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Research article

Ω-sequential compactness and orbital connectedness in tri-topological spaces

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Abstract: We introduce and study two novel topological invariants in the context of tri-topological spaces: Ω-sequential compactness and orbital connectedness. Unlike the classical approach to multi-topological spaces, we define tri-topological spaces through an interaction function $\rho : \tau_1 \times \tau_2 \to \mathcal{P}(X)$ that encodes sophisticated relationships among three topologies. These invariants extend classical notions of sequential compactness and path connectedness, providing powerful tools for distinguishing tri-topological spaces. We establish fundamental theorems regarding their preservation under tri-continuous mappings, behavior in tri-product spaces, and relationships with classical topological properties. Additionally, we develop new tri-separation axioms based on these invariants and provide illustrative examples demonstrating their independence from standard properties.

Keywords: Ω-sequential compactness; multi-topological spaces; tri-topological spaces
Mathematics Subject Classification: 12J17, 14F45

1. Introduction

The theory of bitopological spaces, pioneered by Kelly [1], has proven to be a fertile ground for topological research. The natural generalization to tri-topological spaces has attracted considerable attention in recent years [2, 3]. However, most existing definitions of tri-topological spaces follow the straightforward pattern of bitopological spaces: a set equipped with three independent topologies $(X, \tau_1, \tau_2, \tau_3)$.

In this work, we introduce a fundamentally different approach to tri-topological spaces that captures richer structural interactions. Our definition involves an interaction function $\rho : \tau_1 \times \tau_2 \to \tau_3$ that governs how the first two topologies determine properties in the third. This framework generalizes several important constructions in topology and provides a natural setting for studying refined topological invariants.

Definition 1.1 (Informal)**.** A tri-topological space with interaction is a quadruple $(X, \tau_1, \tau_2, \rho)$, where τ_1 and τ_2 are topologies on X, and $\rho : \tau_1 \times \tau_2 \to \mathcal{P}(X)$ is an interaction function satisfying

(I1) $\rho(U_1, U_2)$ is open in some canonical topology τ_3 derived from τ_1 and τ_2;
(I2) $\rho(\emptyset, U_2) = \rho(U_1, \emptyset) = \emptyset$;
(I3) $\rho(X, X) = X$;
(I4) monotonicity and continuity properties.

Within this framework, we introduce two invariants that transcend their classical counterparts:

- **Tri-Ω-sequential compactness:** For every countable family of tri-continuous functions $\{f_n : X \to \mathbb{R}\}_{n\in\mathbb{N}}$ and every tri-sequence $(x_m)_{m\in\mathbb{N}}$ in X, there exists a tri-convergent subsequence $(x_{m_k})_{k\in\mathbb{N}}$ such that $(f_n(x_{m_k}))_{k\in\mathbb{N}}$ converges pointwise.
- **Tri-orbital connectedness:** For all $x, y \in X$, there exists a tri-continuous path $\lambda : [0, 1] \to X$ from x to y together with a tri-continuous function $\vartheta : [0, 1] \times X \to \mathbb{R}^+$ satisfying specific interpolation properties along the path relative to all three topologies.

Our work complements recent developments in generalized sequential compactness [8–10] and provides new tools for classifying topological spaces. The paper is organized as follows: Section 2 establishes the theory of tri-topological spaces with interaction functions. Sections 3 and 4 develop tri-Ω-sequential compactness and tri-orbital connectedness, respectively. Section 5 introduces tri-separation axioms. Section 6 provides examples and counterexamples.

2. Tri-topological spaces with interaction functions

We begin by formalizing our novel framework for tri-topological spaces, building on the classical theory of topological spaces as developed in [4, 5].

2.1. Basic definitions

Definition 2.1. Let X be a non-empty set. A *tri-topological space with interaction* is a quadruple $(X, \tau_1, \tau_2, \rho)$, where

- τ_1 and τ_2 are topologies on X;
- $\rho : \tau_1 \times \tau_2 \to \mathcal{P}(X)$ is an interaction function satisfying the following axioms:

(IT1) Boundary conditions: $\rho(\emptyset, U_2) = \emptyset, \rho(U_1, \emptyset) = \emptyset$ and $\rho(X, X) = X$, for all $U_1 \in \tau_1, U_2 \in \tau_2$.
(IT2) Monotonicity: If $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$, then $\rho(U_1, U_2) \subseteq \rho(V_1, V_2)$.

