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Research article

Robust synchronization of Caputo–Hadamard fractional-order T–S fuzzy Lur'e systems with delay: LMI and SOS conditions

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Abstract: In this paper, the robust asymptotic synchronization conditions are obtained for uncertain nonlinear Caputo–Hadamard fractional-order Takagi–Sugeno (T–S) fuzzy Lur'e-type systems with time-varying delay. Matched fuzzy-rule weights represent the master and slave plants, and the slave system is controlled by a static output-error parallel distributed compensation controller that can utilize either the current or past output errors. Norm-bounded structures for the uncertain matrices and incremental sector bounds on the local nonlinearities are considered. Delay-free and delayed synchronization criteria are obtained through the use of a common quadratic Lyapunov function, a Caputo–Hadamard quadratic derivative inequality, the Schur complement, and a Caputo–Hadamard Halanay inequality. The constant-matrix results are further extended to polynomial T–S fuzzy Lur'e systems using a global unconstrained sum-of-squares matrix certificate. For a real-life partial-output scenario (with fewer measured outputs than state dimension), the numerical validation first solves the delayed linear matrix inequalities (LMI) as a max-margin semidefinite program using CVXPY, and then describes the polynomial extension via an explicit matrix sum-of-squares (SOS) Gram certificate and an additional simulation example. The main novelty is that, to the best of my knowledge, this is the first study to address simultaneously, within a single convex LMI/SOS framework, the Caputo–Hadamard (logarithmic-time) fractional order, an unbounded time-varying delay, matched norm-bounded parametric uncertainty, incremental sector nonlinearities, and static partial output-error feedback; unlike integer-order or Riemann–Liouville formulations, this setting cannot rely on the classical chain rule and instead requires a Caputo–Hadamard quadratic derivative inequality together with a logarithmic-time Halanay argument. Throughout, "robust synchronization" is meant in the sense of complete synchronization (asymptotic decay of the error to zero) that is guaranteed for every admissible uncertainty; a numerical study further quantifies how the fractional order α shapes the synchronization transient.

Keywords: Caputo–Hadamard fractional-order systems; Takagi–Sugeno fuzzy systems; Lur'e-type systems; robust synchronization; time-varying delay; output-error PDC control; linear matrix

inequalities; sum-of-squares programming
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1. Introduction

Fractional-order modeling has emerged as a prominent tool for the modeling of dynamical phenomena, which exhibit memory, hereditary effects, and nonlocal evolution. The notion of Hadamard-type fractional operators dates back to the logarithmic-kernel construction proposed by Hadamard, and the analytical study of fractional differential equations is background for existence, stability, and control problems for noninteger order dynamics [1–3]. The Caputo–Hadamard derivative is one of these fractional operators, which is very interesting for systems evolving with respect to logarithmic time scale, due to its Caputo-type initial conditions, which are compatible with classical state-space formulations [4–6]. Recent studies have further extended the stability, finite time stability, and delay dependent analysis of Hadamard systems and Caputo–Hadamard systems: Logarithmic time dynamics require specific inequalities, that is, they cannot be directly extended from Caputo/integer-order cases [7–9]. Related recent developments—including uncertainty analysis, neutral stochastic and averaging principles, boundary-value and fuzzy fractional models, and variable-order or data-driven formulations—further witness the growing importance of Hadamard-type and Caputo-type tools in nonlinear analysis and control [10–12]; in the present paper, these tools are specialized to the master–slave synchronization problem studied below. These advances are accompanied by the general work on finite-time convergence and fractional inequalities for discrete, stochastic, and Caputo-type systems [13, 14]. Among these tools, the fractional Halanay-type and quadratic Lyapunov inequalities are the ones directly exploited in the present paper: The Caputo–Hadamard Halanay inequality and the Caputo–Hadamard quadratic derivative inequality recalled in Section 2 are precisely the technical devices on which the delayed synchronization criteria of Section 4 rest, so the literature surveyed here is invoked only insofar as it supplies these specific inequalities.

Synchronization is an important issue of nonlinear control, secure communication, neural dynamics, and complex systems. Master-slave synchronization has been extensively studied in Lur'e-type systems, which are described as a combination of a linear dynamical component and a static nonlinear component with bounded nonlinearities, since it offers a convenient description of a variety of nonlinear systems. For delayed Lur'e systems, the classical results demonstrated the role of feedback design, estimate for delay, and sector information in obtaining the synchronization conditions [15–17]. It has been proved that the convergence mechanism in fractional-order systems is fundamentally different from the exponential behavior of integer-order systems; in the fractional-order framework, Mittag–Leffler stability, and fractional Lur'e synchronization have been achieved [18, 19]. In many real applications, time delay and memory effects coexist, and in this case, the Halanay-type inequalities are still important for the proof of the asymptotic convergence under delayed feedback, and the Caputo–Hadamard versions are particularly well-suited for the logarithmic time fractional dynamics [6, 20]. This direction has been reinforced by several recent works on uncertain Caputo–Hadamard systems and delayed fuzzy neural networks, including studies of synchronizing output feedback under uncertain parameters and delays [21, 22]. The combination of fuzzy modeling, delay analysis, and fractional synchronization tools further confirms the importance of the complete synchronization of Caputo–Hadamard fuzzy competitive neural networks and related fractional T–S fuzzy neural models [23–25]. Closely related recent advances also concern the global stability and synchronization of discrete-time and variable-order fractional-order delayed (quaternion-valued) neural networks [26], which further motivate convex synchronization certificates for fractional fuzzy models.

T–S fuzzy modeling can represent nonlinear systems efficiently by convex combinations of local models. After the first fuzzy identification framework proposed by Takagi and Sugeno, many stability and control approaches have been proposed for nonlinear fuzzy systems with Lyapunov methods and parallel distributed compensation (PDC) [27–29]. The observer-based fuzzy control, relaxed stability conditions, and robust fuzzy design have also significantly contributed to deal with unmeasured states and parametric uncertainties [30–32]. It is seen from the systematic LMI-based design methodology for fuzzy systems and the related publications surveyed that T–S models are very suitable for converting nonlinear control problems to computationally tractable matrix conditions [33, 34]. However, most of the works about T–S fuzzy synchronization of fractional-order systems are for the standard Caputo fractional-order system or for neural network structure, and the fractional-order Caputo–Hadamard Lur'e-type systems with delayed static output-error feedback are still not well developed.

Linear matrix inequalities are a powerful tool for robust control synthesis since they enable the Lyapunov, sector, uncertainty, and delay constraints to be cast in a convex optimization framework [35, 36]. Constant-matrix LMI conditions, however, may be conservative when the local system matrices are polynomial functions of measurable variables. An alternative way of certifying global positivity or negativity of polynomial matrices by semidefinite programming is sum-of-squares programming, which has proven to be a valuable computational tool in robustness analysis and nonlinear polynomial control [37–39]. Most notably, the recent SOS-based results for nonlinear Hadamard fractional-order systems show that polynomial fuzzy models can be analyzed without resorting to a finite set of constant vertices [39]. This gives the incentive to develop global unconstrained matrix-SOS certificates for polynomial T–S fuzzy Lur'e synchronization.

But there are still a number of issues that are outstanding. First, there are few results available that address the strong asymptotic synchronization for uncertain nonlinear Caputo–Hadamard fractional-order T–S fuzzy Lur'e-type systems with time-varying delay. Secondly, lots of fractional synchronization results in the literature fail to consider norm-bounded uncertainties, incremental sector nonlinearities, partial-output feedback, and delayed output-error control all in a unified LMI framework. Third, the classical chain rule cannot be used in the proof; rather, a valid Caputo–Hadamard quadratic inequality of the derivatives and a proper logarithmic-time Halanay argument are required for Caputo–Hadamard systems. Fourth, polynomial fuzzy Lur'e extensions are necessarily set up in a global SOS matrix form, even if operating-set restrictions on local semialgebraic forms are not desired.

Based on the above observation, this paper proposes robust asymptotic synchronization criterion for uncertain nonlinear Caputo–Hadamard fractional-order T–S fuzzy Lur'e-type systems with time-varying delay. The matched fuzzy-rule weights are used to describe the master and slave systems, and the slave is controlled by a static output-error parallel distributed compensation law, which includes the current output error and delayed output error. The nonlinearities are represented by incremental sector constraints, and the matrix uncertainties are modeled by norm-bounded structures. Delay-free and delayed synchronization criteria are derived and expressed in LMIs using a common quadratic

Lyapunov function, a Caputo–Hadamard quadratic derivative inequality, a Schur-complement, and a Caputo–Hadamard Halanay inequality. Finally, the constant-matrix results are generalized to polynomial T–S fuzzy Lur'e systems by using a global unconstrained SOS matrix certificate. Beyond casting the design as matrix inequalities, a further objective is to clarify how the Caputo–Hadamard fractional order and the fuzzy blending actually shape the synchronization process. Unlike the exponential decay of integer-order Lur'e synchronization, the convergence here is governed by a logarithmic-time fractional Halanay mechanism, and the order α directly controls the speed of the synchronization transient; this dependence is quantified numerically in Section 5, where the fuzzy design is also compared with a fuzzy-free controller.

The main contributions of this paper, stated relative to the existing literature, are summarized as follows:

- Unified robust LMI synchronization criteria (delay-free and delayed) are derived for uncertain Caputo–Hadamard fractional-order T–S fuzzy Lur'e-type systems controlled by a static partial output-error PDC law that blends the present and delayed output errors. In contrast with integer-order Lur'e synchronization [15–17] and with neural-network-specific Caputo–Hadamard fuzzy results [21, 23], the proposed criteria handle norm-bounded uncertainties, incremental sector nonlinearities, and an unbounded time-varying delay simultaneously.

- The delayed criterion is established without the classical chain rule, by means of a Caputo–Hadamard quadratic derivative inequality and a logarithmic-time Halanay argument; this extends the single-model analysis of [22] to the double fuzzy-blended rule-pair setting, where a membership-dependent Lyapunov function is not admissible.

- The constant-matrix design is generalized to polynomial T–S fuzzy Lur'e systems through a global unconstrained matrix sum-of-squares certificate, thereby removing the compact operating-set (semialgebraic-multiplier) restriction inherent in local SOS designs [39].

- The conditions are validated on a genuine partial-output benchmark ($k < n$) through a max-margin semidefinite program and an explicit matrix-SOS Gram certificate; a comparison with a fuzzy-free design and a fractional-order (α) sensitivity study further quantify, respectively, the price of the convex relaxation and the role of the fractional dynamics.

The rest of the paper is organized as follows. In Section 2, we present the necessary notation, Caputo–Hadamard fractional preliminaries, sector conditions, and uncertainty structure. The uncertain T–S fuzzy Lur'e master–slave synchronization problem is formulated in Section 3. Section 4 presents the main results. Numerical examples and simulation results are given in Section 5. Lastly, the paper is concluded in Section 6.

2. Preliminaries

Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ denotes the set of real $n \times m$ matrices. For a symmetric matrix X, the notation $X > 0$ ($X < 0$) means that X is positive definite (negative definite). The symbol $*$ denotes the symmetric block induced by transposition. For a square matrix X, $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denote its smallest and largest eigenvalues when $X = X^T$. The notation diag{·} is used for block-diagonal or diagonal matrices. All dimensions are assumed to be compatible.

Definition 2.1 (Hadamard fractional integral and derivative [1,2]). *Let $x : [a, +\infty) \to \mathbb{R}^n$ be sufficiently smooth and let $\alpha > 0$. The Hadamard fractional integral and derivative of order α are respectively defined by*

$$ {}^{H}D_{a,t}^{-\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\theta}\right)^{\alpha-1} x(\theta) \frac{d\theta}{\theta}, \tag{2.1} $$

$$ {}^{H}D_{a,t}^{\alpha} x(t) = \left(t \frac{d}{dt}\right)^{m} \frac{1}{\Gamma(m-\alpha)} \int_a^t \left(\ln \frac{t}{\theta}\right)^{m-\alpha-1} x(\theta) \frac{d\theta}{\theta}, \tag{2.2} $$

where $m = [\alpha] + 1$ and $\Gamma(\cdot)$ is the Euler gamma function. In particular, for $0 < \alpha < 1$,

$$ {}^{H}D_{a,t}^{\alpha} x(t) = t \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\ln \frac{t}{\theta}\right)^{-\alpha} x(\theta) \frac{d\theta}{\theta}. \tag{2.3} $$

Definition 2.2 (Caputo–Hadamard derivative [2, 4]). *Let $\alpha > 0$ and $m = [\alpha] + 1$. The Caputo–Hadamard derivative of $x(t)$ is defined by*

$$ {}^{CH}D_{a,t}^{\alpha} x(t) = {}^{H}D_{a,t}^{\alpha} \left[x(t) - \sum_{k=0}^{m-1} \frac{\left(t\frac{d}{dt}\right)^{k} x(a)}{k!} \left(\ln \frac{t}{a}\right)^{k} \right], \qquad t \geq a. \tag{2.4} $$

For $0 < \alpha < 1$,

$$ {}^{CH}D_{a,t}^{\alpha} x(t) = {}^{H}D_{a,t}^{\alpha} [x(t) - x(a)]. \tag{2.5} $$

Equivalently, when x is absolutely continuous on compact subintervals of $[a, +\infty)$,

$$ {}^{CH}D_{a,t}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\ln \frac{t}{\theta}\right)^{-\alpha} x'(\theta)\, d\theta. \tag{2.6} $$

Lemma 2.1 (Logarithmic-time representation [2,4]). *Let $0 < \alpha < 1$, $s = \ln(t/a)$, and $X(s) = x(ae^s)$. Then, whenever the derivatives exist,*

$$ {}^{CH}D_{a,t}^{\alpha} x(t) = {}^{C}D_{0,s}^{\alpha} X(s), \qquad s = \ln(t/a), \tag{2.7} $$

where ${}^{C}D_{0,s}^{\alpha}$ is the classical Caputo derivative with respect to the logarithmic time s.

