Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 22380–22400.
DOI: 10.3934/math.2026905
Received: 26 February 2026
Revised: 27 June 2026
Accepted: 22 July 2026
Published: 27 July 2026

Research article

Categorical relationships between L-grill-proximity spaces and related L-topological structures

Anwar J. Fawakhreh*

Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia

* **Correspondence:** Email: foakhrh@qu.edu.sa.

Abstract: This work investigated categorical connections between L-grill spaces and L-pre-proximity spaces and established a Galois connection between the category of L-pre-proximity spaces and the category of stratified L-grill spaces within the framework of complete residuated lattices. Furthermore, it introduced the concept of L-grill-proximity spaces, a new structure derived from L-grills, and studied their fundamental properties as well as their categorical connections with L-closure, L-interior, and L-co-topological spaces. Finally, it demonstrated the applicability of the proposed framework through an application to environmental pollution modeling by integrating multiple environmental indicators for quantitative air-quality assessment and decision-making.

Keywords: fuzzy sets; functors; L-grill spaces; L-topological structures; L-grill-proximity spaces
Mathematics Subject Classification: 54A10, 54A15, 03B52, 54C10, 03E72

1. Introduction

Goguen's paper [1] presented L-sets as a natural expansion of Zadeh's fuzzy sets [2]. Since then, numerous researchers have investigated L-topological spaces through diverse theoretical viewpoints. For instance, Höhle and Šostak [3], as well as Ramadan [4], investigated the fundamental properties of L-topological spaces. Subsequently, various scholars extended L-topological spaces and adjoint operators to full lattices. Bělohlávek [5,6] introduced an axiomatic system for L-interior and L-closure operators. Fang [7] investigated strong L-topologies and L-ordered convergence structures on full residuated lattices. Oh and Kim [8] identified fundamental links between closure and co-topological structures under the notion of L-spaces. Many authors have examined different types of L-topological spaces in great detail (see [9–11]).

The topological framework of proximity structures has several applications in digital image classification, pattern recognition, and related domains. There are at least two primary notions of fuzzy closeness in the context of fuzzy structures. Katsaras [12] introduced fuzzy proximity as a subset of $I^{X\times X}$ satisfying specific axioms and examined its relationship with fuzzy topology. Another definition of fuzzy closeness, proposed by Artico and Moresco [13], is a mapping $I^{X\times X} \to I$. Artico's fuzzy proximity structures were later extended to full lattices by other researchers. The concept of L-proximities on complete residuated lattices was developed by Čimoka and Šostak [14], who also investigated their relationship with L-topological spaces (see also [15]). The interaction of L-filters via L-pre-proximity spaces was examined by Jose and Mathew [16].

Given its connections to other subjects such as proximity spaces and topological spaces, the theory of grills is an effective tool for understanding topological structures. To construct fuzzy proximity spaces, Azad [17] introduced the concept of fuzzy grills. Mukherjee and Das [18] were among the first to use grills to create a fuzzy topology, known as grill fuzzy topology. Subsequently, Roy and Mukherjee [19] studied the relationships between grills and topological spaces. Later, Kandil et al. [20] introduced additional properties concerning the interaction between grills and topological spaces, while Rodyna [21] investigated the relationship between grills and proximity structures. Further features of L-pre-proximities were introduced by Ramadan et al. [22], who examined their interactions with L-filters and L-grills. In more recent work, Ramadan and Fawakhreh [23–25] conducted a comprehensive study of the related categories of L-primal, L-grill, L-ideal, L-pre-proximity, L-topology, and L-co-topological spaces. They established Galois correspondences among the associated categories and clarified the relationships among these topological structures. Furthermore, Fawakhreh [26] recently established Galois connections between the category of L-grill spaces and L-quasi-enclosed spaces, and between L-filter spaces and L-quasi-internal spaces. Additional studies on various properties of fuzzy grill topology can be found in [27–29].

The aim of this paper is to establish categorical connections between L-grill and L-pre-proximity spaces, emphasizing the structural relationships between these categories. Beyond the introduction and preliminaries, the paper is organized into five sections. Section 3 identifies the categorical interconnections between L-grill and L-pre-proximity spaces by defining a concrete functor that links these categories. Section 4 introduces the notion of L-gril-proximity spaces, studies some of their properties, and demonstrates their relationships with L-closure, L-interior, and L-co-topological spaces. Finally, Section 5 applies the proposed framework to environmental pollution modeling, employing L-neighborhood systems, L-grill, and L-pre-proximity spaces to represent graded and spatially distributed pollution phenomena and to integrate multiple environmental indicators for quantitative urban air-quality assessment and decision-making.

2. Preliminaries

Throughout this study, we assume that X is a nonempty set. A Heyting algebra $(L, \wedge, \vee, \to, \bot, \top)$ is regarded as a frame. The underlying complete lattice is denoted by $(L, \leq)$, where $\bot$ and $\top$ denote the least and greatest elements of L, respectively. The following distributive law holds:

$$\kappa \wedge (\bigvee_{l\in\Gamma} \aleph_l) = \bigvee_{l\in\Gamma} (\kappa \wedge \aleph_l); \quad \forall \kappa, \aleph_l \in L, \;\; l \in \Gamma.$$

For any $\kappa, \aleph, \delta \in L$, the residuation property $\kappa \wedge \aleph \leq \delta$ if and only if $\aleph \leq \kappa \to \delta$ defines the residual implication operation $\to$. It is further assumed that $(L, \leq, \wedge, ^*)$ is a frame equipped with a mapping $^* : L \to L$ that is both involutive ($\kappa^{**} = \kappa$) and order-reversing, defined by $\kappa \vee \aleph = \kappa^* \to \aleph, \quad \kappa^* = \kappa \to \bot$. We list some basic properties of this structure in the following lemma.

Lemma 2.1 (cf. [3,30,31]). *For all $\kappa, \aleph, \delta, \kappa_l, \aleph_l, \gamma \in L$ and $l \in \Gamma$, the following properties hold:*

(1) $\kappa \to \aleph = \bigvee\{\delta \in L \mid \kappa \wedge \delta \leq \aleph\}$.
(2) $\top \to \aleph = \aleph$, *and* $\kappa \leq \aleph$ *if and only if* $\kappa \to \aleph = \top$.
(3) *If* $\aleph \leq \delta$, *then* $\kappa \to \aleph \leq \kappa \to \delta$ *and* $\delta \to \kappa \leq \aleph \to \kappa$.
(4) $(\bigwedge_l \aleph_l)^* = \bigvee_l \aleph_l^*$ *and* $(\bigvee_l \aleph_l)^* = \bigwedge_l \aleph_l^*$.
(5) $\kappa \to (\bigwedge_l \aleph_l) = \bigwedge_l(\kappa \to \aleph_l)$ *and* $(\bigvee_l \kappa_l) \to \aleph = \bigwedge_l(\kappa_l \to \aleph)$.
(6) $(\kappa \wedge \aleph) \to \delta = \kappa \to (\aleph \to \delta) = \aleph \to (\kappa \to \delta)$.
(7) $\kappa \wedge \aleph = (\kappa \to \aleph^*)^*$ *and* $\kappa \to \aleph = \aleph^* \to \kappa^*$.
(8) $\kappa \to \aleph \leq (\kappa \wedge \delta) \to (\aleph \wedge \delta)$ *and* $(\kappa \to \aleph) \wedge (\aleph \to \delta) \leq \kappa \to \delta$.
(9) $(\kappa \to \aleph) \wedge (\delta \to \gamma) \leq (\kappa \wedge \delta) \to (\aleph \wedge \gamma)$.
(10) $(\kappa \to \aleph) \wedge (\delta \to \gamma) \leq (\kappa \vee \delta) \to (\aleph \vee \gamma)$.
(11) $(\kappa \to \aleph) \vee (\delta \to \gamma) \leq (\kappa \wedge \delta) \to (\aleph \vee \gamma)$.
(12) $\kappa \wedge (\kappa \to \aleph) \leq \aleph$ *and* $\aleph \leq \kappa \to (\kappa \wedge \aleph)$.
(13) $\kappa \leq \aleph \to \delta$ *if and only if* $\kappa \wedge \aleph \leq \delta$.
(14) $\bigvee_l \kappa_l \to \bigvee_l \aleph_l \geq \bigwedge_l(\kappa_l \to \aleph_l)$ *and* $\bigwedge_l \kappa_l \to \bigwedge_l \aleph_l \geq \bigwedge_l(\kappa_l \to \aleph_l)$.
(15) $\aleph \to \kappa \leq (\kappa \to \delta) \to (\aleph \to \delta)$ *and* $\aleph \to \kappa \leq (\delta \to \aleph) \to (\delta \to \kappa)$.

For $\kappa \in L$, the constant mapping $\kappa : X \to L$ is defined by $\kappa(r) = \kappa, \ \forall\, r \in X$.

On X, an L-subset is defined as a function $\xi : X \to L$. Additionally, $\top_r : X \to L$ is a mapping given by

$$\top_r(\hbar) = \begin{cases} \top, & \text{if } \hbar = r, \\ \bot, & \text{otherwise.} \end{cases}$$

Additionally, the degree of subsethood of ω with respect to ξ on X is given by the mapping $S_{th} : L^X \times L^X \to L$, defined as $S_{th}(\omega, \xi) = \bigwedge\limits_{c \in X} (\omega(c) \to \xi(c))$.

The following lemma presents several properties of the degree of subsethood between two L-subsets.

Lemma 2.2 (cf. [7,30]). *For all $\omega, \xi, \kappa, \rho \in L^X$ and $\kappa \in L$, the following properties hold:*

(1) $\omega \leq \xi \iff S_{th}(\omega, \xi) = \top$.
(2) $S_{th}(\omega, \xi) \wedge S_{th}(\xi, \kappa) \leq S_{th}(\omega, \kappa)$.
(3) *If* $\omega \leq \xi$, *then* $S_{th}(\kappa, \omega) \leq S_{th}(\kappa, \xi)$ *and* $S_{th}(\omega, \kappa) \geq S_{th}(\xi, \kappa)$.
(4) $S_{th}(\omega, \xi) \wedge S_{th}(\kappa, \rho) \leq S_{th}(\omega \vee \kappa, \xi \vee \rho)$ *and* $S_{th}(\omega, \xi) \wedge S_{th}(\rho, \kappa) \leq S_{th}(\omega \wedge \rho, \xi \wedge \kappa)$.
(5) $S_{th}(\omega, \xi) \leq S_{th}(\rho, \omega) \to S_{th}(\rho, \xi)$ *and* $S_{th}(\omega, \xi) \leq S_{th}(\xi, \rho) \to S_{th}(\omega, \rho)$.
(6) $S_{th}(\omega, \kappa \wedge \xi) \geq \kappa \wedge S_{th}(\omega, \xi)$ *and* $S_{th}(\omega, \xi) \wedge \omega \leq \xi$.
(7) $\omega \leq S_{th}(\omega, \xi) \to \xi$ *and* $S_{th}(\omega^*, \xi^*) = S_{th}(\xi, \omega)$.
(8) *For all* $\varepsilon, \varrho \in L^Y$ *and any mapping* $\eth : X \to Y$, *we have*

$$S_{th}(\varepsilon, \varrho) \leq S_{th}(\eth^{\leftarrow}(\varepsilon), \eth^{\leftarrow}(\varrho)), \quad \textit{where} \quad \eth^{\leftarrow}(\varepsilon)(r) = \varepsilon(\eth(r)).$$

A Galois correspondence provides a natural relationship between two concrete categories through a pair of functors acting in opposite directions. It generalizes the classical notion of a Galois connection between partially ordered sets and is widely used to compare different mathematical structures.

Recall that if $\mathcal{C}$ and $\mathcal{D}$ are concrete categories over a set **X**, and $\varphi : \mathcal{C} \to \mathcal{D}$ and $\psi : \mathcal{D} \to \mathcal{C}$ are concrete functors over **X**, then (φ, ψ) is called a Galois correspondence [32] between $\mathcal{C}$ and $\mathcal{D}$ provided that $\psi \circ \varphi \leq id_{\mathcal{C}}$ and $id_{\mathcal{D}} \leq \varphi \circ \psi$.

Intuitively, the functor φ maps objects of $\mathcal{C}$ to objects of $\mathcal{D}$, while ψ maps objects in the opposite direction. The above inequalities ensure that these two functors are mutually compatible, thereby establishing a close correspondence between the two categories.

Next, we provide an introduction to several L-topological structures.

