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Research article

Tauberian theorems and Cesàro summability in $\mathscr{L}-$fuzzy normed spaces

Reha Yapalı[1,*] **and Utku Gürdal**[2]

[1] Department of Mathematics, Muş Alparslan University, Muş, Türkiye
[2] Department of Mathematics, Burdur Mehmet Âkif University, Burdur, Türkiye

* **Correspondence:** Email: reha.yapali@alparslan.edu.tr.

Abstract: This paper investigates the Cesàro summability method in the setting of $\mathscr{L}-$fuzzy normed spaces, extending the classical theory of summability to the framework of $\mathscr{L}-$fuzzy norms. We also establish a Tauberian theorem characterizing the conditions under which Cesàro summability implies ordinary convergence in this setting. Furthermore, we derive an analogue of Hardy's classical two-sided Tauberian theorem by employing the notion of q-boundedness in $\mathscr{L}-$fuzzy normed spaces, thereby providing new criteria for analyzing the convergence behavior of sequences and series in this context.

Keywords: $\mathscr{L}-$fuzzy normed spaces; Cesàro summability; Tauberian theorems; slowly oscillating sequences; fuzzy functional analysis
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1. Introduction

Fuzzy set theory, introduced by Zadeh [35], provides a mathematical framework for representing imprecision and uncertainty by allowing elements to have degrees of membership. This theory has become a flexible tool for modeling complex systems and uncertain data.

Building on Zadeh's work, Atanassov [2] introduced intuitionistic fuzzy sets, which involve membership and non-membership degrees together with a hesitation margin. This framework allows uncertainty and ambiguity to be represented in a more refined way.

Motivated by intuitionistic fuzzy metric spaces, intuitionistic fuzzy normed spaces were introduced in [21]. These spaces provide a natural setting for studying convergence and related analytical properties under intuitionistic fuzzy uncertainty.

The investigation of Tauberian theorems has expanded beyond conventional settings to fuzzy frameworks. In the classical Cesàro setting, Móricz established Tauberian conditions under which statistical convergence follows from statistical $(C,1)$ summability [16] and necessary and sufficient conditions under which convergence follows from $(C,1)$ summability [17]. For double sequences of fuzzy numbers, Tauberian conditions have been obtained for ordinary $(C, 1, 1)$ summability, statistical $(C, 1, 1)$ summability, and the weighted-mean $(\bar{N}, p, q)$ method [3, 18, 29]. For fuzzy-number-valued sequences and series, power-series and deferred Cesàro summability methods have likewise been supplemented by conditions ensuring convergence [24, 25]. Statistical summability has also been connected with fuzzy Korovkin-type approximation: The statistical $(C, 1)(E, \mu)$ product method yields approximation results with statistical fuzzy rates, generalized weighted λ-statistical summability yields related approximation theorems, and statistical deferred Nörlund methods have been applied to functions of two variables [5, 7, 20]. The theory has been extended to strong deferred Cesàro summability and deferred statistical convergence for triple sequences of sets [12], deferred statistical convergence of order β for fuzzy variables in credibility space [13], and Zweier $[V, \lambda, \mu]$-summability in neutrosophic n-normed spaces [14]. In intuitionistic fuzzy normed spaces, Savaş and Gürdal introduced I-statistical and I-lacunary statistical convergence and investigated their relationship [23], whereas Talo and Yavuz proved that Cesàro summability together with slow oscillation implies convergence and derived a Hardy-type Tauberian theorem. More recently, Önder et al. obtained statistical-convergence conclusions from statistical Cesàro summability under $q-$boundedness [19], while Debnath investigated the passage from statistical Cesàro summability to statistical convergence in intuitionistic fuzzy n-normed spaces, providing a basis for associated Tauberian results [6].

The motivation for passing from intuitionistic fuzzy normed spaces to $\mathscr{L}-$fuzzy normed spaces comes from the greater flexibility of the lattice-valued framework. Intuitionistic fuzzy normed spaces are based on membership and non-membership degrees, usually represented in a specific ordered structure, whereas $\mathscr{L}-$fuzzy normed spaces allow the underlying degrees of uncertainty to take values in a general complete lattice. This makes it possible to treat different types of ordered uncertainty within a unified setting, including fuzzy, intuitionistic fuzzy, set-valued, and other lattice-valued models at the same time. Therefore, extending Cesàro summability and Tauberian conditions to $\mathscr{L}-$fuzzy normed spaces is not merely a formal generalization, but provides a broader framework in which several fuzzy convergence structures can be studied simultaneously. So, it presents the results of all the separate studies that may have been done later, all under one roof.

Consistent with this motivation, several convergence structures have already been developed in L-fuzzy normed spaces. Statistical convergence was investigated in [31], where it was shown to be more general than convergence. For double sequences, statistical convergence was characterized in relation to Pringsheim convergence and shown to be weaker than convergence [32]. Lacunary statistical convergence was subsequently developed both for sequences [33] and double sequences [30], together with the corresponding notions of lacunary statistical Cauchyness, boundedness, and completeness. In fuzzy number spaces, statistical $(C, 1, 1)$ summability was introduced for double sequences and Tauberian conditions were established [34]. These results provide a foundation for extending summability methods from fuzzy setting to the broader L-fuzzy framework considered here.

The aim of this paper is to study Cesàro summability and related Tauberian theorems in $\mathscr{L}-$fuzzy normed spaces. We derive criteria that allow one to determine when Cesàro summability yields ordinary convergence in this setting.

The main contribution of this work is the formulation of the Cesàro summability method in $\mathscr{L}-$fuzzy normed spaces. We establish one-sided and two-sided Tauberian conditions characterizing when Cesàro summability implies ordinary convergence. These criteria clarify the relation between Cesàro summability and ordinary convergence in $\mathscr{L}-$fuzzy environments.

We also study slowly oscillating sequences in $\mathscr{L}-$fuzzy normed spaces. This concept is central in classical Tauberian theory and is investigated here in comparison with its corresponding concept in normed spaces. We also clarify the role of slowly oscillating sequences in Tauberian theory for $\mathscr{L}-$fuzzy normed spaces.

We extend Tauberian conditions for $\mathscr{L}-$fuzzy normed spaces with the aid of the notions of slowly oscillating sequences and $q-$boundedness. More precisely, we establish both slowly oscillating type and Hardy's two-sided type criteria, which characterize when Cesàro summability implies ordinary convergence within this framework.

This paper provides an initial study of Cesàro summability, slowly oscillating sequences, and their implications for Tauberian theory in $\mathscr{L}-$fuzzy normed spaces. The results contribute to fuzzy functional analysis by identifying conditions under which convergence can be recovered from summability in a non-classical setting.

2. Preliminaries

In this section we give some preliminaries on $\mathscr{L}-$fuzzy normed spaces.

Definition 2.1. [26] Given a complete lattice $\mathscr{L} = (L, \leq)$ and a set X (the universe), an $\mathscr{L}-$fuzzy set (or an $\mathscr{L}-$set for short) over X is defined to be a function

$$A : X \to L.$$

We denote the family of all $\mathscr{L}-$sets over a set X by L^X.

The intersection and the union of two $\mathscr{L}-$sets over the same universe are given by

$$\begin{aligned}(A \cap B)(x) &:= A(x) \wedge B(x),\\ (A \cup B)(x) &:= A(x) \vee B(x),\end{aligned}$$

for all $x \in X$. Similarly, the intersection and the union of a family $\{A_i : i \in I\}$ of such objects are given by

$$\begin{aligned}\Big(\bigcap_{i\in I} A_i\Big)(x) &:= \bigwedge_{i\in I} A_i(x),\\ \Big(\bigcup_{i\in I} A_i\Big)(x) &:= \bigvee_{i\in I} A_i(x).\end{aligned}$$

We denote the smallest and the greatest elements of a complete lattice $\mathscr{L} = (L, \leq)$ as usual by 0_L and 1_L, respectively. We also use the symbols $\geq, \prec$, and $\succ$ in the obvious meanings, that is, $a \geq b :\Longleftrightarrow b \leq a$, or $a \prec b :\Longleftrightarrow\ a \leq b$ and $a \neq b$.

Whenever a complete lattice $(L, \leq)$ is given, we assume the order topology on it, given with the subbase elements in the forms $[0_L, a) = \{x \in L : x \prec a\}$ and $(a, 1_L] = \{x \in L : a \prec x\}$, where $a \in L \setminus \{0_L, 1_L\}$. Then the frequently used expression $\lim a_n = 1_L$ for a sequence (a_n) on L means that for each $a \in L$, there exists an $n_0 \in \mathbb{N}$, such that $a \prec x_n$ for all $n > n_0$. Moreover, for such a sequence, the values $\liminf a_n$ and $\limsup a_n$ are always defined, and in particular, the following equalities hold:

$$\liminf a_n = \bigvee_{n} \bigwedge_{k\geq n} a_k \quad \text{and} \quad \limsup a_n = \bigwedge_{n} \bigvee_{k\geq n} a_k.$$

Definition 2.2. [26] A triangular norm (t-norm) on a complete lattice $\mathscr{L} = (L, \leq)$ is a monotone associative commutative binary operation with the identity 1_L. Dually, a triangular conorm (t-conorm or s-norm) is a monotone associative commutative binary operation with the identity 0_L. More precisely, while a t-norm $\mathscr{T} : L \times L \to L$ satisfies the conditions

(1) $\mathscr{T}(a, b) = \mathscr{T}(b, a)$;
(2) $\mathscr{T}(\mathscr{T}(a, b), c) = \mathscr{T}(a, \mathscr{T}(b, c))$;
(3) $\mathscr{T}(a, 1_L) = a$;
(4) $a \leq b$ and $c \leq d \implies \mathscr{T}(a, c) \leq \mathscr{T}(b, d)$;

for all $a, b, c, d \in L$, a t-conorm $\mathscr{S} : L \times L \to L$ satisfies

(1) $\mathscr{S}(a, b) = \mathscr{S}(b, a)$;
(2) $\mathscr{S}(\mathscr{S}(a, b), c) = \mathscr{S}(a, \mathscr{S}(b, c))$;
(3) $\mathscr{S}(a, 0_L) = a$;
(4) $a \leq b$ and $c \leq d \implies \mathscr{S}(a, c) \leq \mathscr{S}(b, d)$.