Lemma 2.2 (Quadratic Caputo–Hadamard derivative inequality [5,22]). *Let $x(t) \in \mathbb{R}^n$ be differentiable and let $Q = Q^T > 0$. For every $\alpha \in (0, 1)$,*

$$ {}^{CH}D_{a,t}^{\alpha} (x^T(t) Q x(t)) \leq x^T(t) Q\, {}^{CH}D_{a,t}^{\alpha} x(t) + ({}^{CH}D_{a,t}^{\alpha} x(t))^T Q x(t). \tag{2.8} $$

Equivalently,

$$ {}^{CH}D_{a,t}^{\alpha} (x^T(t) Q x(t)) \leq 2 x^T(t) Q\, {}^{CH}D_{a,t}^{\alpha} x(t). \tag{2.9} $$

Lemma 2.3 (Caputo–Hadamard fractional Halanay inequality [5, 22])**.** *Let $\alpha \in (0,1)$ and let $V(t)$ be continuous and nonnegative on the considered history interval and on $[a,+\infty)$. Assume that $\tau(\cdot) \in C([a,+\infty),\mathbb{R}_+)$, $t-\tau(t) \geq a$, and $t-\tau(t) \to +\infty$ as $t \to +\infty$. If there exist constants $\lambda > \kappa > 0$ such that*

$$ {}^{CH}D^{\alpha}_{a,t}V(t) \leq -\lambda V(t) + \kappa \sup_{-\tau(t)\leq\theta\leq 0} V(t+\theta), \tag{2.10}$$

then

$$ \lim_{t\to+\infty} V(t) = 0. \tag{2.11}$$

Lemma 2.4 (Lyapunov asymptotic stability criterion [5, 22])**.** *Consider the Caputo–Hadamard nonlinear system*

$$ {}^{CH}D^{\alpha}_{a,t}x(t) = f(t,x(t)), \qquad 0 < \alpha < 1. \tag{2.12}$$

If there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ and positive constants $c_1, c_2, c_3, \rho, \sigma$ such that

$$ c_1 \|x(t)\|^{\rho} \leq V(x(t)) \leq c_2 \|x(t)\|^{\rho\sigma}, \tag{2.13}$$

$$ {}^{CH}D^{\alpha}_{a,t}V(x(t)) \leq -c_3 \|x(t)\|^{\rho\sigma}, \tag{2.14}$$

then the origin is asymptotically stable.

Lemma 2.5 (Schur complement [35])**.** *Let*

$$ H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix} = H^T. \tag{2.15}$$

Then

$$ H < 0 \tag{2.16}$$

is equivalent to each of the following statements:

$$ H_{11} < 0, \qquad H_{22} - H_{12}^T H_{11}^{-1} H_{12} < 0, \tag{2.17}$$

$$ H_{22} < 0, \qquad H_{11} - H_{12} H_{22}^{-1} H_{12}^T < 0. \tag{2.18}$$

Lemma 2.6 (Norm-bounded uncertainty inequality [35])**.** *Let $F^T(t)F(t) \leq I$ and let X, Y be constant matrices of compatible dimensions. Then, for any scalar $\varepsilon > 0$,*

$$ XF(t)Y + Y^T F^T(t) X^T \leq \varepsilon^{-1} XX^T + \varepsilon Y^T Y. \tag{2.19}$$

In quadratic form, for any vectors ξ and η,

$$ 2\xi^T XF(t)Y\eta \leq \varepsilon^{-1}\xi^T XX^T \xi + \varepsilon \eta^T Y^T Y \eta. \tag{2.20}$$

Lemma 2.7 (T–S convex blending and PDC principle [27, 33])**.** *Let $h_i(z(t))$, $i = 1, \ldots, r$, be normalized membership functions satisfying*

$$ h_i(z(t)) \geq 0, \qquad \sum_{i=1}^{r} h_i(z(t)) = 1. \tag{2.21}$$

Then

$$h_i(t)h_j(t) \geq 0, \qquad \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(t)h_j(t) = 1. \tag{2.22}$$

Consequently, if symmetric matrices Φ_{ij} *satisfy* $\Phi_{ij} < 0$ *for all rule pairs* (i, j)*, then*

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(t)h_j(t)\Phi_{ij} < 0. \tag{2.23}$$

3. Problem formulation

This section generalizes the uncertain nonlinear Caputo–Hadamard synchronization framework of [22] to a T–S fuzzy Lur'e-type setting. Let r be the number of fuzzy rules, and let $z(t)$ denote a measurable premise vector. The normalized membership functions are denoted by $h_i(t) = h_i(z(t))$, $i = 1, \dots, r$, and satisfy

$$h_i(t) \geq 0, \qquad \sum_{i=1}^{r} h_i(t) = 1. \tag{3.1}$$

Assumption 3.1 (Matched premises, matched uncertainties, delay regularity, and well-posedness)**.** *The following standing assumptions are imposed throughout the paper.*

(i) The fuzzy premise variables are matched in the master and slave systems. Equivalently, either $z(t)$ *is exogenous and measurable by both systems, or the master-side premise vector is transmitted to the slave and controller. Hence* $h_i^M(t) = h_i^S(t) = h_i(t)$ *for all i. Without this restriction, additional mismatch terms containing* $h_i(z_x(t)) - h_i(z_y(t))$ *appear in the error system and are not covered by the LMIs below.*

(ii) The uncertainty channels are matched between the master and slave systems. Thus the same admissible matrices $\Delta A_i(t)$*,* $\Delta D_i(t)$*, and* $\Delta W_i(t)$ *act on both systems. Independent master–slave uncertainty channels would generate additional mismatch terms that are not included in the error model* (3.10)*.*

(iii) The functions $h_i(z(t))$ *are continuous or piecewise continuous and satisfy* (3.1) *for all* $t \geq a$*.*

(iv) The delay is continuous and satisfies

$$\tau(\cdot) \in C([a, +\infty), \mathbb{R}_+), \qquad t - \tau(t) \geq a, \qquad t - \tau(t) \to +\infty \quad \textit{as } t \to +\infty. \tag{3.2}$$

Thus no state value before the initial instant a is required. When a bounded prehistory formulation is used, the initial functions are assumed continuous on the corresponding history interval.

(v) The nonlinear mappings f_i *are sufficiently regular, and for every admissible initial condition and every admissible uncertainty satisfying* (3.6)*, the master–slave system admits a unique solution.*

Remark 3.1 (On the delay condition (3.2) and bounded or constant delays)**.** *Condition* (3.2) *only requires that the delayed argument never precedes the initial instant,* $t - \tau(t) \geq a$*, and that it tends to infinity,* $t - \tau(t) \to +\infty$*. It does not force* $\tau(\cdot)$ *to vanish for all time. At* $t = a$*, it gives* $\tau(a) = 0$*, which merely expresses that no state value before the initial instant a is invoked when the analysis is started* *at $t = a$; it is an initialization convention, not a smallness condition. A genuinely bounded or constant delay $\tau(t) \equiv \bar{\tau} > 0$ is also covered: It suffices to prescribe a continuous initial history of the error on the bounded interval $[a - \bar{\tau}, a]$ (equivalently, to start the Halanay comparison at $t = a + \bar{\tau}$). Then $t - \tau(t) = t - \bar{\tau} \geq a$ and $t - \tau(t) \to +\infty$ hold for $t \geq a + \bar{\tau}$, and the Caputo–Hadamard Halanay inequality applies verbatim because that inequality only uses the values of V on the history interval. Hence the criteria are not restricted to vanishing delays, and* (3.2) *is a regularity/initialization requirement rather than a differentiability or boundedness restriction on $\tau(\cdot)$.*

For $0 < \alpha < 1$, consider the uncertain Caputo–Hadamard fractional-order T–S fuzzy master system

$$\begin{cases} {}^{CH}D^{\alpha}_{a,t}x(t) = \displaystyle\sum_{i=1}^{r} h_i(t)\Big[(A_i + \Delta A_i(t))x(t) + (D_i + \Delta D_i(t))x(t - \tau(t)) \\ \qquad\qquad\qquad + (W_i + \Delta W_i(t))f_i(C_i x(t))\Big], \\ \qquad p(t) = Fx(t), \end{cases} \tag{3.3}$$

and the slave system

$$\begin{cases} {}^{CH}D^{\alpha}_{a,t}y(t) = \displaystyle\sum_{i=1}^{r} h_i(t)\Big[(A_i + \Delta A_i(t))y(t) + (D_i + \Delta D_i(t))y(t - \tau(t)) \\ \qquad\qquad\qquad + (W_i + \Delta W_i(t))f_i(C_i y(t))\Big] + u(t), \\ \qquad q(t) = Fy(t), \end{cases} \tag{3.4}$$

where $x(t), y(t) \in \mathbb{R}^n$, $p(t), q(t) \in \mathbb{R}^k$, $A_i, D_i \in \mathbb{R}^{n\times n}$, $W_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{m\times n}$, and $F \in \mathbb{R}^{k\times n}$. The nonlinearities $f_i : \mathbb{R}^m \to \mathbb{R}^m$ are diagonal mappings. The uncertainties are norm-bounded and structured as

$$\Delta A_i(t) = E_{ai}F_{ai}(t)H_{ai}, \qquad \Delta D_i(t) = E_{di}F_{di}(t)H_{di}, \qquad \Delta W_i(t) = E_{wi}F_{wi}(t)H_{wi}, \tag{3.5}$$

where

$$F_{ai}^T(t)F_{ai}(t) \leq I, \qquad F_{di}^T(t)F_{di}(t) \leq I, \qquad F_{wi}^T(t)F_{wi}(t) \leq I. \tag{3.6}$$

The matrices in (3.5) are common to the master and slave models, in accordance with the matched-uncertainty assumption.

The parallel distributed output-feedback controller is selected as

$$u(t) = \sum_{j=1}^{r} h_j(t)\Big[K_{1j}(p(t) - q(t)) + K_{2j}(p(t - \tau(t)) - q(t - \tau(t)))\Big], \tag{3.7}$$

where $K_{1j}, K_{2j} \in \mathbb{R}^{n\times k}$ are gain matrices to be designed. The delayed-output term is deliberately blended using the current premise value $h_j(t)$, not $h_j(t - \tau(t))$. This convention is implementable under Assumption 3.1 because $z(t)$ is available at the controller and avoids additional delayed membership products.

Remark 3.2 (On the role of the output matrix F)**.** *No observability or detectability hypothesis on F is postulated a priori. The output-error feedback acts only through the measured channel $p(t) - q(t) =$*

$Fe(t)$, so synchronization is achievable only if the error is steerable to zero through this channel; the feasibility of the LMIs in Theorems 4.1 and 4.2 is precisely the constructive certificate that the chosen F admits a stabilizing static output-error PDC law. Two natural sufficient structural conditions make this feasibility nondegenerate: F should have full row rank (no measured output is redundant), and the pairs associated with the local closed-loop dynamics should be detectable through F. If $F = 0$ (no measurement), the gains enter only through $Y_{1j}F$ and $Y_{2j}F$, which vanish, and the inequalities reduce to an open-loop stability test that fails for the unstable local matrices considered here. In the numerical study, $F = \begin{bmatrix} 1 & -1 \end{bmatrix}$ has full row rank with $k = 1 < n = 2$, i.e., a genuine partial measurement.

