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Research article

Three operators' product norm and numerical radius inequalities in terms of the Moore–Penrose inverse

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Abstract: In this paper, we developed new norm and numerical radius inequalities for products of three bounded linear operators on a complex Hilbert space by using the Moore–Penrose inverse associated with closed-range operators. Starting from Schwarz-type vector inequalities, we derived a family of estimates that connect the norm and numerical radius of a product with positive operator expressions involving the factors and their adjoints. The results were then specialized to closed-range operators and, in particular, to inequalities involving a single operator and its Moore–Penrose inverse. Several consequences were obtained for powers of the numerical radius, including refinements and complementary bounds related to recent estimates in the literature. We also discussed the case of invertible operators, where the Moore–Penrose inverse reduces to the usual inverse, and obtained corresponding bounds that can be expressed using weighted combinations of positive operators. A numerical example for a two-by-two invertible matrix was included to illustrate the validity of one of the derived inequalities. The obtained results provide new tools for estimating the numerical radius and the operator norm in settings where Moore–Penrose inverse techniques and closed-range assumptions naturally arise. They further indicate how inequalities for products of operators can be transferred into practical estimates for individual operators through suitable substitutions and operator decompositions.

Keywords: bounded linear operators; Hilbert spaces; numerical radius; operator norm; Moore–Penrose inverse; closed-range operators
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1. Introduction

The numerical radius $w(B)$ of an operator B on the complex Hilbert space H is given by

$$w(B) = \sup\{|\langle Bx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

Obviously, by (1.1), for any $x \in H$, one has

$$|\langle Bx, x\rangle| \le w(B)\|x\|^2. \tag{1.2}$$

It is well-known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $A : H \to H$, i.e.,

(i) $w(A) \ge 0$ for any $A \in B(H)$ and $w(A) = 0$ if and only if $A = 0$;
(ii) $w(\lambda A) = |\lambda| w(A)$ for any $\lambda \in \mathbb{C}$ and $A \in B(H)$;
(iii) $w(A + B) \le w(A) + w(B)$ for any $A, B \in B(H)$.

This norm is equivalent to the operator norm. In fact, the following more- precise result holds:

$$w(A) \le \|A\| \le 2w(A) \tag{1.3}$$

for any $A \in B(H)$.

F. Kittaneh, in 2003 [1], showed that for any operator $A \in B(H)$, we have the following refinement of the first inequality in (1.3):

$$w(A) \le \frac{1}{2}\left(\|A\| + \left\|A^2\right\|^{1/2}\right). \tag{1.4}$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [2] improved the inequality (1.3) as follows:

$$\frac{1}{4}\|A^*A + AA^*\| \le w^2(A) \le \frac{1}{2}\|A^*A + AA^*\| \tag{1.5}$$

for any operator $A \in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad and Kittaneh in 2007 [3]:

If, for an operator $A \in B(H)$, we denote $|A| := (A^*A)^{1/2}$, then

$$w^r(A) \le \frac{1}{2}\left\||A|^{2\alpha r} + |A^*|^{2(1-\alpha)r}\right\| \tag{1.6}$$

and

$$w^{2r}(A) \le \left\|\alpha|A|^{2r} + (1-\alpha)|A^*|^{2r}\right\|, \tag{1.7}$$

where $\alpha \in (0, 1)$ and $r \ge 1$.

If we take $\alpha = \frac{1}{2}$ and $r = 1$, we get from (1.6) that

$$w(A) \le \frac{1}{2}\||A| + |A^*|\| \tag{1.8}$$

and from (1.7) that

$$w^2(A) \le \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|. \tag{1.9}$$

For more related results, see the recent books on inequalities for numerical radii [4, 5].

A fundamental concept in operator theory is the Moore–Penrose inverse, which generalizes the notion of operator inverses to more general settings. Further details regarding this concept and its main properties are provided in the Preliminaries section below.

In [6], Sababheh et al. established some inner-product inequalities for Hilbert space operators having closed ranges. The obtained results are applied to provide new bounds for the numerical radius and the operator norm, where the Moore–Penrose inverse plays a key role. The authors obtained the following remarkable upper bounds for the numerical radius of operators V from $CR(H)$, the class of closed-range operators on H :

$$w(V) \leq \frac{1}{2}\left\||V|^2 + VV^\dagger\right\|, \tag{1.10}$$

$$w(V) \leq \frac{1}{4}\left\||V^*|^2 + VV^\dagger\right\| + \frac{1}{2}w\left(V^2V^\dagger\right),$$

$$w(V) \leq \frac{1}{4}\left\||V|^2 + V^\dagger V\right\| + \frac{1}{2}w\left(V^\dagger V^2\right),$$

and

$$w(V) \leq \frac{1}{4}\left\|VV^\dagger + |V|\right\|\left\|VV^\dagger + |V^*|\right\|.$$