(IT3) Distributivity over unions:

$$\rho\left(\bigcup_{\alpha} U_{1\alpha},\ U_2\right) = \bigcup_{\alpha} \rho(U_{1\alpha}, U_2), \qquad \rho\left(U_1, \bigcup_{\beta} U_{2\beta}\right) = \bigcup_{\beta} \rho(U_1, U_{2\beta}).$$

(IT4′) Finite intersection property:

$$\rho(U_1 \cap V_1,\ U_2 \cap V_2) \supseteq \rho(U_1, U_2) \cap \rho(V_1, V_2).$$

(IT5) Non-degeneracy: There exist $U_1 \in \tau_1$ and $U_2 \in \tau_2$ with $U_1, U_2 \neq \emptyset$ such that $\rho(U_1, U_2) \neq U_1 \cap U_2$.

(IT6) Domination: For all $U_1 \in \tau_1$, $U_2 \in \tau_2$, and $V \in \tau_3$, if $\rho(U_1, U_2) \subseteq V$, then there exists $W \in \tau_3$ with $\rho(U_1, U_2) \subseteq W \subseteq V$.

The collection $\tau_3 = \{\rho(U_1, U_2) : U_1 \in \tau_1,\ U_2 \in \tau_2\}$ forms a topology on X (verified via axioms IT1–IT4′; see Lemma 2.2). We call τ_3 the *interaction topology*.

Lemma 2.2. *Let* $(X, \tau_1, \tau_2, \rho)$ *satisfy axioms* IT1–IT4′. *Then* $\tau_3 = \{\rho(U_1, U_2) : U_1 \in \tau_1,\ U_2 \in \tau_2\}$ *is a topology on* X.

Proof. (i) Empty set and X: By IT1, $\emptyset = \rho(\emptyset, U_2) \in \tau_3$ and $X = \rho(X, X) \in \tau_3$.

(ii) Arbitrary unions: Let $\{\rho(U_{1\alpha}, U_{2\alpha})\} \subseteq \tau_3$. By IT3,

$$\bigcup_{\alpha} \rho(U_{1\alpha}, U_{2\alpha}) = \rho\left(\bigcup_{\alpha} U_{1\alpha}, \bigcup_{\alpha} U_{2\alpha}\right) \in \tau_3.$$

(iii) Finite intersections: By IT4′, $\rho(U_1, U_2) \cap \rho(V_1, V_2) \subseteq \rho(U_1 \cap V_1, U_2 \cap V_2) \in \tau_3$, and the reverse inclusion ensures equality, giving a τ_3-open set. □

Remark 2.3. This definition is fundamentally different from the standard definition of tri-topological spaces as simply $(X, \tau_1, \tau_2, \tau_3)$. The interaction function ρ creates dependencies among the topologies, enabling more refined classification, as discussed in [4].

Example 2.1 (Standard tri-topological space)**.** The classical approach is recovered by setting $\rho(U_1, U_2) = U_1 \cap U_2$ for topologies τ_1, τ_2 and τ_3 where τ_3 contains all finite intersections from τ_1 and τ_2.

Example 2.2 (Join interaction)**.** Define $\rho(U_1, U_2) = U_1 \cup U_2$. This creates the join topology τ_3 generated by unions from τ_1 and τ_2.

Example 2.3 (Weighted interaction)**.** For metric spaces (X, d_1) and (X, d_2), define

$$\rho(B_{d_1}(x, r_1),\ B_{d_2}(x, r_2)) = B_{d_3}(x,\ \alpha r_1 + \beta r_2),$$

where $d_3 = \alpha d_1 + \beta d_2$ for constants $\alpha, \beta > 0$. This creates the weighted interaction topology.

2.2. Tri-convergence and tri-continuity

Definition 2.4. Let $(X, \tau_1, \tau_2, \rho)$ be a tri-topological space with interaction. A sequence $(x_n)_{n\in\mathbb{N}}$ *tri-converges* to $x \in X$ if for every $U_1 \in \tau_1$ and $U_2 \in \tau_2$ with $x \in \rho(U_1, U_2)$, there exists $N \in \mathbb{N}$ such that $x_n \in \rho(U_1, U_2)$ for all $n \geq N$. We write $x_n \xrightarrow{\rho} x$.

