kappa \neq \theta\delta$	3	3	0
$\delta \neq 0,\ \kappa = 0$	4	3	1
$\delta \neq 0,\ \kappa = \theta\delta$	4	3	1
$\delta = 0,\ \kappa \neq 0$	4	3	1
$\delta = 0,\ \kappa = 0$	5	3	2

In particular, every derivation of B_2 is inner in the generic case $\delta \neq 0,\ \kappa \neq 0,\ \kappa \neq \theta\delta$. In the exceptional cases listed above, non-inner derivations occur.

Proof. The dimensions of $\mathrm{Der}(\mathsf{B}_2)$ follow immediately from Theorem 3.10 by counting the free scalar parameters in each case. Proposition 3.11 gives $\dim_{\mathbb{F}} \mathrm{Inn}(\mathsf{B}_2) = 3$. Therefore $\dim_{\mathbb{F}} HH^1(\mathsf{B}_2, \mathsf{B}_2) = \dim_{\mathbb{F}} \mathrm{Der}(\mathsf{B}_2) - \dim_{\mathbb{F}} \mathrm{Inn}(\mathsf{B}_2)$, which gives the stated table. □

Remark 3.13. *The rank-two calculation shows why the non-semisimple case must be treated with care. The old claim that all derivations of B_2 are inner for arbitrary δ and κ is not correct. Innerness holds in the generic rank-two case, but exceptional parameter values produce genuine outer derivations.*

4. Action on standard modules

Let $B = \mathsf{B}_n$. Let $\{c^\lambda_{S,T} \mid \lambda \in \Lambda,\ S, T \in \mathcal{T}(\lambda)\}$ be a cellular basis of B. For $\lambda \in \Lambda$, define $B^{\leq\lambda} = \mathrm{span}_{\mathbb{F}}\{c^\mu_{S,T} \mid \mu \leq \lambda\}$, $B^{<\lambda} = \mathrm{span}_{\mathbb{F}}\{c^\mu_{S,T} \mid \mu < \lambda\}$. Thus $B^{<\lambda} \subseteq B^{\leq\lambda}$, and the quotient $J_\lambda = B^{\leq\lambda}/B^{<\lambda}$ is called the λ-cell quotient.

Let $\Delta(\lambda)$ denote the standard, or cell, module corresponding to λ. We write its basis as $\{c^\lambda_S \mid S \in \mathcal{T}(\lambda)\}$. The action of B on $\Delta(\lambda)$ is determined by the cellular multiplication rule $a\,c^\lambda_{S,T} \equiv \sum_{S' \in \mathcal{T}(\lambda)} r_a(S', S) c^\lambda_{S',T} \pmod{B^{<\lambda}}$, where the coefficients $r_a(S', S) \in \mathbb{F}$ are independent of T. Hence $a\,c^\lambda_S = \sum_{S' \in \mathcal{T}(\lambda)} r_a(S', S) c^\lambda_{S'}$.

4.1. Induced map on the cell quotient

A derivation $D \in \mathrm{Der}(B)$ is called *cellular* if $D(B^{\le\lambda}) \subseteq B^{\le\lambda}$ and $D(B^{<\lambda}) \subseteq B^{<\lambda}$ $(\lambda \in \Lambda)$. Thus a cellular derivation preserves the cellular filtration.

Proposition 4.1. *Let $D \in \mathrm{Der}(B)$ be a cellular derivation. Then, for every $\lambda \in \Lambda$, the formula $\overline{D}_\lambda(x + B^{<\lambda}) = D(x) + B^{<\lambda}$ $(x \in B^{\le\lambda})$ defines a well-defined $\mathbb{F}$-linear map $\overline{D}_\lambda : J_\lambda \longrightarrow J_\lambda$. Moreover, $\overline{D}_\lambda(a\xi) = D(a)\xi + a\overline{D}_\lambda(\xi)$ $(a \in B,\ \xi \in J_\lambda)$.*

Proof. Suppose that $x + B^{<\lambda} = y + B^{<\lambda}$. Then $x - y \in B^{<\lambda}$. Since D is cellular, we have $D(x - y) \in B^{<\lambda}$. Therefore $D(x) + B^{<\lambda} = D(y) + B^{<\lambda}$. Thus $\overline{D}_\lambda$ is well defined.

Now let $a \in B$ and let $\xi = x + B^{<\lambda} \in J_\lambda$. Since J_λ is a left B-module, we have $a\xi = ax + B^{<\lambda}$. Hence $\overline{D}_\lambda(a\xi) = D(ax) + B^{<\lambda}$. Using the Leibniz rule in B, we obtain $D(ax) + B^{<\lambda} = D(a)x + aD(x) + B^{<\lambda}$. This is exactly $D(a)\xi + a\overline{D}_\lambda(\xi)$. The proof is complete. □

Remark 4.2. *The map $\overline{D}_\lambda$ is naturally defined on the cell quotient J_λ. It does not automatically define a map on the standard module $\Delta(\lambda)$. To obtain a map on $\Delta(\lambda)$, an additional compatibility condition is required.*

4.2. D-connections on standard modules

Let $\rho_\lambda : B \longrightarrow \mathrm{End}_{\mathbb{F}}(\Delta(\lambda))$ be the representation afforded by the standard module $\Delta(\lambda)$.

Definition 4.3. *Let $D \in \mathrm{Der}(B)$. An $\mathbb{F}$-linear map $\nabla_\lambda : \Delta(\lambda) \longrightarrow \Delta(\lambda)$ is called a D-*connection *on $\Delta(\lambda)$ if $\nabla_\lambda(av) = D(a)v + a\nabla_\lambda(v)$ $(a \in B,\ v \in \Delta(\lambda))$. Equivalently, $\nabla_\lambda\rho_\lambda(a) - \rho_\lambda(a)\nabla_\lambda = \rho_\lambda(D(a))$ $(a \in B)$.*

Since B is generated by $U_1, \dots, U_{n-1}, e$, it is enough to verify $\nabla_\lambda\rho_\lambda(U_i) - \rho_\lambda(U_i)\nabla_\lambda = \rho_\lambda(D(U_i))$ $(1 \le i \le n-1)$ and $\nabla_\lambda\rho_\lambda(e) - \rho_\lambda(e)\nabla_\lambda = \rho_\lambda(D(e))$.