Definition 2.3 ([5, 8])**.** *On a set X, the mapping $\mathcal{C} : L^X \to L^X$ is referred to as an L-closure provided that the next four conditions hold for any $\omega, \xi \in L^X$.*

(C1) $\mathcal{C}(\perp_X) = \perp_X$.
(C2) $\mathcal{C}(\omega) \geq \omega$.
(C3) $S_{th}(\omega, \xi) \leq S_{th}(\mathcal{C}(\omega), \mathcal{C}(\xi))$.
(C4) $\mathcal{C}(\omega \vee \xi) \leq \mathcal{C}(\omega) \vee \mathcal{C}(\xi)$.

We refer to $(X, \mathcal{C})$ as an L-closure space. Additionally, $(X, \mathcal{C})$ will be called:

(St) Stratified, *if for each $\kappa \in L$, $\mathcal{C}(\kappa \wedge \omega) \geq \kappa \wedge \mathcal{C}(\omega)$.*
(AL) Alexandrov, *if for each $\omega_l \in L^X$ and $l \in \Gamma$, $\mathcal{C}(\bigvee_{l \in \Gamma} \omega_l) = \bigvee_{l \in \Gamma} \mathcal{C}(\omega_l)$.*

An L-closure mapping is a mapping $\eth : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ that achieves

$$\eth^{\leftarrow}(\mathcal{C}_Y(\rho)) \geq \mathcal{C}_X(\eth^{\leftarrow}(\rho)), \quad \forall \ \rho \in L^Y.$$

We write **L-CL** for the category of L-closure spaces whose morphisms are L-closure mappings.

Definition 2.4 ([5, 10])**.** *On a set X, the mapping $\mathcal{I} : L^X \to L^X$ is referred to as an L-*interior *provided that, for any $\omega, \xi \in L^X$, the next four conditions hold.*

(I1) $\mathcal{I}(\top_X) = \top_X$.
(I2) $\mathcal{I}(\omega) \leq \omega$.
(I3) $S_{th}(\omega, \xi) \leq S_{th}(\mathcal{I}(\omega), \mathcal{I}(\xi))$.
(I4) $\mathcal{I}(\omega \wedge \xi) \geq \mathcal{I}(\omega) \wedge \mathcal{I}(\xi)$.

*We refer to $(X, \mathcal{I})$ as an L-*interior space. *Additionally, $(X, \mathcal{I})$ will be called:*

(St) Stratified, *if $\mathcal{I}(\kappa \wedge \omega) \geq \kappa \wedge \mathcal{I}(\omega)$, $\forall \ \kappa \in L$.*
(AL) Alexandrov, *if $\mathcal{I}(\bigwedge_{l \in \Gamma} \omega_l) = \bigwedge_{l \in \Gamma} \mathcal{I}(\omega_l)$, $\forall \ \omega_l \in L^X$, $l \in \Gamma$.*

*An L-*interior map *is a mapping $\eth : (X, \mathcal{I}_X) \to (Y, \mathcal{I}_Y)$ that achieves*

$$\eth^{\leftarrow}(\mathcal{I}_Y(\rho)) \leq \mathcal{I}_X(\eth^{\leftarrow}(\rho)), \quad \forall \ \rho \in L^Y.$$

We write **L-INT** for the category of L-interior spaces whose morphisms are L-interior mappings.

Definition 2.5 ([3])**.** *On a set X, the family $\mathcal{N} = \{\mathcal{N}^r : L^X \to L \mid r \in X\}$ is referred to as an L-*neighborhood system *provided that the next four properties hold for any $\kappa, \xi \in L^X$:*

(N1) $\mathcal{N}^r(\top_X) = \top$;

(N2) $\mathcal{N}^r(\xi) \leq \xi(r)$;
(N3) If $\kappa \leq \xi$, then $\mathcal{N}^r(\kappa) \leq \mathcal{N}^r(\xi)$;
(N4) $\mathcal{N}^r(\kappa \wedge \xi) \geq \mathcal{N}^r(\kappa) \wedge \mathcal{N}^r(\xi)$.

We call $(X, \mathcal{N})$ *an* L-neighborhood space. *Furthermore, the space* $(X, \mathcal{N})$ *is said to be* Alexandrov *whenever*

$$\mathcal{N}\left(\bigwedge_{l\in\Gamma} \omega_l\right) = \bigwedge_{l\in\Gamma} \mathcal{N}(\omega_l), \quad \text{for all } \omega_l \in L^X,\ l \in \Gamma.$$

A mapping $\eth : (X, \mathcal{N}_X) \to (Y, \mathcal{N}_Y)$ *between L-neighborhood spaces is said to be continuous at a point* $r \in X$ *if, for any* $\rho \in L^Y$, $\mathcal{N}^{\eth(r)}(\rho) \leq \mathcal{N}^r(\eth^{\leftarrow}(\rho))$, *where* $\mathcal{N}^r \in \mathcal{N}_X$ *and* $\mathcal{N}^{\eth(r)} \in \mathcal{N}_Y$. *If* $\eth$ *is continuous at every point* $r \in X$, *then it is called continuous.*

We write **L-NB** for the category of L-neighborhood spaces whose morphisms are continuous mappings.

Definition 2.6 ([3, 11]). *An L-co-topology on X is a mapping* $\mathcal{CO} : L^X \to L$ *satisfying the next three properties for any* $\omega, \xi \in L^X$:

(CT1) $\mathcal{CO}(\perp_X) = \mathcal{CO}(\top_X) = \top$;
(CT2) $\mathcal{CO}(\omega \vee \xi) \geq \mathcal{CO}(\omega) \wedge \mathcal{CO}(\xi)$, *for every* $\omega, \xi \in L^X$;
(CT3) $\mathcal{CO}(\bigwedge_{l\in\Gamma} \omega_l) \geq \bigwedge_{l\in\Gamma} \mathcal{CO}(\omega_l)$, *for every* $\omega_l \in L^X$ *and* $l \in \Gamma$.

We call $(X, \mathcal{CO})$ *an L-co-topological space. Moreover, the space* $(X, \mathcal{CO})$ *is considered to be:*

(AL) Alexandrov, *whenever for all* $\{\omega_j\}_{l\in\Gamma} \subseteq L^X$, $\mathcal{CO}\left(\bigvee_{l\in\Gamma} \omega_j\right) \geq \bigwedge_{l\in\Gamma} \mathcal{CO}(\omega_j)$.
(R) Enriched, *whenever for all* $\kappa \in L$ *and* $\omega \in L^X$, $\mathcal{CO}(\kappa \to \omega) \geq \mathcal{CO}(\omega)$.

A mapping $\eth : (X, \mathcal{CO}_X) \to (Y, \mathcal{CO}_Y)$ *is said to be* continuous *if, for every* $\rho \in L^Y$,

$$\mathcal{CO}_Y(\rho) \leq \mathcal{CO}_X(\eth^{\leftarrow}(\rho)).$$

We write **L-CTOP** for the category of L-co-topological spaces whose morphisms are continuous mappings.

Definition 2.7 ([22]). *An L-grill on X is a mapping* $\mathcal{GR} : L^X \to L$ *satisfying the next three conditions for any* $\omega, \xi \in L^X$:

(G1) $\mathcal{GR}(\perp_X) = \perp$ *and* $\mathcal{GR}(\top_X) = \top$;
(G2) $S_{th}(\omega, \xi) \leq \mathcal{GR}(\omega) \to \mathcal{GR}(\xi)$;
(G3) $\mathcal{GR}(\omega \vee \xi) \leq \mathcal{GR}(\omega) \vee \mathcal{GR}(\xi)$.

We refer to $(X, \mathcal{GR})$ *as an L-grill space. Furthermore, the space* $(X, \mathcal{GR})$ *is said to be:*

(St) Stratified, *whenever for every* $\kappa \in L$, $\mathcal{GR}(\kappa \to \omega) \leq \kappa \to \mathcal{GR}(\omega)$.
(AL) Alexandrov, *whenever for every* $\omega_l \in L^X$ *and* $l \in \Gamma$, $\mathcal{GR}(\bigvee_{l\in\Gamma} \omega_l) = \bigvee_{l\in\Gamma} \mathcal{GR}(\omega_l)$.

A mapping $\eth : (X, \mathcal{GR}_X) \to (Y, \mathcal{GR}_Y)$ *between L-grill spaces is said to be an* L-grill mapping *if*

$$\mathcal{GR}_X(\eth^{\leftarrow}(\rho)) \leq \mathcal{GR}_Y(\rho), \quad \forall \rho \in L^Y.$$

We write **L-GRL** (respectively, **SL-GRL**) for the category of L-grill (respectively, stratified L-grill) spaces whose morphisms are L-grill mappings. One may notice that in L-grill spaces, if $\omega \leq \xi$, then $\mathcal{GR}(\omega) \leq \mathcal{GR}(\xi)$.

The subsequent lemma characterizes an equivalent form of (G2) for L-grill spaces.

Lemma 2.8 ([22]). *If $(X, \mathcal{GR})$ is an L-grill space, then $\forall\, \xi, \omega \in L^X$ and $\kappa \in L$, the subsequent statements are equivalent:*

(1) $\mathcal{GR}(\kappa \wedge \xi) \geq \kappa \wedge \mathcal{GR}(\xi)$.

(2) $\mathcal{GR}(\kappa \rightarrow \omega) \leq \kappa \rightarrow \mathcal{GR}(\omega)$.

(3) $S_{th}(\omega, \xi) \leq \mathcal{GR}(\omega) \rightarrow \mathcal{GR}(\xi)$.

Definition 2.9 ([22]). *On a set X, the mapping $\mathcal{PR} : L^X \times L^X \rightarrow L$ is referred to as an* L-pre-proximity *provided that the next conditions hold for any $\omega, \xi, \varrho, \omega_1, \omega_2, \xi_1, \xi_2 \in L^X$:*

(LP1) $\mathcal{PR}(\top_X, \perp_X) = \perp$ *and* $\mathcal{PR}(\perp_X, \top_X) = \perp$;

(LP2) $\mathcal{PR}(\omega, \xi) \geq \bigvee_{a \in X} \omega(a) \wedge \xi(a)$;

(LP3) $S_{th}(\omega, \xi) \leq \mathcal{PR}(\varrho, \omega) \rightarrow \mathcal{PR}(\varrho, \xi)$ *and* $S_{th}(\omega, \xi) \leq \mathcal{PR}(\omega, \varrho) \rightarrow \mathcal{PR}(\xi, \varrho)$;

(LP4) $\mathcal{PR}(\omega_1 \wedge \omega_2, \xi_1 \vee \xi_2) \leq \mathcal{PR}(\omega_1, \xi_1) \vee \mathcal{PR}(\omega_2, \xi_2)$.

We refer to $(X, \mathcal{PR})$ *as an* L-pre-proximity space. *Additionally,* $(X, \mathcal{PR})$ *is said to be:*

(St) Stratified, *whenever for every $\kappa \in L$,*

$$\mathcal{PR}(\kappa \wedge \omega, \kappa \rightarrow \xi) \leq \mathcal{PR}(\omega, \xi) \quad or \quad \mathcal{PR}(\kappa \rightarrow \omega, \kappa \wedge \xi) \leq \mathcal{PR}(\omega, \xi).$$

(AL) Alexandrov, *whenever for every $\{\omega_l, \xi_l : l \in \Gamma\} \subseteq L^X$,*

$$\mathcal{PR}(\omega, \bigvee_{l \in \Gamma} \xi_l) \leq \bigvee_{l \in \Gamma} \mathcal{PR}(\omega, \xi_l) \quad and \quad \mathcal{PR}(\bigvee_{l \in \Gamma} \omega_l, \xi) \leq \bigvee_{l \in \Gamma} \mathcal{PR}(\omega_l, \xi).$$

The mapping $\eth : (X, \mathcal{PR}_X) \rightarrow (Y, \mathcal{PR}_Y)$ between L-pre-proximity spaces *is said to be an* L-proximity mapping *whenever*

$$\mathcal{PR}_Y(\omega, \xi) \geq \mathcal{PR}_X(\eth^{\leftarrow}(\omega), \eth^{\leftarrow}(\xi)), \quad for\ all\ \omega, \xi \in L^Y.$$

We write **L-PPROX** for the category consisting of L-pre-proximity spaces whose morphisms are L-proximity mappings.

3. On L-grills and L-pre-proximities

Categorical connections between L-pre-proximity and L-grill spaces are identified in this section. The subsequent theorem defines a concrete functor from **L-GRL** to **L-PRROX**, thereby demonstrating a categorical link between L-grill and L-pre-proximity spaces.