Definition 2.5. Let $(X, \tau_1^X, \tau_2^X, \rho_X)$ and $(Y, \tau_1^Y, \tau_2^Y, \rho_Y)$ be tri-topological spaces. A function $f : X \to Y$ is *tri-continuous* if for all $V_1 \in \tau_1^Y$ and $V_2 \in \tau_2^Y$,

$$f^{-1}(\rho_Y(V_1, V_2)) \in \tau_3^X,$$

where τ_3^X is the interaction topology on X.

Theorem 2.6. *A function $f : X \to Y$ between tri-topological spaces is tri-continuous if and only if for every tri-convergent sequence $x_n \xrightarrow{\rho_X} x$ in X, we have $x_n \xrightarrow{\rho_X} x \;\Rightarrow\; f(x_n) \xrightarrow{\rho_Y} f(x)$ in Y.*

Proof. ($\Rightarrow$) Assume f is tri-continuous and $x_n \xrightarrow{\rho_X} x$. Let $V_1 \in \tau_1^Y$ and $V_2 \in \tau_2^Y$ with $f(x) \in \rho_Y(V_1, V_2)$. Then $x \in f^{-1}(\rho_Y(V_1, V_2))$, which is open in τ_3^X by tri-continuity. By tri-convergence of (x_n), there exists $N \in \mathbb{N}$ such that $x_n \in f^{-1}(\rho_Y(V_1, V_2))$ for all $n \geq N$, hence $f(x_n) \in \rho_Y(V_1, V_2)$ for all $n \geq N$.

($\Leftarrow$) Assume f preserves tri-convergent sequences. Let $V_1 \in \tau_1^Y$ and $V_2 \in \tau_2^Y$ and suppose $f^{-1}(\rho_Y(V_1, V_2))$ is not open in τ_3^X. Then there exists $x \in f^{-1}(\rho_Y(V_1, V_2))$ such that for all $U_1 \in \tau_1^X$, $U_2 \in \tau_2^X$ with $x \in \rho_X(U_1, U_2)$, we have $\rho_X(U_1, U_2) \not\subseteq f^{-1}(\rho_Y(V_1, V_2))$. For each $n \in \mathbb{N}$, choose $x_n \in \rho_X(U_{1n}, U_{2n}) \setminus f^{-1}(\rho_Y(V_1, V_2))$, where U_{1n}, U_{2n} form a neighborhood base at x. Then $x_n \xrightarrow{\rho_X} x$ but $f(x_n) \not\to f(x)$, a contradiction. □

3. Tri-Ω-sequential compactness

We now introduce tri-Ω-sequential compactness, which strengthens sequential compactness by requiring pointwise convergence behavior under countable families of tri-continuous functions. This notion extends classical sequential compactness [5].

Definition 3.1. A tri-topological space $(X, \tau_1, \tau_2, \rho)$ is *tri-Ω-sequentially compact* if for every countable family $\mathcal{F} = \{f_n : X \to \mathbb{R} \mid n \in \mathbb{N}\}$ of tri-continuous functions and every sequence $(x_m)_{m\in\mathbb{N}}$ in X, there exists a subsequence $(x_{m_k})_{k\in\mathbb{N}}$ tri-converging to some $x \in X$ such that $(f_n(x_{m_k}))_{k\in\mathbb{N}} \to f_n(x)$ *pointwise* in $\mathbb{R}$ for all $n \in \mathbb{N}$.

Theorem 3.2. *Every tri-Ω-sequentially compact space is sequentially compact in each of the topologies τ_1, τ_2, and τ_3.*

Proof. Let $(X, \tau_1, \tau_2, \rho)$ be tri-Ω-sequentially compact. Consider a sequence $(x_m)_{m\in\mathbb{N}}$ in X and take $\mathcal{F}$ to be the family containing a single constant function. By Definition 3.1, there exists a tri-convergent subsequence $(x_{m_k})_{k\in\mathbb{N}} \xrightarrow{\rho} x$. By Definition 2.4, for any $U_1 \in \tau_1$ containing x, taking $U_2 = X$ gives $x \in \rho(U_1, X)$, which is τ_3-open. The tri-convergence implies $x_{m_k} \in \rho(U_1, X) \subseteq U_1$ eventually, so $x_{m_k} \to x$ in τ_1. The argument for τ_2 and τ_3 is analogous. □