Proposition 4.4. *Let $D \in \mathrm{Der}(B)$ be cellular. Fix $T_0 \in \mathcal{T}(\lambda)$. Suppose that for every $S \in \mathcal{T}(\lambda)$ there exist scalars $d_\lambda(S', S) \in \mathbb{F}$ such that $D(c^\lambda_{S,T_0}) \equiv \sum_{S'\in\mathcal{T}(\lambda)} d_\lambda(S', S)c^\lambda_{S',T_0} \pmod{B^{<\lambda}}$. Then the formula $\nabla_\lambda(c^\lambda_S) = \sum_{S'\in\mathcal{T}(\lambda)} d_\lambda(S', S)c^\lambda_{S'}$ defines a D-connection on $\Delta(\lambda)$.*

Proof. Let $a \in B$. By the cellular multiplication rule, we have $a\,c^\lambda_{S,T_0} \equiv \sum_{R\in\mathcal{T}(\lambda)} r_a(R, S)c^\lambda_{R,T_0} \pmod{B^{<\lambda}}$. Applying D and using the cellularity of D, we obtain $D(a)c^\lambda_{S,T_0} + aD(c^\lambda_{S,T_0}) \equiv \sum_{R\in\mathcal{T}(\lambda)} r_a(R, S)D(c^\lambda_{R,T_0}) \pmod{B^{<\lambda}}$. Passing to the standard module gives $D(a)c^\lambda_S + a\nabla_\lambda(c^\lambda_S) = \nabla_\lambda(ac^\lambda_S)$. Therefore $\nabla_\lambda(ac^\lambda_S) = D(a)c^\lambda_S + a\nabla_\lambda(c^\lambda_S)$ for every basis vector c^λ_S. By linearity, the same identity holds for every $v \in \Delta(\lambda)$. Hence ∇_λ is a D-connection. □

4.3. The cellular bilinear form and radical stability

Let $\phi_\lambda : \Delta(\lambda) \times \Delta(\lambda) \longrightarrow \mathbb{F}$ be the cellular bilinear form. It is characterized by $c^\lambda_{S,T}c^\lambda_{U,V} \equiv \phi_\lambda(c^\lambda_T, c^\lambda_U)c^\lambda_{S,V} \pmod{B^{<\lambda}}$. The radical of $\Delta(\lambda)$ is $\mathrm{rad}\,\Delta(\lambda) = \{v \in \Delta(\lambda) \mid \phi_\lambda(v, w) = 0 \text{ for all } w \in \Delta(\lambda)\}$. Whenever $\phi_\lambda \neq 0$, the corresponding simple module is $L(\lambda) = \Delta(\lambda)/\,\mathrm{rad}\,\Delta(\lambda)$.

Lemma 4.5. *For all $a \in B$ and $v, w \in \Delta(\lambda)$, one has $\phi_\lambda(av, w) = \phi_\lambda(v, \iota(a)w)$, where ι is the cellular anti-involution.*

Proof. It is enough to prove the identity of the basis elements. Let $v = c^\lambda_S$ and $w = c^\lambda_U$. By the cellular multiplication rule, the coefficients describing the action of a on the first variable are the same as the coefficients describing the action of $\iota(a)$ on the second variable, because $\iota(c^\lambda_{S,T}) = c^\lambda_{T,S}$. Therefore $\phi_\lambda(ac^\lambda_S, c^\lambda_U) = \phi_\lambda(c^\lambda_S, \iota(a)c^\lambda_U)$. By bilinearity, the result follows for all $v, w \in \Delta(\lambda)$. □

Theorem 4.6. *Let $D \in \mathrm{Der}(B)$, and suppose that $\Delta(\lambda)$ admits a D-connection $\nabla_\lambda : \Delta(\lambda) \longrightarrow \Delta(\lambda)$. Then $\nabla_\lambda(\mathrm{rad}\,\Delta(\lambda)) \subseteq \mathrm{rad}\,\Delta(\lambda)$ if and only if $\phi_\lambda(\nabla_\lambda r, w) = 0$ $(r \in \mathrm{rad}\,\Delta(\lambda),\ w \in \Delta(\lambda))$. In particular, the stronger identity $\phi_\lambda(\nabla_\lambda v, w) + \phi_\lambda(v, \nabla_\lambda w) = 0$ $(v, w \in \Delta(\lambda))$ implies $\nabla_\lambda(\mathrm{rad}\,\Delta(\lambda)) \subseteq \mathrm{rad}\,\Delta(\lambda)$.*

Proof. By definition, we have $r \in \mathrm{rad}\,\Delta(\lambda)$ if and only if $\phi_\lambda(r, w) = 0$ $(w \in \Delta(\lambda))$. Therefore $\nabla_\lambda r \in \mathrm{rad}\,\Delta(\lambda)$ if and only if $\phi_\lambda(\nabla_\lambda r, w) = 0$ $(w \in \Delta(\lambda))$. This proves the equivalence.

Now assume that $\phi_\lambda(\nabla_\lambda v, w) + \phi_\lambda(v, \nabla_\lambda w) = 0$ $(v, w \in \Delta(\lambda))$. Let $r \in \mathrm{rad}\,\Delta(\lambda)$. Then $\phi_\lambda(r, \nabla_\lambda w) = 0$ $(w \in \Delta(\lambda))$. Hence $\phi_\lambda(\nabla_\lambda r, w) = -\phi_\lambda(r, \nabla_\lambda w) = 0$. Thus $\nabla_\lambda r \in \mathrm{rad}\,\Delta(\lambda)$. Therefore, the radical is ∇_λ-stable. □

Theorem 4.7. *Assume that $D \circ \iota = \iota \circ D$. Let $\nabla_\lambda : \Delta(\lambda) \longrightarrow \Delta(\lambda)$ be a D-connection. Define $\psi_\lambda : \Delta(\lambda) \times \Delta(\lambda) \longrightarrow \mathbb{F}$ by $\psi_\lambda(v, w) = \phi_\lambda(\nabla_\lambda v, w) + \phi_\lambda(v, \nabla_\lambda w)$. Then $\psi_\lambda(av, w) = \psi_\lambda(v, \iota(a)w)$ $(a \in B,\ v, w \in \Delta(\lambda))$. Thus ψ_λ is a contravariant bilinear form on $\Delta(\lambda)$.*

Proof. Let $a \in B$ and $v, w \in \Delta(\lambda)$. Since ∇_λ is a D-connection, $\nabla_\lambda(av) = D(a)v + a\nabla_\lambda(v)$. Therefore $\psi_\lambda(av, w) = \phi_\lambda(D(a)v, w) + \phi_\lambda(a\nabla_\lambda(v), w) + \phi_\lambda(av, \nabla_\lambda(w))$. Using Lemma 4.5, we get $\phi_\lambda(D(a)v, w) = \phi_\lambda(v, \iota(D(a))w)$, $\phi_\lambda(a\nabla_\lambda(v), w) = \phi_\lambda(\nabla_\lambda(v), \iota(a)w)$, and $\phi_\lambda(av, \nabla_\lambda(w)) = \phi_\lambda(v, \iota(a)\nabla_\lambda(w))$. Since $D \circ \iota = \iota \circ D$, we have $\iota(D(a)) = D(\iota(a))$. Thus $\psi_\lambda(av, w) = \phi_\lambda(v, D(\iota(a))w) + \phi_\lambda(\nabla_\lambda(v), \iota(a)w) + \phi_\lambda(v, \iota(a)\nabla_\lambda(w))$. Again using the D-connection rule, we have $\nabla_\lambda(\iota(a)w) = D(\iota(a))w + \iota(a)\nabla_\lambda(w)$. Hence $\psi_\lambda(av, w) = \phi_\lambda(\nabla_\lambda(v), \iota(a)w) + \phi_\lambda(v, \nabla_\lambda(\iota(a)w))$. This is precisely $\psi_\lambda(v, \iota(a)w)$. The theorem follows. □