Define the synchronization error

$$e(t) = x(t) - y(t), \qquad e_\tau(t) = e(t - \tau(t)). \tag{3.8}$$

For each rule i, define the incremental nonlinearity

$$g_i(t) = g_i(C_i e(t), y(t)) = f_i(C_i(e(t) + y(t))) - f_i(C_i y(t)). \tag{3.9}$$

Then the error system induced by (3.3)–(3.7) is

$$\begin{aligned} {}^{CH}D^{\alpha}_{a,t} e(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(t)h_j(t)\Big[&(A_i + \Delta A_i(t) - K_{1j}F)e(t) \\ &+ (D_i + \Delta D_i(t) - K_{2j}F)e_\tau(t) \\ &+ (W_i + \Delta W_i(t))g_i(t)\Big]. \end{aligned} \tag{3.10}$$

Assumption 3.2 (Incremental sector condition)**.** *For each rule i, there exist known diagonal matrices*

$$L_{1i} = \mathrm{diag}\{l_{1i1}, \ldots, l_{1im}\}, \qquad L_{2i} = \mathrm{diag}\{l_{2i1}, \ldots, l_{2im}\}, \tag{3.11}$$

with $L_{2i} - L_{1i} \geq 0$, such that for every $e(t)$ and $y(t)$,

$$(g_{i\ell}(t) - l_{1i\ell}c_{i\ell}^T e(t))(g_{i\ell}(t) - l_{2i\ell}c_{i\ell}^T e(t)) \leq 0, \quad \ell = 1, \ldots, m, \tag{3.12}$$

where $C_i = \mathrm{col}\{c_{i1}^T, \ldots, c_{im}^T\}$. Equivalently, for every diagonal matrix $\Gamma_i = \Gamma_i^T > 0$,

$$(g_i(t) - L_{1i}C_i e(t))^T \Gamma_i (g_i(t) - L_{2i}C_i e(t)) \leq 0. \tag{3.13}$$

Definition 3.1 (Complete synchronization)**.** *The master system* (3.3) *and the slave system* (3.4) *are completely synchronized if*

$$\lim_{t\to+\infty} \|x(t) - y(t)\| = \lim_{t\to+\infty} \|e(t)\| = 0. \tag{3.14}$$

Definition 3.2 (Robust complete synchronization)**.** *The master system* (3.3) *and the slave system* (3.4) *are robustly completely synchronized if they are completely synchronized for every admissible uncertainty satisfying* (3.5)–(3.6)*. This is the property certified by the criteria below, and it is the precise meaning of "robust synchronization" in the title: Robustness refers to uncertainty-independent complete synchronization, not to a separate notion of synchronization. Thus the word "complete" describes the limiting behavior of the error, while "robust" describes its invariance over the whole admissible uncertainty set.*

Remark 3.3. *Assumption 3.2 is stated in incremental form. This is the form directly required by the synchronization error equation, because the nonlinear term is the difference $f_i(C_i(e+y)) - f_i(C_i y)$. It reduces to the standard sector formulation used for Lur'e-type synchronization when the nonlinearities are globally slope-restricted.*

Remark 3.4 (Summary of the standing assumptions and their scope)**.** *For clarity, the modeling hypotheses and their rationale are collected here. (a) Matched premises and matched uncertainties (Assumption 3.1(i)–(ii)) keep the error system* (3.10) *free of the mismatch terms $h_i(z_x) - h_i(z_y)$ and of independent master/slave uncertainty channels; this is what makes a single common quadratic Lyapunov certificate possible. (b) The admissible uncertainty class is exactly the matched, structured, norm-bounded family* (3.5)–(3.6)*: It covers all time-varying parametric perturbations of A_i, D_i, and W_i that factor through the fixed range and coefficient matrices $E_{(\cdot)i}, H_{(\cdot)i}$ with $F^T_{(\cdot)i}(t)F_{(\cdot)i}(t) \leq I$. It does not claim to cover unstructured, unmatched, or state-dependent-nonlinear uncertainties, which would require additional multipliers and lie outside the present scope. (c) The nonlinearities are diagonal and incrementally sector-bounded (Assumption 3.2); nonlinearities that are not slope-restricted are not admissible. (d) The delay $\tau(\cdot)$ is only required to be continuous and to satisfy* (3.2)*; in particular, it need not be differentiable, and neither an upper bound nor a derivative bound has to be known, because convergence is obtained from the Caputo–Hadamard Halanay inequality rather than from a Lyapunov–Krasovskii functional with delay-derivative terms (see Remark 3.1 for bounded/constant delays). These choices trade generality for a fully convex and numerically tractable formulation, and the resulting conservatism is discussed in the Conclusion.*

4. Main results

In the LMIs below, I_{ai}, I_{di}, and I_{wi} are identity matrices of appropriate dimensions, matching the uncertainty channels $F_{ai}(t)$, $F_{di}(t)$, and $F_{wi}(t)$, respectively.

4.1. Delay-free fuzzy synchronization criterion

First, we consider the delay-free case $D_i = 0$, $\Delta D_i(t) = 0$, and $K_{2j} = 0$, in which the error dynamics become

$$^{CH}D^{\alpha}_{a,t}e(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(t)h_j(t)\Big[(A_i + \Delta A_i(t) - K_jF)e(t) + (W_i + \Delta W_i(t))g_i(t)\Big]. \tag{4.1}$$

Here, the controller (3.7) reduces to the static output-error PDC law that uses only the present output error,

$$u(t) = \sum_{j=1}^{r} h_j(t)\, K_j(p(t) - q(t)) = \sum_{j=1}^{r} h_j(t)\, K_j\, Fe(t), \tag{4.2}$$

so that K_j denotes the present-output gain and coincides with the gain K_{1j} of the delayed law (3.7) (with $K_{2j} = 0$). Throughout this subsection, $K_j \equiv K_{1j}$, and the change of variables used below is $Y_j = PK_j$.

Theorem 4.1 (Robust delay-free T–S fuzzy synchronization)**.** *Assume that Assumptions 3.1 and 3.2 hold. The master and slave T–S fuzzy Caputo–Hadamard systems without delay are completely synchronized if there exist a symmetric matrix $P = P^T > 0$, diagonal matrices $\Gamma_i = \Gamma_i^T > 0$, matrices*

$Y_j \in \mathbb{R}^{n\times k}$, and positive scalars $\varepsilon_{ai}, \varepsilon_{wi}$ such that, for all $i, j \in \{1, \dots, r\}$,

$$\Omega_{ij}^{(0)} = \begin{bmatrix} \Xi_{ij}^{(0)} & PW_i + C_i^T(L_{1i} + L_{2i})\Gamma_i & PE_{ai} & PE_{wi} \\ * & \varepsilon_{wi} H_{wi}^T H_{wi} - 2\Gamma_i & 0 & 0 \\ * & * & -\varepsilon_{ai} I_{ai} & 0 \\ * & * & * & -\varepsilon_{wi} I_{wi} \end{bmatrix} < 0, \tag{4.3}$$

where

$$\Xi_{ij}^{(0)} = PA_i + A_i^T P - Y_j F - F^T Y_j^T + \varepsilon_{ai} H_{ai}^T H_{ai} - 2C_i^T L_{1i}\Gamma_i L_{2i} C_i. \tag{4.4}$$

The controller gains are obtained from

$$K_j = P^{-1} Y_j, \qquad j = 1, \dots, r. \tag{4.5}$$

Proof. Let

$$V(e(t)) = e^T(t) P e(t), \tag{4.6}$$

where $P = P^T > 0$. Then

$$\lambda_{\min}(P) \|e(t)\|^2 \le V(e(t)) \le \lambda_{\max}(P) \|e(t)\|^2. \tag{4.7}$$

By the quadratic Caputo–Hadamard derivative inequality,

$${}^{CH}D_{a,t}^{\alpha} V(e(t)) \le 2e^T(t) P \, {}^{CH}D_{a,t}^{\alpha} e(t). \tag{4.8}$$

Substituting (4.1) into (4.8) gives

$${}^{CH}D_{a,t}^{\alpha} V(e(t)) \le \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(t) h_j(t) Q_{ij}^{(0)}(t), \tag{4.9}$$

where

$$Q_{ij}^{(0)}(t) = 2e^T P(A_i - K_j F)e + 2e^T P W_i g_i + 2e^T P \Delta A_i(t) e + 2e^T P \Delta W_i(t) g_i. \tag{4.10}$$

The first term is symmetrized as

$$2e^T P(A_i - K_j F)e = e^T(PA_i + A_i^T P - PK_j F - F^T K_j^T P)e. \tag{4.11}$$

Using the substitutions, $Y_j = PK_j$ yields

$$PK_j F + F^T K_j^T P = Y_j F + F^T Y_j^T. \tag{4.12}$$

From the uncertainty inequality and (3.5)–(3.6), for every $\varepsilon_{ai} > 0$ and $\varepsilon_{wi} > 0$,

$$2e^T P \Delta A_i(t) e = 2e^T P E_{ai} F_{ai}(t) H_{ai} e \le \varepsilon_{ai}^{-1} e^T P E_{ai} E_{ai}^T P e + \varepsilon_{ai} e^T H_{ai}^T H_{ai} e, \tag{4.13}$$

$$2e^T P \Delta W_i(t) g_i = 2e^T P E_{wi} F_{wi}(t) H_{wi} g_i \le \varepsilon_{wi}^{-1} e^T P E_{wi} E_{wi}^T P e + \varepsilon_{wi} g_i^T H_{wi}^T H_{wi} g_i. \tag{4.14}$$

Furthermore, the sector inequality (3.13) implies

$$0 \le -2(g_i - L_{1i} C_i e)^T \Gamma_i (g_i - L_{2i} C_i e)$$

$$= -2g_i^T\Gamma_i g_i + 2e^T C_i^T(L_{1i} + L_{2i})\Gamma_i g_i - 2e^T C_i^T L_{1i}\Gamma_i L_{2i}C_i e. \tag{4.15}$$

Adding the nonnegative right-hand side in (4.15) to the upper bound of (4.10) and using (4.13)–(4.14), one obtains

$$Q_{ij}^{(0)}(t) \le \zeta_i^T(t)\Pi_{ij}^{(0)}\zeta_i(t), \qquad \zeta_i(t) = \mathrm{col}\{e(t), g_i(t)\}, \tag{4.16}$$

where

$$\Pi_{ij}^{(0)} = \begin{bmatrix} \bar{\Xi}_{ij}^{(0)} & PW_i + C_i^T(L_{1i} + L_{2i})\Gamma_i \\ * & \varepsilon_{wi}H_{wi}^T H_{wi} - 2\Gamma_i \end{bmatrix}, \tag{4.17}$$

and

$$\begin{aligned} \bar{\Xi}_{ij}^{(0)} &= PA_i + A_i^T P - Y_j F - F^T Y_j^T + \varepsilon_{ai}H_{ai}^T H_{ai} - 2C_i^T L_{1i}\Gamma_i L_{2i}C_i \\ &\quad + \varepsilon_{ai}^{-1}PE_{ai}E_{ai}^T P + \varepsilon_{wi}^{-1}PE_{wi}E_{wi}^T P. \end{aligned} \tag{4.18}$$

By applying the Schur complement to (4.3), one obtains

$$\Omega_{ij}^{(0)} < 0 \quad \Longrightarrow \quad \Pi_{ij}^{(0)} < 0. \tag{4.19}$$

Since the number of rule pairs is finite, define

$$\rho_0 = \min_{1\le i,j\le r} \lambda_{\min}\big(-\Pi_{ij}^{(0)}\big) > 0. \tag{4.20}$$

Then

$$\zeta_i^T\Pi_{ij}^{(0)}\zeta_i \le -\rho_0\,\|\zeta_i\|^2 \le -\rho_0\,\|e\|^2, \qquad \forall i, j. \tag{4.21}$$

Using $h_i(t)h_j(t) \ge 0$ and $\sum_{i,j} h_i(t)h_j(t) = 1$, from (4.9)–(4.16), we get

$${}^{CH}D_{a,t}^{\alpha}V(e(t)) \le -\rho_0\,\|e(t)\|^2. \tag{4.22}$$

Together with (4.7), this verifies the hypotheses of the Caputo–Hadamard Lyapunov stability criterion. Hence $\lim_{t\to+\infty} e(t) = 0$, and the master and slave systems are completely synchronized. □

4.2. T–S fuzzy synchronization with unbounded time-varying delay

We now treat the delayed error system (3.10). The result below is the T–S fuzzy counterpart of the delayed LMI criterion developed for the single-model Caputo–Hadamard system in [22]. The proof follows the same Lyapunov–Halanay structure but must handle the double fuzzy blending induced by the local plant rules and the PDC controller rules.