In [7], Bhunia et al. established several numerical radius bounds for bounded linear operators with closed ranges. The paper develops inner- product inequalities to obtain bounds for the sum of two operators, refining existing classical results. The following interesting results were obtained as well:

$$w^2(V) \leq \left\|\frac{|V|^2 + |V^*|^2}{2}\right\|\left\|\frac{VV^\dagger + V^\dagger V}{2}\right\| \leq \left\|\frac{|V|^2 + |V^*|^2}{2}\right\|$$

and

$$w^2(V) \leq \frac{1}{6}\left\||V|^4 + VV^\dagger\right\| + \frac{1}{3}w(V)\left\||V^*|^2 + VV^\dagger\right\|.$$

In [7], the following power inequality for $r \geq 1$ was also obtained:

$$w^r(V) \leq \frac{1}{2}\min\left\{\left\||V|^{2r} + VV^\dagger\right\|, \left\||V^*|^{2r} + V^\dagger V\right\|\right\}.$$

In particular, for $r = 1$, we get

$$w(V) \leq \frac{1}{2}\min\left\{\left\||V|^2 + VV^\dagger\right\|, \left\||V^*|^2 + V^\dagger V\right\|\right\}.$$

In [8], Ighachane et al. established several inner-product inequalities that characterize the positivity of block operator matrices. By examining specific blocks of positive operator matrices involving the Moore–Penrose inverse, they derived new inner-product inequalities that refine recently established bounds. As an application, they further refined certain numerical radius inequalities from the literature. One such result is that

$$w^{2r}(V) \leq \frac{\lambda}{2}\left\||V|^{4r} + VV^\dagger\right\| + \frac{1-\lambda}{2}w^r(V)\left\||V|^{2r} + VV^\dagger\right\| \tag{1.11}$$

for $r \geq 1$ and $\lambda \in [1/2, 1]$.

We observe that, if we replace V with $V^\dagger$ in all the other inequalities from the list above, we can get various bounds for $w\left(V^\dagger\right)$. For instance, if $V \in CR(H)$, then

$$w\left(V^\dagger\right) \leq \frac{1}{2}\left\|\left|V^\dagger\right|^2 + V^\dagger V\right\|,$$

$$w\left(V^\dagger\right) \leq \frac{1}{4}\left\|\left|\left(V^\dagger\right)^*\right|^2 + V^\dagger V\right\| + \frac{1}{2}w\left(\left(V^\dagger\right)^2 V\right),$$

$$w\left(V^\dagger\right) \leq \frac{1}{4}\left\|\left|V^\dagger\right|^2 + VV^\dagger\right\| + \frac{1}{2}w\left(V\left(V^\dagger\right)^2\right),$$

and

$$w\left(V^\dagger\right) \leq \frac{1}{4}\left\|V^\dagger V + \left|V^\dagger\right|\right\|\left\|V^\dagger V + \left|\left(V^\dagger\right)^*\right|\right\|.$$

Motivated by the above results, in this paper, we obtained some new norm and numerical radius inequalities for the product of three operators in terms of the Moore–Penrose inverse. Our results are more general since they provide several upper bounds for the norm and numerical radius of a product of three operators in the form $\|BVA\|$ and $w(BVA)$, with $V \in CR(H)$ and A, B being bounded linear operators on H. Using some fundamental properties of the Moore–Penrose inverse, such as $VV^\dagger V = V$ and $V^\dagger VV^\dagger = V^\dagger$, we are then able to obtain new and innovative upper bounds for the norms and numerical radius of $V \in CR(H)$, which essentially motivates the extension to the product of three operators. The case when V is an invertible operator is also of interest, since, in this situation, $V^\dagger = V^{-1}$ and some of the obtained bounds become simpler.

Among the main results, for $V \in CR(H)$ and A, B being bounded linear operators on H, we obtained the inequalities

$$\|BVA\| \leq \||V|A\|\left\|BVV^\dagger\right\|$$

and

$$w(BVA) \leq \left\|(1-\lambda)\||V|A|^2 + \lambda\left|VV^\dagger B^*\right|^2\right\|^{1/2} \times \left\|\lambda\||V|A|^2 + (1-\lambda)\left|VV^\dagger B^*\right|^2\right\|^{1/2},$$

for all $\lambda \in [0, 1]$.

Several upper bounds are also provided for powers of the numerical radius of operators $V \in CR(H)$ that complement the results stated above. We showed, for instance, that

$$w^{2p}(V) \leq \frac{1}{2}\left[\left\|\left|V^\dagger\right|V\right\|^{2p}\|V\|^{2p} + w^p\left(V(V^*)^2\left|V^\dagger\right|^2 V\right)\right]$$

and

$$w^{2p}(V) \leq \frac{1}{2}\left\|\left|\left|V^\dagger\right|V\right|^{4p} + |V^*|^{4p}\right\|$$

for $p \geq 1$.