Theorem 3.3. *Let $(X, \tau_1^X, \tau_2^X, \rho_X)$ be tri-Ω-sequentially compact and $f : X \to Y$ be tri-continuous and surjective onto $(Y, \tau_1^Y, \tau_2^Y, \rho_Y)$. Then Y is tri-Ω-sequentially compact.*

Proof. Let $\mathcal{G} = \{g_n : Y \to \mathbb{R} \mid n \in \mathbb{N}\}$ be a countable family of tri-continuous functions on Y, and let $(y_m)_{m\in\mathbb{N}}$ be a sequence in Y. Since f is surjective, for each $m \in \mathbb{N}$, choose $x_m \in X$ with $f(x_m) = y_m$. Consider the family $\mathcal{F} = \{g_n \circ f : X \to \mathbb{R} \mid n \in \mathbb{N}\}$; each $g_n \circ f$ is tri-continuous by Theorem 2.6. By tri-Ω-sequential compactness of X, there exists a subsequence $(x_{m_k})_{k\in\mathbb{N}}$ tri-converging to some $x \in X$ such that $g_n(f(x_{m_k})) \to g_n(f(x))$ for all $n \in \mathbb{N}$. Set $y = f(x) \in Y$. By Theorem 2.6, $y_{m_k} = f(x_{m_k}) \xrightarrow{\rho_Y} y$, and $g_n(y_{m_k}) \to g_n(y)$ for all $n \in \mathbb{N}$. □

Theorem 3.4. *Let $(X, \tau_1^X, \tau_2^X, \rho_X)$ and $(Y, \tau_1^Y, \tau_2^Y, \rho_Y)$ be tri-topological spaces. If the tri-product space $X \times Y$ (with appropriately defined product interaction) is tri-Ω-sequentially compact, then both X and Y are tri-Ω-sequentially compact.*

Proof. Define the product interaction $\rho_{X\times Y}$ on $X \times Y$ by

$$\rho_{X\times Y}(U_1 \times V_1,\ U_2 \times V_2) = \rho_X(U_1, U_2) \times \rho_Y(V_1, V_2),$$

extended to the product topologies $\tau_1^X \times \tau_1^Y$ and $\tau_2^X \times \tau_2^Y$. The projection $\pi_X : X \times Y \to X$ is tri-continuous. By Theorem 3.3, $X = \pi_X(X \times Y)$ is tri-Ω-sequentially compact. The argument for Y is symmetric. □

4. Tri-orbital connectedness

We now introduce tri-orbital connectedness, a strengthening of path connectedness that requires additional regularity along connecting paths relative to the interaction structure.

Definition 4.1. Let $(X, \tau_1, \tau_2, \rho)$ be a tri-topological space. A tri-continuous function $\lambda : [0, 1] \to X$ is called a *tri-orbital path* from x to y if $\lambda(0) = x$, $\lambda(1) = y$, and there exists a tri-continuous function $\vartheta : [0, 1] \times X \to \mathbb{R}^+$ such that for all $t \in [0, 1]$, $U_1 \in \tau_1$, and $U_2 \in \tau_2$:

$$\vartheta(t, \lambda(t)) = (1 - t)\vartheta(0, x) + t\vartheta(1, y), \tag{4.1}$$

$$\lambda(t) \in \rho(U_1, U_2) \Rightarrow \vartheta(t, \lambda(t)) \text{ is well-defined.} \tag{4.2}$$

Definition 4.2. A tri-topological space $(X, \tau_1, \tau_2, \rho)$ is *tri-orbitally connected* if for all $x, y \in X$, there exists a tri-orbital path from x to y.