Theorem 4.2 (Robust delayed T–S fuzzy synchronization)**.** *Assume that Assumptions 3.1 and 3.2 hold. Let $\lambda > \kappa > 0$ be prescribed constants. The master system* (3.3) *and the slave system* (3.4) *under the controller* (3.7) *are completely synchronized if there exist a symmetric matrix $P = P^T > 0$, diagonal matrices $\Gamma_i = \Gamma_i^T > 0$, matrices $Y_{1j}, Y_{2j} \in \mathbb{R}^{n\times k}$, and positive scalars $\varepsilon_{ai}, \varepsilon_{di}, \varepsilon_{wi}$ such that, for all $i, j \in \{1, \ldots, r\}$,*

$$\Omega_{ij} = \begin{bmatrix} \Theta_{ij} & PD_i - Y_{2j}F & PW_i + C_i^T(L_{1i} + L_{2i})\Gamma_i & PE_{ai} & PE_{di} & PE_{wi} \\ * & \varepsilon_{di}H_{di}^T H_{di} - \kappa P & 0 & 0 & 0 & 0 \\ * & * & \varepsilon_{wi}H_{wi}^T H_{wi} - 2\Gamma_i & 0 & 0 & 0 \\ * & * & * & -\varepsilon_{ai}I_{ai} & 0 & 0 \\ * & * & * & * & -\varepsilon_{di}I_{di} & 0 \\ * & * & * & * & * & -\varepsilon_{wi}I_{wi} \end{bmatrix} < 0, \tag{4.23}$$

where

$$\Theta_{ij} = PA_i + A_i^T P - Y_{1j}F - F^T Y_{1j}^T + \varepsilon_{ai} H_{ai}^T H_{ai} - 2C_i^T L_{1i}\Gamma_i L_{2i} C_i + \lambda P. \tag{4.24}$$

The static output-error feedback gains are recovered as

$$K_{1j} = P^{-1}Y_{1j}, \qquad K_{2j} = P^{-1}Y_{2j}, \qquad j = 1, \ldots, r. \tag{4.25}$$

Proof. Choose the common quadratic Lyapunov function

$$V(e(t)) = e^T(t)Pe(t), \qquad P = P^T > 0. \tag{4.26}$$

Then

$$\lambda_{\min}(P)\left\|e(t)\right\|^2 \le V(e(t)) \le \lambda_{\max}(P)\left\|e(t)\right\|^2. \tag{4.27}$$

Using the Caputo–Hadamard quadratic derivative inequality gives

$${}^{CH}D_{a,t}^{\alpha} V(e(t)) \le 2e^T(t)P\,{}^{CH}D_{a,t}^{\alpha} e(t). \tag{4.28}$$

Substituting the error system (3.10) into (4.28) and using $e_\tau(t) = e(t - \tau(t))$, we obtain

$${}^{CH}D_{a,t}^{\alpha} V(e(t)) \le \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(t)h_j(t)Q_{ij}(t), \tag{4.29}$$

where

$$\begin{aligned} Q_{ij}(t) = {} & 2e^T P(A_i - K_{1j}F)e + 2e^T P(D_i - K_{2j}F)e_\tau + 2e^T PW_i g_i \\ & + 2e^T P\Delta A_i(t)e + 2e^T P\Delta D_i(t)e_\tau + 2e^T P\Delta W_i(t)g_i. \end{aligned} \tag{4.30}$$

The nominal present-state term is written as

$$2e^T P(A_i - K_{1j}F)e = e^T(PA_i + A_i^T P - PK_{1j}F - F^T K_{1j}^T P)e. \tag{4.31}$$

Introducing the variables $Y_{1j} = PK_{1j}$ and $Y_{2j} = PK_{2j}$ gives

$$PK_{1j}F + F^T K_{1j}^T P = Y_{1j}F + F^T Y_{1j}^T, \tag{4.32}$$

$$P(D_i - K_{2j}F) = PD_i - Y_{2j}F. \tag{4.33}$$

For the three uncertainty terms, Lemma 2.6 gives the following estimates:

$$2e^T P\Delta A_i(t)e \le \varepsilon_{ai}^{-1} e^T PE_{ai}E_{ai}^T Pe + \varepsilon_{ai} e^T H_{ai}^T H_{ai} e, \tag{4.34}$$

$$2e^T P\Delta D_i(t)e_\tau \le \varepsilon_{di}^{-1} e^T PE_{di}E_{di}^T Pe + \varepsilon_{di} e_\tau^T H_{di}^T H_{di} e_\tau, \tag{4.35}$$

$$2e^T P\Delta W_i(t)g_i \le \varepsilon_{wi}^{-1} e^T PE_{wi}E_{wi}^T Pe + \varepsilon_{wi} g_i^T H_{wi}^T H_{wi} g_i. \tag{4.36}$$

Moreover, the incremental sector inequality (3.13) yields

$$\begin{aligned} 0 &\le -2(g_i - L_{1i}C_i e)^T \Gamma_i (g_i - L_{2i}C_i e) \\ &= -2g_i^T \Gamma_i g_i + 2e^T C_i^T (L_{1i} + L_{2i})\Gamma_i g_i - 2e^T C_i^T L_{1i}\Gamma_i L_{2i} C_i e. \end{aligned} \tag{4.37}$$

Combining (4.30)–(4.37), we arrive at

$$Q_{ij}(t) \le \eta_i^T(t)\bar{\Pi}_{ij}\eta_i(t), \qquad \eta_i(t) = \mathrm{col}\{e(t), e_\tau(t), g_i(t)\}, \tag{4.38}$$

where

$$\bar{\Pi}_{ij} = \begin{bmatrix} \bar{\Theta}_{ij}^0 & PD_i - Y_{2j}F & PW_i + C_i^T(L_{1i} + L_{2i})\Gamma_i \\ * & \varepsilon_{di}H_{di}^T H_{di} & 0 \\ * & * & \varepsilon_{wi}H_{wi}^T H_{wi} - 2\Gamma_i \end{bmatrix}, \tag{4.39}$$

and

$$\begin{aligned} \bar{\Theta}_{ij}^0 = {} & PA_i + A_i^T P - Y_{1j}F - F^T Y_{1j}^T + \varepsilon_{ai}H_{ai}^T H_{ai} - 2C_i^T L_{1i}\Gamma_i L_{2i}C_i \\ & + \varepsilon_{ai}^{-1}PE_{ai}E_{ai}^T P + \varepsilon_{di}^{-1}PE_{di}E_{di}^T P + \varepsilon_{wi}^{-1}PE_{wi}E_{wi}^T P. \end{aligned} \tag{4.40}$$

Now add $\lambda V(e(t))$ and subtract the Halanay supremum term. Since

$$V(e_\tau(t)) = e_\tau^T(t)Pe_\tau(t) \le \sup_{-\tau(t)\le\theta\le 0} V(e(t+\theta)), \tag{4.41}$$

we have

$$-\kappa \sup_{-\tau(t)\le\theta\le 0} V(e(t+\theta)) \le -\kappa e_\tau^T(t)Pe_\tau(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(t)h_j(t)[-\kappa e_\tau^T(t)Pe_\tau(t)], \tag{4.42}$$

because $\sum_{i,j} h_i(t)h_j(t) = 1$. Consequently,

$$\begin{aligned} & {}^{CH}D_{a,t}^{\alpha}V(e(t)) + \lambda V(e(t)) - \kappa \sup_{-\tau(t)\le\theta\le 0} V(e(t+\theta)) \\ & \le \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(t)h_j(t)\eta_i^T(t)\Pi_{ij}\eta_i(t), \end{aligned} \tag{4.43}$$

where

$$\Pi_{ij} = \begin{bmatrix} \bar{\Theta}_{ij} & PD_i - Y_{2j}F & PW_i + C_i^T(L_{1i} + L_{2i})\Gamma_i \\ * & \varepsilon_{di}H_{di}^T H_{di} - \kappa P & 0 \\ * & * & \varepsilon_{wi}H_{wi}^T H_{wi} - 2\Gamma_i \end{bmatrix}, \tag{4.44}$$

with

$$\begin{aligned} \bar{\Theta}_{ij} = {} & PA_i + A_i^T P - Y_{1j}F - F^T Y_{1j}^T + \varepsilon_{ai}H_{ai}^T H_{ai} - 2C_i^T L_{1i}\Gamma_i L_{2i}C_i + \lambda P \\ & + \varepsilon_{ai}^{-1}PE_{ai}E_{ai}^T P + \varepsilon_{di}^{-1}PE_{di}E_{di}^T P + \varepsilon_{wi}^{-1}PE_{wi}E_{wi}^T P. \end{aligned} \tag{4.45}$$

Applying the Schur complement to (4.23) shows that

$$\Omega_{ij} < 0 \quad \Longrightarrow \quad \Pi_{ij} < 0, \qquad i, j = 1, \ldots, r. \tag{4.46}$$

Consequently,

$$\eta_i^T(t)\Pi_{ij}\eta_i(t) < 0 \tag{4.47}$$

for every nonzero $\eta_i(t)$ and for every rule pair (i, j). Since $h_i(t)h_j(t) \ge 0$ and $\sum_{i,j} h_i(t)h_j(t) = 1$, inequality (4.43) gives

$${}^{CH}D_{a,t}^{\alpha}V(e(t)) + \lambda V(e(t)) - \kappa \sup_{-\tau(t)\le\theta\le 0} V(e(t+\theta)) \le 0. \tag{4.48}$$

Therefore,

$$ {}^{CH}D^{\alpha}_{a,t}V(e(t)) \leq -\lambda V(e(t)) + \kappa \sup_{-\tau(t)\leq\theta\leq 0} V(e(t+\theta)). \tag{4.49} $$

Because $\lambda > \kappa > 0$, the Caputo–Hadamard fractional Halanay inequality implies

$$ \lim_{t\to+\infty} V(e(t)) = 0. \tag{4.50} $$

Using (4.27), we get

$$ \lim_{t\to+\infty} \|e(t)\| = 0. \tag{4.51} $$

Thus the master and slave systems are completely synchronized. □

Corollary 4.1 (Uncertainty-free delayed T–S fuzzy synchronization)**.** *Assume that* $\Delta A_i(t) = 0$, $\Delta D_i(t) = 0$, *and* $\Delta W_i(t) = 0$ *for all i. Let* $\lambda > \kappa > 0$. *The delayed T–S fuzzy master and slave systems are completely synchronized if there exist* $P = P^T > 0$, *diagonal matrices* $\Gamma_i = \Gamma_i^T > 0$, *and matrices* $Y_{1j}, Y_{2j} \in \mathbb{R}^{n\times k}$ *such that, for all* $i, j = 1, \ldots, r$,

$$ \begin{bmatrix} \Theta^c_{ij} & PD_i - Y_{2j}F & PW_i + C_i^T(L_{1i} + L_{2i})\Gamma_i \\ * & -\kappa P & 0 \\ * & * & -2\Gamma_i \end{bmatrix} < 0, \tag{4.52} $$

where

$$ \Theta^c_{ij} = PA_i + A_i^T P - Y_{1j}F - F^T Y_{1j}^T - 2C_i^T L_{1i}\Gamma_i L_{2i} C_i + \lambda P. \tag{4.53} $$

The gains are given by $K_{1j} = P^{-1}Y_{1j}$ *and* $K_{2j} = P^{-1}Y_{2j}$.

Proof. The proof is obtained from Theorem 4.2 by setting $E_{ai} = 0$, $E_{di} = 0$, and $E_{wi} = 0$ for all rules and deleting the corresponding uncertainty blocks. The remaining Schur complement is exactly (4.52). □

Remark 4.1 (Reduction to the base article)**.** *If* $r = 1$, *then* $h_1(t) = 1$ *and the double fuzzy sum disappears. The LMI in Theorem 4.2 recovers the single-model structure of Hong and Thanh [22] as a special case, up to the notation used for the rule-dependent matrices and local nonlinearities.*

Remark 4.2 (Why a common quadratic function is used)**.** *The proof deliberately uses* $V(e) = e^T Pe$ *instead of a membership-dependent function* $\sum_i h_i(t)e^T P_i e$. *For Caputo–Hadamard derivatives, an ordinary Leibniz rule is not available in general. A membership-dependent Lyapunov function would introduce fractional derivatives of* $h_i(t)$ *and additional memory terms that cannot be handled by the LMI* (4.23) *without extra assumptions. The common quadratic choice is conservative but mathematically clean and directly compatible with the Caputo–Hadamard derivative inequality.*

Remark 4.3 (Relation to delayed fuzzy Hadamard models)**.** *Unlike existing delayed fuzzy Hadamard fractional-order results that mainly address finite-time stability or boundedness of a single T–S fuzzy plant [7], and unlike neural-network-specific Caputo–Hadamard fuzzy synchronization models [21], the present work studies robust master–slave synchronization of uncertain Caputo–Hadamard fractional-order T–S fuzzy Lur'e-type systems under static output-error PDC control. The derived LMIs explicitly handle norm-bounded uncertainties, sector-bounded local nonlinearities, and unbounded time-varying delays through a Caputo–Hadamard Halanay inequality.*

Remark 4.4 (On the technical difficulties and the role of the fractional order and the fuzzy blending)**.** *It is worth making explicit why the Caputo–Hadamard fractional order and the fuzzy structure are not mere notational additions to a standard LMI argument; the proof above hinges on three obstacles that are absent from integer-order or single-model Lur'e synchronization. (i) No chain rule. The Caputo–Hadamard derivative does not obey the classical Leibniz/chain rule, so $V(e) = e^T Pe$ cannot be differentiated directly. The step* (4.28) *therefore relies on the quadratic Caputo–Hadamard derivative inequality (Lemma 2.3), which replaces the integer-order identity $\dot{V} = 2e^T P\dot{e}$ by an inequality and is what allows the uncertainty, sector, and Halanay terms to be assembled into a single LMI. (ii) Logarithmic-time memory. The dynamics evolve on the logarithmic scale $s = \ln(t/a)$, so the usual exponential Lyapunov and Halanay estimates do not transfer; the asymptotic decay is obtained from the Caputo–Hadamard Halanay inequality (Lemma 2.5) together with the nonconstant logarithmic delay map, which has no integer-order counterpart. (iii) Double fuzzy blending. The blending $\sum_i \sum_j h_i(t)h_j(t)$ produces the rule-pair inequalities* (4.23) *and, as noted above, precludes a membership-dependent Lyapunov function: A term $\sum_i h_i(t)e^T P_i e$ would require a fractional derivative of the products $h_i(t)$, for which no usable Leibniz rule exists. The common quadratic certificate is thus a necessity here, not a convenience. These three points are precisely the difficulties the construction overcomes. How the order α and the fuzzy blending then influence the synchronization behavior, as opposed to the derivation, is quantified numerically in Section 5.*

4.3. Polynomial T–S fuzzy Lur'e systems and SOS synchronization conditions

The LMI criteria in Theorems 4.1 and 4.2 assume rule-dependent constant matrices. In many nonlinear applications, however, the local dynamics contain polynomial state-dependent coefficients. Polynomial fuzzy models and SOS methods have been used for Caputo–Hadamard fractional-order control design, especially when the local matrices depend polynomially on measured variables [39]. This subsection shows that the same Caputo–Hadamard Lyapunov–Halanay argument can be extended to polynomial T–S fuzzy Lur'e systems by replacing the constant matrix inequality $\Omega_{ij} < 0$ with a global polynomial matrix inequality $\Omega_{ij}(\xi) < 0$ certified through an unconstrained matrix-SOS condition. In contrast with local SOS certificates based on semialgebraic multipliers, the condition below is imposed for all $\xi \in \mathbb{R}^{N_\xi}$; consequently, the resulting synchronization statement is global with respect to the polynomial variables.