Theorem 3.1. *We are given an L-grill space $(X, \mathcal{GR})$ and a mapping $\mathcal{PR}_{\mathcal{GR}} : L^X \times L^X \rightarrow L$ specified by $\mathcal{PR}_{\mathcal{GR}}(\xi, \omega) = \bigvee_{r \in X} \Big(\xi(r) \wedge (\omega(r) \vee \mathcal{GR}(\omega))\Big)$, $\forall\, \xi, \omega \in L^X$. Then,*

(1) *the space $(X, \mathcal{PR}_{\mathcal{GR}})$ is L-pre-proximity. Furthermore, $(X, \mathcal{PR}_{\mathcal{GR}})$ is Alexandrov whenever $(X, \mathcal{GR})$ is Alexandrov;*

(2) *if $(X, \mathcal{GR})$ is stratified, then $\mathcal{PR}_{\mathcal{GR}}(\xi, \kappa \wedge \omega) \geq \kappa \wedge \mathcal{PR}_{\mathcal{GR}}(\xi, \omega)$ and $(X, \mathcal{PR}_{\mathcal{GR}})$ is also stratified;*

(3) *$\eth : (X, \mathcal{PR}_{\mathcal{GR}_X}) \rightarrow (Y, \mathcal{PR}_{\mathcal{GR}_Y})$ is an L-proximity mapping whenever $\eth : (X, \mathcal{GR}_X) \rightarrow (Y, \mathcal{GR}_Y)$ is an L-grill mapping.*

Proof. (1) (LP1) Obvious.

(LP2) $\mathcal{PR}_{\mathcal{GR}}(\psi,\xi) = \bigvee_{r\in X}\Big(\psi(r)\wedge(\xi(r)\vee\mathcal{GR}(\xi))\Big)\geq\bigvee_{r\in X}\psi(r)\wedge\xi(r).$

(LP3) Let $\psi,\omega,\varrho\in L^X$, and using the residuation properties and the definition of the subsethood degree, we have

$$\begin{aligned}\mathcal{PR}_{\mathcal{GR}}(\psi,\varrho)\to\mathcal{PR}_{\mathcal{GR}}(\omega,\varrho) &= \bigvee_{r\in X}\Big(\psi(r)\wedge(\varrho(r)\vee\mathcal{GR}(\varrho))\Big)\to\bigvee_{r\in X}\Big(\omega(r)\wedge(\varrho(r)\vee\mathcal{GR}(\varrho))\Big)\\ &\geq\bigwedge_{r\in X}(\psi(r)\to\omega(r))=S_{th}(\psi,\omega).\end{aligned}$$

For the second inequality, using the property of L-grills that $S_{th}(\psi,\omega)\leq\mathcal{GR}(\psi)\to\mathcal{GR}(\omega)$, we obtain

$$\begin{aligned}\mathcal{PR}_{\mathcal{GR}}(\varrho,\psi)\to\mathcal{PR}_{\mathcal{GR}}(\varrho,\omega) &= \bigvee_{r\in X}\Big(\varrho(r)\wedge(\psi(r)\vee\mathcal{GR}(\psi))\Big)\to\bigvee_{r\in X}\Big(\varrho(r)\wedge(\omega(r)\vee\mathcal{GR}(\omega))\Big)\\ &\geq\bigwedge_{r\in X}\Big(\psi(r)\vee\mathcal{GR}(\psi)\to\omega(r)\vee\mathcal{GR}(\omega)\Big)\\ &\geq\bigwedge_{r\in X}(\psi(r)\to\omega(r))\wedge(\mathcal{GR}(\psi)\to\mathcal{GR}(\omega))\\ &= S_{th}(\psi,\omega)\wedge(\mathcal{GR}(\psi)\to\mathcal{GR}(\omega))=S_{th}(\psi,\omega).\end{aligned}$$

(LP4) For every $\xi_1,\xi_2,\omega_1,\omega_2\in L^X$, we have

$$\begin{aligned}&\mathcal{PR}_{\mathcal{GR}}(\xi_1,\omega_1)\vee\mathcal{PR}_{\mathcal{GR}}(\xi_2,\omega_2)\\ &=\bigvee_{r\in X}\Big(\xi_1(r)\wedge(\omega_1(r)\vee\mathcal{GR}(\omega_1))\Big)\vee\bigvee_{r\in X}\Big(\xi_2(r)\wedge(\omega_2(r)\vee\mathcal{GR}(\omega_2))\Big)\\ &\geq\bigvee_{r\in X}\Big(\xi_1(r)\wedge(\omega_1(r)\vee\mathcal{GR}(\omega_1))\vee(\xi_2(r)\wedge(\omega_2(r)\vee\mathcal{GR}(\omega_2)))\Big)\\ &\geq\bigvee_{r\in X}(\xi_1(r)\wedge\xi_2(r))\wedge\Big((\omega_1(r)\vee\mathcal{GR}(\omega_1))\vee(\omega_2(r)\vee\mathcal{GR}(\omega_2))\Big)\\ &=\bigvee_{r\in X}(\xi_1(r)\wedge\xi_2(r))\wedge\Big((\omega_1(r)\vee\omega_2(r))\vee\mathcal{GR}(\omega_1\vee\omega_2)\Big)\\ &=\bigvee_{r\in X}(\xi_1\wedge\xi_2)(r)\wedge\Big((\omega_1\vee\omega_2)(r)\vee\mathcal{GR}(\omega_1\vee\omega_2)\Big)\\ &=\mathcal{PR}_{\mathcal{GR}}(\xi_1\wedge\xi_2,\omega_1\vee\omega_2).\end{aligned}$$

(AL) $\forall\ \xi,\omega_l\in L^X$, $l\in\Gamma$, we have

$$\begin{aligned}\mathcal{PR}_{\mathcal{GR}}\Big(\xi,\bigvee_{l\in\Gamma}\omega_l\Big) &= \bigvee_{r\in X}\Big(\xi(r)\wedge\Big(\bigvee_{l\in\Gamma}\omega_l(r)\vee\mathcal{GR}\Big(\bigvee_{l\in\Gamma}\omega_l\Big)\Big)\Big)\\ &=\bigvee_{r\in X}\Big(\xi(r)\wedge\bigvee_{l\in\Gamma}(\omega_l(r)\vee\mathcal{GR}(\omega_l))\Big)\\ &=\bigvee_{l\in\Gamma}\bigvee_{r\in X}\Big(\xi(r)\wedge(\omega_l(r)\vee\mathcal{GR}(\omega_l))\Big)\\ &=\bigvee_{l\in\Gamma}\mathcal{PR}_{\mathcal{GR}}(\xi,\omega_l).\end{aligned}$$

(2) Since $(X,\mathcal{GR})$ is stratified, then $\mathcal{GR}(\kappa\wedge\varpi)\geq\kappa\wedge\mathcal{GR}(\varpi)$. Thus

$$\begin{aligned}\mathcal{PR}_{\mathcal{GR}}(\xi,\kappa\wedge\varpi) &= \bigvee_{r\in X}\Big(\xi(r)\wedge(\kappa\wedge\varpi(r)\vee\mathcal{GR}(\kappa\wedge\varpi))\Big)\\ &\geq\bigvee_{r\in X}\Big(\xi(r)\wedge(\kappa\wedge\varpi(r)\vee\kappa\wedge\mathcal{GR}(\varpi))\Big)\\ &=\bigvee_{r\in X}\Big(\xi(r)\wedge(\kappa\wedge(\varpi(r)\vee\mathcal{GR}(\varpi)))\Big)\\ &=\kappa\wedge\mathcal{PR}_{\mathcal{GR}}(\xi,\varpi).\end{aligned}$$

Next,

$$\begin{aligned}
\mathcal{PR}_{\mathcal{GR}}(\kappa \wedge \xi, \kappa \to \varpi) &= \bigvee_{r\in X} \Big((\kappa \wedge \xi(r)) \wedge ((\kappa \to \varpi(r)) \vee \mathcal{GR}(\kappa \to \varpi))\Big) \\
&\leq \bigvee_{r\in X} \Big[(\kappa \wedge \xi(r)) \wedge \Big((\kappa \to \varpi(r)) \vee (\kappa \to \mathcal{GR}(\varpi))\Big)\Big] \\
&\leq \bigvee_{r\in X} \Big[(\kappa \wedge \xi(r)) \wedge \Big(\kappa \to (\varpi(r) \vee \mathcal{GR}(\varpi))\Big)\Big] \\
&\leq \bigvee_{r\in X} \Big(\xi(r) \wedge (\varpi(r) \vee \mathcal{GR}(\varpi))\Big) \\
&= \mathcal{PR}_{\mathcal{GR}}(\xi, \varpi).
\end{aligned}$$

Therefore, $(X, \mathcal{PR}_{\mathcal{GR}})$ is stratified.

(3) For every $\xi, \varpi \in L^Y$, we obtain

$$\begin{aligned}
\mathcal{PR}_{\mathcal{GR}_X}(ð^{\leftarrow}(\xi), ð^{\leftarrow}(\varpi)) &= \bigvee_{r\in X} \Big(ð^{\leftarrow}(\xi)(r) \wedge (ð^{\leftarrow}(\varpi)(r) \vee \mathcal{GR}_X(ð^{\leftarrow}(\varpi))\Big) \\
&\leq \bigvee_{r\in X} \Big(\xi(ð(r)) \wedge (\varpi(ð(r)) \vee \mathcal{GR}_Y(\varpi))\Big) \\
&\leq \bigvee_{\hbar\in Y} \Big(\xi(\hbar) \wedge (\varpi(\hbar) \vee \mathcal{GR}_Y(\varpi))\Big) = \mathcal{PR}_{\mathcal{GR}_Y}(\xi, \varpi).
\end{aligned}$$

□

The correspondence $(X, \mathcal{GR}_X) \mapsto (X, \mathcal{PR}_{\mathcal{GR}_X})$ established in the aforementioned theorem induces a functor Ψ from **L-GRL** to **L-PPROX** given by $\Psi(X, \mathcal{GR}_X) = (X, \mathcal{PR}_{\mathcal{GR}_X})$, $\Psi(ð) = ð$. The next corollary gives a similar result.

Corollary 3.2. *Given an L-grill space $(X, \mathcal{GR})$ and a mapping $\mathcal{PR}_{\mathcal{GR}} : L^X \times L^X \to L$ specified by*

$$\mathcal{PR}_{\mathcal{GR}}(\omega, \xi) = \bigvee_{r\in X} \Big(\omega(r) \wedge (\xi(r) \vee \mathcal{GR}(\xi))\Big),\ \forall\, \xi, \omega \in L^X.$$

Then,

(1) *the space $(X, \mathcal{PR}_{\mathcal{GR}})$ is L-pre-proximity. Additionally, $(X, \mathcal{PR}_{\mathcal{GR}})$ is Alexandrov whenever $(X, \mathcal{GR})$ is Alexandrov.*

(2) *$\mathcal{PR}_{\mathcal{GR}}(\xi, \kappa \wedge \omega) \geq \kappa \wedge \mathcal{PR}_{\mathcal{GR}}(\xi, \omega)$ and $(X, \mathcal{PR}_{\mathcal{GR}})$ is stratified whenever $(X, \mathcal{GR})$ is stratified.*

The next theorem defines a a categorical relationship between L-pre-proximity and stratified L-grill spaces.

Theorem 3.3. *We are given the L-pre-proximity space $(X, \mathcal{PR})$. If the mapping $\mathcal{GR}_{\mathcal{PR}} : L^X \longrightarrow L$ is specified by*

$$\mathcal{GR}_{\mathcal{PR}}(\xi) = \bigvee_{r\in X} \Big(\bigwedge_{\varrho\in L^X} (\mathcal{PR}^*(\varrho, \varrho^*) \wedge S_{th}(\varrho, \xi^*)) \to \varrho^*\Big)(r),\ \forall \xi \in L^X,$$

then

(1) *the space $(X, \mathcal{GR}_{\mathcal{PR}})$ is a stratified L-grill space.*

(2) *$ð : (X, \mathcal{GR}_{\mathcal{PR}_X}) \to (Y, \mathcal{GR}_{\mathcal{PR}_Y})$ is an L-grill mapping whenever $ð : (X, \mathcal{PR}_X) \to (Y, \mathcal{PR}_Y)$ is an L-proximity mapping.*

Proof. (1) (G1) Obvious.