According to this definition, the only difference between the concepts of t-norm and t-conorm lies in the fact that their identity elements are different.

Example 2.3. [26] The functions $\mathscr{T}_1$, $\mathscr{T}_2$, and $\mathscr{T}_3$ given with

$\mathscr{T}_1(x, y) = \min\{x, y\}$,
$\mathscr{T}_2(x, y) = xy$,
$\mathscr{T}_3(x, y) = \max\{x + y - 1, 0\}$

are some well-known examples of t-norms on the complete lattice $([0, 1], \leq)$, while

$\mathscr{S}_1(x, y) = \max\{x, y\}$,
$\mathscr{S}_2(x, y) = x + y - xy$,
$\mathscr{S}_3(x, y) = \min\{x + y, 1\}$

are examples of t-conorms. Intersection of sets defines a t-norm on a powerset lattice $(\mathcal{P}, \subseteq)$, and union defines a t-conorm. t-definers and t-codefiners [4] correspond respectively to t-norms and t-conorms on the lattice $([0, \infty], \geq)$, which is notable as the underlying lattice of the Lawvere quantale [10].

Remark 2.4. By the associativity property, for any $x, y, z \in L$, the following relation holds for any t-norm:

$$\mathscr{T}(\mathscr{T}(x, y), z) = \mathscr{T}(x, \mathscr{T}(y, z)) = \mathscr{T}(x, y, z).$$

Then for any $x_1, x_2, ..., x_n \in L$, the notation can easily be extended as

$$\mathscr{T}(x_1, x_2, ..., x_n) := \mathscr{T}(\mathscr{T}(x_1, x_2, ..., x_{n-1}), \mathscr{T}(x_n))$$

for $n \geq 3$. This represents the iterative application of the t-norm $\mathscr{T}$ across n elements in L. In fact, in the special case of $([0, 1], \leq)$, it is shown that this can be extended to the case of infinite multisets [9]. A similar argument is also valid for t-conorms.

In light of Remark 2.4, given a t-norm $\mathscr{T}$ or a t-conorm $\mathscr{S}$, we use the shorter notations

$$\mathscr{T}_{m \leq i \leq n}(a_i) := \mathscr{T}(a_m, a_{m+1}, \ldots, a_n)$$

and

$$\mathscr{S}_{m \leq i \leq n}(a_i) := \mathscr{S}(a_m, a_{m+1}, \ldots, a_n).$$

Definition 2.5. [26] A $t-$norm $\mathscr{T}$ on a complete lattice $\mathscr{L} = (L, \preceq)$ is called continuous, if for every pair of sequences (a_n) and (b_n) on L such that $(a_n) \to a \in L$ and $(b_n) \to b \in L$, one has the property that $\mathscr{T}(a_n, b_n) \to \mathscr{T}(a, b)$ with respect to the order topology on L.

Definition 2.6. [26] A mapping $\mathscr{N} : L \to L$ is called a negator on $\mathscr{L} = (L, \preceq)$ if
N_1) $\mathscr{N}(0_L) = 1_L$;
N_2) $\mathscr{N}(1_L) = 0_L$;
N_3) $x \preceq y$ implies $\mathscr{N}(y) \preceq \mathscr{N}(x)$ for all $x, y \in L$.
If in addition,
N_4) $\mathscr{N}(\mathscr{N}(x)) = x$ for all $x \in L$;
then the negator $\mathscr{N}$ is said to be involutive.

(N_4) and (N_3) together imply injectivity by $\mathscr{N}(x) = \mathscr{N}(y) \Rightarrow \mathscr{N}(x) \preceq \mathscr{N}(y)$ and $\mathscr{N}(y) \preceq \mathscr{N}(x) \Rightarrow \mathscr{N}(\mathscr{N}(y)) \preceq \mathscr{N}(\mathscr{N}(x))$ and $\mathscr{N}(\mathscr{N}(x)) \preceq \mathscr{N}(\mathscr{N}(x)) \Rightarrow y \preceq x$ and $x \preceq y \Rightarrow x = y$. This also suggests that an involutive negator necessarily satisfies the property
N_3') $x \prec y$ implies $\mathscr{N}(y) \prec \mathscr{N}(x)$ for all $x, y \in L$,
which is stronger than (N_3).

Example 2.7. On the lattice $([0, 1], \leq)$, the function $\mathscr{N}_s : [0, 1] \to [0, 1]$ defined as $\mathscr{N}_s(x) = 1 - x$ is an example of an involutive negator, called the standard negator on $[0, 1]$, which is used in the theory of fuzzy sets. Similarly, given the lattice $([0, 1]^2, \preceq)$ with the order

$$(\mu_1, \nu_1) \preceq (\mu_2, \nu_2) \iff \mu_1 \leq \mu_2 \quad \textit{and} \quad \nu_1 \geq \nu_2$$

for all $(\mu_i, \nu_i) \in [0, 1]^2$, $i = 1, 2$, then the mapping $\mathscr{N}_\iota : [0, 1]^2 \to [0, 1]^2$,

$$\mathscr{N}_\iota(\mu, \nu) = (\nu, \mu),$$

is an involutive negator used in the theory of intuitionistic fuzzy sets in the sense of Atanassov [2]. For a powerset lattice $(\mathcal{P}(X), \subseteq)$, the complementation $\mathscr{N}_c(A) = A^c$ corresponds to an involutive negator. In the case of the lattice $([0, \infty], \geq)$, $\mathscr{N}_i(x) = \frac{1}{x}$ is an involutive negator.

Given an involutive negator $\mathscr{N}$ on a lattice $(L, \preceq)$, each t-norm $\mathscr{T}$, corresponds to a dual t-conorm $\mathscr{S}$ defined by

$$\mathscr{S}(a, b) := \mathscr{N}(\mathscr{T}(\mathscr{N}(a), \mathscr{N}(b))).$$

Equivalently, we have $\mathscr{N}(\mathscr{S}(a, b)) = \mathscr{T}(\mathscr{N}(a), \mathscr{N}(b))$. In Example 2.3, each t-conorm $\mathscr{S}_i$ is the dual of the t-norm $\mathscr{T}_i$ with respect to the standard negation on $([0, 1], \leq)$ for $i \in \{1, 2, 3\}$.

Remark 2.8. In general, for a given continuous t-norm $\mathscr{T}$ and a negator $\mathscr{N}$, it is not always feasible to find an element $r \in L \setminus \{0_L\}$ for each $\varepsilon \in L \setminus \{0_L\}$, such that

$$\mathscr{T}(\mathscr{N}(r), \mathscr{N}(r)) \succ \mathscr{N}(\varepsilon),$$

or equivalently

$$\mathscr{S}(r, r) \prec \varepsilon, \tag{2.1}$$

where $\mathscr{S}$ is the dual t-conorm. For further details and specific cases in which this inequality holds, refer to [22].

In this study, we assume the existence of a continuous t-norm $\mathscr{T}$, an involutive negator $\mathscr{N}$, and the dual conorm $\mathscr{S}$ of $\mathscr{T}$ with respect to the negator $\mathscr{N}$, with the property that, for each $\varepsilon \in L \setminus \{0_L, 1_L\}$, there exists an $r \in L \setminus \{0_L, 1_L\}$ such that $\mathscr{T}(\mathscr{N}(r), \mathscr{N}(r)) \succ \mathscr{N}(\varepsilon)$, and equivalently $\mathscr{S}(r, r) \prec \varepsilon$.

Specifically, for a given r, it is possible to determine s such that $\mathscr{T}(\mathscr{N}(s), \mathscr{N}(s)) \succ \mathscr{N}(r)$ and $\mathscr{S}(s, s) \prec r$. Consequently, it follows that $\mathscr{T}(\mathscr{N}(r), \mathscr{N}(s), \mathscr{N}(s)) \succ \mathscr{N}(\varepsilon)$ and $\mathscr{T}(r, s, s) \prec \varepsilon$.

More generally, for any given $\varepsilon \succ 0_L$, there exists values $r_1, r_2, ..., r_n$ such that

$$\mathscr{T}(\mathscr{N}(r_1), \mathscr{N}(r_2), ..., \mathscr{N}(r_n), \mathscr{N}(r_n)) \succ \mathscr{N}(\varepsilon),$$

and equivalently

$$\mathscr{S}(r_1, r_2, ..., r_n, r_n) \prec \varepsilon. \tag{2.2}$$

This establishes that the t-conorm $\mathscr{S}$, when applied to appropriately chosen r_i, can yield a value smaller than ε, demonstrating its consistency with the defined inequalities.