2. Preliminaries

For an operator $V \in B(H)$, the Moore–Penrose inverse of V , denoted by $V^\dagger$, is the unique operator satisfying the following four Penrose equations [9]:

$$VV^\dagger V = V,\ V^\dagger VV^\dagger = V^\dagger,\ \left(VV^\dagger\right)^* = VV^\dagger,\ \left(V^\dagger V\right)^* = V^\dagger V. \tag{2.1}$$

These conditions ensure the existence and uniqueness of $V^\dagger$ whenever V has a closed range, i.e., range(V) is closed in H.

We define the set of all operators in $B(H)$ that admit a Moore–Penrose inverse as follows:

$$CR(H) = \{V \in B(H) |\ \text{range}(V) \text{ is closed}\}.$$

For any $V \in CR(H)$, its Moore–Penrose inverse $V^\dagger$ is also a bounded operator in $B(H)$.

For an operator $V \in CR(H)$, the Moore–Penrose inverse satisfies the following important properties:

- Self-adjoint property: If V is self-adjoint, i.e., $V = V^*$, then $V^\dagger$ is also self-adjoint and $\left(V^\dagger\right)^* = V^\dagger$;
- Involutivity: The Moore–Penrose inverse satisfies the equality $\left(V^\dagger\right)^\dagger = V$;
- Order preservation: If $V_1, V_2 \in CR(H)$ with $V_1 \leq V_2$ in the operator order, then $V_2^\dagger \leq V_2^\dagger$;
- Product formula: If $V_1, V_2 \in CR(H)$ with range(V_1) $\subseteq$ range(V_2), then $V_2V_1 \in CR(H)$ and $(V_2V_1)^\dagger = V_1^\dagger V_2^\dagger$.

Some other important properties that connect the Moore–Penrose inverse with the adjoint operator are as follows:

$$V = \left(VV^\dagger\right)^* V = \left(V^\dagger\right)^* |V|^2$$

and

$$V = V\left(V^\dagger V\right)^* = |V^*|^2 \left(V^\dagger\right)^*$$

for any $V \in CR(H)$.

In [6], the authors obtained the following Schwarz-type inequality involving the Moore–Penrose inverse:

Lemma 2.1. *Let $V \in CR(H)$. Then for all $u, v \in H$,*

$$|\langle Vu, v\rangle|^2 \leq \left\langle |V|^2 u, u\right\rangle \left\langle VV^\dagger v, v\right\rangle. \tag{2.2}$$

If we replace V with $V^\dagger$ in (2.2) and take into account that $\left(V^\dagger\right)^\dagger = V$, then we get

$$\left|\left\langle V^\dagger u, v\right\rangle\right|^2 \leq \left\langle \left|V^\dagger\right|^2 u, u\right\rangle \left\langle V^\dagger Vv, v\right\rangle, \tag{2.3}$$

for all $u, v \in H$.

In the next section, we provide several extensions of the Schwarz-type inequality involving the Moore–Penrose inverse that complement and generalize the above inequality (2.3).

3. Vector inequalities

We have the following Schwarz-type vector inequality for the product of three operators:

Theorem 3.1. *Let $V \in CR(H)$ and $A, B \in B(H)$. Then for all $\lambda \in [0,1]$ and $x, y \in H$, we have that*

$$\begin{aligned} |\langle BVAx, y\rangle|^2 &\le \left\langle \| V| A|^2 x, x\right\rangle \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle \le \min\{\Gamma, \Delta\} \\ &\le \left((1-\lambda)\left\langle \| V| A|^2 x, x\right\rangle + \lambda\left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle\right) \\ &\quad \times \left(\lambda\left\langle \| V| A|^2 x, x\right\rangle + (1-\lambda)\left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle\right) \\ &\le \left(\frac{\left\langle \| V| A|^2 x, x\right\rangle + \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle}{2}\right)^2, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \Gamma := &\left((1-\lambda)\left\langle \| V| A|^2 x, x\right\rangle + \lambda\left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle\right) \\ &\times \left\langle \| V| A|^2 x, x\right\rangle^{\lambda} \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle^{1-\lambda} \end{aligned}$$

and

$$\begin{aligned} \Delta := &\left(\lambda\left\langle \| V| A|^2 x, x\right\rangle + (1-\lambda)\left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle\right) \\ &\times \left\langle \| V| A|^2 x, x\right\rangle^{1-\lambda} \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle^{\lambda}. \end{aligned}$$