(G2) Let $\xi, \omega \in L^X$, and then

$$\begin{aligned}
&\mathcal{GR}_{\mathcal{PR}}(\xi) \to \mathcal{GR}_{\mathcal{PR}}(\omega)\\
&= \bigvee_{r\in X}\bigwedge_{\varrho\in L^X}\Big((\mathcal{PR}^*(\varrho,\varrho^*)\wedge S_{th}(\varrho,\xi^*))\to \varrho^*\Big)(r)\\
&\to \bigvee_{r\in X}\bigwedge_{\mu\in L^X}\Big((\mathcal{PR}^*(\mu,\mu^*)\wedge S_{th}(\mu,\omega^*))\to \mu^*\Big)(r)\\
&\geq \bigwedge_{r\in X}\Big[\bigwedge_{\varrho\in L^X}\Big((\mathcal{PR}^*(\varrho,\varrho^*)\wedge S_{th}(\varrho,\xi^*)\to \varrho^*(r))\Big)\\
&\to \bigwedge_{\mu\in L^X}\Big((\mathcal{PR}^*(\mu^*,\mu)\wedge S_{th}(\mu,\omega^*))\to \mu^*(r))\Big)\Big]\\
&\geq \bigwedge_{r\in X}\bigwedge_{\varrho\in L^X}\Big[\Big((\mathcal{PR}^*(\varrho^*,\varrho)\wedge S_{th}(\varrho,\xi^*))\to \varrho^*(r)\Big)\to\Big(\mathcal{PR}^*(\varrho^*,\varrho)\wedge S_{th}(\varrho,\omega^*)\to \varrho^*(r)\Big)\Big]\\
&\geq \bigwedge_{r\in X}\bigwedge_{\varrho\in L^X}\Big(S_{th}(\varrho,\omega^*)\to S_{th}(\varrho,\xi^*)\Big)\geq S_{th}(\omega^*,\xi^*)=S_{th}(\xi,\omega).
\end{aligned}$$

(G3) For each $\xi, \omega \in L^X$, we obtain

$$\begin{aligned}
&\mathcal{GR}_{\mathcal{PR}}(\xi)\vee\mathcal{GR}_{\mathcal{PR}}(\omega)\\
&= \bigvee_{r\in X}\bigwedge_{\varrho\in L^X}\Big((\mathcal{PR}^*(\varrho,\varrho^*)\wedge S_{th}(\varrho,\xi^*))\to \varrho^*\Big)(r)\vee\bigvee_{r\in X}\bigwedge_{\mu\in L^X}\Big((\mathcal{PR}^*(\mu,\mu^*)\wedge S_{th}(\mu,\omega^*))\to \mu^*\Big)(r)\\
&\geq \bigvee_{r\in X}\bigwedge_{\varrho,\mu\in L^X}\Big[\Big((\mathcal{PR}^*(\varrho,\varrho^*)\wedge S_{th}(\varrho,\xi^*))\to \varrho^*\Big)(r)\vee\Big((\mathcal{PR}^*(\mu,\mu^*)\wedge S_{th}(\mu,\omega^*))\to \mu^*\Big)(r)\Big]\\
&\geq \bigvee_{r\in X}\bigwedge_{\varrho,\mu\in L^X}\Big[(\mathcal{PR}^*(\mu,\mu^*)\wedge\mathcal{PR}^*(\varrho,\varrho^*)\wedge S_{th}(\varrho,\xi^*)\wedge S_{th}(\mu,\omega^*))\to \varrho^*\vee\mu^*\Big](r)\\
&\geq \bigvee_{r\in X}\bigwedge_{\varrho,\mu\in L^X}\Big[(\mathcal{PR}^*(\varrho\wedge\mu,\varrho^*\vee\mu^*))\wedge S_{th}(\varrho\wedge\mu,\xi^*\wedge\omega^*))\to \varrho^*\vee\mu^*\Big](r)\\
&= \bigvee_{r\in X}\bigwedge_{\varrho,\mu\in L^X}\Big[(\mathcal{PR}^*(\varrho\wedge\mu,(\varrho\wedge\mu)^*)\wedge S_{th}(\varrho\wedge\mu,(\xi\vee\omega)^*)\to (\varrho\wedge\mu)^*\Big](r)\\
&= \mathcal{GR}_{\mathcal{PR}}(\xi\vee\omega).
\end{aligned}$$

Since $(X, \mathcal{GR}_{\mathcal{PR}})$ is an L-grill space, by (1), it satisfies axiom (G2). Thus,

$$S_{th}(\omega,\xi)\leq \mathcal{GR}_{\mathcal{PR}}(\omega)\to\mathcal{GR}_{\mathcal{PR}}(\xi),\quad \forall\ \omega,\xi\in L^X.$$

Therefore, by Lemma 2.8, this is equivalent to the stratification condition

$$\mathcal{GR}_{\mathcal{PR}}(\kappa\to\omega)\leq \kappa\to\mathcal{GR}_{\mathcal{PR}}(\omega),\quad \forall\kappa\in L,\ \omega\in L^X.$$

Hence, $(X, \mathcal{GR}_{\mathcal{PR}})$ is a stratified L-grill space.
(2) For every $\xi \in L^Y$,

$$\begin{aligned}
\mathcal{GR}_{\mathcal{PR}_Y}(\xi) &= \bigvee_{y\in Y}\bigwedge_{\nu\in L^Y}\Big((\mathcal{PR}_Y^*(\nu,\nu^*)\wedge S_{th}(\nu,\xi^*))\to \nu^*(\hbar)\Big)\\
&\geq \bigvee_{\eth(r)\in Y}\bigwedge_{\nu\in L^Y}\Big((\mathcal{PR}_Y^*(\nu,\nu^*)\wedge S_{th}(\nu,\xi^*))\to \nu^*(\eth(r))\Big)\\
&\geq \bigvee_{r\in X}\bigwedge_{\eth^{\leftarrow}(\nu)\in L^X}(\mathcal{PR}_X^*(\eth^{\leftarrow}(\nu),\eth^{\leftarrow}(\nu^*))\wedge S_{th}(\eth^{\leftarrow}(\nu),\eth^{\leftarrow}(\xi^*))\to \eth^{\leftarrow}(\nu^*)(r))\\
&\geq \mathcal{GR}_{\mathcal{PR}_X}(\eth^{\leftarrow}(\xi)).
\end{aligned}$$

□

The correspondence $(X, \mathcal{PR}_X) \mapsto (X, \mathcal{GR})$ established in the aforementioned theorem induces a functor Ω : **L-PPROX** $\to$ **SL-GRL** defined by

$$\Omega(X, \mathcal{PR}_X) = (X, \mathcal{GR}_{\mathcal{PR}_X}), \quad \Omega(\eth) = \eth.$$

Proposition 3.4. *Given $(X, \mathcal{PR})$ as an L-pre-proximity spaces, hence $\mathcal{PR}_{\mathcal{GR}_{\mathcal{PR}}} \leq \mathcal{PR}$.*

Proof. Since

$$\mathcal{GR}_{\mathcal{PR}}(\xi) = \bigvee_{b\in X}\Big(\bigwedge_{\varrho\in L^X}(\mathcal{PR}^*(\varrho,\varrho^*) \wedge S_{th}(\varrho,\xi^*)) \to \varrho^*\Big)(b),$$

then by taking $\varrho = \top_X$, we get $\mathcal{GR}_{\mathcal{PR}}(\xi) \leq \bigvee_{b\in X} \xi(b)$. Thus,

$$\begin{aligned}
\mathcal{PR}_{\mathcal{GR}_{\mathcal{PR}}}(\xi,\varrho) &= \bigvee_{b\in X}\Big(\xi(b) \wedge (\varrho(b) \vee \mathcal{GR}_{\mathcal{PR}}(\varrho))\Big) \leq \bigvee_{b\in X}\Big(\xi(b) \wedge (\varrho(b) \vee \bigvee_{b\in X}\varrho(b))\Big) \\
&= \bigvee_{b\in X}\Big(\xi(b) \wedge \bigvee_{b\in X}\varrho(b))\Big) = \bigvee_{b\in X}(\xi(b) \wedge \varrho(b)) \leq \mathcal{PR}(\xi,\varrho).
\end{aligned}$$

□

Proposition 3.5. *Given as a stratified L-grill space $(X, \mathcal{GR})$, hence $\mathcal{GR}_{\mathcal{PR}_{\mathcal{GR}}} \geq \mathcal{GR}$.*

Proof. Since $\mathcal{PR}_{\mathcal{GR}}(\varrho,\varrho^*) = \bigvee_{r\in X}\Big(\varrho(r) \wedge (\varrho^*(r) \vee \mathcal{GR}(\varrho^*))\Big)$, then

$$\begin{aligned}
\mathcal{PR}^*_{\mathcal{GR}}(\varrho,\varrho^*) &= \bigwedge_{r\in X}\Big(\varrho(r) \to (\varrho^*(r) \vee \mathcal{GR}(\varrho^*))^*\Big) \\
&= S_{th}\Big(\varrho, (\varrho \wedge \mathcal{GR}^*(\varrho^*))\Big) \leq S_{th}\Big(\varrho, \mathcal{GR}^*(\varrho^*)\Big).
\end{aligned}$$

Now,

$$\begin{aligned}
\mathcal{GR}_{\mathcal{PR}_{\mathcal{GR}}}(\xi) &= \bigvee_{r\in X}\bigwedge_{\varrho\in L^X}\Big((\mathcal{PR}^*_{\mathcal{GR}}(\varrho,\varrho^*) \wedge S_{th}(\varrho,\xi^*)) \to \varrho^*\Big)(r) \\
&\geq \bigvee_{r\in X}\bigwedge_{\varrho\in L^X}\Big[(S_{th}(\varrho,\mathcal{GR}^*(\varrho^*)) \wedge S_{th}(\varrho,\xi^*)) \to \varrho^*\Big](r) \\
&\geq \bigvee_{r\in X}\bigwedge_{\varrho\in L^X}\Big[(S_{th}(\varrho,\mathcal{GR}^*(\varrho)) \wedge S_{th}(\mathcal{GR}(\xi),\mathcal{GR}(\varrho^*))) \to \varrho^*\Big](r) \\
&= \bigvee_{r\in X}\bigwedge_{\varrho\in L^X}\Big[(S_{th}(\varrho,\mathcal{GR}^*(\varrho^*)) \wedge S_{th}(\mathcal{GR}^*(\varrho^*),\mathcal{GR}^*(\xi))) \to \varrho^*\Big](r) \\
&\geq \bigvee_{r\in X}\bigwedge_{\varrho\in L^X}\Big[S_{th}(\varrho,\mathcal{GR}^*(\xi)) \to \varrho^*\Big](r) \\
&= \bigvee_{r\in X}\bigwedge_{\varrho\in L^X}\Big[S_{th}(\mathcal{GR}(\xi),\varrho^*) \to \varrho^*\Big](r) \\
&\geq \mathcal{GR}(\xi).
\end{aligned}$$

□

From Theorems 3.1 and 3.3, together with Propositions 3.4 and 3.5, one obtains the following corollary.

Corollary 3.6. (Ω, Ψ) *forms a Galois connection.*

4. On L-grill-proximities and L-topological structures

Based on the L-grill notion, we provide the idea of L-grill-proximity spaces, and study some of their properties. We further explore its connection to L-topological structures.

Definition 4.1. *On a set X, the mapping $\mathcal{GP}_{\mathcal{GR}} : L^X \times L^X \to L$ is referred to as an L-*grill-proximity *provided that the subsequent conditions hold for any $\omega, \mu, \kappa \in L^X$:*

(LP1) $\mathcal{GP}_{\mathcal{GR}}(\omega, \mu) = \mathcal{GP}_{\mathcal{GR}}(\mu, \omega)$;

(LP2) $S_{th}(\omega, \mu) \leq \mathcal{GP}_{\mathcal{GR}}(\omega, \kappa) \to \mathcal{GP}_{\mathcal{GR}}(\mu, \kappa)$;

(LP3) $\mathcal{GP}_{\mathcal{GR}}(\omega \vee \mu, \kappa) \leq \mathcal{GP}_{\mathcal{GR}}(\omega, \kappa) \vee \mathcal{GP}_{\mathcal{GR}}(\mu, \kappa)$;

(LP4) $\mathcal{GP}_{\mathcal{GR}}(\omega, \mu) \leq \mathcal{GR}(\omega) \wedge \mathcal{GR}(\mu)$;

(LP5) $\mathcal{GP}_{\mathcal{GR}}(\omega, \mu) \geq \mathcal{GR}(\omega \wedge \mu)$.