Definition 2.9. [26] Let V be a real vector space, $\mathscr{L} = (L, \preceq)$ be a complete lattice, $\mathscr{T}$ be a continuous t-norm on $\mathscr{L}$, and ρ be an $\mathscr{L}-$set on $V \times (0, \infty)$ satisfying the following:

(L_1) $\rho(x, t) \succ 0_L$ for all $x \in V, t > 0$;
(L_2) $\rho(x, t) = 1_L$ for all $t > 0$ if and only if $x = \theta$;
(L_3) $\rho(\alpha x, t) = \rho(x, \frac{t}{|\alpha|})$ for all $x \in V$, $t > 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$;
(L_4) $\mathscr{T}(\rho(x, t), \rho(y, s)) \preceq \rho(x + y, t + s)$ for all $x, y \in V$ and $t, s > 0$;
(L_5) $\lim_{t\to\infty} \rho(x, t) = 1_L$ and $\lim_{t\to 0} \rho(x, t) = 0_L$ for all $x \in V \setminus \{\theta\}$;
(L_6) The mappings $f_x : (0, \infty) \to L$ given by $f(t) = \rho(x, t)$ are continuous.

In this case, the triple $(V, \rho\ \mathscr{T})$ is called an $\mathscr{L}-$fuzzy normed space or $\mathscr{L}-$normed space.

Since we assume the existence of an involutive negator and the dual t-conorm, each $\mathscr{L}-$fuzzy norm ρ corresponds to an $\mathscr{L}-$set ς on $V \times (0, \infty)$ defined by

$$\varsigma(x, t) := \mathscr{N}(\rho(x, t)),$$

of which we make use for a bit shorter proofs. We prefer the formulation involving ς to that involving ρ in many situations, since it both reduces the use of negation and yields cleaner statements, while also providing more intuitive analogies for readers with a classical analysis background, particularly in the equality to 0_L appearing in condition (L_2'), the direction of the triangle inequality in condition (L_4'), and in the statements involving being smaller than a given $\varepsilon \succ 0_L$. Under this convention, the axioms in Definition 2.9 becomes as follows:

(L_1') $\varsigma(x, t) \prec 1_L$ for all $x \in V, t > 0$;
(L_2') $\varsigma(x, t) = 0_L$ for all $t > 0$ if and only if $x = \theta$;
(L_3') $\varsigma(\alpha x, t) = \varsigma(x, \frac{t}{|\alpha|})$ for all $x \in V$, $t > 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$;
(L_4') $\varsigma(x + y, t + s) \preceq \mathscr{S}(\varsigma(x, t), \varsigma(y, s))$ for all $x, y \in V$ and $t, s > 0$;
(L_5') $\lim_{t\to\infty} \varsigma(x, t) = 0_L$ and $\lim_{t\to 0} \varsigma(x, t) = 1_L$ for all $x \in V \setminus \{\theta\}$;
(L_6') The mappings $f_x : (0, \infty) \to L$ given by $f(t) = \varsigma(x, t)$ are continuous.

Additionally, we assume that the $\mathscr{L}-$fuzzy norm ρ defined in this context satisfies the subadditivity property, an essential characteristic that extends the classical normed space framework into the

$\mathscr{L}-$fuzzy setting. Specifically, the subadditivity property in the $\mathscr{L}-$fuzzy norm space is defined as follows:

For any x, y in the space and for any $t > 0$, the $\mathscr{L}-$fuzzy norm ρ satisfies

$$\mathscr{T}(\rho(x,t), \rho(y,t)) \preceq \rho(x+y,t),$$

or equivalently

$$\varsigma(x+y,t) \preceq \mathscr{S}(\varsigma(x,t), \varsigma(y,t)).$$

This subadditivity property ensures that the $\mathscr{L}-$fuzzy normed space adheres to a generalization of the triangle inequality, which is fundamental in establishing structural properties such as convergence, boundedness, and continuity within the $\mathscr{L}-$fuzzy context.

Here, intuitively, the value of $\rho(x,t)$ can be interpreted as the truth value of the proposition "the length of the vector x is less than (or less than or equal to) t", whereas $\varsigma(x,t)$ can be interpreted as the truth value of the proposition "the length of the vector x is greater than (or greater than or equal to) t". Although this interpretation may not be formally precise, it is suitable for obtaining a general intuition and for making sense of the relations. This interpretation suggests that when $s < t$, we should have $\rho(x,s) \preceq \rho(x,t)$, that is, $\varsigma(x,s) \succeq \varsigma(x,t)$. Indeed, if $s < t$, then $t - s > 0$, and we have

$$\rho(x,s) = \mathscr{T}(\rho(x,s), 1_L) \preceq \mathscr{T}(\rho(x,s), \rho(\theta, t-s)) \preceq \rho(x+\theta, s+t-s) = \rho(x,t).$$

Definition 2.10. [26] A sequence (x_n) in an $\mathscr{L}-$fuzzy normed space $(V, \rho, \mathscr{T})$ is said to be convergent to ℓ if for each $\varepsilon \in L \setminus \{0_L\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\rho(x_n - \ell, t) \succ \mathscr{N}(\varepsilon),$$

that is

$$\varsigma(x_n - \ell, t) \prec \varepsilon.$$

In this case, we denote this convergence by $\mathscr{L} - \lim_n x_n = \ell$.

Definition 2.11. [26] A sequence (x_n) in an $\mathscr{L}-$fuzzy normed space $(V, \rho, \mathscr{T})$ is called a Cauchy sequence if for each $\varepsilon \in L \setminus \{0_L\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$,

$$\rho(x_n - x_m, t) \succ \mathscr{N}(\varepsilon),$$

or equivalently

$$\varsigma(x_n - x_m, t) \prec \varepsilon.$$

Definition 2.12. Let $(V, \rho, \mathscr{T})$ be an $\mathscr{L}-$fuzzy normed space. Then a sequence $x = (x_k)$ is said to be bounded with respect to fuzzy norm ρ, provided that for each $r \in L \setminus \{0_L\}$, there exists a $t > 0$ such that for all $k \in \mathbb{N}$,

$$\rho(x_k, t) \succ \mathscr{N}(r),$$

that is

$$\varsigma(x_k, t) \prec r.$$

3. Cesàro summability in $\mathscr{L}-$ fuzzy normed spaces

Now, we define the weighted mean summability method in the context of $\mathscr{L}-$fuzzy normed spaces and establish the associated Tauberian theorems. For an in-depth discussion on summability theory and Tauberian theory, the reader is referred to [1, 8, 15].

Definition 3.1. Let $(V, \rho, \mathscr{T})$ be an $\mathscr{L}-$ fuzzy normed space, where V is a vector space. Suppose that (x_n) is a sequence in V. In this case, (x_n) is said to be Cesàro summable to a limit $\ell \in V$, if the sequence of Cesàro means

$$y_n = \frac{1}{n+1} \sum_{k=0}^{n} x_k$$

converges to ℓ. In other words, for each $\varepsilon \in L \setminus \{0_L\}$ and $t > 0$, there exists an integer n_0 such that for all $n > n_0$,

$$\varsigma(y_n - \ell, t) \prec \varepsilon.$$

Additionally, Cesàro sums are frequently denoted by σ_n in the literature. Therefore, the arithmetic means σ_n of (x_n) is defined by

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} x_k.$$

Consequently, the sequence (x_n) is Cesàro summable to $\ell \in V$ in $\mathscr{L}-$ fuzzy normed space if

$$\mathscr{L} - \lim_{n\to\infty} \sigma_n = \ell.$$

Initially, we will demonstrate that the Cesàro summability method possesses regularity within the framework of an $\mathscr{L}-$fuzzy normed space. This will involve establishing the necessary properties and conditions under which the Cesàro method maintains regular summability, providing a foundation for analyzing the convergence behavior. By thoroughly examining these properties, we can ascertain the consistency of the Cesàro summability method with respect to $\mathscr{L}-$fuzzy norms, thereby contributing to the broader understanding of summability theory in fuzzy normed spaces.

Theorem 3.2. Let $(V, \rho, \mathscr{T})$ be an $\mathscr{L}-$fuzzy normed space. Suppose that (x_n) is a sequence in V, which converges to $\ell \in V$. Then, the Cesàro mean of the sequence (x_n) also converges to ℓ.