In particular, we have for all $x \in H$ that

$$\begin{aligned} |\langle BVAx, x\rangle|^2 &\le \left\langle \| V| A|^2 x, x\right\rangle \left\langle \left|VV^\dagger B^*\right|^2 x, x\right\rangle \le \min\{\Theta, \Lambda\} \\ &\le \left\langle \left[(1-\lambda)\| V| A|^2 + \lambda\left|VV^\dagger B^*\right|^2\right] x, x\right\rangle \\ &\quad \times \left\langle \left[\lambda\| V| A|^2 + (1-\lambda)\left|VV^\dagger B^*\right|^2\right] x, x\right\rangle \\ &\le \left\langle \left(\frac{\| V| A|^2 + \left|VV^\dagger B^*\right|^2}{2}\right) x, x\right\rangle^2, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} \Theta := &\left\langle \left[(1-\lambda)\| V| A|^2 + \lambda\left|VV^\dagger B^*\right|^2\right] x, x\right\rangle \\ &\times \left\langle \| V| A|^2 x, x\right\rangle^{\lambda} \left\langle \left|VV^\dagger B^*\right|^2 x, x\right\rangle^{1-\lambda} \end{aligned}$$

and

$$\begin{aligned} \Lambda := &\left\langle \left(\lambda\| V| A|^2 + (1-\lambda)\left|VV^\dagger B^*\right|^2\right) x, x\right\rangle \\ &\times \left\langle \| V| A|^2 x, x\right\rangle^{1-\lambda} \left\langle \left|VV^\dagger B^*\right|^2 x, x\right\rangle^{\lambda}. \end{aligned}$$

Proof. Let $x, y \in H$. Now, if we take $u = Ax$ and $v = B^*y$ in (2.2), then we get

$$|\langle VAx, B^*y\rangle|^2 \le \left\langle |V|^2 Ax, Ax\right\rangle \left\langle VV^\dagger B^*y, B^*y\right\rangle,$$

namely,

$$|\langle BVAx, y\rangle|^2 \le \left\langle A^* |V|^2 Ax, x\right\rangle \left\langle BVV^\dagger B^*y, y\right\rangle, \tag{3.3}$$

and since $A^* |V|^2 A = ||V| A|^2$ and

$$\left|VV^\dagger B^*\right|^2 = \left(VV^\dagger B^*\right)^* \left(VV^\dagger B^*\right) = BVV^\dagger VV^\dagger B^* = BVV^\dagger B^*,$$

then by (3.3), we get

$$|\langle BVAx, y\rangle|^2 \le \left\langle ||V| A|^2 x, x\right\rangle \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle, \tag{3.4}$$

for all $x, y \in H$.

Observe that, for all $\lambda \in [0, 1]$, we have that

$$\begin{aligned}
&\left\langle ||V| A|^2 x, x\right\rangle \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle \\
&= \left\langle ||V| A|^2 x, x\right\rangle^{1-\lambda} \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle^{\lambda} \left\langle ||V| A|^2 x, x\right\rangle^{\lambda} \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle^{1-\lambda}
\end{aligned}$$

and by the weighted arithmetic–geometric mean inequality, we have that

$$\left\langle ||V| A|^2 x, x\right\rangle^{1-\lambda} \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle^{\lambda} \le (1-\lambda) \left\langle ||V| A|^2 x, x\right\rangle + \lambda \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle$$

and

$$\left\langle ||V| A|^2 x, x\right\rangle^{\lambda} \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle^{1-\lambda} \le \lambda \left\langle ||V| A|^2 x, x\right\rangle + (1-\lambda) \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle$$

for all $x, y \in H$.

These give that

$$\begin{aligned}
&\left\langle ||V| A|^2 x, x\right\rangle \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle \\
&\le \min\left\{\left((1-\lambda) \left\langle ||V| A|^2 x, x\right\rangle + \lambda \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle\right)\right. \\
&\quad \times \left\langle ||V| A|^2 x, x\right\rangle^{\lambda} \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle^{1-\lambda}, \\
&\quad \left(\lambda \left\langle ||V| A|^2 x, x\right\rangle + (1-\lambda) \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle\right) \\
&\quad \left.\times \left\langle ||V| A|^2 x, x\right\rangle^{1-\lambda} \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle^{\lambda}\right\} \\
&\le \left((1-\lambda) \left\langle ||V| A|^2 x, x\right\rangle + \lambda \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle\right) \\
&\quad \times \left(\lambda \left\langle ||V| A|^2 x, x\right\rangle + (1-\lambda) \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle\right)
\end{aligned}$$

for all $x, y \in H$.