Remark 4.8. *The condition $D \circ \iota = \iota \circ D$ is an equivariance condition: the derivation commutes with the cellular anti-involution. For the blob algebra, ι is diagrammatic reflection, so the condition says that differentiating and reflecting give the same result. It should not be confused with any claim that D "acts trivially on the structure constants" of the cellular basis. The proof of Theorem 4.7 uses only the equivariance identity $\iota(D(a)) = D(\iota(a))$ together with contravariance of ϕ_λ.*

Corollary 4.9. *Assume that $D \circ \iota = \iota \circ D$. Let ∇_λ be a D-connection on $\Delta(\lambda)$. Suppose that the space of contravariant bilinear forms on $\Delta(\lambda)$ is one-dimensional and is spanned by ϕ_λ. Then there exists a scalar $\chi_\lambda(D, \nabla_\lambda) \in \mathbb{F}$ such that $\phi_\lambda(\nabla_\lambda v, w) + \phi_\lambda(v, \nabla_\lambda w) = \chi_\lambda(D, \nabla_\lambda)\phi_\lambda(v, w)$ $(v, w \in \Delta(\lambda))$. Consequently, we have $\nabla_\lambda(\mathrm{rad}\,\Delta(\lambda)) \subseteq \mathrm{rad}\,\Delta(\lambda)$. In particular, if $\chi_\lambda(D, \nabla_\lambda) = 0$, then ∇_λ is skew-adjoint with respect to ϕ_λ.*

Proof. By Theorem 4.7, the bilinear form $\psi_\lambda(v, w) = \phi_\lambda(\nabla_\lambda v, w) + \phi_\lambda(v, \nabla_\lambda w)$ is contravariant. By assumption, the space of contravariant bilinear forms on $\Delta(\lambda)$ is one-dimensional and spanned by ϕ_λ. Therefore, there exists a scalar $\chi_\lambda(D, \nabla_\lambda) \in \mathbb{F}$ such that $\psi_\lambda = \chi_\lambda(D, \nabla_\lambda)\phi_\lambda$. Equivalently, $\phi_\lambda(\nabla_\lambda v, w) + \phi_\lambda(v, \nabla_\lambda w) = \chi_\lambda(D, \nabla_\lambda)\phi_\lambda(v, w)$.

Let $r \in \mathrm{rad}\,\Delta(\lambda)$. Then $\phi_\lambda(r, w) = 0$ $(w \in \Delta(\lambda))$. Substituting $v = r$, we get $\phi_\lambda(\nabla_\lambda r, w) + \phi_\lambda(r, \nabla_\lambda w) = \chi_\lambda(D, \nabla_\lambda)\phi_\lambda(r, w)$. The second term on the left is zero, and the right-hand side is also zero. Therefore $\phi_\lambda(\nabla_\lambda r, w) = 0$ $(w \in \Delta(\lambda))$. Hence $\nabla_\lambda r \in \mathrm{rad}\,\Delta(\lambda)$. Thus the radical is ∇_λ-stable.

If $\chi_\lambda(D, \nabla_\lambda) = 0$, then $\phi_\lambda(\nabla_\lambda v, w) + \phi_\lambda(v, \nabla_\lambda w) = 0$, so ∇_λ is skew-adjoint with respect to ϕ_λ. □

4.4. Descent to simple modules

Theorem 4.10. *Assume $\phi_\lambda \neq 0$, so that $L(\lambda) = \Delta(\lambda)/\operatorname{rad}\Delta(\lambda)$ is the corresponding nonzero simple module. Let $D \in \operatorname{Der}(B)$, and suppose that $\Delta(\lambda)$ admits a D-connection $\nabla_\lambda : \Delta(\lambda) \longrightarrow \Delta(\lambda)$. Then ∇_λ descends to a well-defined $\mathbb{F}$-linear map $\overline{\nabla}_\lambda : L(\lambda) \longrightarrow L(\lambda)$ given by $\overline{\nabla}_\lambda(v+\operatorname{rad}\Delta(\lambda)) = \nabla_\lambda(v)+\operatorname{rad}\Delta(\lambda)$ if and only if $\nabla_\lambda(\operatorname{rad}\Delta(\lambda)) \subseteq \operatorname{rad}\Delta(\lambda)$. When this condition holds, one has $\overline{\nabla}_\lambda(a\overline{v}) = D(a)\overline{v}+a\overline{\nabla}_\lambda(\overline{v})$ ($a \in B$, $\overline{v} \in L(\lambda)$).*

Proof. A linear map on a vector space descends to the quotient by a subspace if and only if that subspace is invariant under the map. Hence ∇_λ descends to $L(\lambda)$ if and only if $\nabla_\lambda(\operatorname{rad}\Delta(\lambda)) \subseteq \operatorname{rad}\Delta(\lambda)$.

Assume that this condition holds. Then the formula $\overline{\nabla}_\lambda(v + \operatorname{rad}\Delta(\lambda)) = \nabla_\lambda(v) + \operatorname{rad}\Delta(\lambda)$ is well defined. Let $\overline{v} = v + \operatorname{rad}\Delta(\lambda)$. Using the D-connection rule on $\Delta(\lambda)$, we obtain $\overline{\nabla}_\lambda(a\overline{v}) = \overline{\nabla}_\lambda(av + \operatorname{rad}\Delta(\lambda)) = \nabla_\lambda(av) + \operatorname{rad}\Delta(\lambda) = D(a)v + a\nabla_\lambda(v) + \operatorname{rad}\Delta(\lambda) = D(a)\overline{v} + a\overline{\nabla}_\lambda(\overline{v})$. This proves the quotient Leibniz rule. ◻

Remark 4.11. *The descent of a D-connection to a simple module is not automatic from the cellularity of D. One first needs a D-connection on the standard module $\Delta(\lambda)$, and then one must verify that the radical $\operatorname{rad}\Delta(\lambda)$ is stable under this connection.*

5. Outer derivations in the fully degenerate case

In this section, we consider the fully degenerate parameter regime $\delta = \kappa = 0$. We write $B_n^0 = \mathsf{B}_n(q, 0, 0)$. Thus the blob generator satisfies $e^2 = 0$, $U_1eU_1 = 0$, and $eU_i = U_ie$ ($2 \leq i \leq n-1$). Throughout this section we assume that q is not a root of unity. Hence the Temperley–Lieb subalgebra $T_n = \mathsf{TL}_n = \langle U_1, \dots, U_{n-1}\rangle$ is semisimple.

The purpose of this section is to explain how outer derivations arise in the degenerate case. We first give a general reduction, and then we give a complete calculation in rank two.

5.1. Reduction to derivations vanishing on the Temperley–Lieb part

Let $D \in \operatorname{Der}(B_n^0)$. Its restriction to T_n is a derivation $D|_{T_n} : T_n \longrightarrow B_n^0$, where B_n^0 is regarded as a T_n-bimodule by left and right multiplication. Since T_n is semisimple, it is separable over $\mathbb{F}$. Consequently, we have $HH^1(T_n, M) = 0$ for every T_n-bimodule M. In particular, for $M = B_n^0$, there exists an element $a \in B_n^0$ such that $D(x) = [a, x]$ ($x \in T_n$). Therefore, after replacing D by $D - \operatorname{ad}_a$, we may assume that $D(U_i) = 0$ ($1 \leq i \leq n-1$).