Proof. Let (x_n) be a sequence in an $\mathscr{L}-$fuzzy normed space $(V, \rho, \mathscr{T})$, and suppose that (x_n) converges to $\ell \in V$. Fix $t > 0$. By (2.1), for any given $\varepsilon \in L \setminus \{0_L\}$, we can choose an $r \in L \setminus \{0_L\}$ such that

$$\mathscr{S}(r, r) \prec \varepsilon.$$

Since (x_n) converges to ℓ, there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\varsigma\Big(x_n - \ell, \frac{t}{2}\Big) \prec \varepsilon.$$

Since

$$\sigma_n - \ell = \Big(\frac{1}{n+1} \sum_{k=0}^{n} x_k\Big) - \ell = \frac{1}{n+1} \sum_{k=0}^{n} (x_k - \ell),$$

it can be split into two parts as

$$\sigma_n - \ell = \frac{1}{n+1}\bigg(\sum_{k=0}^{n_0}(x_k - \ell) + \sum_{k=n_0+1}^{n}(x_k - \ell)\bigg).$$

Additionally, by (L_3'), we have

$$\varsigma\Big(\frac{1}{n+1}\sum_{k=0}^{n}(x_k - \ell), t\Big) = \varsigma\Big(\sum_{k=0}^{n}(x_k - \ell), (n+1)t\Big)$$

for all $t > 0$. Therefore, for the partial sum over the first n_0 terms, there exists $n_1 \in \mathbb{N}$ such that for all $n > n_1$,

$$\varsigma\Big(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\Big) \prec \varepsilon.$$

In other words, we have for the first n_0 terms

$$\mathscr{L} - \lim_{n\to\infty} \varsigma\Big(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\Big) = 0_L.$$

Let

$$A = \sum_{k=0}^{n_0}(x_k - \ell).$$

This is a fixed element of V. Hence, by (L_5'), we have

$$\varsigma\left(A, \frac{(n+1)t}{2}\right) \to 0_L$$

as $n \to \infty$, since $(n+1)t/2 \to \infty$. Thus, there exists $n_1 \in \mathbb{N}$ such that

$$\varsigma\left(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\right) \prec r$$

for all $n > n_1$. Therefore, for $n > \max\{n_0, n_1\}$, we have

$$\begin{aligned}
\varsigma\Big(\frac{1}{n+1}\sum_{k=0}^{n} x_k - \ell, t\Big) &= \varsigma\Big(\frac{1}{n+1}\sum_{k=0}^{n}(x_k - \ell), t\Big) = \varsigma\Big(\sum_{k=0}^{n}(x_k - \ell), (n+1)t\Big) \\
&\leq \mathscr{S}\bigg(\varsigma\Big(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\Big), \varsigma\Big(\sum_{k=n_0+1}^{n}(x_k - \ell), \frac{(n+1)t}{2}\Big)\bigg) \\
&\leq \mathscr{S}\bigg(\varsigma\Big(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\Big), \varsigma\Big(\sum_{k=n_0+1}^{n}(x_k - \ell), \frac{(n-n_0)t}{2}\Big)\bigg) \\
&\geq \mathscr{S}\bigg(\varsigma\Big(\sum_{k=0}^{n_0}(x_k - \ell), \frac{(n+1)t}{2}\Big), \varsigma\Big(x_{n_0+1} - \ell, \frac{t}{2}\Big), \varsigma\Big(x_{n_0+2} - \ell, \frac{t}{2}\Big), \cdots, \varsigma\Big(x_n - \ell, \frac{t}{2}\Big)\bigg) \\
&\prec \mathscr{S}(r_1, r_2, ..., r_{n-n_0}, r_{n-n_0}) \\
&\prec \varepsilon,
\end{aligned}$$

for appropriate r_i as explained in (2.2). This shows that the Cesàro summability method is regular. □

A similar result was established for intuitionistic fuzzy normed spaces in Theorem 3.2 of [28]. The following example states and explains that the theorem proved above is a lossless generalization of the corresponding result in [28].

Example 3.3. Let us consider the subset $\Delta = \{(a_1, a_2) \in [0,1]^2 : a_1 + a_2 \leq 1\}$ of $[0,1] \times [0,1]$, together with the partial order defined on this set by $(a_1, a_2) \leq (b_1, b_2) \iff a_1 \leq b_1$ and $a_2 \geq b_2$. Then $\mathscr{D} = (\Delta, \leq)$ is a complete lattice with the operations $\bigvee_{i\in I}(a_{1i}, a_{2i}) = (\sup_{i\in I} a_{1i}, \inf_{i\in I} a_{2i})$ and $\bigwedge_{i\in I}(a_{1i}, a_{2i}) = (\inf_{i\in I} a_{1i}, \sup_{i\in I} a_{2i})$. As the continuous t-norm, we consider $\mathscr{T} : \Delta \times \Delta \to \Delta$, $\mathscr{T}((a_1, a_2), (b_1, b_2)) = (\min(a_1, b_1), \max(a_2, b_2))$, and as the involutive negator, we consider $\mathscr{N} : \Delta \to \Delta$, $\mathscr{N}(a_1, a_2) = (a_2, a_1)$.

The dual t-conorm $\mathscr{S} : \Delta\times\Delta \to \Delta$ is then given by $\mathscr{S}((a_1, a_2), (b_1, b_2)) = (\max(a_1, b_1), \min(a_2, b_2))$, and the condition explained in Remark 2.8 is satisfied since $0_\Delta = (0, 1) \in \Delta$, and for any $(\varepsilon_1, \varepsilon_2) \in \Delta \setminus \{(0, 1)\}$, taking $(r_1, r_2) = (\frac{\varepsilon_1}{2}, \frac{1+\varepsilon_2}{2})$ readily satisfies $\mathscr{S}((r_1, r_2), (r_1, r_2)) = (r_1, r_2) \prec (\varepsilon_1, \varepsilon_2)$ by $\frac{\varepsilon_1}{2} < \varepsilon_1$, $\frac{1+\varepsilon_2}{2} > \varepsilon_2$, and $(r_1, r_2) \in \Delta \setminus \{(0, 1)\}$ since $0 < \frac{\varepsilon_1}{2} \leq \frac{1}{2}$, $\frac{1}{2} \leq \frac{1+\varepsilon_2}{2} < 1$, and $r_1 + r_2 = \frac{\varepsilon_1}{2} + \frac{1+\varepsilon_2}{2} \leq \frac{2}{2} = 1$.

In this setting, if ρ is an $\mathscr{L}-$set on $V \times (0,\infty)$ for a vector space V, then for all $x \in V$ and $t > 0$, $\rho(x,t) \in \Delta$, hence it has two coordinates. We write them as $\rho(x,t) = (\rho_1(x,t), \rho_2(x,t))$. Especially, for a given intuitionistic fuzzy norm (μ, ν) on V, we define $\rho_1 = \mu$ and $\rho_2 = \nu$, that is, $\rho(x,t) = (\mu(x,t), \nu(x,t))$. Then, under our notation $\varsigma(x,t) := \mathscr{N}(\rho(x,t))$, we should have $\varsigma_1 = \nu$, $\varsigma_2 = \mu$.

The definition of an intuitionistic fuzzy normed space is given in [21] as follows:

(a)	$\mu(x,t) > 0$;	(a')	$\nu(x,t) < 1$;
(b)	$\mu(x,t) = 1 \iff x = 0$;	(b')	$\nu(x,t) = 0 \iff x = 0$;
(c)	$\mu(\alpha x, t) = \mu\Big(x, \dfrac{t}{\lvert\alpha\rvert}\Big)$ for each $\alpha \neq 0$;	(c')	$\nu(\alpha x, t) = \nu\Big(x, \dfrac{t}{\lvert\alpha\rvert}\Big)$ for each $\alpha \neq 0$;
(d)	$\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$;	(d')	$\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y, t+s)$;
(e)	$\mu(x,\cdot) : (0,\infty) \to [0,1]$ is continuous;	(e')	$\nu(x,\cdot) : (0,\infty) \to [0,1]$ is continuous;
(f)	$\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$;	(f')	$\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$,

where V is a vector space, $\mu, \nu : V\times(0,\infty) \to [0,1]$, $\mu(x,t)+\nu(x,t) \leq 1$, $*$ is a continuous t-norm, and $\diamond$ is a continuous t-conorm. However, in [28], the t-norm and t-conorm are fixed to the minimum t-norm and maximum t-norm, and we adopt this convention as well in order to recover the corresponding results therein.

Therefore, comparing these conditions with our Definition 2.9, we see that (L_1) follows from (a) and (a'), (L_2) from (b) and (b'), (L_3) from (c) and (c'), (L_4) from (d) and (d') when the minimum t-norm and maximum t-conorm are chosen, (L_5) from (f) and (f'), and (L_6) from (e) and (e'). Also, in this setting, the corresponding notions of convergence coincide with those in intuitionistic fuzzy normed spaces.

As a result, if we interpret Theorem 3.2 in the special case of the lattice $\mathscr{D}$, we recover Theorem 3.2 of [28].

The purpose of the following examples is to show that the converse of Theorem 3.2 does not hold in general. More precisely, Cesàro summability in an $\mathscr{L}-$fuzzy normed space need not imply ordinary convergence without an additional Tauberian condition.

In Example 3.3, the sequence $((-1)^n)$ is used to illustrate this phenomenon in a simple $\mathscr{L}-$fuzzy normed space. Its Cesàro means tend to 0, whereas the original sequence continues to oscillate and therefore does not converge.

Example 3.4. Let $\mathscr{L} = ([0,\infty], \leq)$, $\mathscr{T}(a,b) = \min(a,b)$, $\mathscr{N}(a) = a^{-1}$ for $a < \infty$, and $\mathscr{N}(\infty) = 0$. Consider the $\mathscr{L}-$fuzzy norm ρ on $\mathbb{R}$, considered as a vector space over itself, given by $\rho(x,t) = \frac{t}{|x|}$ for $x \neq 0$, and $\rho(0,t) = \infty$ for all $t > 0$. In this case, $\varsigma(x,t) = \frac{|x|}{t}$.

For the sequence $(x_n) = ((-1)^n)$,

$$\sigma_n = \begin{cases} \frac{1}{n+1}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

(σ_n) is convergent to 0, since for a given $\varepsilon \in [0,\infty] \setminus \{0\}$, $\varsigma(\sigma_n, t) = \frac{|\sigma_n|}{t} < \varepsilon$ for all $n > \frac{1}{t\varepsilon}$. Then (x_n) is Cesàro summable to 0. Hence, if (x_n) is convergent, it can converge only to 0. However, this is not the case, since for $t = \varepsilon = 1$,

$$\varsigma(x_n, t) = \frac{|(-1)^n|}{t} = \frac{1}{1} < 1 = \varepsilon$$

is not satisfied for any $n \in \mathbb{N}$.