Let

$$\xi(t) = \mathrm{col}\{e(t), e_\tau(t), y(t)\} \in \mathbb{R}^{N_\xi} \tag{4.54}$$

be a measurable polynomial variable and let $A_i(\xi)$, $D_i(\xi)$, $W_i(\xi)$ be polynomial matrices. Consider the polynomial delayed error system

$$\begin{aligned} {}^{CH}D_{a,t}^{\alpha}e(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(t)h_j(t)\Big[&(A_i(\xi) + \Delta A_i(t,\xi) - K_{1j}F)e(t) \\ &+ (D_i(\xi) + \Delta D_i(t,\xi) - K_{2j}F)e_\tau(t) \\ &+ (W_i(\xi) + \Delta W_i(t,\xi))g_i(t)\Big], \end{aligned} \tag{4.55}$$

where the uncertainty channels are

$$\Delta A_i(t,\xi) = E_{ai}F_{ai}(t)H_{ai}(\xi), \quad \Delta D_i(t,\xi) = E_{di}F_{di}(t)H_{di}(\xi), \quad \Delta W_i(t,\xi) = E_{wi}F_{wi}(t)H_{wi}(\xi), \tag{4.56}$$

with the same norm bounds as in (3.6). The matrices $H_{ai}(\xi)$, $H_{di}(\xi)$, and $H_{wi}(\xi)$ may also be polynomial. The polynomial system is understood globally: The bounds (3.6), the sector condition in Assumption 3.2, and the regularity assumptions in Assumption 3.1 are required to hold for all admissible times and for all values of the involved state, delayed-state, and output variables.

A symmetric polynomial matrix $M(\xi)$ is called a matrix SOS, denoted $M(\xi) \in \Sigma^{q\times q}[\xi]$, if there exists a polynomial matrix $R(\xi)$ such that $M(\xi) = R^T(\xi)R(\xi)$. Equivalently, $z^T M(\xi)z$ is an SOS polynomial in the joint variables (ξ, z), where $z \in \mathbb{R}^q$ [37,38]. Therefore, $M(\xi) \in \Sigma^{q\times q}[\xi]$ implies $M(\xi) \geq 0$ for every $\xi \in \mathbb{R}^{N_\xi}$.

Theorem 4.3 (SOS-based global synchronization for polynomial T–S fuzzy Lur'e systems)**.** *Assume that Assumptions 3.1 and 3.2 hold globally for the polynomial error system* (4.55)*. Let $\lambda > \kappa > 0$ be prescribed constants. The master and slave polynomial T–S fuzzy Lur'e systems are globally completely synchronized if there exist $P = P^T > 0$, diagonal matrices $\Gamma_i = \Gamma_i^T > 0$, matrices $Y_{1j}, Y_{2j} \in \mathbb{R}^{n\times k}$, positive scalars $\varepsilon_{ai}, \varepsilon_{di}, \varepsilon_{wi}$, and a scalar $\eta > 0$ such that, for all $i, j = 1, \ldots, r$,*

$$\mathcal{M}_{ij}(\xi) := -\Omega_{ij}(\xi) - \eta I_q \in \Sigma^{q\times q}[\xi], \tag{4.57}$$

where q is the dimension of $\Omega_{ij}(\xi)$ and

$$\Omega_{ij}(\xi) = \begin{bmatrix} \Theta_{ij}(\xi) & PD_i(\xi) - Y_{2j}F & PW_i(\xi) + C_i^T(L_{1i} + L_{2i})\Gamma_i & PE_{ai} & PE_{di} & PE_{wi} \\ * & \varepsilon_{di}H_{di}^T(\xi)H_{di}(\xi) - \kappa P & 0 & 0 & 0 & 0 \\ * & * & \varepsilon_{wi}H_{wi}^T(\xi)H_{wi}(\xi) - 2\Gamma_i & 0 & 0 & 0 \\ * & * & * & -\varepsilon_{ai}I_{ai} & 0 & 0 \\ * & * & * & * & -\varepsilon_{di}I_{di} & 0 \\ * & * & * & * & * & -\varepsilon_{wi}I_{wi} \end{bmatrix}, \tag{4.58}$$

with

$$\Theta_{ij}(\xi) = PA_i(\xi) + A_i^T(\xi)P - Y_{1j}F - F^T Y_{1j}^T + \varepsilon_{ai}H_{ai}^T(\xi)H_{ai}(\xi) - 2C_i^T L_{1i}\Gamma_i L_{2i}C_i + \lambda P. \tag{4.59}$$

The controller gains are recovered from

$$K_{1j} = P^{-1}Y_{1j}, \qquad K_{2j} = P^{-1}Y_{2j}, \qquad j = 1, \ldots, r. \tag{4.60}$$

Proof. The proof is given in detail because the polynomial dependence must not be interpreted as a merely local condition. The decisive point is that (4.57) is an unconstrained SOS identity in the full variable $\xi \in \mathbb{R}^{N_\xi}$. Hence the negativity of the polynomial matrix is certified globally, and no invariance of a compact operating set is needed.

Consider the common quadratic Lyapunov function

$$V(e(t)) = e^T(t)Pe(t), \qquad P = P^T > 0. \tag{4.61}$$

It satisfies

$$\lambda_{\min}(P)\,\|e(t)\|^2 \leq V(e(t)) \leq \lambda_{\max}(P)\,\|e(t)\|^2. \tag{4.62}$$

By the Caputo–Hadamard quadratic derivative inequality,

$${}^{CH}D^{\alpha}_{a,t}V(e(t)) \leq 2e^T(t)P\,{}^{CH}D^{\alpha}_{a,t}e(t). \tag{4.63}$$

Substituting (4.55) into (4.63) gives

$$ {}^{CH}D^{\alpha}_{a,t}V(e(t)) \le \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(t)h_j(t)Q_{ij}(t), \tag{4.64} $$

where, for compactness, the argument $\xi(t)$ is omitted in the polynomial matrices and

$$ \begin{aligned} Q_{ij}(t) &= 2e^T P(A_i(\xi) - K_{1j}F)e + 2e^T P(D_i(\xi) - K_{2j}F)e_\tau + 2e^T PW_i(\xi)g_i \\ &\quad + 2e^T P\Delta A_i(t,\xi)e + 2e^T P\Delta D_i(t,\xi)e_\tau + 2e^T P\Delta W_i(t,\xi)g_i. \end{aligned} \tag{4.65} $$

The nominal terms are symmetrized as

$$ 2e^T P(A_i(\xi) - K_{1j}F)e = e^T\Big(PA_i(\xi) + A_i^T(\xi)P - PK_{1j}F - F^T K_{1j}^T P\Big)e, \tag{4.66} $$

$$ 2e^T P(D_i(\xi) - K_{2j}F)e_\tau = 2e^T(PD_i(\xi) - PK_{2j}F)e_\tau. \tag{4.67} $$

Using the changes of variables

$$ Y_{1j} = PK_{1j}, \qquad Y_{2j} = PK_{2j}, \tag{4.68} $$

which are equivalent to the gain recovery formula (4.60), Eqs (4.66) and (4.67) become

$$ 2e^T P(A_i(\xi) - K_{1j}F)e = e^T\Big(PA_i(\xi) + A_i^T(\xi)P - Y_{1j}F - F^T Y_{1j}^T\Big)e, \tag{4.69} $$

$$ 2e^T P(D_i(\xi) - K_{2j}F)e_\tau = 2e^T(PD_i(\xi) - Y_{2j}F)e_\tau. \tag{4.70} $$

We next bound the uncertain terms. For the A_i-channel, using (4.56), (3.6), and $2a^Tb \le \varepsilon^{-1}a^Ta + \varepsilon b^Tb$, we obtain, for every $\varepsilon_{ai} > 0$,

$$ \begin{aligned} 2e^T P\Delta A_i(t,\xi)e &= 2e^T PE_{ai}F_{ai}(t)H_{ai}(\xi)e \\ &\le \varepsilon_{ai}^{-1}e^T PE_{ai}E_{ai}^T Pe + \varepsilon_{ai}e^T H_{ai}^T(\xi)F_{ai}^T(t)F_{ai}(t)H_{ai}(\xi)e \\ &\le \varepsilon_{ai}^{-1}e^T PE_{ai}E_{ai}^T Pe + \varepsilon_{ai}e^T H_{ai}^T(\xi)H_{ai}(\xi)e. \end{aligned} \tag{4.71} $$

Similarly,

$$ 2e^T P\Delta D_i(t,\xi)e_\tau \le \varepsilon_{di}^{-1}e^T PE_{di}E_{di}^T Pe + \varepsilon_{di}e_\tau^T H_{di}^T(\xi)H_{di}(\xi)e_\tau, \tag{4.72} $$

$$ 2e^T P\Delta W_i(t,\xi)g_i \le \varepsilon_{wi}^{-1}e^T PE_{wi}E_{wi}^T Pe + \varepsilon_{wi}g_i^T H_{wi}^T(\xi)H_{wi}(\xi)g_i. \tag{4.73} $$

Because Assumption 3.2 is imposed globally, for each diagonal matrix $\Gamma_i > 0$ and for all $e(t), y(t)$,

$$ \begin{aligned} 0 &\le -2(g_i - L_{1i}C_ie)^T\Gamma_i(g_i - L_{2i}C_ie) \\ &= -2g_i^T\Gamma_ig_i + 2e^TC_i^T(L_{1i} + L_{2i})\Gamma_ig_i - 2e^TC_i^TL_{1i}\Gamma_iL_{2i}C_ie. \end{aligned} \tag{4.74} $$

Adding the nonnegative expression in (4.74) to the upper bound of (4.65) and using (4.69)–(4.73) yields

$$ Q_{ij}(t) \le \eta_i^T(t)\bar{\Pi}_{ij}(\xi(t))\eta_i(t), \qquad \eta_i(t) = \mathrm{col}\{e(t), e_\tau(t), g_i(t)\}, \tag{4.75} $$

where

$$ \bar{\Pi}_{ij}(\xi) = \begin{bmatrix} \bar{\Theta}_{ij}^0(\xi) & PD_i(\xi) - Y_{2j}F & PW_i(\xi) + C_i^T(L_{1i} + L_{2i})\Gamma_i \\ * & \varepsilon_{di}H_{di}^T(\xi)H_{di}(\xi) & 0 \\ * & * & \varepsilon_{wi}H_{wi}^T(\xi)H_{wi}(\xi) - 2\Gamma_i \end{bmatrix}, \tag{4.76} $$

and

$$\bar{\Theta}^0_{ij}(\xi) = PA_i(\xi) + A_i^T(\xi)P - Y_{1j}F - F^TY_{1j}^T + \varepsilon_{ai}H_{ai}^T(\xi)H_{ai}(\xi) - 2C_i^TL_{1i}\Gamma_iL_{2i}C_i$$
$$+ \varepsilon_{ai}^{-1}PE_{ai}E_{ai}^TP + \varepsilon_{di}^{-1}PE_{di}E_{di}^TP + \varepsilon_{wi}^{-1}PE_{wi}E_{wi}^TP. \tag{4.77}$$

We now introduce the Halanay terms. Since

$$V(e_\tau(t)) = e_\tau^T(t)Pe_\tau(t) \le \sup_{-\tau(t)\le\theta\le 0} V(e(t+\theta)), \tag{4.78}$$

one has

$$-\kappa \sup_{-\tau(t)\le\theta\le 0} V(e(t+\theta)) \le -\kappa e_\tau^T(t)Pe_\tau(t). \tag{4.79}$$

Furthermore, because $\sum_{i,j} h_i(t)h_j(t) = 1$, the terms $\lambda V(e(t))$ and $-\kappa e_\tau^TPe_\tau$ may be distributed over the same convex double sum. Combining (4.64), (4.75), and (4.79), we get

$$^{CH}D^\alpha_{a,t}V(e(t)) + \lambda V(e(t)) - \kappa \sup_{-\tau(t)\le\theta\le 0} V(e(t+\theta))$$
$$\le \sum_{i=1}^r\sum_{j=1}^r h_i(t)h_j(t)\eta_i^T(t)\Pi_{ij}(\xi(t))\eta_i(t), \tag{4.80}$$

where

$$\Pi_{ij}(\xi) = \begin{bmatrix} \bar{\Theta}_{ij}(\xi) & PD_i(\xi) - Y_{2j}F & PW_i(\xi) + C_i^T(L_{1i} + L_{2i})\Gamma_i \\ * & \varepsilon_{di}H_{di}^T(\xi)H_{di}(\xi) - \kappa P & 0 \\ * & * & \varepsilon_{wi}H_{wi}^T(\xi)H_{wi}(\xi) - 2\Gamma_i \end{bmatrix}, \tag{4.81}$$

with

$$\bar{\Theta}_{ij}(\xi) = PA_i(\xi) + A_i^T(\xi)P - Y_{1j}F - F^TY_{1j}^T + \varepsilon_{ai}H_{ai}^T(\xi)H_{ai}(\xi) - 2C_i^TL_{1i}\Gamma_iL_{2i}C_i + \lambda P$$
$$+ \varepsilon_{ai}^{-1}PE_{ai}E_{ai}^TP + \varepsilon_{di}^{-1}PE_{di}E_{di}^TP + \varepsilon_{wi}^{-1}PE_{wi}E_{wi}^TP. \tag{4.82}$$