We refer to $(X, \mathcal{GP}_{\mathcal{GR}})$ as an L-grill-proximity space. Additionally, $(X, \mathcal{GP}_{\mathcal{GR}})$ is said to be:

(St) stratified*, if $\forall\ \kappa \in L$ and $\omega, \mu \in L^X$,*

$$\mathcal{GP}_{\mathcal{GR}}(\omega, \kappa_X \to \mu) \leq \kappa \to \mathcal{GP}_{\mathcal{GR}}(\omega, \mu),$$

or equivalently, $\mathcal{GP}_{\mathcal{GR}}(\omega, \kappa_X \wedge \mu) \geq \kappa \wedge \mathcal{GP}_{\mathcal{GR}}(\omega, \mu)$.

(AL) Alexandrov*, if for each subfamily $\{\omega_l : l \in \Gamma\} \subseteq L^X$,*

$$\mathcal{GP}_{\mathcal{GR}}(\bigvee_{l \in \Gamma} \omega_l, \mu) = \bigvee_{l \in \Gamma} \mathcal{GP}_{\mathcal{GR}}(\omega_l, \mu).$$

*The mapping $\eth : (X, \mathcal{GP}_{\mathcal{GR}_X}) \to (Y, \mathcal{GP}_{\mathcal{GR}_Y})$ between L-grill-proximity spaces is called an L-*proximity mapping *provided that $\forall\ \omega, \mu \in L^Y$,*

$$\mathcal{GP}_{\mathcal{GR}_X}(\eth^{\leftarrow}(\omega), \eth^{\leftarrow}(\mu)) \leq \mathcal{GP}_{\mathcal{GR}_Y}(\omega, \mu).$$

The category that has L-grill-proximity spaces as objects and L-proximity mappings as morphisms is referred to as **L-GPROX**.

Example 4.2. *Consider the L-grill-proximity space $(X, \mathcal{GP}_{\mathcal{GR}})$. If $\mathcal{GP}_{\mathcal{GR}}(\omega, \eta) = \mathcal{GR}(\omega) \wedge \mathcal{GR}(\eta)$, then $\mathcal{GP}_{\mathcal{GR}}$ is the largest L-grill-proximity on X. If $\mathcal{GR}(\omega) = \bigvee_{r \in X} \omega(r)$, then*

$$\mathcal{GP}_{\mathcal{GR}}(\omega, \eta) = \bigvee_{r \in X} \omega(r) \wedge \bigvee_{r \in X} \eta(r).$$

Example 4.3. *Consider the L-grill-proximity space $(X, \mathcal{GP}_{\mathcal{GR}})$. If $\mathcal{GP}_{\mathcal{GR}}(\omega, \eta) = \mathcal{GR}(\omega \wedge \eta)$, then $\mathcal{GP}_{\mathcal{GR}}$ is the smallest L-grill-proximity on X. If $\mathcal{GR}(\omega) = \bigvee_{r \in X} \omega(r)$, then*

$$\mathcal{GP}_{\mathcal{GR}}(\omega, \eta) = \bigvee_{r \in X} (\omega \wedge \eta)(r).$$

Theorem 4.4. *Given an L-grill-proximity space $(X, \mathcal{GP}_{\mathcal{GR}})$. For any $\omega \in L^X$, define the mapping $\Phi^{\omega} : L^X \to L$ as $\Phi^{\omega}(\mu) = \mathcal{GP}_{\mathcal{GR}}(\omega, \mu)$, $\forall \mu \in L^X$. Then Φ^{ω} is an L-grill mapping on X.*

Proof. We omit the proof since it is simple. □

Definition 4.5. *Given an L-grill-proximity space $(X, \mathcal{GP}_{\mathcal{GR}})$. For any $\omega \in L^X$, we define*

$$\omega^{\mathcal{GP}_{\mathcal{GR}}}(r) = \mathcal{GP}_{\mathcal{GR}}(\top_r, \omega) \qquad \textit{and} \qquad \omega_{\mathcal{GP}_{\mathcal{GR}}}(r) = \mathcal{GP}^*_{\mathcal{GR}}(\top_r, \omega^*).$$

The subsequent theorem establishes a concrete functor inducing L-closure spaces from L-grill-proximity spaces.

Theorem 4.6. *Given an L-grill-proximity space $(X, \mathcal{GP}_{\mathcal{GR}})$ and $\omega^{\mathcal{GP}_{\mathcal{GR}}}(r) = \mathcal{GP}_{\mathcal{GR}}(\top_r, \omega)$, where $\omega \in L^X$. If a mapping $C_{\mathcal{GP}_{\mathcal{GR}}} : L^X \to L^X$ is given as:*

$$C_{\mathcal{GP}_{\mathcal{GR}}}(\omega) = \omega \vee \omega^{\mathcal{GP}_{\mathcal{GR}}}, \ \ \forall\, \omega \in L^X,$$

then

(1) $(X, C_{\mathcal{GP}_{\mathcal{GR}}})$ is an L-closure space;

(2) $(X, C_{\mathcal{GP}_{\mathcal{GR}}})$ is stratified (resp., Alexandrov) if $(X, \mathcal{GP}_{\mathcal{GR}})$ is stratified (resp., Alexandrov);

(3) $\eth : (X, C_{\mathcal{GP}_{\mathcal{GR}_X}}) \to (Y, C_{\mathcal{GP}_{\mathcal{GR}_Y}})$ is an L-closure mapping if $\eth : (X, \mathcal{GP}_{\mathcal{GR}_X}) \to (Y, \mathcal{GP}_{\mathcal{GR}_Y})$ is an L-proximity mapping.

Proof. (1) (C1) Obvious.

(C2) $C_{\mathcal{GP}_{\mathcal{GR}}}(\omega) = \omega \vee \omega^{\mathcal{GP}_{\mathcal{GR}}} \geq \omega$.

(C3)

$$\begin{aligned}
S_{th}(C_{\mathcal{GP}_{\mathcal{GR}}}(\omega), C_{\mathcal{GP}_{\mathcal{GR}}}(\mu)) &= \bigwedge_{r\in X} \Big(C_{\mathcal{GP}_{\mathcal{GR}}}(\omega)(r) \to C_{\mathcal{GP}_{\mathcal{GR}}}(\mu)(r)\Big) \\
&= \bigwedge_{r\in X} \Big((\omega(r) \vee \omega^{\mathcal{GP}_{\mathcal{GR}}}(r)) \to (\mu(r) \vee \mu^{\mathcal{GP}_{\mathcal{GR}}}(r))\Big) \\
&\geq \bigwedge_{r\in X} \Big((\omega(r) \to \mu(r)) \wedge (\mu^{\mathcal{GP}_{\mathcal{GR}}}(r) \to \mu^{\mathcal{GP}_{\mathcal{GR}}}(r))\Big) \\
&\geq \bigwedge_{r\in X} \Big(\omega(r) \to \mu(r)\Big) \wedge \bigwedge_{r\in X} \Big(\omega^{\mathcal{GP}_{\mathcal{GR}}}(r) \to \mu^{\mathcal{GP}_{\mathcal{GR}}}(r)\Big) \\
&= \bigwedge_{r\in X} \Big(\omega(r) \to \mu(r)\Big) \wedge \bigwedge_{r\in X} \Big(\mathcal{GP}_{\mathcal{GR}}(\top_r, \omega) \to \mathcal{GP}_{\mathcal{GR}}(\top_r, \mu)\Big) \\
&\geq S_{th}(\omega, \mu) \wedge S_{th}(\omega, \mu) \\
&= S_{th}(\omega, \mu).
\end{aligned}$$

(C4)

$$\begin{aligned}
C_{\mathcal{GP}_{\mathcal{GR}}}(\omega)(r) \vee C_{\mathcal{GP}_{\mathcal{GR}}}(\mu)(r) &= (\omega \vee \mathcal{GP}_{\mathcal{GR}}(\top_r, \omega)) \vee (\mu \vee \mathcal{GP}_{\mathcal{GR}}(\top_r, \mu)) \\
&= (\omega \vee \mu) \vee (\mathcal{GP}_{\mathcal{GR}}(\top_r, \omega) \vee \mathcal{GP}_{\mathcal{GR}}(\top_r, \mu)) \\
&\geq (\omega \vee \mu) \vee (\mathcal{GP}_{\mathcal{GR}}(\top_r, \omega \vee \mu)) \\
&= C_{\mathcal{GP}_{\mathcal{GR}}}(\omega \vee \mu)(r).
\end{aligned}$$

(2) For every $\omega \in L^X$ and $\varrho \in L$, we get

$$\begin{aligned}
C_{\mathcal{GP}_{\mathcal{GR}}}(\varrho \to \omega) &= (\varrho \to \omega) \vee (\varrho \to \omega)^{\mathcal{GP}_{\mathcal{GR}}} \\
&= (\varrho \to \omega) \vee \mathcal{GP}_{\mathcal{GR}}(\top_r, \varrho \to \omega) \leq (\varrho \to \omega) \vee (\varrho \to \mathcal{GP}_{\mathcal{GR}}(\top_r, \omega)) \\
&\leq (\varrho \wedge \varrho) \to (\omega \vee \mathcal{GP}_{\mathcal{GR}}(\top_r, \omega)) = \varrho \to (\omega \vee \omega^{\mathcal{GP}_{\mathcal{GR}}}) = \varrho \to C_{\mathcal{GP}_{\mathcal{GR}}}(\omega).
\end{aligned}$$

(3) For every $\omega \in L^X$, we get

$$\eth^{\leftarrow}(C_{\mathcal{GP}_{\mathcal{GR}_Y}}(\omega))(r) = C_{\mathcal{GP}_{\mathcal{GR}_Y}}(\omega)(\eth(r))$$

$$
\begin{aligned}
&= \omega(\eth(r)) \vee \omega^{\mathcal{GP}_{\mathcal{GR}_Y}}(\eth(r)) = \eth^{\leftarrow}(\omega)(r) \vee \mathcal{GP}_{\mathcal{GR}_Y}(\top_{\eth(r)}, \omega) \\
&\geq \eth^{\leftarrow}(\omega)(r) \vee \mathcal{GP}_{\mathcal{GR}_X}(\eth^{\leftarrow}(\top_{\eth(r)}), \eth^{\leftarrow}(\omega)) \\
&\geq \eth^{\leftarrow}(\omega)(r) \vee \mathcal{GP}_{\mathcal{GR}_X}(\top_r, \eth^{\leftarrow}(\omega)) \\
&= C_{\mathcal{GP}_{\mathcal{GR}_X}}(\eth^{\leftarrow}(\omega))(r).
\end{aligned}
$$

This completes the proof. □

From the above theorem, we deduce that if $(X, \mathcal{GP}_{\mathcal{GR}})$ is an L-grill-proximity space and Γ : **L-GPROX** $\to$ **L-CL** is defined by $\Gamma((X, \mathcal{GP}_{\mathcal{GR}})) = (X, C_{\mathcal{GP}_{\mathcal{GR}}})$, then Γ is a functor from **L-GPROX** to **L-CL**.

The following theorem describes a categorical relationship between L-grill-proximity and L-interior spaces.

Theorem 4.7. *We are given an L-grill-proximity space* $(X, \mathcal{GP}_{\mathcal{GR}})$ *and* $\omega_{\mathcal{GP}_{\mathcal{GR}}}(r) = \mathcal{GP}^*_{\mathcal{GR}}(\top_r, \omega^*)$, *where* $\omega \in L^X$. *If a mapping* $\mathcal{I}_{\mathcal{GP}_{\mathcal{GR}}} : L^X \to L^X$ *is given as:*

$$
\mathcal{I}_{\mathcal{GP}_{\mathcal{GR}}}(\omega) = \omega \wedge \omega_{\mathcal{GP}_{\mathcal{GR}}}, \ \ \forall \omega \in L^X,
$$

then

(1) *the space* $(X, \mathcal{I}_{\mathcal{GP}_{\mathcal{GR}}})$ *is L-interior;*

(2) $(X, \mathcal{I}_{\mathcal{GP}_{\mathcal{GR}}})$ *is stratified (resp., Alexandrov) if* $(X, \mathcal{GP}_{\mathcal{GR}})$ *is stratified (resp., Alexandrov);*

(3) $\eth : (X, \mathcal{I}_{\mathcal{GP}_{\mathcal{GR}_X}}) \to (Y, \mathcal{I}_{\mathcal{GP}_{\mathcal{GR}_Y}})$ *is an L-interior mapping if* $\eth : (X, \mathcal{GP}_{\mathcal{GR}_X}) \to (Y, \mathcal{GP}_{\mathcal{GR}_Y})$ *is an L-proximity mapping.*

Proof. (1) (I1) Obvious.

(I2) $\mathcal{I}_{\mathcal{GP}_{\mathcal{GR}}}(\omega) \leq \omega$.