In Example 3.4, the same phenomenon is demonstrated in the function space $C[0,1]$. The sequence $f_n(x) = (-x)^n$ is not convergent, since the distance between two consecutive terms remains uniformly separated. On the other hand, its Cesàro means converge uniformly to the zero function, as shown by the estimate

$$|\sigma_n(x)| = \left| \frac{1}{n+1} \sum_{k=0}^{n} (-x)^k \right| \leq \frac{2}{n+1}$$

for all $x \in [0,1]$. This uniform estimate directly implies convergence to zero in the given $\mathscr{L}-$fuzzy norm.

Example 3.5. For the lattice $\mathscr{L} = ([0,1], \leq)$, define

$$\mathscr{T}(a,b) = \min(a,b), \ \mathscr{N}(a) = 1 - a.$$

Then $\rho : C[0,1] \times (0,\infty) \to [0,1]$ given by

$$\rho(h,t) = t(t + \sup_{x\in[0,1]} |h(x)|)^{-1}$$

is an $\mathscr{L}-$fuzzy norm on the vector space $C[0,1]$. Then

$$\varsigma(h,t) = \sup_{x\in[0,1]} |h(x)| \cdot (t + \sup_{x\in[0,1]} |h(x)|)^{-1}.$$

Consider the sequence (f_n), given by $f_n(x) = (-x)^n$. To see that (f_n) is not convergent, observe that

$$\sup_{x\in[0,1]} \left|(-x)^{n+1} - (-x)^n\right| = \sup_{x\in[0,1]} \left|(-1)^{n+1}(x^{n+1} + x^n)\right| = 2$$

for all $n \in \mathbb{N}$ so that

$$\varsigma(f_{n+1} - f_n, t) = \frac{2}{t+2}.$$

Therefore, there is no $n_0 \in \mathbb{N}$ such that the condition $\varsigma(f_m - f_n, t) < \varepsilon$ for $t = 2$, $\varepsilon = \frac{1}{2}$. However, it is Cesàro summable to the zero function. In fact,

$$\sigma_n(x) = \frac{1}{n+1}\sum_{k=0}^{n}(-x)^k \leq \frac{2}{n+1},$$

since $x^k \leq 1$ and

$$-x^{2k+1} + x^{2k} = x(-x+1) \leq 1$$

for all $x \in [0,1]$. This means that

$$\rho(\sigma_n, t) \geq t(t + \frac{2}{n+1})^{-1} = \frac{nt+t}{nt+t+2},$$

or equivalently

$$\varsigma(\sigma_n, t) = 1 - \rho(\sigma_n, t) \leq \frac{2}{nt+t+2}.$$

Then for any $\varepsilon \in [0,1] \setminus \{0\}$ and $t > 0$, one has $\varsigma(\sigma_n, t) < \varepsilon$ for all $n > \frac{2-2\varepsilon-t\varepsilon}{t\varepsilon}$.

In Example 3.5, we give an analogue in the Hilbert space ℓ^2. The standard unit sequence (e_n) has no limit in ℓ^2 because its mass moves along the coordinates and does not approach any fixed vector. However, the Cesàro means spread this mass over the first $n+1$ coordinates with size $1/(n+1)$. Hence, their ℓ^2-norm tends to zero, which yields Cesàro summability to the zero vector in the corresponding $\mathscr{L}-$fuzzy norm.

Example 3.6. Consider the lattice $\mathscr{L} = ([0,1], \leq)$ with

$$\mathscr{T}(a,b) = ab,\ \mathscr{N}(a) = 1 - a.$$

On the vector space ℓ^2, consider the $\mathscr{L}-$fuzzy norm ρ, given by

$$\rho((x_n), t) = \sqrt[-t]{\exp\sqrt{\sum_{k=0}^{\infty}|x_k|^2}}.$$

Then,

$$\varsigma((x_n), t) = 1 - \sqrt[-t]{\exp\sqrt{\sum_{k=0}^{\infty}|x_k|^2}}.$$

Now consider the sequence (e_n) on ℓ^2, where

$$\begin{aligned} e_0 &= (0,0,0,0,\ldots),\\ e_1 &= (1,0,0,0,\ldots),\\ e_2 &= (0,1,0,0,\ldots),\\ e_3 &= (0,0,1,0,\ldots), \end{aligned}$$

and so on. (e_n) cannot be convergent to any $a \in \ell^2$. To see this, assume that (e_n) converges to a point $a = (a_0, a_1, a_2, \ldots) \in \ell^2$. By $a \in \ell^2 \subseteq c_0$, the sequence (a_n) must converge to 0, so that for $\varepsilon = 0.3$ and $t = \frac{1}{2}$, there is an $n_0 \in \mathbb{N}$ such that $|a_n| < \frac{1}{2}$ for all $n \geq n_0$. Then

$$\varsigma(e_n - a, \frac{1}{2}) = 1 - \left(\exp\sqrt{\sum_{k=0}^{\infty}|e_{nk} - a_k|^2}\right)^{-1}$$

$$\begin{aligned}
&\leq 1 - (\exp|e_{nn} - a_n|)^{-1} \\
&\leq 1 - \left(\exp\frac{1}{2}\right)^{-1} \\
&= 1 - \frac{1}{\sqrt{e}} \approx 0.3935 \nless \varepsilon.
\end{aligned}$$

However, (e_n) is Cesàro summable to the zero vector. The reason for Cesàro convergence is that the averaging process distributes the unit mass of e_k over the first $n+1$ coordinates. Indeed, let $\varepsilon \in [0, 1]\backslash\{0\}$ and $t > 0$,

$$\begin{aligned}
\sigma_0 &= (0, 0, 0, 0, \ldots), \\
\sigma_1 &= \left(\frac{1}{2}, 0, 0, 0, \ldots\right), \\
\sigma_2 &= \left((\frac{1}{3}, \frac{1}{3}, 0, 0, \ldots\right), \\
\sigma_3 &= (\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, \ldots\right),
\end{aligned}$$

and in general,

$$\sigma_{nk} = \begin{cases} \frac{1}{1+n}, & \text{if } k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Then, setting

$$n_0 = \frac{1}{(t\ln(1-\varepsilon))^2} - 1$$

for $n > n_0$, we have

$$\ln(1-\varepsilon) < \frac{-1}{t\sqrt{n+1}},$$

so that

$$\varsigma((\sigma_n), t) = 1 - \sqrt[-t]{\exp\sqrt{\sum_{k=0}^{\infty}|\sigma_{nk}|^2}} = 1 - \sqrt[-t]{\exp\sqrt{\sum_{k=0}^{n}\frac{1}{(n+1)^2}}} = 1 - e^{\frac{-1}{t\sqrt{n+1}}} < \varepsilon.$$

Lemma 3.7. [27] Let us define $\langle\lambda\rangle$ for every $\lambda > 0$ by $\langle\lambda\rangle := \lambda - \lfloor\lambda\rfloor$. Then, the following statements hold:

(i) If $\lambda > 1$, then $\lambda_n > n$ for each $n \in \mathbb{N} \setminus \{0\}$ with $n \geq \langle\lambda\rangle^{-1}$.
(ii) If $0 < \lambda < 1$, then $\lambda_n < n$ for each $n \in \mathbb{N} \setminus \{0\}$.

Lemma 3.8. [27] We have the following statements:

(i) Let $\lambda > 1$. For each $n \in \mathbb{N} \setminus \{0\}$ with $n \geq (3\lambda - 1)/\lambda(\lambda - 1)$, we have

$$\frac{\lambda}{\lambda - 1} < \frac{\lambda_n + 1}{\lambda_n - n} < \frac{2\lambda}{\lambda - 1}.$$

(ii) If $0 < \lambda < 1$, for each $n \in \mathbb{N} \setminus \{0\}$ with $n > \lambda^{-1}$ we have

$$0 < \frac{\lambda_n + 1}{n - \lambda_n} < \frac{2\lambda}{1 - \lambda}.$$

We now state a lemma due to Móricz that will be used in the proof of the main theorem.

Lemma 3.9. [17] The following propositions are established as fundamental components of our theoretical framework:

(i) For all $\lambda > 1$ and large enough n, that is, $\lambda_n > n$,

$$x_n - \sigma_n = \frac{\lambda_n + 1}{\lambda_n - n}(\sigma_{\lambda_n} - \sigma_n) - \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n).$$

(ii) For all $0 < \lambda < 1$ and large enough n, that is, $\lambda_n < n$,

$$x_n - \sigma_n = \frac{\lambda_n + 1}{n - \lambda_n}(\sigma_n - \sigma_{\lambda_n}) - \frac{1}{n - \lambda_n}\sum_{k=\lambda_n+1}^{n}(x_n - x_k).$$

We now present the main result of this paper.

Theorem 3.10. Let $(V, \rho, \mathscr{T})$ be an $\mathscr{L}-$fuzzy normed space, and suppose that (x_n) is Cesàro summable to ℓ. Then (x_n) converges to ℓ, if and only if

$$\inf_{\lambda>1}\limsup_{n\to\infty} \varsigma\Big(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), t\Big) = 0_L \tag{3.1}$$

for all $t > 0$.

Proof. In $\mathscr{L}-$fuzzy normed space $(V, \rho, \mathscr{T})$, consider a sequence (x_n), which is Cesàro summable to $\ell \in V$.