It remains to connect the SOS condition to the negativity of $\Pi_{ij}(\xi)$. Since $\mathcal{M}_{ij}(\xi)$ is a matrix SOS, for every $\xi \in \mathbb{R}^{N_\xi}$ and every vector $z \in \mathbb{R}^q$,

$$z^T\mathcal{M}_{ij}(\xi)z \ge 0. \tag{4.83}$$

Using (4.57), this is equivalent to

$$z^T\Omega_{ij}(\xi)z \le -\eta z^Tz, \qquad \forall z \in \mathbb{R}^q, \qquad \forall \xi \in \mathbb{R}^{N_\xi}. \tag{4.84}$$

Hence

$$\Omega_{ij}(\xi) \le -\eta I_q < 0, \qquad \forall \xi \in \mathbb{R}^{N_\xi}, \qquad i,j = 1,\dots,r. \tag{4.85}$$

Partition $\Omega_{ij}(\xi)$ as

$$\Omega_{ij}(\xi) = \begin{bmatrix} \Omega^{11}_{ij}(\xi) & \Omega^{12}_{ij} \\ * & \Omega^{22}_{ij} \end{bmatrix}, \tag{4.86}$$

where $\Omega^{11}_{ij}(\xi)$ is the leading 3×3 block matrix associated with the variables (e, e_τ, g_i) and

$$\Omega^{22}_{ij} = -\operatorname{diag}\{\varepsilon_{ai}I_{ai}, \varepsilon_{di}I_{di}, \varepsilon_{wi}I_{wi}\} < 0. \tag{4.87}$$

The Schur complement of Ω_{ij}^{22} in (4.86) is precisely

$$\Omega_{ij}^{11}(\xi) - \Omega_{ij}^{12}(\Omega_{ij}^{22})^{-1}(\Omega_{ij}^{12})^T = \Pi_{ij}(\xi). \tag{4.88}$$

Indeed, the inverse of (4.87) contributes the three positive terms

$$\varepsilon_{ai}^{-1} P E_{ai} E_{ai}^T P, \qquad \varepsilon_{di}^{-1} P E_{di} E_{di}^T P, \qquad \varepsilon_{wi}^{-1} P E_{wi} E_{wi}^T P, \tag{4.89}$$

which are exactly the additional terms appearing in (4.82). Therefore, from (4.85), (4.87), and the Schur complement, we obtain

$$\Pi_{ij}(\xi) < 0, \qquad \forall \xi \in \mathbb{R}^{N_\xi}, \qquad i, j = 1, \dots, r. \tag{4.90}$$

Consequently,

$$\eta_i^T(t)\Pi_{ij}(\xi(t))\eta_i(t) \leq 0, \qquad \forall t \geq a, \qquad i, j = 1, \dots, r. \tag{4.91}$$

Because the membership products satisfy $h_i(t)h_j(t) \geq 0$ and $\sum_{i,j} h_i(t)h_j(t) = 1$, (4.80) and (4.91) imply

$${}^{CH}D_{a,t}^{\alpha} V(e(t)) \leq -\lambda V(e(t)) + \kappa \sup_{-\tau(t) \leq \theta \leq 0} V(e(t + \theta)). \tag{4.92}$$

Since $\lambda > \kappa > 0$ and the delay satisfies Assumption 3.1, the Caputo–Hadamard fractional Halanay inequality gives

$$\lim_{t \to +\infty} V(e(t)) = 0. \tag{4.93}$$

Finally, (4.62) yields

$$\lim_{t \to +\infty} \|e(t)\| = 0. \tag{4.94}$$

Thus the master and slave polynomial T–S fuzzy Lur'e systems are completely synchronized. Since no restriction has been placed on $\xi(t)$ or on the magnitude of the initial history beyond the standing well-posedness assumptions, the synchronization conclusion is global. □

Remark 4.5. *The SOS condition* (4.57) *is deliberately written without semialgebraic multipliers. Hence the polynomial matrix inequality is certified on the whole space* $\mathbb{R}^{N_\xi}$*, exactly as in an unconstrained SOS formulation. This makes the result global, but it is also more conservative than a local certificate with domain multipliers.*

5. Numerical examples

The first example validates Theorem 4.2 through a two-rule uncertain delayed T–S fuzzy Caputo–Hadamard system. The second example illustrates the SOS-based polynomial extension of Theorem 4.3 by constructing an explicit global matrix-SOS certificate. In response to the output-feedback concern, the measured output in both examples is genuinely partial, namely $k = 1 < n = 2$, with

$$\alpha = 0.85, \qquad a = 1, \qquad r = 2, \qquad n = 2, \qquad m = 1, \qquad F = \begin{bmatrix} 1 & -1 \end{bmatrix}. \tag{5.1}$$

The premise functions are

$$h_1(t) = 0.5 + 0.4\sin(0.7 \ln t), \qquad h_2(t) = 1 - h_1(t), \qquad t \geq 1. \tag{5.2}$$

Therefore $0.1 \le h_i(t) \le 0.9$ and $h_1(t) + h_2(t) = 1$. The delay is selected as

$$\tau(t) = 0.25(t-1), \qquad t \ge 1, \tag{5.3}$$

so that

$$t - \tau(t) = 0.75t + 0.25 \ge 1, \qquad t - \tau(t) \to +\infty. \tag{5.4}$$

The local matrices are

$$A_1 = \begin{bmatrix} 0.35 & 0.25 \\ -0.10 & -0.80 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0.20 & -0.20 \\ 0.15 & -0.90 \end{bmatrix}, \tag{5.5}$$

$$D_1 = \begin{bmatrix} 0.020 & 0.010 \\ 0 & 0.015 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0.015 & -0.005 \\ 0.010 & 0.020 \end{bmatrix}, \tag{5.6}$$

$$W_1 = \begin{bmatrix} 0.020 \\ 0.010 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} -0.015 \\ 0.020 \end{bmatrix}, \tag{5.7}$$

$$C_1 = C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{5.8}$$

The nominal local matrices A_1 and A_2 each have one unstable eigenvalue; consequently, the uncontrolled synchronization error diverges in the simulation below. Thus, the comparison tests the necessity of feedback rather than only its ability to accelerate an already stable response. The nonlinearities are

$$f_1(v) = f_2(v) = \tanh(v). \tag{5.9}$$

Since $0 \le d(\tanh v)/dv \le 1$, the incremental sector condition holds with

$$L_{1i} = 0, \qquad L_{2i} = 1, \qquad i = 1, 2. \tag{5.10}$$

The uncertainty matrices are

$$E_{a1} = 0.015 I_2, \qquad E_{a2} = 0.018 I_2, \tag{5.11}$$

$$E_{d1} = 0.008 I_2, \qquad E_{d2} = 0.009 I_2, \tag{5.12}$$

$$E_{w1} = \begin{bmatrix} 0.006 \\ 0 \end{bmatrix}, \qquad E_{w2} = \begin{bmatrix} 0 \\ 0.006 \end{bmatrix}, \tag{5.13}$$

$$H_{a1} = \begin{bmatrix} 0.20 & 0.06 \\ 0.06 & 0.15 \end{bmatrix}, \qquad H_{a2} = \begin{bmatrix} 0.18 & -0.04 \\ -0.04 & 0.16 \end{bmatrix}, \tag{5.14}$$

$$H_{d1} = 0.08 I_2, \qquad H_{d2} = \begin{bmatrix} 0.08 & 0.015 \\ 0 & 0.07 \end{bmatrix}, \tag{5.15}$$

$$H_{w1} = \begin{bmatrix} 0.08 \end{bmatrix}, \qquad H_{w2} = \begin{bmatrix} 0.09 \end{bmatrix}. \tag{5.16}$$

5.1. Example 1: CVXPY solution of the constant-matrix LMIs

The delayed LMIs (4.23) were solved using CVXPY as a max-margin semidefinite program. The constants were fixed as

$$\lambda = 0.8, \qquad \kappa = 0.3, \tag{5.17}$$

and the optimization problem was

$$\begin{aligned}
\max_{P,\Gamma_i,Y_{1j},Y_{2j},\varepsilon_{ai},\varepsilon_{di},\varepsilon_{wi},\mu} \quad & \mu \\
\text{subject to} \quad & \operatorname{trace}(P) = 1, \quad P \geq \mu I_2, \quad \Gamma_i \geq \mu I_1, \\
& \varepsilon_{ai} \geq \mu, \quad \varepsilon_{di} \geq \mu, \quad \varepsilon_{wi} \geq \mu, \quad \Omega_{ij} \leq -\mu I, \quad i, j = 1, 2.
\end{aligned} \tag{5.18}$$

The upper bounds $\Gamma_i, \varepsilon_{ai}, \varepsilon_{di}, \varepsilon_{wi} \leq 50$ were imposed only to avoid artificial scaling, while trace(P) = 1 fixed the Lyapunov scale. The semidefinite programming (SDP) was implemented in CVXPY 1.9.0 and solved with CLARABEL 0.11.1 using `tol_gap_abs`= 10^{-8}, `tol_feas`= 10^{-8}, and `max_iter`= 10000. The solver status was `optimal`, and the optimal LMI margin was

$$\mu^* = 0.075906754606. \tag{5.19}$$

The optimal decision variables were

$$P = \begin{bmatrix} 0.2703684359 & -0.0789559418 \\ -0.0789559418 & 0.7296315641 \end{bmatrix}, \tag{5.20}$$

$$\Gamma_1 = 0.0868189695, \qquad \Gamma_2 = 0.0870173138, \tag{5.21}$$

$$\varepsilon_{a1} = 0.1063470818, \quad \varepsilon_{a2} = 0.1421111876, \quad \varepsilon_{d1} = 0.1371524348, \quad \varepsilon_{d2} = 0.0855146647, \tag{5.22}$$

$$\varepsilon_{w1} = 0.0902255148, \qquad \varepsilon_{w2} = 0.1193856082. \tag{5.23}$$

The optimized variables Y_{1j}, Y_{2j} and the recovered gains are

$$Y_{11} = \begin{bmatrix} 2.1357375015 \\ -1.5831485109 \end{bmatrix}, \qquad Y_{12} = \begin{bmatrix} 2.1357375017 \\ -1.5831485101 \end{bmatrix}, \tag{5.24}$$

$$Y_{21} = \begin{bmatrix} -0.0038163844 \\ 0.0112470794 \end{bmatrix}, \qquad Y_{22} = \begin{bmatrix} -0.0038163846 \\ 0.0112470796 \end{bmatrix}, \tag{5.25}$$

$$K_{11} = \begin{bmatrix} 7.5028148291 \\ -1.3578862929 \end{bmatrix}, \qquad K_{12} = \begin{bmatrix} 7.5028148303 \\ -1.3578862916 \end{bmatrix}, \tag{5.26}$$

$$K_{21} = \begin{bmatrix} -0.0099276457 \\ 0.0143404334 \end{bmatrix}, \qquad K_{22} = \begin{bmatrix} -0.0099276462 \\ 0.0143404335 \end{bmatrix}. \tag{5.27}$$

The largest eigenvalue of each optimized LMI matrix is strictly negative, as shown in Table 1. Therefore, the hypotheses of Theorem 4.2 are satisfied.

Table 1. Eigenvalue verification of the optimized LMI matrices Ω_{ij}.

Rule pair (i, j)	$\lambda_{\min}(\Omega_{ij})$	$\lambda_{\max}(\Omega_{ij})$
$(1, 1)$	-7.504922586266	-0.075906756712
$(1, 2)$	-7.504922585065	-0.075906756716
$(2, 1)$	-7.678379222052	-0.075906755316
$(2, 2)$	-7.678379220858	-0.075906755315

For simulation, the logarithmic-time transformation $s = \ln t$ is used, so that the Caputo–Hadamard system is simulated as a Caputo system in s. The physical delay is not a constant delay in logarithmic time. Since

$$t = e^s, \qquad \tau(t) = 0.25(t-1), \tag{5.28}$$

the delayed physical time and its logarithmic counterpart are

$$t - \tau(t) = 0.75e^s + 0.25, \qquad s_\tau(s) = \ln(0.75e^s + 0.25). \tag{5.29}$$

Thus the delayed state is evaluated as $e(t-\tau(t)) = E(s_\tau(s))$ by linear interpolation on the logarithmic grid. The step size is $\Delta s = 0.01$, and the final deterministic horizon is $s = 12$, i.e., $t = e^{12}$.