(I3)

$$
\begin{aligned}
S_{th}(\mathcal{I}_{\mathcal{GP}_{\mathcal{GR}}}(\omega), \mathcal{I}_{\mathcal{GP}_{\mathcal{GR}}}(\mu)) &= \bigwedge_{r\in X} \Big(\mathcal{I}_{\mathcal{GP}_{\mathcal{GR}}}(\omega)(r) \to \mathcal{I}_{\mathcal{GP}_{\mathcal{GR}}}(\mu)(r)\Big) \\
&= \bigwedge_{r\in X} \Big((\omega(r) \wedge \omega_{\mathcal{GP}_{\mathcal{GR}}}(r)) \to (\mu(r) \wedge \mu_{\mathcal{GP}_{\mathcal{GR}}}(r))\Big) \\
&= \bigwedge_{r\in X} \Big((\omega(r) \wedge \mathcal{GP}^*_{\mathcal{GR}}(\top_r, \omega^*)) \to (\mu(r) \wedge \mathcal{GP}^*_{\mathcal{GR}}(\top_r, \mu^*))\Big) \\
&\geq \bigwedge_{r\in X} \Big((\omega(r) \to \mu(r)) \wedge \bigwedge_{r\in X} (\mathcal{GP}_{\mathcal{GR}}(\top_r, \mu^*) \to \mathcal{GP}_{\mathcal{GR}}(\top_r, \omega^*))\Big) \\
&\geq S_{th}(\omega, \mu) \wedge S_{th}(\mu^*, \omega^*) = S_{th}(\omega, \mu).
\end{aligned}
$$

(I4)

$$
\begin{aligned}
\mathcal{I}_{\mathcal{GP}_{\mathcal{GR}}}(\omega \wedge \xi)(r) &= (\omega \wedge \xi)(r) \wedge (\omega \wedge \xi)_{\mathcal{GP}_{\mathcal{G}}}(r) \\
&= (\omega \wedge \xi)(r) \wedge \mathcal{GP}^*_{\mathcal{GR}}(\top_r, \omega^* \vee \xi^*) \\
&\geq (\omega \wedge \xi)(r) \wedge (\mathcal{GP}^*_{\mathcal{GR}}(\top_r, \omega^*) \wedge \mathcal{GP}^*_{\mathcal{GR}}(\top_r, \xi^*)) \\
&= (\omega \wedge \mathcal{GP}^*_{\mathcal{GR}}(\top_r, \omega^*)) \wedge (\xi \wedge \mathcal{GP}^*_{\mathcal{GR}}(\top_r, \xi^*)) \\
&= \mathcal{I}_{\mathcal{GP}_{\mathcal{GR}}}(\omega)(r) \wedge \mathcal{I}_{\mathcal{GP}_{\mathcal{GR}}}(\xi)(r).
\end{aligned}
$$

(2) For every $\omega \in L^X$ and $\varrho \in L$, we get

$$
\begin{aligned}
\mathcal{I}_{\mathcal{GP}_{\mathcal{GR}}}(\varrho \wedge \omega)(r) &= (\varrho \wedge \omega)(r) \wedge (\varrho \wedge \omega)_{\mathcal{GP}_{\mathcal{GR}}}(r) \\
&= (\varrho \wedge \omega(r)) \wedge \mathcal{GP}^*_{\mathcal{GR}}(\top_r, \varrho \to \omega^*) \\
&\geq (\varrho \wedge \omega(r)) \wedge (\varrho \wedge \mathcal{GP}^*_{\mathcal{GR}}(\top_r, \omega^*)) \\
&= \varrho \wedge \mathcal{I}_{\mathcal{GP}_{\mathcal{GR}}}(\omega)(r).
\end{aligned}
$$

(3) For each $\varpi \in L^Y$, we get

$$\begin{aligned}
ð^{\leftarrow}(\mathcal{I}_{\mathcal{GP}_{\mathcal{GR}_Y}}(\varpi))(r) &= \mathcal{I}_{\mathcal{GP}_{\mathcal{GR}_Y}}(\varpi)(ð(r)) \\
&= \varpi(ð(r)) \wedge \mathcal{GP}^*_{\mathcal{GR}_Y}(\top_{ð(r)}, \varpi^*) \\
&\leq ð^{\leftarrow}(\varpi)(r) \wedge \mathcal{GP}^*_{\mathcal{GR}_X}(ð^{\leftarrow}(\top_{ð(r)}), ð^{\leftarrow}(\varpi^*)) \\
&\leq ð^{\leftarrow}(\varpi)(r) \wedge \mathcal{GP}^*_{\mathcal{GR}_X}(\top_r, ð^{\leftarrow}(\varpi^*)) = \mathcal{I}_{\mathcal{GP}_{\mathcal{GR}_X}}(ð^{\leftarrow}(\varpi)(r)).
\end{aligned}$$

This completes the proof. □

From the above theorem, we deduce that if $(X, \mathcal{GP}_{\mathcal{GR}})$ is an L-grill-proximity space and Δ : **L-GPROX** $\rightarrow$ **L-INT** is defined by $\Delta((X, \mathcal{GP}_{\mathcal{GR}})) = (X, \mathcal{I}_{\mathcal{GP}_{\mathcal{GR}}})$, then Δ is a functor from **L-GPROX** to **L-INT**.

Next, we investigate a categorical relationship between Alexandrov L-grill-proximity spaces and Alexandrov L-co-topological space.

Theorem 4.8. *Let $(X, \mathcal{GP}_{\mathcal{GR}})$ be an Alexandrov L-grill-proximity space. If the mapping $\mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}} : L^X \rightarrow L$ is given by $\mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}}(\omega) = \mathcal{GP}^*_{\mathcal{GR}}(\omega^*, \omega)$, where $\omega \in L^X$, then*

(1) *the space $(X, \mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}})$ is an Alexandrov L-co-topological space;*

(2) *if $(X, \mathcal{GP}_{\mathcal{GR}})$ is stratified and $\mathcal{GP}_{\mathcal{GR}}(\varrho \wedge \omega, \mu) = \varrho \wedge \mathcal{GP}_{\mathcal{GR}}(\omega, \mu)$, then $(X, \mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}})$ is enriched;*

(3) *$ð : (X, \mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}_X}}) \rightarrow (Y, \mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}_Y}})$ is a continuous mapping if $ð : (X, \mathcal{GP}_{\mathcal{GR}_X}) \rightarrow (Y, \mathcal{GP}_{\mathcal{GR}_Y})$ is an L-proximity mapping.*

Proof. We prove only (1) and (2).

(1) (T1) Since $\mathcal{GP}_{\mathcal{GR}}(\omega, \mu) \leq \mathcal{GR}(\omega) \wedge \mathcal{GR}(\mu)$, we get

$$\mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}}(\top_X) = \mathcal{GP}^*_{\mathcal{GR}}(\bot_X, \top_X) \geq \mathcal{GR}^*(\bot_X) \vee \mathcal{GR}^*(\top_X) = \top \vee \bot = \top,$$

$$\mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}}(\bot_X) = \mathcal{GP}^*_{\mathcal{GR}}(\top_X, \bot_X) \geq \mathcal{GR}^*(\top_X) \vee \mathcal{G}^*(\bot_X) = \bot \vee \top = \top.$$

(T2)

$$\begin{aligned}
\mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}}(\omega \vee \mu) &= \mathcal{GP}^*_{\mathcal{GR}}(\omega^* \wedge \mu^*, \omega \vee \mu) \\
&\geq \mathcal{GP}^*_{\mathcal{GR}}(\omega^*, \omega) \wedge \mathcal{GP}^*_{\mathcal{GR}}(\mu^*, \mu) = \mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}}(\omega) \wedge \mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}}(\mu).
\end{aligned}$$

(T3)

$$\mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}}(\bigwedge_{l \in \Gamma} \omega_l) = \mathcal{GP}^*_{\mathcal{GR}}(\bigvee_{l \in \Gamma} \omega^*_l, \bigwedge_{l \in \Gamma} \omega_l) \geq \bigwedge_{l \in \Gamma} \mathcal{GP}^*_{\mathcal{GR}}(\omega^*_l, \omega_l) = \bigwedge_{l \in \Gamma} \mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}}(\omega_l),$$

$$\mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}}(\bigvee_{l \in \Gamma} \omega_l) = \mathcal{GP}^*_{\mathcal{GR}}(\bigwedge_{l \in \Gamma} \omega^*_l, \bigvee_{l \in \Gamma} \omega_l) \geq \bigwedge_{l \in \Gamma} \mathcal{GP}^*_{\mathcal{GR}}(\omega^*_l, \omega_l) = \bigwedge_{l \in \Gamma} \mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}}(\omega_l).$$

(2)

$$\begin{aligned}
\mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}}(\varrho \rightarrow \omega) &= \mathcal{GP}^*_{\mathcal{GR}}(\varrho \wedge \omega^*, \varrho \rightarrow \omega) = \varrho \rightarrow \mathcal{GP}^*_{\mathcal{GR}}(\omega^*, \varrho_X \rightarrow \omega) \\
&\geq \varrho \rightarrow (\varrho \wedge \mathcal{GP}^*_{\mathcal{GR}}(\omega^*, \omega)) \geq \mathcal{GP}^*_{\mathcal{GR}}(\omega^*, \omega) = \mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}}(\omega).
\end{aligned}$$

□

Hence, we deduce that if $(X, \mathcal{GP}_{\mathcal{GR}})$ is an L-grill-proximity space and Θ : **L-GPROX** $\rightarrow$ **CTOP** is defined by $\Theta((X, \mathcal{GP}_{\mathcal{GR}})) = (X, \mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}})$, then Θ is a functor from Alexandrov **L-GPROX** to Alexandrov **L-CTOP**.

Theorem 4.9. *Given an L-closure space $(X, C_{\mathcal{GP}_{\mathcal{GR}}})$. For any $\omega \in L^X$, define the mapping $\mathcal{CO}_{C_{\mathcal{GP}_{\mathcal{GR}}}} : L^X \to L$ as $\mathcal{CO}_{C_{\mathcal{GP}_{\mathcal{GR}}}}(\omega) = S_{th}(C_{\mathcal{GP}_{\mathcal{GR}}}(\omega), \omega)$. Then,*

(1) *the space $(X, \mathcal{CO}_{C_{\mathcal{GP}_{\mathcal{GR}}}})$ is L-co-topological;*

(2) *$(X, \mathcal{CO}_{C_{\mathcal{GP}_{\mathcal{GR}}}})$ is enriched if $(X, C_{\mathcal{GP}_{\mathcal{GR}}})$ is stratified;*

(3) *$(X, \mathcal{CO}_{C_{\mathcal{GP}_{\mathcal{GR}}}})$ is Alexandrov if $(X, C_{\mathcal{GP}_{\mathcal{GR}}})$ is Alexandrov;*

(4) *$\mathcal{CO}_{C_{\mathcal{GP}_{\mathcal{GR}}}} \le \mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}}$ if $(X, \mathcal{GP}_{\mathcal{GR}})$ is Alexandrov and $\mathcal{GP}_{\mathcal{GR}}(\kappa \wedge \omega, \mu) = \kappa \wedge \mathcal{GP}_{\mathcal{GR}}(\omega, \mu)$;*

(5) *the mapping $\eth : (X, \mathcal{CO}_{C_{\mathcal{GP}_{\mathcal{GR}_X}}}) \to (Y, \mathcal{CO}_{C_{\mathcal{GP}_{\mathcal{GR}_Y}}})$ is continuous if $\eth : (X, C_{\mathcal{GP}_{\mathcal{GR}_X}}) \to (Y, C_{\mathcal{GP}_{\mathcal{GR}_Y}})$ is an L-closure mapping.*

Proof. Since the proofs of (1)–(3) are straightforward, we only provide the proofs of (4) and (5).