Necessity. Suppose that (x_n) converges to ℓ. Fix $t > 0$. For any $\lambda > 1$ by Lemma 3.9, we have

$$x_n - \sigma_n = \frac{\lambda_n + 1}{\lambda_n - n}(\sigma_{\lambda_n} - \sigma_n) - \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n).$$

By Lemma 3.8, for $n \geq (3\lambda - 1)/\lambda(\lambda - 1)$, we have

$$\varsigma\left(\frac{\lambda_n + 1}{\lambda_n - n}(\sigma_{\lambda_n} - \sigma_n), t\right) = \varsigma\left(\sigma_{\lambda_n} - \sigma_n, \frac{t}{\frac{\lambda_n+1}{\lambda_n-n}}\right) \preceq \varsigma\left(\sigma_{\lambda_n} - \sigma_n, \frac{t}{\frac{2\lambda}{\lambda-1}}\right).$$

Since (σ_n) is Cauchy,

$$\lim_{n\to\infty} \varsigma\left(\frac{\lambda_n + 1}{\lambda_n - n}(\sigma_{\lambda_n} - \sigma_n), t\right) = 0_L,$$

which means that $\frac{\lambda_n+1}{\lambda_n-n}(\sigma_{\lambda_n} - \sigma_n) \to 0$. Therefore, we have

$$\lim_{n\to\infty} \varsigma\left(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), t\right) = 0_L.$$

Sufficiency. Suppose that the equality (3.1) is satisfied. Fix $t > 0$. Then for any given $\varepsilon \in L \setminus \{0_L\}$, there exist $r_1, r_2 \in L \setminus \{0_L\}$ by (2.2) such that $\mathscr{S}(r_1, r_2, r_2) \prec \varepsilon$. In this case, we have the following:

- There exist $\lambda > 1$ and $n_0 \in \mathbb{N}$ such that

$$\varsigma\Big(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), \frac{t}{3}\Big) \prec r_1$$

for all $n > n_0$.
- There exists $n_1 \in \mathbb{N}$ such that $\varsigma\left(\sigma_n - \ell, \frac{t}{3}\right) \prec r_2$ for $n > n_1$.
- There exists $n_2 \in \mathbb{N}$ such that

$$\varsigma\left(\frac{\lambda_n + 1}{\lambda_n - n}(\sigma_{\lambda_n} - \sigma_n), \frac{t}{3}\right) \prec r_2$$

for $n > n_2$, in view of the fact that $\frac{\lambda_n+1}{\lambda_n-n}(\sigma_{\lambda_n} - \sigma_n) \to 0$.

Hence, we get

$$\begin{aligned}
\varsigma(x_n - \ell, t) &= \varsigma\,(x_n - \sigma_n + \sigma_n - \ell, t) \\
&= \varsigma\Big(\frac{\lambda_n + 1}{\lambda_n - n}(\sigma_{\lambda_n} - \sigma_n) - \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n) + \sigma_n - \ell, t\Big) \\
&\leq \mathscr{S}\left(\varsigma\Big(\frac{\lambda_n + 1}{\lambda_n - n}(\sigma_{\lambda_n} - \sigma_n), \frac{t}{3}\Big), \varsigma\Big(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), \frac{t}{3}\Big), \varsigma\Big(\sigma_n - \ell, \frac{t}{3}\Big)\right) \\
&\prec \mathscr{S}(r_1, r_2, r_2) \\
&\prec \varepsilon
\end{aligned}$$

for $n > \max\{n_0, n_1, n_2\}$, which completes the proof. □

Remark 3.11. In the context of Example 3.3, if we reinterpret Theorem 3.10, then in particular condition (3.1), namely

$$\inf_{\lambda>1}\limsup_{n\to\infty} \varsigma\Big(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), t\Big) = 0_\Delta,$$

splits into two coordinatewise parts

$$\inf_{\lambda>1}\limsup_{n\to\infty} \nu\Big(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), t\Big) = 0,$$

and

$$\sup_{\lambda>1}\liminf_{n\to\infty} \mu\Big(\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), t\Big) = 1,$$

as $0_\Delta = (0, 1) \in \Delta$, $\varsigma(x, t) = (\varsigma_1(x, t), \varsigma_2(x, t))$, and $\varsigma_1 = \nu$, $\varsigma_2 = \mu$. These exactly correspond to Eqs (3.1) and (3.2) in [28], thereby reducing to Theorem 3.6 therein.

Building upon the framework established in the preceding theorem, it is possible to derive a subsequent theorem by altering the parameter condition from $\lambda > 1$ to $0 < \lambda < 1$. This modification not only broadens the applicability of the theorem but also highlights its versatility under varying constraints of the parameter λ. The resulting theorem, as presented below, demonstrates how these adjustments yield analogous conclusions within the modified parameter space. In the context of Theorem 3.10, by using the same argument for $0 < \lambda < 1$, we obtain the following result.

Theorem 3.12. Let $(V, \rho, \mathscr{T})$ be an $\mathscr{L}$−fuzzy normed space, and suppose that (x_n) is Cesàro summable to $\ell \in V$. Then (x_n) converges to ℓ if and only if

$$\inf_{0<\lambda<1} \limsup_{n\to\infty} \varsigma\Big(\frac{1}{n-\lambda_n}\sum_{k=\lambda_n+1}^{n}(x_n - x_k), t\Big) = 0_L \tag{3.2}$$

for all $t > 0$.

Proof. Necessity. Let (x_n) converge to ℓ. Fix $t > 0$. By Lemma 3.9, for all $0 < \lambda < 1$ and large enough n, that is, $\lambda_n < n$, we have

$$x_n - \sigma_n = \frac{\lambda_n + 1}{n - \lambda_n}(\sigma_n - \sigma_{\lambda_n}) - \frac{1}{n-\lambda_n}\sum_{k=\lambda_n+1}^{n}(x_n - x_k).$$

By Lemma 3.8, we have for $n \geq (3\lambda - 1)/\lambda(\lambda - 1)$ that

$$\varsigma\Big(\frac{\lambda_n + 1}{n-\lambda_n}(\sigma_n - \sigma_{\lambda_n}), t\Big) = \varsigma\Big(\sigma_n - \sigma_{\lambda_n}, \frac{t}{\frac{\lambda_n+1}{n-\lambda_n}}\Big) \leq \varsigma\Big(\sigma_n - \sigma_{\lambda_n}, \frac{t}{\frac{2\lambda}{1-\lambda}}\Big).$$

Since (σ_n) is a Cauchy sequence,

$$\lim_{n\to\infty} \varsigma\left(\frac{\lambda_n + 1}{n-\lambda_n}(\sigma_n - \sigma_{\lambda_n}), t\right) = 0_L,$$

which means that $\frac{\lambda_n+1}{n-\lambda_n}(\sigma_n - \sigma_{\lambda_n}) \to 0$. Therefore, we have

$$\lim_{n\to\infty} \varsigma\Big(\frac{1}{n-\lambda_n}\sum_{k=\lambda_n+1}^{n}(x_n - x_k), t\Big) = 0_L.$$

Sufficiency. Suppose that (3.2) is satisfied. Fix $t > 0$. Then for all $\varepsilon \in L\backslash\{0_L\}$, there are $r_1, r_2 \in L\backslash\{0_L\}$ such that $\mathscr{S}(r_1, r_2, r_2) \prec \varepsilon$. Then we have the following:

- There exist $0 < \lambda < 1$ and $n_0 \in \mathbb{N}$ such that

$$\varsigma\Big(\frac{1}{n-\lambda_n}\sum_{k=\lambda_n+1}^{n}(x_n - x_k), \frac{t}{3}\Big) \prec r_1$$

  for all $n > n_0$.
- There exists $n_1 \in \mathbb{N}$ such that $\varsigma\left(\sigma_n - \ell, \frac{t}{3}\right) \prec r_2$ for all $n > n_1$.

- There exists $n_2 \in \mathbb{N}$ such that

$$\varsigma\left(\frac{\lambda_n + 1}{n - \lambda_n}(\sigma_n - \sigma_{\lambda_n}), \frac{t}{3}\right) \prec r_2$$

for all $n > n_2$, by the fact that $\frac{\lambda_n+1}{n-\lambda_n}(\sigma_n - \sigma_{\lambda_n}) \to 0$.

Hence, we get

$$\begin{aligned}
\varsigma(x_n - \ell, t) &= \varsigma\left(x_n - \sigma_n + \sigma_n - \ell, t\right) \\
&= \varsigma\Big(\frac{\lambda_n + 1}{n - \lambda_n}(\sigma_n - \sigma_{\lambda_n}) - \frac{1}{n - \lambda_n}\sum_{k=\lambda_n+1}^{n}(x_n - x_k) + \sigma_n - \ell, t\Big) \\
&\leq \mathscr{S}\left(\varsigma\Big(\frac{\lambda_n + 1}{n - \lambda_n}(\sigma_n - \sigma_{\lambda_n}), \frac{t}{3}\Big), \varsigma\Big(\frac{1}{n - \lambda_n}\sum_{k=\lambda_n+1}^{n}(x_n - x_k), \frac{t}{3}\Big), \varsigma\Big(\sigma_n - \ell, \frac{t}{3}\Big)\right) \\
&\prec \mathscr{S}(r_1, r_2, r_2) \\
&\prec \varepsilon
\end{aligned}$$

for all $n > \max\{n_0, n_1, n_2\}$. □

We now introduce slowly oscillating sequences in $\mathscr{L}-$fuzzy normed spaces and establish several related results.