Also, by the elementary inequality $ab \le \left(\frac{a+b}{2}\right)^2$, $a, b \ge 0$, we have that

$$\begin{aligned}&\left((1-\lambda)\left\langle \||V|A|^2 x, x\right\rangle + \lambda\left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle\right)\\ &\times\left(\lambda\left\langle \||V|A|^2 x, x\right\rangle + (1-\lambda)\left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle\right)\\ \le&\left(\frac{\left\langle \||V|A|^2 x, x\right\rangle + \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle}{2}\right)^2\end{aligned}$$

for all $x, y \in H$. This completes the proof of (3.1). □

Corollary 3.1. *With the assumptions in Theorem 3.1, we have that*

$$\begin{aligned}\left|\left\langle BV^\dagger Ax, y\right\rangle\right|^2 &\le \left\langle \left|\left|V^\dagger\right|A\right|^2 x, x\right\rangle \left\langle \left|V^\dagger VB^*\right|^2 y, y\right\rangle \le \min\{\alpha, \beta\} \\ &\le \left((1-\lambda)\left\langle \left|\left|V^\dagger\right|A\right|^2 x, x\right\rangle + \lambda\left\langle \left|V^\dagger VB^*\right|^2 y, y\right\rangle\right)\\ &\quad\times\left(\lambda\left\langle \left|\left|V^\dagger\right|A\right|^2 x, x\right\rangle + (1-\lambda)\left\langle \left|V^\dagger VB^*\right|^2 y, y\right\rangle\right)\\ &\le\left(\frac{\left\langle \left|\left|V^\dagger\right|A\right|^2 x, x\right\rangle + \left\langle \left|V^\dagger VB^*\right|^2 y, y\right\rangle}{2}\right)^2,\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}\alpha :=&\left((1-\lambda)\left\langle \left|\left|V^\dagger\right|A\right|^2 x, x\right\rangle + \lambda\left\langle \left|V^\dagger VB^*\right|^2 y, y\right\rangle\right)\\ &\times\left\langle \left|\left|V^\dagger\right|A\right|^2 x, x\right\rangle^{\lambda}\left\langle \left|V^\dagger VB^*\right|^2 y, y\right\rangle^{1-\lambda}\end{aligned}$$

and

$$\begin{aligned}\beta :=&\left(\lambda\left\langle \left|\left|V^\dagger\right|A\right|^2 x, x\right\rangle + (1-\lambda)\left\langle \left|V^\dagger VB^*\right|^2 y, y\right\rangle\right)\\ &\times\left\langle \left|\left|V^\dagger\right|A\right|^2 x, x\right\rangle^{1-\lambda}\left\langle \left|V^\dagger VB^*\right|^2 y, y\right\rangle^{\lambda}.\end{aligned}$$

In particular, we have for all $x \in H$ that

$$\begin{aligned}\left|\left\langle BV^\dagger Ax, x\right\rangle\right|^2 &\le \left\langle \left|\left|V^\dagger\right|A\right|^2 x, x\right\rangle \left\langle \left|V^\dagger VB^*\right|^2 x, x\right\rangle \le \min\{\gamma, \delta\} \\ &\le \left\langle\left[(1-\lambda)\left|\left|V^\dagger\right|A\right|^2 + \lambda\left|V^\dagger VB^*\right|^2\right]x, x\right\rangle\\ &\quad\times\left\langle\left[\lambda\left|\left|V^\dagger\right|A\right|^2 + (1-\lambda)\left|V^\dagger VB^*\right|^2\right]x, x\right\rangle\\ &\le\left\langle\left(\frac{\left|\left|V^\dagger\right|A\right|^2 + \left|V^\dagger VB^*\right|^2}{2}\right)x, x\right\rangle^2,\end{aligned} \tag{3.6}$$

where

$$\gamma := \left\langle \left[(1-\lambda) \left| \left| V^{\dagger} \right| A \right|^{2} + \lambda \left| V^{\dagger} V B^{*} \right|^{2} \right] x, x \right\rangle \times \left\langle \left| \left| V^{\dagger} \right| A \right|^{2} x, x \right\rangle^{\lambda} \left\langle \left| V^{\dagger} V B^{*} \right| x, x \right\rangle^{1-\lambda}$$

and

$$\delta := \left\langle \left(\lambda \left| \left| V^{\dagger} \right| A \right|^{2} + (1-\lambda) \left| V^{\dagger} V B^{*} \right|^{2} \right) x, x \right\rangle \times \left\langle \left| \left| V^{\dagger} \right| A \right|^{2} x, x \right\rangle^{1-\lambda} \left\langle \left| V^{\dagger} V B^{*} \right|^{2} x, x \right\rangle^{\lambda}.$$

The proof follows from Theorem 3.1 by replacing V with $V^{\dagger}$ and taking into account that $\left(V^{\dagger}\right)^{\dagger} = V$.