The deterministic uncertainty profile is

$$F_{ai}(t) = \sin(0.41\ln t)I_2, \qquad F_{di}(t) = \cos(0.37\ln t)I_2, \qquad F_{wi}(t) = \sin(0.29\ln t), \tag{5.30}$$

which satisfies all norm bounds. The initial values are

$$x(1) = \begin{bmatrix} 1.2 \\ -0.8 \end{bmatrix}, \qquad y(1) = \begin{bmatrix} -0.5 \\ 0.7 \end{bmatrix}, \tag{5.31}$$

so $e(1) = \begin{bmatrix} 1.7 & -1.5 \end{bmatrix}^T$ and $\|e(1)\| = 2.267156809751$. Because $t - \tau(t) \geq 1$, no pre-a history is required in this example; equivalently, the initial history is constant at the initial value when interpolation requests $s \leq 0$.

Figure 1 shows that both error components converge toward zero. Figure 2 compares the controlled and uncontrolled error norms under the same uncertainty profile. In contrast with the previous stable-comparison case, the uncontrolled error grows because of the unstable local dynamics, whereas the proposed PDC controller drives the error norm downward. Quantitatively,

s	$\|\|e(s)\|\|$ controlled	$\|\|e(s)\|\|$ uncontrolled
4	2.4020×10^{-1}	5.4572×10^{0}
6	1.4687×10^{-1}	8.3548×10^{0}
8	1.0220×10^{-1}	1.1725×10^{1}
12	6.0613×10^{-2}	3.3651×10^{1}

(5.32)

Finally, Figure 3 reports a Monte Carlo test with 50 smooth random uncertainty profiles satisfying $F^T(t)F(t) \leq I$. The worst final controlled norm at $s = 10$ is 7.7333×10^{-2}, and the mean final controlled norm is 7.7048×10^{-2}, confirming the robustness predicted by the LMI condition.

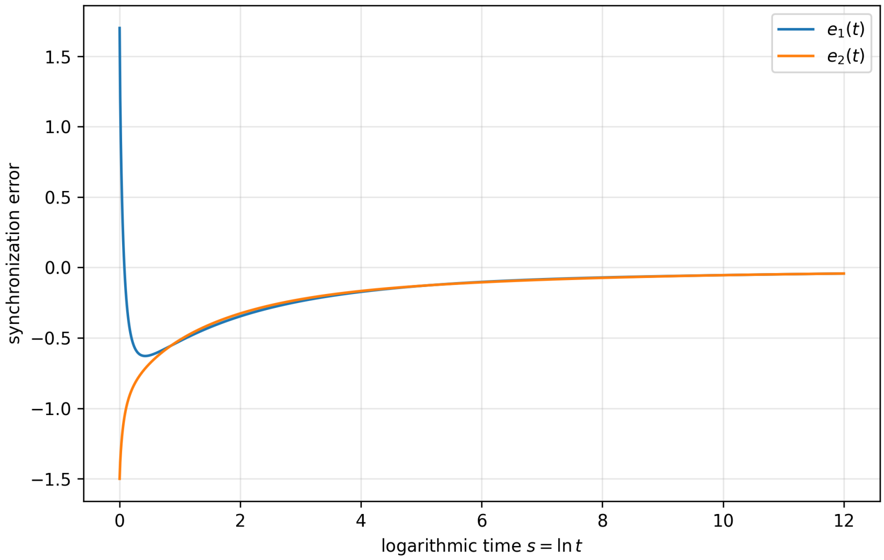

Figure 1. Controlled synchronization error trajectories with the partial output $p - q = Fe$, $F = \begin{bmatrix} 1 & -1 \end{bmatrix}$.

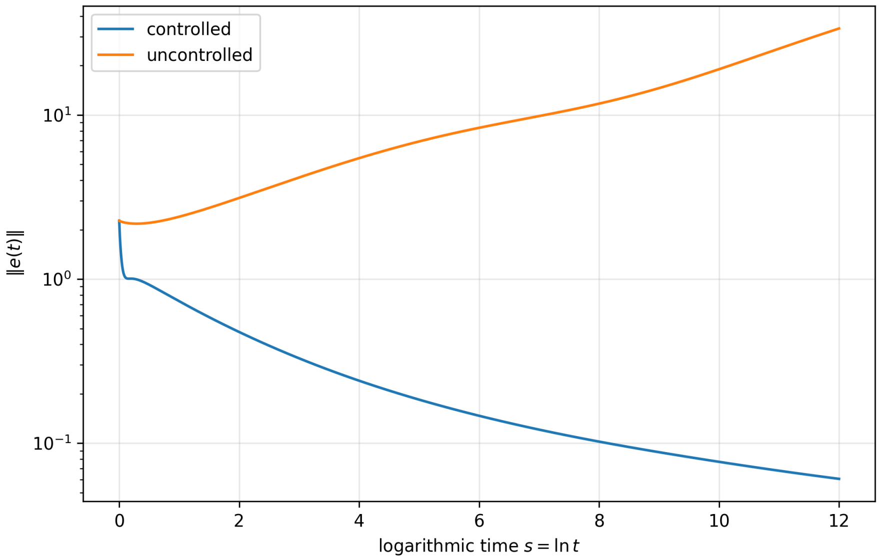

Figure 2. Controlled versus uncontrolled synchronization-error norms under the deterministic admissible uncertainty profile.

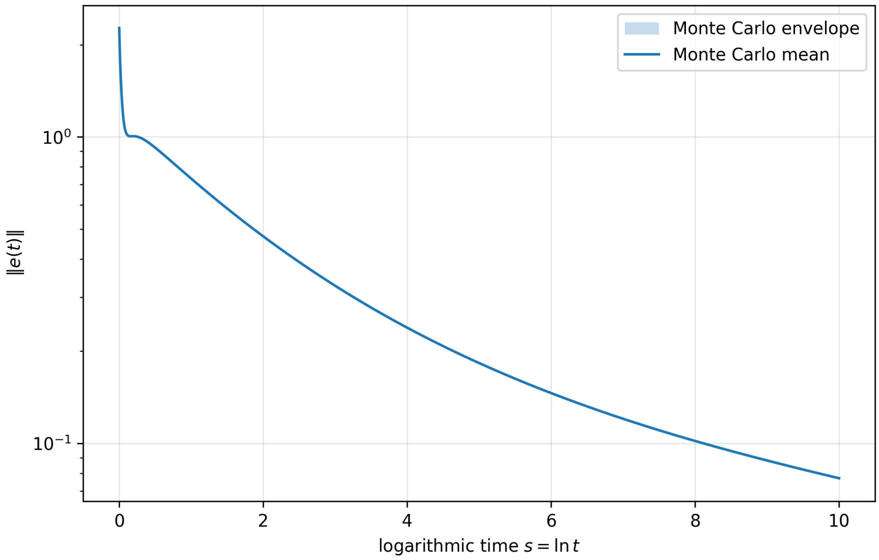

Figure 3. Monte Carlo envelope of the controlled synchronization-error norm over 50 admissible uncertainty profiles.

5.2. Comparison with a fuzzy-free design and effect of the fractional order

To address the request for a comparison with mainstream control methods that do not exploit the fuzzy/PDC structure, and for a clearer account of the parameter settings, two fuzzy-free baselines are designed on the same partial-output benchmark and with the same constants $\lambda = 0.8$, $\kappa = 0.3$, $\alpha = 0.85$. They are the two standard non-fuzzy output-feedback approaches:

(i) a non-fuzzy common-gain robust design, obtained by forcing a single gain pair $Y_{1j} \equiv Y_1$, $Y_{2j} \equiv Y_2$ in the robust LMIs (4.23) (the PDC freedom is switched off while all robustness terms are kept); this is the classical quadratic-stabilization approach, in which one common Lyapunov matrix and one controller robustly stabilize all rules; and

(ii) a nominal non-fuzzy design, a single static output-error controller computed for the averaged model $A_{\text{avg}} = \frac{1}{2}\sum_i A_i$, D_{avg}, W_{avg} without uncertainty robustification; this is the standard nominal (uncertainty-agnostic) output-feedback design.

The max-margin SDP returns $\mu^\star = 7.5907 \times 10^{-2}$ for the proposed rule-dependent PDC and $\mu = 7.5907 \times 10^{-2}$ for the common-gain design: The two margins coincide to nine significant digits because, for this scalar-output benchmark, the optimal PDC gains are rule-independent ($K_{11} = K_{12}$). This is structural rather than accidental: The proposed fuzzy LMIs contain the common-gain LMIs as the special case $Y_{1j} \equiv Y_1$, $Y_{2j} \equiv Y_2$, so the certified robust margin of the fuzzy design can never be smaller than that of any fuzzy-free design, $\mu_{\text{fuzzy}} \geq \mu_{\text{common}}$. On benign benchmarks, this inequality is tight; a strict gain appears when the local rules differ enough that no common gain certifies all rule pairs, and, most importantly, in the polynomial setting of Section 5.3, where a single constant-linear time-invariant controller admits no global negativity certificate at all.

Table 2 reports the synchronization-error norm of the three controllers on the true two-rule uncertain plant. The open-loop error diverges, reaching 3.37×10^1 at $s = 12$ because of the unstable local matrices, whereas every stabilizing design drives the error to a small value; the proposed robust fuzzy PDC matches the best baseline while additionally certifying robustness. A robust analysis LMI confirms this guarantee: With the proposed gains fixed, the maximal feasible margin is 7.59×10^{-2} against 7.48×10^{-2} for the nominal design, while the open loop is not certifiable. We deliberately refrain from claiming a large numerical superiority on this minimal example, whose purpose is reproducibility; the honest conclusion is that the value of the framework lies in the guarantee of robust synchronization, in its strict generalization of fuzzy-free designs, and in the polynomial extension, rather than in a performance gap on a two-rule scalar-output plant.

Table 2. Synchronization-error norm $\|e(s)\|$ on the true uncertain plant under three designs (deterministic admissible profile, $\alpha = 0.85$).

s	uncontrolled	nominal non-fuzzy	proposed robust fuzzy PDC
4	5.4572×10^0	2.4216×10^{-1}	2.4020×10^{-1}
6	8.3548×10^0	1.4673×10^{-1}	1.4687×10^{-1}
8	1.1725×10^1	1.0191×10^{-1}	1.0220×10^{-1}
12	3.3651×10^1	6.0699×10^{-2}	6.0613×10^{-2}

Finally, Table 3 and Figure 4 isolate the effect of the fractional order by re-running the proposed controller for $\alpha \in \{0.70, 0.85, 0.95\}$ on the same data. A larger order produces a markedly faster decay of the error: At $s = 12$, the controlled norm falls from 1.5348×10^{-1} for $\alpha = 0.70$ to 6.0613×10^{-2} for $\alpha = 0.85$ and to 1.6838×10^{-2} for $\alpha = 0.95$. This is the logarithmic-time fractional analogue of the order-dependent convergence rate familiar from Mittag–Leffler stability, and it shows concretely how the Caputo–Hadamard fractional-order shapes the synchronization transient, beyond merely entering the notation. The computations of this subsection are reproduced by `scripts/compare_methods.py`.

Table 3. Effect of the Caputo–Hadamard fractional order α on the controlled error norm $\|e(s)\|$ (proposed PDC).

s	$\alpha = 0.70$	$\alpha = 0.85$	$\alpha = 0.95$
4	3.5050×10^{-1}	2.4020×10^{-1}	1.5998×10^{-1}
6	2.6186×10^{-1}	1.4687×10^{-1}	7.1527×10^{-2}
8	2.1125×10^{-1}	1.0220×10^{-1}	3.8650×10^{-2}
12	1.5348×10^{-1}	6.0613×10^{-2}	1.6838×10^{-2}

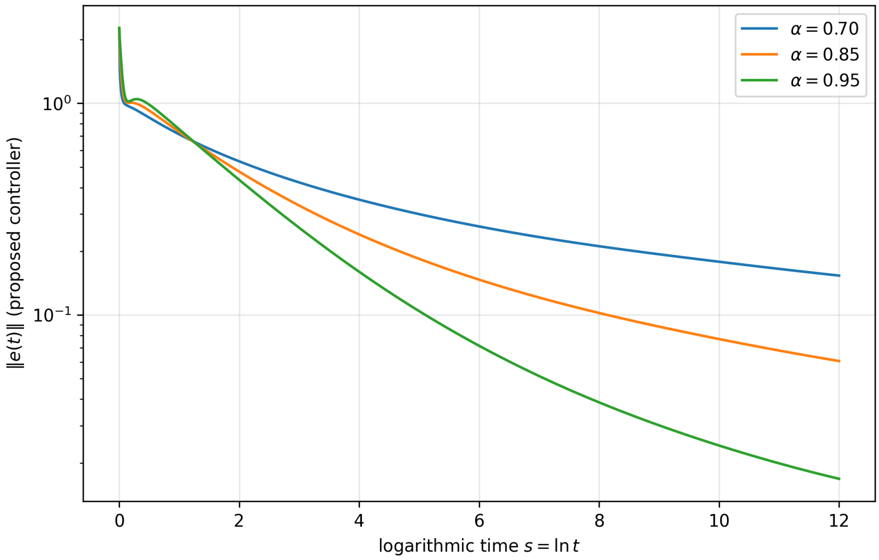

Figure 4. Effect of the fractional order α on the controlled synchronization-error norm. A larger α yields a faster decay of the error in logarithmic time $s = \ln t$.