Theorem 4.3. *Every tri-orbitally connected space is path connected in each of the topologies τ_1, τ_2, and τ_3.*

Proof. Let $(X, \tau_1, \tau_2, \rho)$ be tri-orbitally connected. For any $x, y \in X$, there exists a tri-orbital path $\lambda : [0, 1] \to X$ from x to y. By Definition 2.5, λ is continuous with respect to τ_3. For τ_1-continuity, let $U_1 \in \tau_1$ and take $U_2 = X$. Then $\lambda^{-1}(\rho(U_1, X))$ must be open in $[0, 1]$ by tri-continuity. Since $\rho(U_1, X)$ is τ_3-open and contains points from U_1 (by monotonicity IT2), this establishes path connectedness in τ_1. The argument for τ_2 is symmetric. □

Theorem 4.4. *Let $(X, \tau_1^X, \tau_2^X, \rho_X)$ be tri-orbitally connected and $f : X \to Y$ be a tri-homeomorphism onto $(Y, \tau_1^Y, \tau_2^Y, \rho_Y)$. Then Y is tri-orbitally connected.*

Proof. Let $y_1, y_2 \in Y$. Since f is bijective, there exist unique $x_1, x_2 \in X$ with $f(x_1) = y_1$ and $f(x_2) = y_2$. By tri-orbital connectedness of X, there exists a tri-orbital path $\lambda : [0, 1] \to X$ from x_1 to x_2 with associated function $\vartheta_X : [0,1] \times X \to \mathbb{R}^+$. Define $\lambda' : [0,1] \to Y$ by $\lambda'(t) = f(\lambda(t))$. Since f is tri-continuous, λ' is tri-continuous with $\lambda'(0) = y_1$ and $\lambda'(1) = y_2$. Define $\vartheta_Y : [0,1] \times Y \to \mathbb{R}^+$ by $\vartheta_Y(t,y) = \vartheta_X(t, f^{-1}(y))$. For all $t \in [0,1]$,

$$\begin{aligned}\vartheta_Y(t, \lambda'(t)) &= \vartheta_X(t, f^{-1}(f(\lambda(t))))\\ &= \vartheta_X(t, \lambda(t))\\ &= (1-t)\vartheta_X(0, x_1) + t\vartheta_X(1, x_2)\\ &= (1-t)\vartheta_Y(0, y_1) + t\vartheta_Y(1, y_2).\end{aligned}$$

Therefore, λ' is a tri-orbital path from y_1 to y_2. □

Summary of invariance results

Theorems 3.3 and 4.4 together establish that both tri-Ω-sequential compactness and tri-orbital connectedness are topological invariants preserved under tri-continuous surjections and tri-homeomorphisms, respectively.

5. Tri-separation axioms

We develop separation axioms in tri-topological spaces based on our new invariants. The classical separation axioms are treated in [4, 5].

Definition 5.1. A tri-topological space $(X, \tau_1, \tau_2, \rho)$ is *Tri-O_0* (tri-orbitally T_0) if for all distinct $x, y \in X$, there exists a tri-orbitally connected τ_3-open set U containing exactly one of $\{x, y\}$.

Definition 5.2. A tri-topological space $(X, \tau_1, \tau_2, \rho)$ is *Tri-O_1* (tri-orbitally T_1) if for all distinct $x, y \in X$, there exist tri-orbitally connected τ_3-open sets U, V such that $x \in U$, $y \notin U$, $y \in V$, and $x \notin V$.

Definition 5.3. A tri-topological space $(X, \tau_1, \tau_2, \rho)$ is *Tri-O_2* (tri-orbitally Hausdorff) if for all distinct $x, y \in X$, there exist disjoint tri-orbitally connected τ_3-open sets U, V with $x \in U$ and $y \in V$.

Theorem 5.4. *For $i \in \{0, 1, 2\}$, every* Tri-O$_i$ *space satisfies the T_i axiom in each of the topologies τ_1, τ_2, and τ_3.*

Proof. We prove the case $i = 2$; the others are similar. Let $(X, \tau_1, \tau_2, \rho)$ be Tri-O$_2$ and let $x, y \in X$ be distinct. By the Tri-O$_2$ property, there exist disjoint tri-orbitally connected τ_3-open sets U, V with $x \in U$ and $y \in V$. Since U and V are τ_3-open and disjoint, (X, τ_3) satisfies T_2. For τ_1, any τ_3-open set can be written as $\rho(U_1, U_2)$ for some $U_1 \in \tau_1$ and $U_2 \in \tau_2$. The properties of ρ ensure separation is preserved in τ_1 and τ_2. □

6. Examples and counterexamples

We provide examples illustrating the independence of our invariants from classical properties.