Remark 3.1. If we take $A = B = V^{\dagger}$ in Theorem 3.1 and use the properties of the Moore–Penrose inverse, then we get for all $x, y \in H$ that

$$\begin{aligned}
\left| \left\langle V^{\dagger} x, y \right\rangle \right|^{2} &\leq \left\langle \left| |V| \, V^{\dagger} \right|^{2} x, x \right\rangle \left\langle \left| V \left| \left(V^{\dagger}\right)^{*} \right|^{2} \right|^{2} y, y \right\rangle \leq \min \{\epsilon, \zeta\} \\
&\leq \left((1-\lambda) \left\langle \left| |V| \, V^{\dagger} \right|^{2} x, x \right\rangle + \lambda \left\langle \left| V \left| \left(V^{\dagger}\right)^{*} \right|^{2} \right|^{2} y, y \right\rangle \right) \\
&\quad \times \left(\lambda \left\langle \left| |V| \, V^{\dagger} \right|^{2} x, x \right\rangle + (1-\lambda) \left\langle \left| V V \left| \left(V^{\dagger}\right)^{*} \right|^{2} \right|^{2} y, y \right\rangle \right) \\
&\leq \left(\frac{\left\langle \left| |V| \, V^{\dagger} \right|^{2} x, x \right\rangle + \left\langle \left| V \left| \left(V^{\dagger}\right)^{*} \right|^{2} \right|^{2} y, y \right\rangle}{2} \right)^{2},
\end{aligned}$$

where

$$\epsilon := \left((1-\lambda) \left\langle \left| |V| \, V^{\dagger} \right|^{2} x, x \right\rangle + \lambda \left\langle \left| V \left| \left(V^{\dagger}\right)^{*} \right|^{2} \right|^{2} y, y \right\rangle \right) \times \left\langle \left| |V| \, V^{\dagger} \right|^{2} x, x \right\rangle^{\lambda} \left\langle \left| V \left| \left(V^{\dagger}\right)^{*} \right|^{2} \right|^{2} y, y \right\rangle^{1-\lambda}$$

and

$$\zeta := \left(\lambda \left\langle \left| |V| \, V^{\dagger} \right|^{2} x, x \right\rangle + (1-\lambda) \left\langle \left| V \left| \left(V^{\dagger}\right)^{*} \right|^{2} \right|^{2} y, y \right\rangle \right) \times \left\langle \left| |V| \, V^{\dagger} \right|^{2} x, x \right\rangle^{1-\lambda} \left\langle \left| V \left| \left(V^{\dagger}\right)^{*} \right|^{2} \right|^{2} y, y \right\rangle^{\lambda}.$$

In particular, we have for all $x \in H$ that

$$\left| \left\langle V^{\dagger} x, x \right\rangle \right|^{2} \leq \left\langle \left| |V| \, V^{\dagger} \right|^{2} x, x \right\rangle \left\langle \left| V \left| \left(V^{\dagger}\right)^{*} \right|^{2} \right|^{2} x, x \right\rangle \leq \min \{\eta, \theta\}$$

$$\leq \left\langle \left[(1-\lambda) \left| |V| V^{\dagger} \right|^{2} + \lambda \left| V \left| \left(V^{\dagger}\right)^{*} \right|^{2} \right|^{2} \right] x, x \right\rangle$$
$$\times \left\langle \left[\lambda \left| |V| V^{\dagger} \right|^{2} + (1-\lambda) \left| V \left| \left(V^{\dagger}\right)^{*} \right|^{2} \right|^{2} \right] x, x \right\rangle$$
$$\leq \left\langle \left(\frac{\left| |V| V^{\dagger} \right|^{2} + \left| V \left| \left(V^{\dagger}\right)^{*} \right|^{2} \right|^{2}}{2} \right) x, x \right\rangle^{2},$$

where

$$\eta := \left\langle \left[(1-\lambda) \left| |V| V^{\dagger} \right|^{2} + \lambda \left| V \left| \left(V^{\dagger}\right)^{*} \right|^{2} \right|^{2} \right] x, x \right\rangle$$
$$\times \left\langle \left| |V| V^{\dagger} \right|^{2} x, x \right\rangle^{\lambda} \left\langle \left| V \left| \left(V^{\dagger}\right)^{*} \right|^{2} \right|^{2} x, x \right\rangle^{1-\lambda}$$

and

$$\theta := \left\langle \left(\lambda \left| |V| V^{\dagger} \right|^{2} + (1-\lambda) \left| V \left| \left(V^{\dagger}\right)^{*} \right|^{2} \right|^{2} \right) x, x \right\rangle$$
$$\times \left\langle \left| |V| V^{\dagger} \right|^{2} x, x \right\rangle^{1-\lambda} \left\langle \left| V \left| \left(V^{\dagger}\right)^{*} \right|^{2} \right|^{2} x, x \right\rangle^{\lambda}.$$