Thus every outer derivation class has a representative which vanishes on the Temperley–Lieb generators. Such representatives are controlled by the possible values of $D(e)$.

Lemma 5.1. *Let $D \in \operatorname{Der}(B_n^0)$ satisfy $D(U_i) = 0$ ($1 \leq i \leq n-1$). Put $x = D(e)$. Then $x \in B_n^0$ satisfies*

$$xe + ex = 0, \tag{5.1}$$

$$U_1xU_1 = 0, \tag{5.2}$$

$$xU_i = U_ix,\ 2 \leq i \leq n-1. \tag{5.3}$$

Conversely, if $x \in B_n^0$ satisfies (5.1)–(5.3)*, then the assignment $D_x(U_i) = 0$ ($1 \leq i \leq n-1$), $D_x(e) = x$ extends uniquely to a derivation of B_n^0.*

Proof. Assume first that $D \in \mathrm{Der}(B_n^0)$ and that $D(U_i) = 0$ for every i. Since $e^2 = 0$, we get $0 = D(e^2) = D(e)e + eD(e)$. Thus $xe + ex = 0$. Next, applying D to $U_1eU_1 = 0$ gives $D(U_1)eU_1 + U_1D(e)U_1 + U_1eD(U_1) = 0$. Since $D(U_1) = 0$, this becomes $U_1xU_1 = 0$. Finally, applying D to $eU_i = U_ie$ $(2 \leq i \leq n-1)$ gives $D(e)U_i + eD(U_i) = D(U_i)e + U_iD(e)$. Again using $D(U_i) = 0$, we obtain $xU_i = U_ix$.

Conversely, suppose that $x \in B_n^0$ satisfies (5.1)–(5.3). Define $D_x(U_i) = 0$ $(1 \leq i \leq n-1)$, $D_x(e) = x$. The derived defining relations of the blob algebra are then satisfied. Indeed, the Temperley–Lieb relations are automatically preserved because $D_x(U_i) = 0$. The relation $e^2 = 0$ gives precisely $xe + ex = 0$. The relation $U_1eU_1 = 0$ gives precisely $U_1xU_1 = 0$. The relations $eU_i = U_ie$ for $i \geq 2$ give precisely $xU_i = U_ix$. Therefore, by the extension criterion for derivations (Theorem 3.1), the assignment extends uniquely to a derivation of B_n^0. □

Define the vector space $\mathcal{N}_n = \{x \in B_n^0 \mid xe + ex = 0,\ U_1xU_1 = 0,\ xU_i = U_ix \text{ for } 2 \leq i \leq n-1\}$. Then Lemma 5.1 gives an isomorphism between $\mathcal{N}_n$ and the space of derivations of B_n^0 which vanish on T_n.

Let $C_n = C_{B_n^0}(T_n) = \{z \in B_n^0 \mid zt = tz \text{ for all } t \in T_n\}$ be the centralizer of the Temperley–Lieb subalgebra in B_n^0. If $z \in C_n$, then the inner derivation ad_z vanishes on T_n, and $\mathrm{ad}_z(e) = ze - ez$. Thus the relative inner derivations correspond to the subspace $[C_n, e] = \{ze - ez \mid z \in C_n\} \subseteq \mathcal{N}_n$.

Theorem 5.2. *Assume that q is not a root of unity. Then every class in* $\mathrm{Out}(B_n^0) = \mathrm{Der}(B_n^0)/\mathrm{Inn}(B_n^0)$ *has a representative D_x satisfying $D_x(U_i) = 0$ $(1 \leq i \leq n-1)$, $D_x(e) = x$ for some $x \in \mathcal{N}_n$. Moreover,* $\mathrm{Out}(B_n^0) \cong \mathcal{N}_n/[C_n, e]$ *as $\mathbb{F}$-vector spaces.*

Proof. Let $D \in \mathrm{Der}(B_n^0)$. As argued before the lemma, the restriction $D|_{T_n}$ is a derivation of T_n with values in the T_n-bimodule B_n^0. Because T_n is semisimple and $\mathbb{F}$ has characteristic zero, the algebra T_n is separable, and therefore its Hochschild cohomology $HH^1(T_n, B_n^0)$ vanishes. Hence there exists an element $a \in B_n^0$ such that $D(t) = [a, t]$ $(t \in T_n)$. Thus the derivation $D' = D - \mathrm{ad}_a$ vanishes on T_n. By Lemma 5.1, $D' = D_x$ for some $x \in \mathcal{N}_n$. This shows that every outer class contains a representative of the required form.

Now define a linear map $\Phi : \mathcal{N}_n \longrightarrow \mathrm{Out}(B_n^0)$, $\Phi(x) = [D_x] = D_x + \mathrm{Inn}(B_n^0)$. We claim that Φ is surjective and that its kernel is exactly $[C_n, e]$.

Surjectivity. Given an outer class $[D]$, pick D_x as above. Then $[D] = [D_x] = \Phi(x)$. Hence Φ is surjective.

Kernel. Suppose $\Phi(x) = 0$. Then D_x is inner, so there exists $z \in B_n^0$ with $D_x = \mathrm{ad}_z$. Since D_x vanishes on T_n, we have $[z, t] = 0$ for all $t \in T_n$, i.e., $z \in C_n$. Evaluating at e yields $x = D_x(e) = \mathrm{ad}_z(e) = ze - ez \in [C_n, e]$. Conversely, if $x = ze - ez$ with $z \in C_n$, then $D_x = \mathrm{ad}_z$ is inner, so $\Phi(x) = 0$. Therefore, $\ker\Phi = [C_n, e]$.

By the first isomorphism theorem, we have $\mathrm{Out}(B_n^0) \cong \mathcal{N}_n/[C_n, e]$. This completes the proof. □

Remark 5.3. *Theorem 5.2 is the correct general statement. One should not claim that* $\mathrm{Out}(B_n^0)$ *is always three-dimensional without an explicit computation of both $\mathcal{N}_n$ and the centralizer C_n. In particular, the element eU_1e is not forced to be zero by the defining relations when $\delta = \kappa = 0$. Hence the separate maps defined by $e \mapsto U_1e$ and $e \mapsto eU_1$ are not, in general, derivations. The correct rank-two calculation is given below.*

5.2. The rank-two fully degenerate case

We now give a complete calculation for $B_2^0 = \mathsf{B}_2(q,0,0)$. Put $\theta = [2]_q = q + q^{-1}$. Since q is not a root of unity, we have $\theta \neq 0$. Write $U = U_1$. Then B_2^0 is generated by U and e, with the defining relations $U^2 = \theta U, e^2 = 0, UeU = 0$. By Lemma 3.9, specialized to $\delta = \kappa = 0$, a basis is $\mathcal{B} = \{1, U, e, Ue, eU, W\},\ W = eUe$.

Theorem 5.4. *Every derivation $D \in \mathrm{Der}(B_2^0)$ is uniquely of the form $D(U) = b\,Ue + c\,eU$, and $D(e) = h\,e + r(Ue - eU) + tW$, where $b, c, h, r, t \in \mathbb{F}$. Hence $\dim_{\mathbb{F}} \mathrm{Der}(B_2^0) = 5$.*

Proof. Let $D(U) = a_0\,1 + a_1U + a_2e + a_3Ue + a_4eU + a_5W$ and $D(e) = b_0\,1 + b_1U + b_2e + b_3Ue + b_4eU + b_5W$.