Definition 3.13. Let $(V, \rho, \mathscr{T})$ be an $\mathscr{L}-$fuzzy normed space. Then a sequence (x_n) on V is said to be slowly oscillating if

$$\inf_{\lambda>1} \limsup_{n\to\infty} \max_{n<k\leq\lambda_n} \varsigma(x_k - x_n, t) = 0_L \tag{3.3}$$

for all $t > 0$, where λ_n denotes the integer part of the product λn, that is, $\lambda_n = \lfloor \lambda n \rfloor$.

Intuitively, this means that the terms of the sequence do not oscillate significantly on short multiplicative intervals of the form $[n, \lambda n]$, once n is sufficiently large. Thus, although the sequence need not be monotone or convergent by definition, its local fluctuations become small in the $\mathscr{L}-$fuzzy normed sense.

The condition above can be equivalently reformulated as follows: A sequence (x_n) in an $\mathscr{L}-$fuzzy normed space is slowly oscillating, if and only if, for every $t > 0$ and for any $\varepsilon \in L \setminus \{0_L\}$, there exist constants $\lambda > 1$ and $n_0 \in \mathbb{N}$ (both dependent on t and ε) such that the inequality

$$\varsigma(x_k - x_n, t) \prec \varepsilon$$

holds for all indices n and k satisfying $n_0 \leq n < k \leq \lambda_n$.

This implies that as the terms of the sequence tends to infinity, the differences between successive terms become closer to each other to exhibit minimal oscillation. The parameters $t > 0$ and $\varepsilon \in L \setminus \{0_L\}$ act as controls for the precision and bounds of the oscillatory behavior, while $\lambda > 1$ and $n_0 \in \mathbb{N}$ ensure that these properties hold uniformly beyond a certain index n_0 as the sequence evolves. This reformulation provides a clear and flexible criterion for identifying slowly oscillating sequences in the context of $\mathscr{L}-$fuzzy normed spaces.

Building upon the discussions in Example 3.3 and Remark 3.11, in the case of the lattice $\mathscr{D} = (\Delta, \leq)$, this definition corresponds exactly to Definition 4.1 in [28].

Theorem 3.14. Let $(V, \rho, \mathscr{T})$ be an $\mathscr{L}-$fuzzy normed space. If (x_n) is Cesàro summable to $\ell \in V$ and slowly oscillating, then for all $t > 0$,

$$\inf_{\lambda>1} \limsup_{n\to\infty} \varsigma\Big(\frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} (x_k - x_n), t\Big) = 0_L.$$

In other words, from Theorem 3.10, (x_n) converges to ℓ.

Proof. Suppose that (x_n) is a slowly oscillating sequence in $\mathscr{L}-$fuzzy normed space $(V, \rho, \mathscr{T})$. Fix $t > 0$. For given $\varepsilon \in L \setminus \{0_L\}$, there exist $\lambda > 1$ and $n_0 \in \mathbb{N}$ such that

$$\varsigma(x_k - x_n) \prec \varepsilon$$

whenever $n_0 \leq n < k \leq \lambda_n$. By slow oscillation, for the chosen $t > 0$ and $\varepsilon \in L \setminus \{0_L\}$, we may choose $\lambda > 1$ and $n_0 \in \mathbb{N}$ such that all differences $x_k - x_n$, with $n_0 \leq n < k \leq \lfloor \lambda n \rfloor$, are small in the $\mathscr{L}-$fuzzy normed sense. Applying the subadditivity property of ς to the finite sum then gives the desired estimate for the averaged difference. Hence,

$$\begin{aligned}
\varsigma\Big(\frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} (x_k - x_n), t\Big) &= \varsigma\Big(\sum_{k=n+1}^{\lambda_n} (x_k - x_n), (\lambda_n - n)t\Big) \\
&\leq \mathscr{S}(\varsigma(x_{n+1} - x_n, t), \varsigma(x_{n+2} - x_n, t), \ldots, \varsigma(x_{\lambda_n} - x_n, t)) \\
&\prec \mathscr{S}(r_1, r_2, \ldots, r_{\lambda_n - n}, r_{\lambda_n - n}) \\
&\prec \varepsilon,
\end{aligned}$$

provided that $n_0 \leq n < k \leq \lambda_n$. □

Alaca and Efe [11] introduced the notion of $q-$boundedness in intuitionistic fuzzy metric spaces. We extend this concept to $\mathscr{L}-$fuzzy normed spaces as follows.

Definition 3.15. Let $(V, \rho, \mathscr{T})$ be an $\mathscr{L}-$fuzzy normed space. A sequence (x_n) is called $q-$bounded in $(V, \rho, \mathscr{T})$, if

$$\mathscr{L} - \lim_{t\to\infty} \sup_{n\in\mathbb{N}} \varsigma(x_n, t) = 0_L,$$

and it is called $\mathscr{S}-$bounded if

$$\mathscr{L} - \lim_{t\to\infty} \sup_{n\geq 2} \sup_{\substack{a_i\in\mathbb{N} \\ 1\leq i\leq n}} \mathscr{S}_{1\leq i\leq n} \varsigma(x_{a_i}, t) = 0_L.$$

It should be noted that in the definition of $\mathscr{S}-$boundedness, the supremum of the t-conorm values is taken over all possible finite sequences of natural numbers, of length at least two. The concept of $q-$boundedness is a special case of $\mathscr{S}-$boundedness, obtained when the t-conorm is chosen as $\mathscr{S}(a, b) = \sup(a, b)$.

Theorem 3.16. Let $(V, \rho, \mathscr{T})$ be an $\mathscr{L}-$fuzzy normed space. If $\{n(x_n - x_{n-1})\}$ is $\mathscr{S}-$bounded, then (x_n) is slowly oscillating.

Proof. From the definition of $\mathscr{S}-$boundedness, for $\varepsilon \in L\setminus\{0_L\}$, there exists $K_\varepsilon > 0$ such that for $t > K_\varepsilon$

$$\sup_{n\geq 2}\sup_{\substack{a_i\in\mathbb{N}\\ 1\leq i\leq n}}\mathscr{S}_{1\leq i\leq n}(\varsigma(a_i(x_{a_i}-x_{a_{i-1}}),t)) \prec \varepsilon.$$

To prove slow oscillation, fix $t > 0$ and choose $\lambda > 1$ sufficiently close to 1 so that $t/(\lambda-1)¿K_\varepsilon$. For indices $n < k \leq \lfloor \lambda n \rfloor$, we write

$$x_k - x_n = \sum_{j=n+1}^{k}(x_j - x_{j-1}).$$

The following estimate uses the subadditivity of ς, the scaling property (L_3'), and the monotonicity of ς with respect to the second variable. For each $t > 0$, if we choose $\lambda < 1 + \frac{t}{K_\varepsilon}$, then

$$\begin{aligned}
\varsigma(x_k - x_n, t) &= \varsigma\Big(\sum_{j=n+1}^{k}(x_j - x_{j-1}), t\Big) \\
&\leq \mathscr{S}_{n+1\leq j\leq k}\Big(\varsigma\Big(x_j - x_{j-1}, \frac{t}{k-n}\Big)\Big) \\
&= \mathscr{S}_{n+1\leq j\leq k}\Big(\varsigma\Big(j(x_j - x_{j-1}), \frac{jt}{k-n}\Big)\Big) \\
&\leq \mathscr{S}_{n+1\leq j\leq k}\Big(\varsigma\Big(j(x_j - x_{j-1}), \frac{nt}{k-n}\Big)\Big) \\
&\leq \mathscr{S}_{n+1\leq j\leq k}\Big(\varsigma\Big(j(x_j - x_{j-1}), \frac{t}{\frac{k}{n}-1}\Big)\Big) \\
&\leq \mathscr{S}_{n+1\leq j\leq k}\Big(\varsigma\Big(j(x_j - x_{j-1}), \frac{t}{\lambda-1}\Big)\Big) \\
&\leq \sup_{n\geq 2}\sup_{\substack{a_i\in\mathbb{N}\\ 1\leq i\leq n}}\mathscr{S}_{1\leq i\leq n}\varsigma\Big(a_i(x_{a_i}-x_{a_{i-1}}), \frac{t}{\lambda-1}\Big) \\
&\prec \varepsilon
\end{aligned}$$

holds for $n_0 < n < k \leq \lambda_n$ and $n+1 \leq j \leq k$, since $\frac{t}{\lambda-1} > K_\varepsilon$ for $\lambda < 1 + \frac{t}{K_\varepsilon}$.

Hence, the sequence (x_n) is slowly oscillating. This means that the fluctuations of the sequence become progressively smaller, and the terms of the sequence stabilize in a controlled manner as n increases. □

Consequently, a Hardy-type two-sided Tauberian result is obtained in the setting of $\mathscr{L}-$fuzzy normed spaces, incorporating the interplay between Cesàro summability, $q-$boundedness, and convergence. This generalization serves to extend the applicability of Hardy's foundational result to these generalized spaces, providing a deeper understanding of summability methods and their relationship to convergence within the framework of fuzzy mathematics.

4. Conclusions

In this paper, we have studied Cesàro summability and related Tauberian theorems in $\mathscr{L}-$fuzzy normed spaces. The results provide a basis for further investigations of summability methods in this setting. In particular, we have examined the relation between Cesàro summability, $q-$boundedness, slow oscillation, and convergence.