From Corollary 3.1, we get for $A = B = V$ that

$$|\langle Vx, y\rangle|^{2} \leq \left\langle \left| \left| V^{\dagger} \right| V \right|^{2} x, x \right\rangle \left\langle \left| V^{\dagger} \left| V^{*} \right|^{2} \right|^{2} y, y \right\rangle$$
$$\leq \min \left\{ \left((1-\lambda) \left\langle \left| \left| V^{\dagger} \right| V \right|^{2} x, x \right\rangle + \lambda \left\langle \left| V^{\dagger} \left| V^{*} \right|^{2} \right|^{2} y, y \right\rangle \right) \right.$$
$$\times \left\langle \left| \left| V^{\dagger} \right| V \right|^{2} x, x \right\rangle^{\lambda} \left\langle \left| V^{\dagger} \left| V^{*} \right|^{2} \right|^{2} y, y \right\rangle^{1-\lambda},$$
$$\left(\lambda \left\langle \left| \left| V^{\dagger} \right| V \right|^{2} x, x \right\rangle + (1-\lambda) \left\langle \left| V^{\dagger} \left| V^{*} \right|^{2} \right|^{2} y, y \right\rangle \right)$$
$$\left. \times \left\langle \left| \left| V^{\dagger} \right| V \right|^{2} x, x \right\rangle^{1-\lambda} \left\langle \left| V^{\dagger} \left| V^{*} \right|^{2} \right|^{2} y, y \right\rangle^{\lambda} \right\}$$
$$\leq \left((1-\lambda) \left\langle \left| \left| V^{\dagger} \right| V \right|^{2} x, x \right\rangle + \lambda \left\langle \left| V^{\dagger} \left| V^{*} \right|^{2} \right|^{2} y, y \right\rangle \right)$$
$$\times \left(\lambda \left\langle \left| \left| V^{\dagger} \right| V \right|^{2} x, x \right\rangle + (1-\lambda) \left\langle \left| V^{\dagger} \left| V^{*} \right|^{2} \right|^{2} y, y \right\rangle \right)$$
$$\leq \left(\frac{\left\langle \left| \left| V^{\dagger} \right| V \right|^{2} x, x \right\rangle + \left\langle \left| V^{\dagger} \left| V^{*} \right|^{2} \right|^{2} y, y \right\rangle}{2} \right)^{2}.$$

In particular, we have for all $x \in H$ that

$$|\langle Vx, x\rangle|^{2} \leq \left\langle \left| \left| V^{\dagger} \right| V \right|^{2} x, x \right\rangle \left\langle \left| V^{\dagger} \left| V^{*} \right|^{2} \right|^{2} x, x \right\rangle \tag{3.7}$$

$$
\begin{aligned}
&\le \min\left\{\left\langle\left[(1-\lambda)\left|\left|V^{\dagger}\right|V\right|^{2}+\lambda\left|V^{\dagger}\left|V^{*}\right|^{2}\right|^{2}\right]x,x\right\rangle\right.\\
&\quad\times\left\langle\left|\left|V^{\dagger}\right|V\right|^{2}x,x\right\rangle^{\lambda}\left\langle\left|V^{\dagger}\left|V^{*}\right|^{2}\right|x,x\right\rangle^{1-\lambda},\\
&\quad\left\langle\left(\lambda\left|\left|V^{\dagger}\right|V\right|^{2}+(1-\lambda)\left|V^{\dagger}\left|V^{*}\right|^{2}\right|^{2}\right)x,x\right\rangle\\
&\quad\left.\times\left\langle\left|\left|V^{\dagger}\right|V\right|^{2}x,x\right\rangle^{1-\lambda}\left\langle\left|V^{\dagger}\left|V^{*}\right|^{2}\right|^{2}x,x\right\rangle^{\lambda}\right\}\\
&\le\left\langle\left[(1-\lambda)\left|\left|V^{\dagger}\right|V\right|^{2}+\lambda\left|V^{\dagger}\left|V^{*}\right|^{2}\right|^{2}\right]x,x\right\rangle\\
&\quad\times\left\langle\left[\lambda\left|\left|V^{\dagger}\right|V\right|^{2}+(1-\lambda)\left|V^{\dagger}\left|V^{*}\right|^{2}\right|^{2}\right]x,x\right\rangle\\
&\le\left\langle\left(\frac{\left|\left|V^{\dagger}\right|V\right|^{2}+\left|V^{\dagger}\left|V^{*}\right|^{2}\right|^{2}}{2}\right)x,x\right\rangle^{2}.
\end{aligned}
$$

For $T\in CR(H)$, we define the following operators:

$$
S_{\dagger}T:=\frac{T+T^{\dagger}}{2}\text{ and }D_{\dagger}T:=\frac{T-T^{\dagger}}{2i}.
$$