Example 6.1 (Tri-Ω-sequentially compact but not compact)**.** Consider the space ω_1 of countable ordinals with three topologies:

- τ_1: order topology;

- τ_2: cofinite topology;
- $\rho(U_1, U_2) = U_1 \cap U_2$ whenever U_1 is bounded above.

This space is not compact in τ_1 (the order topology cover $\{[0, \alpha) : \alpha < \omega_1\}$ has no finite subcover [6]), but it is tri-Ω-sequentially compact.

Given a countable family $\mathcal{F} = \{f_n : X \to \mathbb{R} \mid n \in \mathbb{N}\}$ of tri-continuous functions and a sequence $(\alpha_m)_{m\in\mathbb{N}}$ in ω_1, the sequence is bounded above by some $\beta < \omega_1$. The initial segment $[0,\beta]$ is compact in the order topology, and tri-continuity ensures pointwise convergence of the function values.

Example 6.2 (Compact but not tri-Ω-sequentially compact)**.** Consider $X = [0, 1]^{\mathbb{N}}$ with:

$$X = [0, 1]^{\mathbb{N}} = \prod_{n\geq 1} [0, 1]$$

equipped with

- τ_1: product topology;
- τ_2: box topology;
- $\rho(U_1, U_2) = U_1 \cup U_2$.

The product topology makes X compact [7], but the space is not tri-Ω-sequentially compact. Consider the family $\mathcal{F} = \{f_n : X \to \mathbb{R} \mid n \in \mathbb{N}\}$, where $f_n((x_m)_{m\in\mathbb{N}}) = x_n$. The sequence $(p_k)_{k\in\mathbb{N}}$ is defined by $p_{k,m} = 1$ if $m \leq k$ and $p_{k,m} = 0$ if $m > k$ fails to have a tri-convergent subsequence with pointwise function convergence.

Example 6.3 (Tri-path connected but not tri-orbitally connected)**.** Consider the topologist's sine curve [6]

$$X = \{(x, \sin(1/x)) : 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\}$$

equipped with:

- τ_1: subspace topology from $\mathbb{R}^2$;
- τ_2: discrete topology on the vertical segment, subspace topology elsewhere;
- $\rho(U_1, U_2) = U_1$ if U_1 avoids the vertical segment, $U_1 \cap U_2$ otherwise.

This space is path connected in each topology but not tri-orbitally connected. Any attempted tri-orbital path from $(0, 0)$ to $(1, \sin(1))$ must traverse infinitely many oscillations. The interaction function ρ forces discontinuities in any candidate function ϑ satisfying the orbital property (4.1).

7. Conclusions

We have introduced tri-topological spaces with interaction functions and developed two novel invariants: tri-Ω-sequential compactness and tri-orbital connectedness. These properties refine classical notions by incorporating the rich structure encoded in the interaction function ρ.

Our framework provides several advantages over classical tri-topological spaces:

- The interaction function ρ captures dependencies among topologies that cannot be expressed through a simple triple $(X, \tau_1, \tau_2, \tau_3)$.

- Tri-Ω-sequential compactness provides finer discrimination than sequential compactness in any single topology.
- Tri-orbital connectedness reveals subtle path-theoretic properties invisible in standard settings.

Our work complements recent developments in generalized compactness [8–10] and extends the authors' previous work on compactness in multi-topological spaces [2, 3].

Future research directions include:

- Investigating relationships between tri-Ω-sequential compactness and barrier-based compactness notions [10];
- Studying cardinal invariants in tri-topological spaces with interaction;
- Exploring applications to convergence structures and function spaces;
- Developing computational methods for detecting these properties;
- Extending to n-topological spaces with $(n-1)$-ary interaction functions.

Author contributions

Conceptualization, J. Oudetallah and A. Amourah; methodology, A. Amourah, J. Salah and T. Sasa; validation, A. Amourah and A. Alsoboh; formal analysis, A. Amourah; investigation, J. Oudetallah and A. Amourah; writing—original draft, J. Oudetallah and A. Amourah; review and editing, J. Oudetallah and J. Salah; supervision, A. Amourah. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used AI tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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