5.3. *Example 2: explicit polynomial-matrix SOS certificate*

We now illustrate the SOS extension in Theorem 4.3 by constructing an explicit global matrix-SOS certificate. Starting from the optimized matrices of Example 1, define the polynomial local matrices

$$A_i(\xi) = A_i - \rho_i \xi^2 I_2, \qquad \rho_1 = 0.08, \qquad \rho_2 = 0.10, \tag{5.33}$$

with

$$\xi(t) = Fe(t). \tag{5.34}$$

The remaining matrices D_i, W_i, C_i, E_{ai}, E_{di}, E_{wi}, H_{ai}, H_{di}, and H_{wi} are kept as in Example 1. Thus the error system contains a genuine polynomial matrix coefficient while the same static output-error PDC gains are used.

For this example, no semialgebraic multiplier is needed, because the certificate is global. Let $\eta_s = 0.03$. Since

$$PA_i(\xi) + A_i^T(\xi)P = PA_i + A_i^T P - 2\rho_i \xi^2 P, \tag{5.35}$$

the polynomial LMI matrix satisfies

$$\Omega_{ij}(\xi) = \Omega_{ij}(0) + \xi^2 R_i, \qquad R_i = \mathrm{diag}\{-2\rho_i P, 0, 0, 0, 0, 0\}. \tag{5.36}$$

Therefore

$$-\Omega_{ij}(\xi) - \eta_s I = Q_{0,ij} + \xi^2 Q_{2,i}, \tag{5.37}$$

where

$$Q_{0,ij} = -\Omega_{ij}(0) - \eta_s I, \qquad Q_{2,i} = \mathrm{diag}\{2\rho_i P, 0, 0, 0, 0, 0\}. \tag{5.38}$$

The matrices $Q_{0,ij}$ and $Q_{2,i}$ are positive semidefinite; hence, with Cholesky or spectral factors $Q_{0,ij} = R_{0,ij}^T R_{0,ij}$ and $Q_{2,i} = R_{2,i}^T R_{2,i}$,

$$-\Omega_{ij}(\xi) - \eta_s I = R_{0,ij}^T R_{0,ij} + (\xi R_{2,i})^T (\xi R_{2,i}), \tag{5.39}$$

which is an explicit matrix-SOS certificate. Table 4 reports the numerical margins of this certificate.

Table 4. Explicit matrix-SOS certificate for the polynomial matrices $A_i(\xi) = A_i - \rho_i \xi^2 I_2$.

Rule pair (i, j)	$\lambda_{\max}(\Omega_{ij}(0))$	$\lambda_{\min}(Q_{0,ij})$	$\lambda_{\min}(Q_{2,i})$	$\lambda_{\max}(Q_{2,i})$
$(1, 1)$	$-7.590675671190 \times 10^{-2}$	$4.590675671190 \times 10^{-2}$	0	$1.188522356737 \times 10^{-1}$
$(1, 2)$	$-7.590675671576 \times 10^{-2}$	$4.590675671577 \times 10^{-2}$	0	$1.188522356737 \times 10^{-1}$
$(2, 1)$	$-7.590675531613 \times 10^{-2}$	$4.590675531613 \times 10^{-2}$	0	$1.485652945922 \times 10^{-1}$
$(2, 2)$	$-7.590675531456 \times 10^{-2}$	$4.590675531456 \times 10^{-2}$	0	$1.485652945922 \times 10^{-1}$

The simulation uses the same logarithmic-time Adams product-integration scheme and the same deterministic admissible uncertainty profile as Example 1. Figure 5 shows the controlled and uncontrolled norms. Thus, the example demonstrates that a constant-LMI controller can remain valid for a polynomially damped extension when the additional polynomial terms admit an SOS negativity certificate. The polynomial damping term prevents numerical blow-up of the uncontrolled error, but the uncontrolled synchronization error remains large and does not converge to zero, whereas the proposed PDC controller enforces synchronization. The quantitative values are

s	$\|e(s)\|$ controlled	$\|e(s)\|$ uncontrolled
4	2.3643×10^{-1}	1.6801×10^{0}
6	1.4515×10^{-1}	1.6516×10^{0}
8	1.0128×10^{-1}	1.6444×10^{0}
12	6.0275×10^{-2}	1.9023×10^{0}

(5.40)

At $s = 12$, the controlled error vector is $[-4.2474 \times 10^{-2}, -4.2767 \times 10^{-2}]^T$, whereas the uncontrolled error vector is $[1.8996, -1.0215 \times 10^{-1}]^T$. This confirms the practical effectiveness of the polynomial SOS condition. To complement the norm plots, Figure 6 gives a phase-portrait view of the synchronization error for several initial error vectors. In the uncontrolled case, the trajectories do not approach the origin, while the controlled trajectories are driven toward $(e_1, e_2) = (0, 0)$. The corresponding controlled error components are displayed in Figure 7; both components decay toward zero, in agreement with the norm plots.

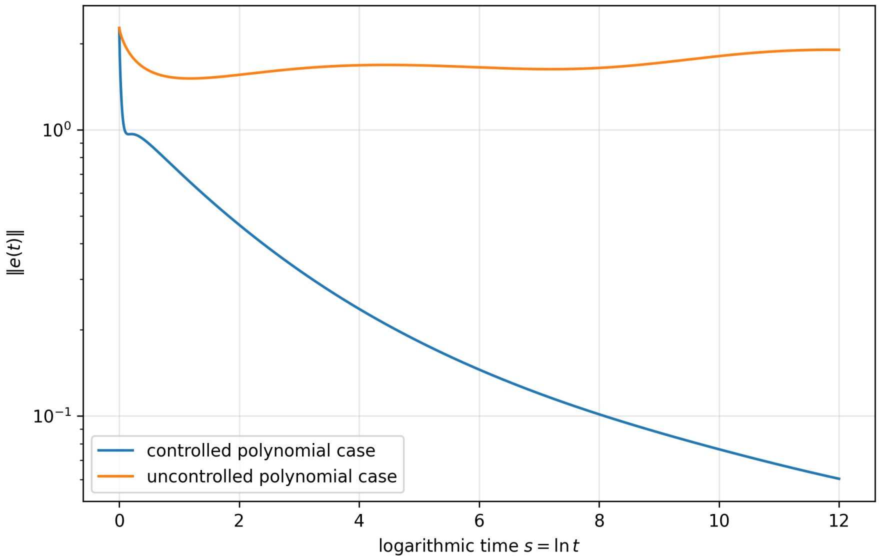

Figure 5. Controlled versus uncontrolled synchronization-error norms for the polynomial-matrix SOS example.

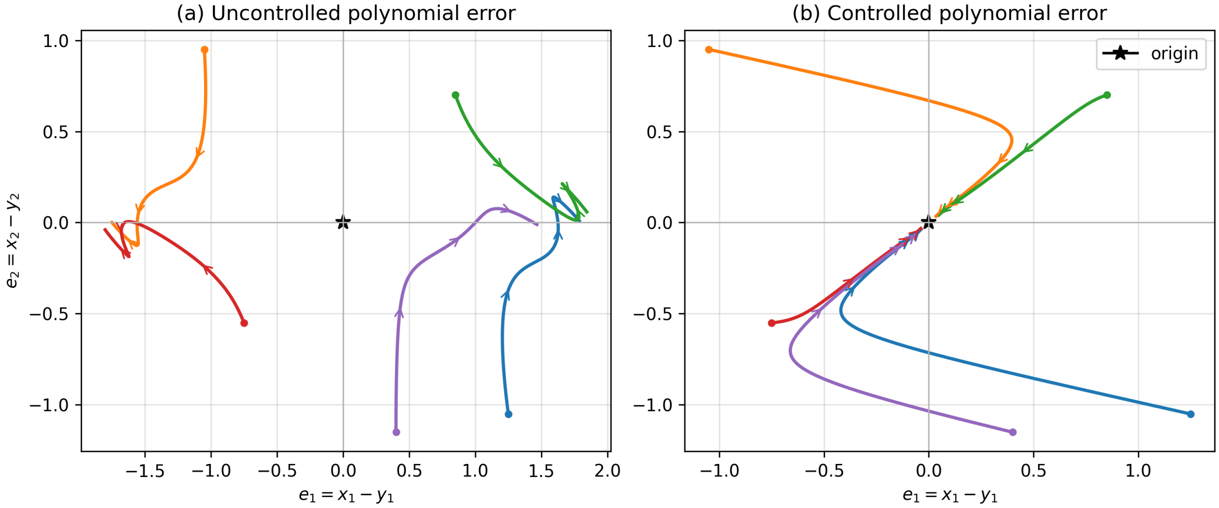

Figure 6. Phase portraits of the synchronization error for the polynomial-matrix SOS example: uncontrolled error dynamics and controlled error dynamics under the proposed SOS-based static output-error PDC controller. The arrows indicate the direction of evolution in logarithmic time $s = \ln t$.

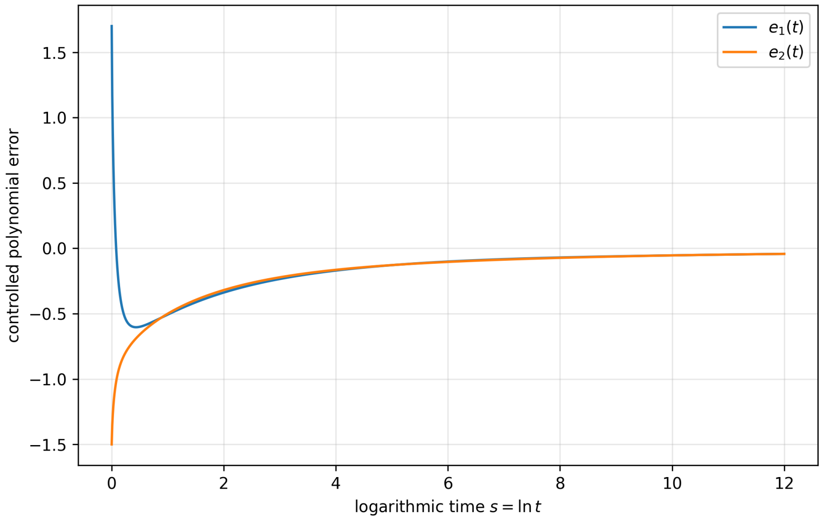

Figure 7. Controlled synchronization-error components in the polynomial-matrix SOS example.

6. Conclusions

In this paper, suitable and effective asymptotic synchronization conditions have been formulated for an uncertain nonlinear Caputo–Hadamard fractional-order T–S fuzzy Lur'e-type system with time-varying delay. In the terminology of Definition 3.2, these conditions certify robust complete synchronization: the error decays to zero for every admissible norm-bounded uncertainty, which is the precise sense in which the synchronization established here is robust. The proposed framework took matched/common premise variables of the master and slave systems and used a static output-error parallel distributed compensation controller, which utilized current and past output-error information. Delay-free and delayed synchronization conditions were obtained in the form of tractable rule-pair LMIs by using a common quadratic Lyapunov function, a valid Caputo–Hadamard quadratic derivative inequality, incremental sectors constraints, norm-bounded uncertainty estimates, and Schur-complement arguments. The delayed criterion was derived by a Caputo–Hadamard Halanay inequality, thus it could be applied to admissible time-varying delays that could be unbounded, but the effective logarithmic-time memory window had to grow as time went to infinity. The feasibility and effectiveness of the proposed LMI conditions were verified by the numerical example with a constant matrix. The LMIs were, in particular, solved as a max-margin semidefinite program via CVXPY, and the resulting controller was validated for a real partial-output situation where the number of measured outputs is less than the state dimension. The simulations revealed the divergence of the uncontrolled synchronization error, convergence of the controlled synchronization error, and robustness of the proposed controller by a 50-profile Monte Carlo uncertainty test. A comparison with a fuzzy-free design confirmed that the proposed controller attains the same synchronization as the best non-fuzzy baseline while additionally guaranteeing a certified robustness margin, and a fractional-order sensitivity study showed that a larger order α yields a faster decay of the synchronization error. Furthermore, the polynomial-matrix example demonstrated the global SOS extension by providing an explicit polynomial-matrix SOS Gram certificate and by simulating the convergence of the controlled polynomial synchronization error to zero and the uncontrolled polynomial error to a large error, thereby showing the potential of the global SOS extension. Conservatism is the major cost of the proposed tractable formulation. The conservatism is primarily because of using a common Lyapunov matrix, matched/common premise variables, norm-bounded uncertainty estimates, and quadratic bounding of sector nonlinearities. Future research could then focus on asynchronous premise variables, membership-dependent/fuzzy Lyapunov functions, output-error PDC synchronization with event-triggered, relaxed fuzzy LMI/SOS formulations, and less conservative polynomial Lyapunov designs of Caputo–Hadamard fractional-order systems based on carefully justified fractional composition or Lyapunov-derivative inequalities.

Use of Generative-AI tools declaration

The author declares that no artificial intelligence (AI) tools were used in the creation of this article.

Conflict of interest

The author declares that there is no conflict of interest.

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