Future work may consider other summability methods, such as Abel, Borel, logarithmic, and statistical summability, in $\mathscr{L}-$fuzzy normed spaces. Such extensions may further clarify the relationships between summability, boundedness, oscillation, and stability. They may also provide useful tools for problems involving uncertainty, including fuzzy systems, optimization, and computational mathematics. These possible applications are motivated by the fact that averaging procedures naturally appear in iterative algorithms, approximation processes, and computational schemes involving uncertain or imprecise data. In such situations, ordinary convergence may be too restrictive or difficult to verify, whereas Cesàro summability provides a weaker convergence notion. Tauberian conditions then become important because they determine when this weaker convergence is sufficient to recover ordinary convergence. This makes the present framework relevant for further studies on fuzzy systems, optimization methods, and numerical processes in $\mathscr{L}-$fuzzy environments.

These methods may also lead to new Tauberian theorems connecting summability and convergence. This would contribute to a more systematic treatment of summability theory in fuzzy frameworks. Possible applications include fuzzy decision-making, control theory, and data analysis, where fuzzy norms can be used to model imprecision.

Overall, the present work provides a starting point for further study of summability methods in $\mathscr{L}-$fuzzy normed spaces. We expect that these directions will support further development of Tauberian theory in fuzzy mathematical analysis.

Author contributions

R. Yapalı: Conceptualization, formal analysis, investigation, validation, writing – original draft; U. Gürdal: Formal analysis, investigation, validation, writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

References

1. A. Aasma, H. Dutta, P. N. Natarajan, *An introductory course in summability theory*, John Wiley & Sons, Inc., 2017. http://dx.doi.org/10.1002/9781119397786
2. K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets Syst.*, **20** (1986), 87–96. http://dx.doi.org/10.1016/S0165-0114(86)80034-3
3. İ. Çanak, Ü. Totur, Z. Önder, A Tauberian theorem for $(C, 1, 1)$ summable double sequences of fuzzy numbers, *Iran. J. Fuzzy Syst.*, **14** (2017), 61–75.

4. S. Çetin, Unidefiners, *J. Nat. Appl. Sci.*, **29** (2025), 220–227. http://dx.doi.org/10.19113/sdufenbed.1666339
5. A. A. Das, S. K. Paikray, T. Pradhan, H. Dutta, Statistical $(C,1)(E,\mu)$-summability and associated fuzzy approximation theorems with statistical fuzzy rates, *Soft Comput.*, **24** (2020), 10883–10892. http://dx.doi.org/10.1007/s00500-019-04591-2
6. P. Debnath, Statistical Cesàro summability in intuitionistic fuzzy n-normed linear spaces leading towards Tauberian theorems, *Axioms*, **13** (2024), 557. http://dx.doi.org/10.3390/axioms13080557
7. H. Dutta, J. Gogoi, Weighted λ-statistical convergence connecting a statistical summability of sequences of fuzzy numbers and Korovkin-type approximation theorems, *Soft Comput.*, **23** (2019), 12883–12895. http://dx.doi.org/10.1007/s00500-019-03846-2
8. H. Dutta, B. E. Rhoades, *Current topics in summability theory and applications*, Singapore: Springer, 2016. http://dx.doi.org/10.1007/978-981-10-0913-6
9. U. Gürdal, Ö. Oğur, S. Çetin, Unima on $[0,1]$, *Fuzzy Sets Syst.*, **516** (2025), 109439. http://dx.doi.org/10.1016/j.fss.2025.109439
10. U. Gürdal, E. U. Ekici, S. Çetin, Modulative categories and fixed point theorems on modulative metric spaces, *Hacet. J. Math. Stat.*, **55** (2026), 525–545. http://dx.doi.org/10.15672/hujms.1643663
11. H. Efe, C. Alaca, Compact and bounded sets in intuitionistic fuzzy metric spaces, *Demonstr. Math.*, **40** (2007), 449–456. http://dx.doi.org/10.1515/dema-2007-0216
12. Ö. Kişi, M. Gürdal, On deferred Cesàro summability and statistical convergence for the sets of triple sequences, *Ann. Fuzzy Math. Inform.*, **24** (2022), 115–127. http://dx.doi.org/10.30948/afmi.2022.24.2.115
13. Ö. Kişi, M. Gürdal, Certain aspects of deferred statistical convergence of fuzzy variables in credibility space, *J. Anal.*, **32** (2024), 2057–2075. http://dx.doi.org/10.1007/s41478-023-00583-6
14. Ö. Kişi, R. Savaş, On the structure of Zweier (λ,μ)-statistical convergence in neutrosophic n-normed space, *J. Nat. Appl. Sci.*, **29** (2025), 649–660. http://dx.doi.org/10.19113/sdufenbed.1793560
15. J. Korevaar, *Tauberian theory: A century of developments*, Heidelberg: Springer, 2004. http://dx.doi.org/10.1007/978-3-662-10225-1
16. F. Móricz, Tauberian conditions under which statistical convergence follows from statistical summability $(C,1)$, *J. Math. Anal. Appl.*, **275** (2002), 277–287. http://dx.doi.org/10.1016/S0022-247X(02)00338-4
17. F. Móricz, Necessary and sufficient Tauberian conditions, under which convergence follows from summability $(C,1)$, *Bull. Lond. Math. Soc.*, **26** (1994), 288–294. http://dx.doi.org/10.1112/blms/26.3.288
18. Z. Önder, İ. Çanak, Ü. Totur, Tauberian theorems for statistically $(C,1,1)$ summable double sequences of fuzzy numbers, *Open Math.*, **15** (2017), 157–178. http://dx.doi.org/10.1515/math-2017-0006
19. Z. Önder, S. Karakahya, İ. Çanak, Tauberian theorems for the statistical Cesàro summability method in intuitionistic fuzzy normed spaces, *Soft Comput.*, **28** (2024), 87–104. http://dx.doi.org/10.1007/s00500-023-09335-x

20. P. Parida, B. B. Jena, S. K. Paikray, M. Mursaleen, On fuzzy approximation theorems for functions of two variables via statistical deferred Nörlund summability, *Iran. J. Fuzzy Syst.*, **23** (2026), 17–35.
21. R. Saadati, J. H. Park, On the intuitionistic fuzzy topological spaces, *Chaos Solitons Fract.*, **27** (2006), 331–344. http://dx.doi.org/10.1016/j.chaos.2005.03.019
22. R. Saadati, A. Razani, H. Adibi, A common fixed point theorem in L-fuzzy metric spaces, *Chaos Solitons Fract.*, **33** (2007), 358–363. http://dx.doi.org/10.1016/j.chaos.2006.01.023
23. E. Savaş, M. Gürdal, Certain summability methods in intuitionistic fuzzy normed spaces, *J. Intell. Fuzzy Syst.*, **27** (2014), 1621–1629. http://dx.doi.org/10.3233/IFS-141128
24. S. A. Sezer, İ. Çanak, Power series methods of summability for series of fuzzy numbers and related Tauberian theorems, *Soft Comput.*, **21** (2017), 1057–1064. http://dx.doi.org/10.1007/s00500-015-1840-0
25. S. A. Sezer, İ. Çanak, Deferred Cesàro means of fuzzy number-valued sequences with applications to Tauberian theorems, *Filomat*, **37** (2023), 5993–6004. http://dx.doi.org/10.2298/FIL2318993S
26. S. Shakeri, R. Saadati, C. Park, Stability of the quadratic functional equation in non-Archimedean L-fuzzy normed spaces, *Int. J. Nonlinear Anal. Appl.*, **1** (2010), 72–83. http://dx.doi.org/10.22075/ijnaa.2010.77
27. Ö. Talo, F. Başar, Necessary and sufficient Tauberian conditions for the A^r method of summability, *Math. J. Okayama Univ.*, **60** (2018), 209–219.
28. Ö. Talo, E. Yavuz, Cesàro summability of sequences in intuitionistic fuzzy normed spaces and related Tauberian theorems, *Soft Comput.*, **25** (2021), 2315–2323. http://dx.doi.org/10.1007/s00500-020-05301-z
29. Ü. Totur, İ. Çanak, Tauberian theorems for $(\bar{N}; p; q)$ summable double sequences of fuzzy numbers, *Soft Comput.*, **24** (2020), 2301–2310. http://dx.doi.org/10.1007/s00500-019-04060-w
30. R. Yapalı, H. Çoşkun, Lacunary statistical convergence for double sequences on $\mathscr{L}$-fuzzy normed space, *J. Math. Sci. Model.*, **6** (2023), 24–31. http://dx.doi.org/10.33187/jmsm.1127905
31. R. Yapalı, H. Çoşkun, U. Gürdal, Statistical convergence on L-fuzzy normed space, *Filomat*, **37** (2023), 2077–2085. http://dx.doi.org/10.2298/FIL2307077Y
32. R. Yapalı, U. Gürdal, Pringsheim and statistical convergence for double sequences on L-fuzzy normed space, *AIMS Mathematics*, **6** (2021), 13726–13733. https://doi.org/10.3934/math.2021796
33. R. Yapalı, E. Korkmaz, M. Çınar, H. Çoşkun, Lacunary statistical convergence on L-fuzzy normed space, *J. Intell. Fuzzy Syst.*, **46** (2022), 1985–1993. https://doi.org/10.3233/JIFS-222695
34. R. Yapalı, Ö. Talo, Tauberian conditions for double sequences which are statistically summable $(C, 1, 1)$ in fuzzy number space, *J. Intell. Fuzzy Syst.*, **33** (2017), 947–956. http://dx.doi.org/10.3233/JIFS-162211
35. L. A. Zadeh, Fuzzy sets, *Inform. Control*, **8** (1965), 338–353. http://dx.doi.org/10.1016/S0019-9958(65)90241-X