(4)

$$\begin{aligned}
\mathcal{CO}_{C_{\mathcal{GP}_{\mathcal{GR}}}}(\varpi) &= S_{th}(C_{\mathcal{GP}_{\mathcal{GR}}}(\varpi), \varpi) \\
&= \bigwedge_{\hbar \in X} (C_{\mathcal{GP}_{\mathcal{GR}}}(\varpi)(\hbar) \to \varpi(\hbar)) = \bigwedge_{\hbar \in X} (\varpi(\hbar) \vee \mathcal{GP}_{\mathcal{GR}}(\top_\hbar, \varpi) \to \varpi(\hbar)) \\
&\le \bigwedge_{\hbar \in X} (\mathcal{GP}_{\mathcal{GR}}(\top_\hbar, \varpi) \to \varpi(\hbar)) = \bigwedge_{\hbar \in X} (\varpi^*(\hbar) \to \mathcal{GP}^*_{\mathcal{GR}}(\top_\hbar, \varpi)) \\
&= \mathcal{GP}^*_{\mathcal{GR}}(\bigvee_{\hbar \in X} (\varpi^* \wedge \top_\hbar)(\hbar), \varpi)) = \mathcal{GP}^*_{\mathcal{GR}}(\varpi^*, \varpi) = \mathcal{CO}_{\mathcal{GP}_{\mathcal{GR}}}(\varpi).
\end{aligned}$$

(5)

$$\begin{aligned}
\mathcal{CO}_{C_{\mathcal{GP}_{\mathcal{GR}_Y}}}(\varpi) &= S_{th}(C_{\mathcal{GP}_{\mathcal{GR}_Y}}(\varpi), \varpi) \le S_{th}(\eth^{\leftarrow}(C_{\mathcal{GP}_{\mathcal{GR}_Y}}(\varpi)), \eth^{\leftarrow}(\varpi)) \\
&\le S_{th}(C_{\mathcal{GP}_{\mathcal{GR}_X}}(\eth^{\leftarrow}(\varpi)), \eth^{\leftarrow}(\varpi)) = \mathcal{CO}_{C_{\mathcal{GP}_{\mathcal{GR}_X}}}(\eth^{\leftarrow}(\varpi)).
\end{aligned}$$

□

Hence, the above correspondence $(X, C_{\mathcal{GP}_{\mathcal{GR}}}) \mapsto (X, \mathcal{CO}_{C_{\mathcal{GP}_{\mathcal{GR}}}})$ demonstrates a functor from **L-CL** to **L-CTOP**.

Definition 4.10. *Let $\mathcal{GP}_{\mathcal{GR}}$ be an L-grill-proximity on X. For every $\omega \in L^X$, define the mapping $\mathcal{N}^{\omega}_{\mathcal{GP}_{\mathcal{GR}}} : L^X \to L$ as $\mathcal{N}^{\omega}_{\mathcal{GP}_{\mathcal{GR}}}(\mu) = \mathcal{GP}^*_{\mathcal{GR}}(\omega, \mu^*), \forall\, \mu \in L^X$. Then $\mathcal{N}^{\omega}_{\mathcal{GP}_{\mathcal{GR}}}$ is said to be an L-grill-proximal neighborhood. If $\mathcal{N}^{\omega}_{\mathcal{GP}_{\mathcal{GR}}}(\mu) = \top$, then we say that μ is an L-grill-proximal neighborhood of ω. In particular, $\mathcal{N}^{\top_r}_{\mathcal{GP}_{\mathcal{GR}}}(\mu) = \mathcal{GP}^*_{\mathcal{GR}}(\top_r, \mu^*)$.*

The following theorem establishes the fundamental properties of the L-grill-proximal neighborhood.

Theorem 4.11. *Let $\mathcal{GP}_{\mathcal{GR}}$ be an L-grill-proximity on X and $\mathcal{N}^{\omega}_{\mathcal{GP}_{\mathcal{GR}}}$ be an L-grill-proximal neighborhood. Then for any $\omega, \mu, \varrho \in L^X$, we have*

(1) *$\mathcal{N}^{\top_X}_{\mathcal{GP}_{\mathcal{GR}}}(\top_X) = \top$ and $\mathcal{N}^{\top_X}_{\mathcal{GP}_{\mathcal{GR}}}(\perp_X) = \perp$;*

(2) *$\mathcal{N}^{\omega}_{\mathcal{GP}_{\mathcal{GR}}}(\mu) \le S_{th}(\omega, \mu)$;*

(3) *$S_{th}(\mu, \varrho) \le \mathcal{N}^{\omega}_{\mathcal{GP}_{\mathcal{GR}}}(\mu) \to \mathcal{N}^{\omega}_{\mathcal{GP}_{\mathcal{GR}}}(\varrho)$;*

(4) *$\mathcal{N}^{\omega}_{\mathcal{GP}_{\mathcal{GR}}}(\mu \wedge \varrho) \ge \mathcal{N}^{\omega}_{\mathcal{GP}_{\mathcal{GR}}}(\mu) \wedge \mathcal{N}^{\omega}_{\mathcal{GP}_{\mathcal{GR}}}(\varrho)$.*

Proof. The proofs of (1) and (2) are straightforward; therefore, they are omitted.

(3)

$$\begin{aligned}
\mathcal{N}^{\omega}_{\mathcal{GP}_{\mathcal{GR}}}(\mu) \to \mathcal{N}^{\omega}_{\mathcal{GP}_{\mathcal{GR}}}(\varrho) &= \mathcal{GP}^*_{\mathcal{GR}}(\omega, \mu^*) \to \mathcal{GP}^*_{\mathcal{GR}}(\omega, \varrho^*) \\
&= \mathcal{GP}_{\mathcal{GR}}(\omega, \varrho^*) \to \mathcal{GP}_{\mathcal{GR}}(\omega, \mu^*) \\
&\ge S_{th}(\varrho^*, \mu^*) = S_{th}(\mu, \varrho).
\end{aligned}$$

(4)

$$\begin{aligned}\mathcal{N}^{\omega}_{\mathcal{GP}_{\mathcal{GR}}}(\mu)\wedge\mathcal{N}^{\omega}_{\mathcal{GP}_{\mathcal{GR}}}(\varrho) &= \mathcal{GP}^{*}_{\mathcal{GR}}(\omega,\mu^{*})\wedge\mathcal{GP}^{*}_{\mathcal{GR}}(\omega,\varrho^{*})\\ &\leq \mathcal{GP}^{*}_{\mathcal{GR}}(\omega,\mu^{*}\vee\varrho^{*}) = \mathcal{GP}^{*}_{\mathcal{GR}}(\omega,(\mu\wedge\varrho)^{*})\\ &= \mathcal{N}^{\omega}_{\mathcal{GP}_{\mathcal{GR}}}(\mu\wedge\varrho).\end{aligned}$$

□

Theorem 4.12. *Given an L-grill-proximity space $(X,\mathcal{GP}_{\mathcal{GR}})$. For any $\omega,\varrho\in L^X$ and $r\in X$, we define the mapping $\Phi^r_{\mathcal{GR}}: L^X\to L^X$ as $\Phi^r_{\mathcal{GR}}(\omega)=\mathcal{GR}(\omega\wedge\varrho)\wedge\mathcal{N}^{\top_r}_{\mathcal{GP}_{\mathcal{GR}}}(\varrho)$. Then*
(1) $S_{th}(\omega,\mu)\leq S_{th}(\Phi^r_{\mathcal{GR}}(\omega),\Phi^r_{\mathcal{GR}}(\mu))$;
(2) $\Phi^r_{\mathcal{GR}}(\omega\vee\mu)\leq\Phi^r_{\mathcal{GR}}(\omega)\vee\Phi^r_{\mathcal{GR}}(\mu)$;
(3) $\Phi^r_{\mathcal{GR}}(\omega\wedge\mu)\leq\Phi^r_{\mathcal{GR}}(\omega)\wedge\Phi^r_{\mathcal{GR}}(\mu)$.

Proof. (1)

$$\begin{aligned}S_{th}(\Phi^r_{\mathcal{GR}}(\omega),\Phi^r_{\mathcal{GR}}(\mu)) &= \bigwedge_{r\in X}(\Phi^r_{\mathcal{GR}}(\omega)\to\Phi^r_{\mathcal{GR}}(\mu))\\ &= \bigwedge_{r\in X}\Big((\mathcal{GR}(\omega\wedge\varrho)\wedge\mathcal{N}^{\top_r}_{\mathcal{GP}_{\mathcal{GR}}}(\varrho))\to(\mathcal{GR}(\mu\wedge\varrho)\wedge\mathcal{N}^{\top_r}_{\mathcal{GP}_{\mathcal{GR}}}(\varrho))\Big)\\ &\geq \bigwedge_{r\in X}\Big((\mathcal{GR}(\omega\wedge\varrho)\to\mathcal{GR}(\mu\wedge\varrho))\wedge(\mathcal{N}^{\top_r}_{\mathcal{GP}_{\mathcal{GR}}}(\varrho)\to\mathcal{N}^{\top_r}_{\mathcal{GP}_{\mathcal{GR}}}(\varrho))\Big)\\ &\geq S_{th}(\omega\wedge\varrho,\mu\wedge\varrho)\geq S_{th}(\omega,\mu)\wedge S_{th}(\varrho,\varrho)=S_{th}(\omega,\mu).\end{aligned}$$

(2)

$$\begin{aligned}\Phi^r_{\mathcal{GR}}(\omega\vee\mu) &= \mathcal{GR}((\omega\vee\mu)\wedge\varrho)\wedge\mathcal{N}^{\top_r}_{\mathcal{GP}_{\mathcal{GR}}}(\varrho)\\ &= \mathcal{GR}((\omega\wedge\varrho)\vee(\mu\wedge\varrho))\wedge\mathcal{N}^{\top_r}_{\mathcal{GP}_{\mathcal{GR}}}(\varrho)\\ &\leq \Big(\mathcal{GR}(\omega\wedge\varrho)\vee\mathcal{GR}(\mu\wedge\varrho)\Big)\wedge\mathcal{N}^{\top_r}_{\mathcal{GP}_{\mathcal{GR}}}(\varrho)\\ &= \Big(\mathcal{GR}(\omega\wedge\varrho)\wedge\mathcal{N}^{\top_r}_{\mathcal{GP}_{\mathcal{GR}}}(\varrho)\Big)\vee\Big(\mathcal{GR}(\mu\wedge\varrho)\wedge\mathcal{N}^{\top_r}_{\mathcal{GP}_{\mathcal{GR}}}(\varrho)\Big)\\ &= \Phi^r_{\mathcal{GR}}(\omega)\vee\Phi^r_{\mathcal{GR}}(\mu).\end{aligned}$$

(3) The proof is similar to that of (2). □

5. Application: environmental pollution assessment

In this application, we employ the theoretical framework of L-grills, L-neighborhood systems, and L-pre-proximities to model environmental pollution phenomena in a graded and spatially distributed manner by integrating multiple environmental indicators for effective quantitative monitoring and decision-making in urban air-quality management.

Let $X=\{x_1,x_2,x_3,x_4,x_5\}$ denote a set of five air-quality monitoring locations, and let $L=[0,1]$ denote the complete Heyting algebra equipped with the usual order. Pollution at each location is represented as an L-subset $\omega: X\to L$, where $\omega(r)$ indicates the measured pollution intensity at location $r\in X$.

We consider three pollution indicators, modeled as L-subsets $\lambda,\mu,\nu\in L^X$, defined, respectively, by

$$\lambda=\mathrm{PM}_{2.5}\text{ concentration},\quad \mu=\mathrm{NO}_2\text{ levels},\quad \nu=\mathrm{O}_3\text{ concentration}.$$

These indicators correspond to fine particulate matter, nitrogen dioxide, and ground-level ozone, and jointly describe particulate, gaseous, and photochemical pollution components.

The normalized measurements of the indicators at each monitoring station are given in Table 1.

Table 1. Normalized values of the pollution indicators at the monitoring stations.

Indicators	Stations				
	x_1	x_2	x_3	x_4	x_5
λ	0.8	0.6	0.4	0.7	0.5
μ	0.7	0.5	0.6	0.6	0.4
ν	0.5	0.4	0.7	0.3	0.6

The global evaluation that is given by the L-grill mapping $\mathcal{GR}_{\mathcal{G}} : L^X \to L$ is defined as:

$$\mathcal{GR}_{\mathcal{G}}(\alpha) = \bigvee_{x\in X} \alpha(x) = \sup_{x\in X} \alpha(x), \quad \alpha \in L^X.$$

The computations of the L-grill mapping with the given pollution indicators are:

$$\begin{aligned}\mathcal{GR}_{\mathcal{G}}(\lambda) &= \sup\{0.8, 0.6, 0.4, 0.7, 0.5\} = 0.8,\\ \mathcal{GR}_{\mathcal{G}}(\mu) &= \sup\{0.7, 0.5, 0.6, 0.6, 0.4\} = 0.7,\\ \mathcal{GR}_{\mathcal{G}}(\nu) &= \sup\{0.5, 0.4, 0.7, 0.3, 0.6\} = 0.7.\end{aligned}$$

This indicates that $PM_{2.5}$ (λ) is the globally dominant pollutant.