Since $U^2 = \theta U$, applying D gives $D(U)U + UD(U) = \theta D(U)$. Using the basis $\mathcal{B}$ and the relations $U^2 = \theta U,\ e^2 = 0,\ UeU = 0$, one obtains $a_0 = 0,\ a_1 = 0,\ a_2 = 0,\ a_5 = 0$. Therefore $D(U) = a_3Ue + a_4eU$. Put $b = a_3,\ c = a_4$. Thus $D(U) = bUe + ceU$.

Next, since $e^2 = 0$, applying D gives $D(e)e + eD(e) = 0$. Expanding this identity in the basis $\mathcal{B}$ yields $b_0 = 0,\ b_1 = 0,\ b_3 + b_4 = 0$. Therefore $D(e) = b_2e + b_3(Ue - eU) + b_5W$. Put $h = b_2,\ r = b_3,\ t = b_5$. Then $D(e) = he + r(Ue - eU) + tW$.

Finally, applying D to $UeU = 0$ gives $D(U)eU + UD(e)U + UeD(U) = 0$. Substituting the above expressions for $D(U)$ and $D(e)$, every term contains either e^2 or UeU, and hence the identity is automatically satisfied. Thus every derivation has the displayed form.

Conversely, any assignment $D(U) = bUe + ceU,\ D(e) = he + r(Ue - eU) + tW$ satisfies the derived relations coming from $U^2 = \theta U,\ e^2 = 0,\ UeU = 0$. Hence, by the extension criterion, it extends uniquely to a derivation of B_2^0. The five scalars b, c, h, r, t are free, so $\dim_{\mathbb{F}} \mathrm{Der}(B_2^0) = 5$. □

Proposition 5.5. *The inner derivations of B_2^0 form a three-dimensional subspace of $\mathrm{Der}(B_2^0)$. More precisely, they are spanned by $\mathrm{ad}_e, \mathrm{ad}_U$, and ad_{Ue}. Their values on the generators are $\mathrm{ad}_e(U) = eU - Ue,\ \mathrm{ad}_e(e) = 0, \mathrm{ad}_U(U) = 0,\ \mathrm{ad}_U(e) = Ue - eU$, and $\mathrm{ad}_{Ue}(U) = -\theta Ue,\ \mathrm{ad}_{Ue}(e) = -W$. Consequently, we have $\dim_{\mathbb{F}} \mathrm{Inn}(B_2^0) = 3$.*

Proof. The abovementioned formulas follow directly from $\mathrm{ad}_x(y) = xy - yx$ and the defining relations $U^2 = \theta U,\ e^2 = 0,\ UeU = 0$. Consider, $\mathrm{ad}_e(U) = eU - Ue$, and $\mathrm{ad}_U(e) = Ue - eU$. Moreover, $\mathrm{ad}_{Ue}(U) = UeU - U^2e = 0 - \theta Ue = -\theta Ue$, and $\mathrm{ad}_{Ue}(e) = Uee - eUe = 0 - W = -W$.

These three inner derivations are linearly independent. Indeed, ad_e is the only one among them with a nonzero value $eU - Ue$ on U, ad_U is the only one with a nonzero $Ue - eU$ component on e and zero value on U, and ad_{Ue} has a nonzero W-component on e. Therefore $\dim_{\mathbb{F}} \mathrm{Inn}(B_2^0) \geq 3$. On the other hand, direct expansion of ad_x for $x = x_0\,1 + x_1U + x_2e + x_3Ue + x_4eU + x_5W$ shows that every inner derivation is a linear combination of these three. Hence $\dim_{\mathbb{F}} \mathrm{Inn}(B_2^0) = 3$. □

Theorem 5.6. *For the fully degenerate rank-two blob algebra $B_2^0 = \mathsf{B}_2(q,0,0)$, one has $\dim_{\mathbb{F}} \mathrm{Out}(B_2^0) = 2$. A basis of $\mathrm{Out}(B_2^0)$ is represented by the two derivations ∂_1, ∂_2 defined by $\partial_1(U) = 0,\ \partial_1(e) = e$, and $\partial_2(U) = 0,\ \partial_2(e) = W = eUe$.*

Proof. By Theorem 5.4, we have $\dim_{\mathbb{F}} \mathrm{Der}(B_2^0) = 5$. By Proposition 5.5, we have $\dim_{\mathbb{F}} \mathrm{Inn}(B_2^0) = 3$. Therefore $\dim_{\mathbb{F}} \mathrm{Out}(B_2^0) = \dim_{\mathbb{F}} \mathrm{Der}(B_2^0) - \dim_{\mathbb{F}} \mathrm{Inn}(B_2^0) = 2$.

It remains to identify convenient representatives. By Theorem 5.4, a derivation is given by $D(U) = bUe + ceU,\ D(e) = he + r(Ue - eU) + tW$. The inner derivations ad_e and ad_{Ue} allow us to remove the Ue and eU components of $D(U)$, while ad_U removes the $Ue - eU$ component of $D(e)$. Thus every outer class has a representative of the form $D(U) = 0$, $D(e) = he + tW$. The two choices $(h, t) = (1, 0)$ and $(h, t) = (0, 1)$ give the derivations ∂_1 and ∂_2. To see directly that these classes are non-inner, let $x = x_0 1 + x_1 U + x_2 e + x_3 Ue + x_4 eU + x_5 W$ and suppose that $\mathrm{ad}_x(U) = 0$. Using Proposition 3.11 with $\delta = \kappa = 0$, comparison of the Ue- and eU-coefficients gives $x_3 = x_4 = -\frac{x_2}{\theta}$. Substitution in $\mathrm{ad}_x(e)$ then yields $\mathrm{ad}_x(e) = x_1(Ue - eU)$. Thus every inner derivation that vanishes on U takes e into $\mathbb{F}(Ue - eU)$, whereas e and W are linearly independent from that subspace by Lemma 3.9. Hence the classes of ∂_1 and ∂_2 are linearly independent in $\mathrm{Out}(B_2^0)$. Since the outer derivation space has dimension two, they form a basis. □

5.3. *Lie algebra structure in rank two*

Theorem 5.7. *The Lie algebra* $\mathrm{Out}(B_2^0)$ *is two-dimensional and non-Abelian. If* $[\partial_i]$ *denotes the class of* ∂_i *modulo inner derivations, then* $[[\partial_1], [\partial_2]] = [\partial_2] \neq 0$.

Proof. By Theorem 5.6 and Proposition 3.6, we have $\mathrm{Out}(B_2^0) = \mathbb{F}[\partial_1] \oplus \mathbb{F}[\partial_2]$, where $[\partial_i]$ denotes the class of ∂_i modulo inner derivations. For brevity, the displayed direct sum uses the conventional notation $\mathbb{F}[\partial_i]$ for the span of that class. The chosen representatives satisfy $\partial_1(U) = 0$, $\partial_1(e) = e$, and $\partial_2(U) = 0$, $\partial_2(e) = W = eUe$. We compute their commutator in $\mathrm{Der}(B_2^0)$. Since both derivations vanish on U, their commutator also vanishes on U. Moreover, $[\partial_1, \partial_2](e) = \partial_1(\partial_2(e)) - \partial_2(\partial_1(e))$. Now $\partial_2(e) = W = eUe$. Using $\partial_1(U) = 0$ and $\partial_1(e) = e$, we get $\partial_1(W) = \partial_1(eUe) = \partial_1(e)Ue + e\partial_1(U)e + eU\partial_1(e) = eUe + eUe = 2W$. We also have, $\partial_2(\partial_1(e)) = \partial_2(e) = W$. Therefore $[\partial_1, \partial_2](e) = 2W - W = W$. Thus in $\mathrm{Der}(B_2^0)$, $[\partial_1, \partial_2] = \partial_2$. Since ∂_2 is outer (not inner), its class in $\mathrm{Out}(B_2^0)$ is non-zero. Hence the induced bracket on the quotient satisfies $[[\partial_1], [\partial_2]] = [\partial_2] \neq 0$ in $\mathrm{Out}(B_2^0)$. Therefore $\mathrm{Out}(B_2^0)$ is non-Abelian, with the given bracket. □

6. Transfer to the Deguchi–Martin quotient

Alotaibi's basis correspondence is proved over $\mathbb{C}$. In this section, whenever that correspondence is invoked we therefore assume $\mathbb{F} = \mathbb{C}$, or, more generally, that a compatible base change has been fixed so that the same canonical bases and basis bijection are available over $\mathbb{F}$.

In this section, we explain what can, and what cannot, be transferred from the blob algebra to the Deguchi–Martin quotient. The main point is that a combinatorial bijection between cellular bases transfers vector-space data and standard-module data, but it does not by itself transfer the algebra derivations. For derivations, one needs either an algebra homomorphism or a derivation of the Hecke algebra which preserves the defining ideal of the quotient.

6.1. *Combinatorial correspondence in the* (1|1)*-case*

Let $\mathcal{H}_n(q)$ be the Hecke algebra of type A_{n-1}, and let $H_n^{1,1}$ denote the Deguchi–Martin quotient in the (1|1)-case. Let $\pi : \mathcal{H}_n(q) \longrightarrow H_n^{1,1}$ be the canonical quotient map.

In the (1|1)-case, the cell modules of $H_n^{1,1}$ are indexed by the hook partitions $\lambda = (r, 1^{n-r})$, $1 \le r \le n$. The corresponding blob-algebra label is $t = t(\lambda) = n + 1 - 2r$. Thus $t \in \Lambda_{n-1} = \{\, n-1, n-3, \ldots, 3-n, 1-n \,\}$. We denote the corresponding blob cell module by $\Delta_{n-1}(t)$, and the corresponding Deguchi–Martin cell module by R_n^{λ}.

The cellular bases of these two modules are indexed by the same combinatorial objects. More precisely, the canonical half-diagram basis of $\Delta_{n-1}(t)$ is in bijection with the standard word basis of R_n^λ. We write this bijection as $\Phi_\lambda : \Delta_{n-1}(t) \longrightarrow R_n^\lambda$.

Theorem 6.1 (Alotaibi's basis correspondence)**.** *Let $\lambda = (r, 1^{n-r})$ and $t = n+1-2r$. Then the canonical basis of the blob cell module $\Delta_{n-1}(t)$ is in bijection with the canonical word basis of the Deguchi–Martin cell module R_n^λ. Consequently, this basis bijection extends uniquely to a $\mathbb{F}$-linear isomorphism $\Phi_\lambda : \Delta_{n-1}(t) \longrightarrow R_n^\lambda$.*

Proof. The basis bijection is Alotaibi's Theorem 3.4.1 [13], with the parameters related by $t = n+1-2r$. Under the field convention stated at the beginning of this section, extending the basis bijection linearly gives the asserted $\mathbb{F}$-linear isomorphism. No algebra-isomorphism assertion is used here. □

Remark 6.2. *The map Φ_λ is a vector-space isomorphism between cell modules. It is not, by itself, an algebra isomorphism between the blob algebra and the Deguchi–Martin quotient. Therefore, one cannot define a derivation of $H_n^{1,1}$ merely by conjugating a derivation of the blob algebra through Φ_λ. For that, an actual algebra map is required.*

6.2. Descent of derivations to the Deguchi–Martin quotient

We now state the correct quotient-level derivation result. Let $I_{1,1} = \ker \pi$ be the defining two-sided ideal of the Deguchi–Martin quotient $H_n^{1,1} = \mathcal{H}_n(q)/I_{1,1}$.

Theorem 6.3. *Let $d : \mathcal{H}_n(q) \longrightarrow \mathcal{H}_n(q)$ be an $\mathbb{F}$-linear derivation. Then d induces a well-defined derivation $\overline{d} : H_n^{1,1} \longrightarrow H_n^{1,1}$ if and only if $d(I_{1,1}) \subseteq I_{1,1}$. In that case, we have $\overline{d}(\pi(h)) = \pi(d(h))$ $(h \in \mathcal{H}_n(q))$.*

Proof. Assume first that $d(I_{1,1}) \subseteq I_{1,1}$. Define $\overline{d}(\pi(h)) = \pi(d(h))$. We show that this is well defined. Suppose that $\pi(h) = \pi(h')$. Then $h - h' \in I_{1,1}$. By the hypothesis, $d(h - h') \in I_{1,1}$. Therefore $\pi(d(h)) = \pi(d(h'))$. Hence $\overline{d}$ is well defined.

For $h_1, h_2 \in \mathcal{H}_n(q)$, we have $\overline{d}(\pi(h_1)\pi(h_2)) = \overline{d}(\pi(h_1h_2)) = \pi(d(h_1h_2))$. Since d is a derivation, we have $d(h_1h_2) = d(h_1)h_2 + h_1d(h_2)$. Thus $\overline{d}(\pi(h_1)\pi(h_2)) = \pi(d(h_1))\pi(h_2) + \pi(h_1)\pi(d(h_2))$. Therefore $\overline{d}(ab) = \overline{d}(a)b + a\overline{d}(b)$ $(a, b \in H_n^{1,1})$. Hence $\overline{d}$ is a derivation.

Conversely, suppose that d induces a well-defined derivation $\overline{d} : H_n^{1,1} \longrightarrow H_n^{1,1}$ with $\overline{d}(\pi(h)) = \pi(d(h))$. If $x \in I_{1,1}$, then $\pi(x) = 0$. Hence $0 = \overline{d}(0) = \overline{d}(\pi(x)) = \pi(d(x))$. Therefore $d(x) \in I_{1,1}$. Thus $d(I_{1,1}) \subseteq I_{1,1}$. This proves the equivalence. □

6.3. Transport through an algebra isomorphism

The previous theorem concerns derivations of the Hecke algebra which descend to the Deguchi–Martin quotient. We now record the separate situation in which a derivation of a blob algebra can be transported to the Deguchi–Martin quotient.