Now, the L-pre-proximity induced by $\mathcal{G}$ is defined by

$$\delta_{\mathcal{G}}(\alpha, \beta) = \mathcal{GR}_{\mathcal{G}}(\gamma), \quad \gamma = \alpha \wedge \beta \text{ (pointwise minimum)},$$

and its station-by-station computations are given by:

$$\begin{aligned}\lambda \wedge \mu &= (0.7, 0.5, 0.4, 0.6, 0.4), & \delta_{\mathcal{G}}(\lambda, \mu) &= \sup(0.7, 0.5, 0.4, 0.6, 0.4) = 0.7,\\ \lambda \wedge \nu &= (0.5, 0.4, 0.4, 0.3, 0.5), & \delta_{\mathcal{G}}(\lambda, \nu) &= \sup(0.5, 0.4, 0.4, 0.3, 0.5) = 0.5,\\ \mu \wedge \nu &= (0.5, 0.4, 0.6, 0.3, 0.4), & \delta_{\mathcal{G}}(\mu, \nu) &= \sup(0.5, 0.4, 0.6, 0.3, 0.4) = 0.6.\end{aligned}$$

For each station x_i, the L-neighborhood system induced by $\mathcal{G}$ can be computed by:

$$\mathcal{N}_{\mathcal{G}}(x_i)(\alpha) = \mathcal{GR}_{\mathcal{G}}(\alpha \wedge 1_{x_i}) = \alpha(x_i),$$

where 1_{x_i} is the L-subset equal to 1 at x_i and 0 elsewhere. The L-neighborhood evaluations to the station level are given in Table 2.

Table 2. *L*-neighborhood evaluations of pollutants at each station.

L-neighborhoods	Stations				
	x_1	x_2	x_3	x_4	x_5
$\mathcal{N}_{\mathcal{G}}(x)(\lambda)$	0.8	0.6	0.4	0.7	0.5
$\mathcal{N}_{\mathcal{G}}(x)(\mu)$	0.7	0.5	0.6	0.6	0.4
$\mathcal{N}_{\mathcal{G}}(x)(\nu)$	0.5	0.4	0.7	0.3	0.6

Table 3 summarizes the evaluation of pollution indicators: global L-grill, pairwise L-pre-proximity, and max station values.

Table 3. Combined evaluation of pollution indicators: global L-grill, pairwise L-pre-proximity, and max station values.

Pollutant	Global L-grill $\mathcal{GR}_{\mathcal{G}}$	Pairwise L-pre-proximity $\delta_{\mathcal{GR}}$	Max station value
λ ($PM_{2.5}$)	0.8	$\delta_{\mathcal{G}}(\lambda,\mu) = 0.7$, $\delta_{\mathcal{G}}(\lambda,\nu) = 0.5$	$x_1 : 0.8$
μ (NO_2)	0.7	$\delta_{\mathcal{G}}(\mu,\lambda) = 0.7$, $\delta_{\mathcal{G}}(\mu,\nu) = 0.6$	$x_1 : 0.7$
ν (O_3)	0.7	$\delta_{\mathcal{G}}(\nu,\lambda) = 0.5$, $\delta_{\mathcal{G}}(\nu,\mu) = 0.6$	$x_3 : 0.7$

Interpretation

According to Table 3, we conclude the following:

- The global maximum values $\mathcal{GR}$ indicate $PM_{2.5}$ is the globally dominant pollutant, particularly at station x_1.
- The L-pre-proximity values highlight moderate to strong correlations among pollutants. The highest pairwise L-pre-proximity $\delta_{\mathcal{GR}} = 0.7$ shows strong correlations between $PM_{2.5}$ and NO_2, suggesting linked pollution sources.
- Stations x_1 and x_3 are locally critical due to high neighborhood values, indicating stations needing immediate attention: x_1 (high $PM_{2.5}$ and NO_2), x_3 (high O_3).
- The combined analysis demonstrates how L-grill and neighborhood structures yield graded, spatially informed pollution insights.

6. Conclusions

In this study, we have successfully established categorical connections between L-grill and L-pre-proximity spaces. By constructing concrete functors and proving the existence of a Galois connection between these categories, we have clarified the structural relationships between them. Additionally, we introduced the notion of L-grill-proximity spaces, a structure derived from L-grills, and investigated their fundamental properties together with their categorical relationships to L-closure, L-interior, and L-co-topological spaces. In particular, we showed that L-grill-proximity spaces are categorically connected to L-closure, L-interior, and L-co-topological spaces through a Galois correspondence.

The application of this framework to environmental pollution modeling demonstrates that L-grills and L-pre-proximities provide an effective quantitative tool for integrating multiple environmental indicators and analyzing spatially distributed phenomena, thereby supporting air-quality assessment and informed decision-making.

We believe that this work opens new avenues for future research by laying a foundation for further investigations, including:

(1) Investigating the categorical properties of L-primal-proximities and characterizing the concrete functors connecting them to L-topological structures.
(2) Exploring the relationships connecting L-grill spaces and L-topogenous structures.
(3) Inducing L-neighborhoods from L-pre-proximity structures, with a particular emphasis on identifying new practical applications.
(4) Examining the role of L-neighborhoods and their applications in L-grill approximation spaces.

Use of Generative-AI tools declaration

The author declares that no Artificial Intelligence (AI) tools were used in the creation of this article.

Acknowledgments

The author would like to express sincere gratitude to Professor Ahmed A. Ramadan of Beni-Suef University for his valuable communications, insightful suggestions, and constructive comments. His thoughtful input significantly enhanced the quality and clarity of this paper, and his support is greatly appreciated.

Funding

The Researcher would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University (www.qu.edu.sa) for financial support (QU-APC-2026).

Conflict of interest

The author declares no conflict of interest.

References

1. J. A. Goguen, *L*-fuzzy sets, *J. Math. Anal. Appl.*, **8** (1967), 145–174. https://doi.org/10.1016/0022-247X(67)90189-8
2. L. A. Zadeh, Fuzzy sets, *Inf. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
3. U. Höhle, A. Šostak, Axiomatic foundations of fixed-basis fuzzy topology, In: U. Höhle, S. E. Rodabaugh, *Mathematics of fuzzy sets*, The Handbooks of Fuzzy Sets Series, Springer, Boston, **3** (1999), 123–272. https://doi.org/10.1007/978-1-4615-5079-2_5
4. A. A. Ramadan, Smooth topological spaces, *Fuzzy Sets Syst.*, **48** (1992), 371–375. https://doi.org/10.1016/0165-0114(92)90352-5
5. R. Bělohlávek, Fuzzy closure operators, *J. Math. Anal. Appl.*, **262** (2001), 473–489. https://doi.org/10.1006/jmaa.2000.7456
6. R. Bělohlávek, T. Funioková, Fuzzy interior operators, *Int. J. Gen. Syst.*, **33** (2004), 415–430. https://doi.org/10.1080/03081070410001679724
7. J. Fang, Relationships between *L*-ordered convergence structures and strong *L*-topologies, *Fuzzy Sets Syst.*, **161** (2010), 2923–2944. https://doi.org/10.1016/j.fss.2010.07.010
8. J. M. Oh, Y. C. Kim, *L*-Fuzzy closure operators and *L*-fuzzy cotopologies, *J. Math. Comput. Sci.*, **9** (2019), 131–145.
9. A. A. Ramadan, A. J. Fawakhreh, E. H. Elkordy, Novel categorical relations between $\mathcal{L}$-fuzzy co-topologies and $\mathcal{L}$-fuzzy ideals, *AIMS Math.*, **9** (2024), 20572–20587. https://doi.org/10.3934/math.2024999

10. F. G. Shi, *L*-fuzzy interiors and *L*-fuzzy closures, *Fuzzy Sets Syst.*, **160** (2009), 1218–1232. https://doi.org/10.1016/j.fss.2008.09.002

11. D. Zhang, An enriched category approach to many valued topology, *Fuzzy Sets Syst.*, **158** (2007), 349–366. https://doi.org/10.1016/j.fss.2006.10.001

12. A. K. Katsaras, Fuzzy proximity spaces, *J. Math. Anal. Appl.*, **68** (1979), 100–110. https://doi.org/10.1016/0022-247X(79)90102-1

13. G. Artico, R. Moresco, Fuzzy proximities and totally bounded fuzzy uniformities, *J. Math. Anal. Appl.*, **99** (1984), 320–337. https://doi.org/10.1016/0022-247X(84)90221-X

14. D. Čimoka, A. Šostak, *L*-fuzzy syntopogenous structures, Part I: Fundamentals and application to *L*-fuzzy topologies, *L*-fuzzy proximities and *L*-fuzzy uniformities, *Fuzzy Sets Syst.*, **232** (2013), 74–97. https://doi.org/10.1016/j.fss.2013.04.009

15. Y. C. Kim, K. C. Min, *L*-fuzzy proximities and *L*-fuzzy topologies, *Inf. Sci.*, **173** (2005), 93–113. https://doi.org/10.1016/j.ins.2004.07.005

16. M. Jose, S. C. Mathew, On the categorical connections of *L*-*G*-filter spaces and a Galois correspondence with *L*-fuzzy pre-proximity spaces, *New Mat. Nat. Comput.*, **19** (2023), 33–49. https://doi.org/10.1142/S1793005722500478

17. K. K. Azad, Fuzzy grills and a characterization of fuzzy proximity, *J. Math. Anal. Appl.*, **79** (1981), 13–17. https://doi.org/10.1016/0022-247X(81)90003-2

18. M. N. Mukherjee, S. Das, Fuzzy grills and induced fuzzy topology, *Math. Vesnk*, **62** (2010), 285–297.

19. B. Roy, M. N. Mukherjee, On a typical topology induced by a grill, *Soochow J. Math.*, **33** (2007), 771.

20. A. Kandil, S. A. El-Sheikh, M. Abdelhakem, S. A. Hazza, On ideals and grills in topological spaces, *South Asian J. Math.*, **5** (2015), 233–238.

21. A. R. Hosny Proximity structures and grills, *Int. J. Eng. Res.*, **4** (2013), 427–450.

22. A. A. Ramadan, M. A. Usama, A. A. Abd El-Latif, *L*-fuzzy pre-proximities, *L*-fuzzy filters and *L*-fuzzy grills, *J. Egypt. Math. Soc.*, **28** (2020), 47. https://doi.org/10.1186/s42787-020-00105-4

23. A. A. Ramadan, A. J. Fawakhreh, On *L*-fuzzy preproximity spaces and *L*-fuzzy ideals, *New Math. Nat. Comput.*, **21** (2025), 867–886. https://doi.org/10.1142/S1793005725500437

24. A. A. Ramadan, A. J. Fawakhreh, Categories of *L*-primals, *L*-pre-proximities, and *L*-topologies, *Axioms*, **14** (2025), 541. https://doi.org/10.3390/axioms14070541

25. A. A. Ramadan, A. J. Fawakhreh, On the categories of *LF*-ideals, *LF*-grills, and *LF*-topological spaces, *Axioms*, **15** (2026), 68. https://doi.org/10.3390/axioms15010068

26. A. J. Fawakhreh, Connections of *L*-quasi-internal and *L*-quasi-enclosed spaces with *L*-grills, *L*-filters and *L*-neighborhood spaces, *Axioms*, **15** (2026), 462. https://doi.org/10.3390/axioms15060462

27. S. E. Abbas, M. A. Hebeshi, Construction of stratified *L*-fuzzy topological structures, *Appl. Math. Lett.*, **24** (2011), 1062–1068. https://doi.org/10.1016/j.aml.2011.01.023

28. I. Ibedou, S. E. Abbas, Fuzzy topological concepts via ideals and grills, *Ann. Fuzzy Math. Inf.*, **15** (2018), 137–148. https://doi.org/10.30948/afmi.2018.15.2.137

29. A. Kandil, S. A. El-Sheikh, E. Said, Generalized G-proximity spaces, *South Asian J. Math.*, **7** (2017), 140–147.

30. R. Bělohlávek, *Fuzzy relational systems: foundations and principles*, IFSR International Series in Systems Science and Systems Engineering, Vol. 20, Springer New York, NY, 2002. https://doi.org/10.1007/978-1-4615-0633-1

31. S. Gottwald, Many-valued logic and fuzzy set theory, In: U. Höhle, S. E. Rodabaugh, *Mathematics of fuzzy sets*, The Handbooks of Fuzzy Sets Series, Springer, Boston, **3** (1999), 5–89. https://doi.org/10.1007/978-1-4615-5079-2_2

32. J. Adámek, H. Herrlich, G. E. Strecker, *Abstract and concrete categories*, John Wiley and Sons, Inc., 1990.