Suppose there is an algebra isomorphism $\Psi : B \longrightarrow H_n^{1,1}$, where B is the appropriate blob-algebra realization of the $(1|1)$-sector. Then any derivation of B can be transported to $H_n^{1,1}$.

Theorem 6.4. *Let $\Psi : B \longrightarrow H_n^{1,1}$ be an algebra isomorphism. If $D \in \mathrm{Der}(B)$, then $D^\Psi = \Psi \circ D \circ \Psi^{-1}$ is a derivation of $H_n^{1,1}$. Moreover, if D is inner, then D^Ψ is inner.*

Proof. Let $a, b \in H_n^{1,1}$. Put $x = \Psi^{-1}(a)$, $y = \Psi^{-1}(b)$. Then $D^{\Psi}(ab) = \Psi D \Psi^{-1}(ab) = \Psi D(xy)$. Since D is a derivation, we have $D(xy) = D(x)y + xD(y)$. Therefore $D^{\Psi}(ab) = \Psi(D(x)y + xD(y))$. Using the fact that Ψ is an algebra homomorphism, we obtain $D^{\Psi}(ab) = \Psi(D(x))\Psi(y) + \Psi(x)\Psi(D(y))$. Hence $D^{\Psi}(ab) = D^{\Psi}(a)b + aD^{\Psi}(b)$. Thus D^{Ψ} is a derivation.

Now suppose that $D = \mathrm{ad}_u$ for some $u \in B$. Then, for $a \in H_n^{1,1}$, we have $D^{\Psi}(a) = \Psi(u\Psi^{-1}(a) - \Psi^{-1}(a)u) = \Psi(u)a - a\Psi(u)$. Hence $D^{\Psi} = \mathrm{ad}_{\Psi(u)}$. Therefore, inner derivations are transported to inner derivations. □

Remark 6.5. *Theorem 6.4 requires an actual algebra isomorphism. The basis bijection of Theorem 6.1 is not sufficient for this purpose. Moreover, the existence of such an isomorphism is not asserted here for arbitrary δ, κ. The small-rank algebra isomorphisms studied in [13] use a specific normalization and specific parameter values; any application of the theorem must first match those parameters with the presentation of $\mathsf{B}_n(q, \delta, \kappa)$ used in this article.*

6.4. Transport of D-connections on standard modules

A basis bijection alone does not transport algebra derivations. The following statement records the precise additional hypotheses under which a connection can be transported.

Theorem 6.6. *Let $\Psi : B \longrightarrow H_n^{1,1}$ be an algebra isomorphism and let $D \in \mathrm{Der}(B)$. Put $\overline{d} = \Psi \circ D \circ \Psi^{-1} \in \mathrm{Der}(H_n^{1,1})$. Assume that $\Phi_\lambda : \Delta_{n-1}(t) \longrightarrow R_n^\lambda$ is an $\mathbb{F}$-linear isomorphism which intertwines the module actions, namely*

$$\Phi_\lambda(bv) = \Psi(b)\Phi_\lambda(v) \; (b \in B,\ v \in \Delta_{n-1}(t)). \tag{6.1}$$

If $\nabla_\lambda : \Delta_{n-1}(t) \longrightarrow \Delta_{n-1}(t)$ is a D-connection, then $\widetilde{\nabla}_\lambda = \Phi_\lambda \circ \nabla_\lambda \circ \Phi_\lambda^{-1}$ is an $\overline{d}$-connection on R_n^λ; equivalently, $\widetilde{\nabla}_\lambda(aw) = \overline{d}(a)w + a\widetilde{\nabla}_\lambda(w)$ $(a \in H_n^{1,1},\ w \in R_n^\lambda)$.

Proof. Let $a \in H_n^{1,1}$ and $w \in R_n^\lambda$. Since Ψ and Φ_λ are surjective, write $a = \Psi(b)$, $w = \Phi_\lambda(v)$ for some $b \in B$ and $v \in \Delta_{n-1}(t)$. By (6.1),

$$\begin{aligned}\widetilde{\nabla}_\lambda(aw) &= \Phi_\lambda \nabla_\lambda \Phi_\lambda^{-1}(\Psi(b)\Phi_\lambda(v)) \\ &= \Phi_\lambda(\nabla_\lambda(bv)).\end{aligned}$$

Because ∇_λ is a D-connection, $\nabla_\lambda(bv) = D(b)v + b\nabla_\lambda(v)$. Applying Φ_λ and using (6.1) twice gives

$$\begin{aligned}\widetilde{\nabla}_\lambda(aw) &= \Psi(D(b))\Phi_\lambda(v) + \Psi(b)\Phi_\lambda(\nabla_\lambda(v)) \\ &= \overline{d}(\Psi(b))w + a\widetilde{\nabla}_\lambda(w) \\ &= \overline{d}(a)w + a\widetilde{\nabla}_\lambda(w).\end{aligned}$$

Hence $\widetilde{\nabla}_\lambda$ is an $\overline{d}$-connection. □

6.5. Compatibility with cellular bilinear forms and radicals

Let $\phi_t : \Delta_{n-1}(t) \times \Delta_{n-1}(t) \longrightarrow \mathbb{F}$ be the cellular bilinear form on the blob cell module, and let $\widetilde{\phi}_\lambda : R_n^\lambda \times R_n^\lambda \longrightarrow \mathbb{F}$ be the cellular bilinear form on the Deguchi–Martin cell module.

Proposition 6.7. *Assume that the basis bijection Φ_λ identifies the cellular Gram matrices of ϕ_t and $\widetilde{\phi}_\lambda$ up to a nonzero scalar. That is, assume that there exists $\gamma_\lambda \in \mathbb{F}^\times$ such that $\widetilde{\phi}_\lambda(\Phi_\lambda(v), \Phi_\lambda(w)) = \gamma_\lambda \phi_t(v, w)$ $(v, w \in \Delta_{n-1}(t))$. Then $\Phi_\lambda(\operatorname{rad} \Delta_{n-1}(t)) = \operatorname{rad} R_n^\lambda$.*

Proof. Let $v \in \operatorname{rad} \Delta_{n-1}(t)$. Then $\phi_t(v, w) = 0$ $(w \in \Delta_{n-1}(t))$. For every $z \in R_n^\lambda$, there exists $w \in \Delta_{n-1}(t)$ such that $z = \Phi_\lambda(w)$. Hence $\widetilde{\phi}_\lambda(\Phi_\lambda(v), z) = \widetilde{\phi}_\lambda(\Phi_\lambda(v), \Phi_\lambda(w)) = \gamma_\lambda \phi_t(v, w) = 0$. Thus $\Phi_\lambda(v) \in \operatorname{rad} R_n^\lambda$. Therefore $\Phi_\lambda(\operatorname{rad} \Delta_{n-1}(t)) \subseteq \operatorname{rad} R_n^\lambda$.

Conversely, let $z \in \operatorname{rad} R_n^\lambda$. Write $z = \Phi_\lambda(v)$ for some $v \in \Delta_{n-1}(t)$. For every $w \in \Delta_{n-1}(t)$, we have $0 = \widetilde{\phi}_\lambda(z, \Phi_\lambda(w)) = \widetilde{\phi}_\lambda(\Phi_\lambda(v), \Phi_\lambda(w)) = \gamma_\lambda \phi_t(v, w)$. Since $\gamma_\lambda \neq 0$, it follows that $\phi_t(v, w) = 0$ $(w \in \Delta_{n-1}(t))$. Hence $v \in \operatorname{rad} \Delta_{n-1}(t)$. Therefore $z \in \Phi_\lambda(\operatorname{rad} \Delta_{n-1}(t))$. This proves the equality. □

Theorem 6.8. *Assume the hypotheses of Proposition 6.7. Let $\nabla_\lambda : \Delta_{n-1}(t) \longrightarrow \Delta_{n-1}(t)$ be a linear map, and define $\widetilde{\nabla}_\lambda = \Phi_\lambda \circ \nabla_\lambda \circ \Phi_\lambda^{-1}$. Then $\nabla_\lambda(\operatorname{rad} \Delta_{n-1}(t)) \subseteq \operatorname{rad} \Delta_{n-1}(t)$ if and only if $\widetilde{\nabla}_\lambda(\operatorname{rad} R_n^\lambda) \subseteq \operatorname{rad} R_n^\lambda$.*

Proof. By Proposition 6.7, we have $\Phi_\lambda(\operatorname{rad} \Delta_{n-1}(t)) = \operatorname{rad} R_n^\lambda$. Therefore, $\nabla_\lambda(\operatorname{rad} \Delta_{n-1}(t)) \subseteq \operatorname{rad} \Delta_{n-1}(t)$ if and only if $\Phi_\lambda \nabla_\lambda \Phi_\lambda^{-1}(\operatorname{rad} R_n^\lambda) \subseteq \operatorname{rad} R_n^\lambda$. Since $\widetilde{\nabla}_\lambda = \Phi_\lambda \nabla_\lambda \Phi_\lambda^{-1}$, the assertion follows. □

6.6. Rank-three illustration

We finish with a rank-three illustration. Let $n = 3$. By the convention above, $H_3^{1,1} = \mathcal{H}_3(q)$, because the obstruction $(2, 2)$ has size four. Its cell modules are indexed by the hook partitions (3), $(2, 1)$, $(1, 1, 1)$. For $\lambda = (r, 1^{3-r})$, the corresponding blob label for B_2 is $t = 4 - 2r$. Therefore the correspondence is $(3) \longleftrightarrow -2$, $(2, 1) \longleftrightarrow 0$, $(1, 1, 1) \longleftrightarrow 2$.

Example 6.9. *For $n = 3$, the basis bijection identifies the standard basis of each Deguchi–Martin cell module with the standard basis of the corresponding blob cell module listed above. Hence a linear operator on a blob cell module can be transported as a linear operator by conjugation with Φ_λ.*

This basis-level transport must not be confused with transport of algebra derivations. Indeed, $H_3^{1,1} = \mathcal{H}_3(q)$ is generated as an algebra by T_1, T_2. Consequently, if $d \in \operatorname{Der}(H_3^{1,1})$ satisfies $d(T_1) = d(T_2) = 0$, then $d = 0$ on the whole algebra. In particular, no nonzero derivation of $H_3^{1,1}$ can simultaneously satisfy $d(T_1) = d(T_2) = 0$ and take a nonzero algebra element to itself.

Thus a blob derivation such as the rank-two outer derivation $\partial_1(U) = 0$, $\partial_1(e) = e$ does not become a derivation of $H_3^{1,1}$ merely by transporting cell-module basis vectors. If an algebra isomorphism $\Psi : \mathsf{B}_2 \longrightarrow H_3^{1,1}$ is available for a specified normalization and parameter choice, then the correct transported derivation is $\partial_1^\Psi = \Psi \circ \partial_1 \circ \Psi^{-1}$, and its values on T_1, T_2 must be computed from the actual formulas for $\Psi^{-1}(T_1)$ and $\Psi^{-1}(T_2)$. This is precisely the distinction between basis-level correspondence and algebra-level transport.

7. Conclusions and future directions

This article presents a detailed investigation of derivations on the blob algebra $\mathsf{B}_n(q, \delta, \kappa)$, with particular emphasis on their structural and representation-theoretic aspects. By working directly from the defining relations, we formulated a necessary and sufficient condition ensuring that a linear assignment on generators extends to a derivation, thereby translating the Leibniz property into explicit algebraic constraints.

Whenever the algebra is semisimple, we established that every derivation is inner. Consequently, the derivation space does not exhibit additional degrees of freedom beyond those induced by commutators, aligning the behaviour of B_n with that of classical semisimple algebras. A contrasting picture emerges in the singular regime $\delta = \kappa = 0$. In this setting, we provided a general reduction of the outer derivation space and computed it completely in rank two, where it is two-dimensional and non-Abelian. The correspondence with the centralizer of the Temperley–Lieb subalgebra offers a conceptual understanding of this phenomenon and connects the analysis to first Hochschild cohomology.

On the module-theoretic side, we formulated the precise fixed-right-index hypothesis under which a cellular derivation induces a D-connection on a cell module. We then obtained criteria for preservation of the cellular radical and for descent to simple quotients. For the Deguchi–Martin quotient, we kept the combinatorial basis correspondence separate from algebraic transport: Hecke-algebra derivations descend exactly when the defining ideal is stable; a blob-algebra derivation transports through an actual algebra isomorphism; module connections transport when the algebra and module actions intertwine; and radical stability transfers when the two cellular Gram forms correspond up to a nonzero scalar.

The developments in this paper suggest several avenues for further research.

(i) **Computation of higher cohomology groups.** An immediate extension of this work is the determination of higher-degree Hochschild cohomology groups of the blob algebra. Such calculations would provide an insight into extension classes and deformation obstructions beyond the first level.

(ii) **Broader diagram algebra framework.** It would be natural to apply similar methods to other families of diagram algebras, including Brauer, partition, and Birman–Wenzl–Murakami algebras. Investigating whether analogous outer derivation phenomena occur in these settings could reveal common structural patterns.

(iii) **Homological and categorical perspectives.** A deeper understanding may be obtained by interpreting derivations within derived or triangulated categories. Exploring links with quasi-hereditary structures and highest weight theory could unify these results within a broader homological framework.

(iv) **Physical interpretations.** Given the origins of the blob algebra in lattice models of statistical mechanics, it is worth exploring whether derivations correspond to meaningful transformations of physical systems, such as perturbations of transfer matrices or symmetry operations.

(v) **Modular and root-of-unity cases.** The present work assumes an $\mathbb{F}$ of characteristic zero and a q that is not a root of unity. Extending the analysis to positive characteristic and to the case where q is a root of unity is an important direction; the algebra becomes non-semisimple in new ways, and the cellular structure may change. The interplay among derivations, modular representation theory, and the behaviour of the bilinear form in this setting could reveal additional outer phenomena.

These perspectives indicate that the theory of derivations for blob algebras is not only structurally rich but also closely connected to broader themes in modern algebra and mathematical physics.

Author contributions

Muntasir Suhail, Osman Abdalla Adam Osman, Mohammed Rabih, and Mohammad Shane Alam: Conceptualization, Methodology, Formal analysis, Investigation, Validation, Writing–original draft, and Writing–review and editing. All authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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