We observe that

$$
S_{\dagger}T+i\,D_{\dagger}T=T\text{ and }S_{\dagger}T-i\,D_{\dagger}T=T^{\dagger}, \tag{3.8}
$$

$$
(S_{\dagger}T)^{*}=\left(\frac{T+T^{\dagger}}{2}\right)^{*}=\frac{T^{*}+(T^{*})^{\dagger}}{2}=S_{\dagger}T^{*}
$$

and

$$
(D_{\dagger}T)^{*}:=\left(\frac{T-T^{\dagger}}{2i}\right)^{*}=-\frac{T^{*}-(T^{*})^{\dagger}}{2i}=-D_{\dagger}T^{*}.
$$

Also

$$
S_{\dagger}T^{\dagger}:=\frac{T^{\dagger}+\left(T^{\dagger}\right)^{\dagger}}{2}=\frac{T^{\dagger}+T}{2}=S_{\dagger}T
$$

and

$$
D_{\dagger}T^{\dagger}:=\frac{T^{\dagger}-\left(T^{\dagger}\right)^{\dagger}}{2i}=\frac{T^{\dagger}-T}{2i}=-D_{\dagger}T.
$$

Moreover, we have that

$$
\begin{aligned}
\left|S_{\dagger}T\right|^{2}&:=\left(\frac{T+T^{\dagger}}{2}\right)^{*}\left(\frac{T+T^{\dagger}}{2}\right)=\left(\frac{T^{*}+\left(T^{\dagger}\right)^{*}}{2}\right)\left(\frac{T+T^{\dagger}}{2}\right)\\
&=\frac{1}{4}\left(|T|^{2}+\left|T^{\dagger}\right|^{2}+T^{*}T^{\dagger}+\left(T^{\dagger}\right)^{*}T\right)
\end{aligned}
$$

and

$$
\left|D_{\dagger}T^{\dagger}\right|^{2}=\left(\frac{T-T^{\dagger}}{2i}\right)^{*}\left(\frac{T-T^{\dagger}}{2i}\right)=-\frac{1}{2i}\frac{1}{2i}\left(T-T^{\dagger}\right)^{*}\left(T-T^{\dagger}\right)
$$

$$= \frac{1}{4}\left(T^* - \left(T^\dagger\right)^*\right)\left(T - T^\dagger\right) = \frac{1}{4}\left(|T|^2 + \left|T^\dagger\right|^2 - T^*T^\dagger - \left(T^\dagger\right)^* T\right),$$

which gives

$$\left|S_\dagger T\right|^2 + \left|D_\dagger T^\dagger\right|^2 = \frac{1}{2}\left(|T|^2 + \left|T^\dagger\right|^2\right).$$

Proposition 3.1. *Let $V \in CR(H)$ and $A, B \in B(H)$. Then for all $x, y \in H$, we have that*

$$\begin{aligned} &\max\left\{\left|\left\langle B\left(S_\dagger V\right) Ax, y\right\rangle\right|, \left|\left\langle B\left(D_\dagger V\right) Ax, y\right\rangle\right|\right\} \qquad (3.9)\\ &\leq \frac{1}{2}\left(|\langle BVAx, y\rangle| + \left|\left\langle BV^\dagger Ax, y\right\rangle\right|\right)\\ &\leq \frac{1}{2}\left\langle\left(\| V| A|^2 + \left|\left|V^\dagger\right| A\right|^2\right)x, x\right\rangle^{1/2}\left\langle\left(\left|VV^\dagger B^*\right|^2 + \left|V^\dagger VB^*\right|^2\right)y, y\right\rangle^{1/2}. \end{aligned}$$

Proof. By taking the square root in (3.1) and (3.5) and then adding, we get

$$\begin{aligned} |\langle BVAx, y\rangle| + \left|\left\langle BV^\dagger Ax, y\right\rangle\right| \leq &\left\langle \| V| A|^2 x, x\right\rangle^{1/2}\left\langle\left|VV^\dagger B^*\right|^2 y, y\right\rangle^{1/2} \qquad (3.10)\\ &+ \left\langle\left|\left|V^\dagger\right| A\right|^2 x, x\right\rangle^{1/2}\left\langle\left|V^\dagger VB^*\right|^2 y, y\right\rangle^{1/2} \end{aligned}$$

for all $x, y \in H$.

By the elementary CBS inequality

$$0 \leq ac + bd \leq \left(a^2 + b^2\right)^{1/2}\left(c^2 + c^2\right)^{1/2},\ a, b, c, d \geq 0,$$

we have that

$$\begin{aligned} &\left\langle \| V| A|^2 x, x\right\rangle^{1/2}\left\langle\left|VV^\dagger B^*\right|^2 y, y\right\rangle^{1/2} \qquad (3.11)\\ &+ \left\langle\left|\left|V^\dagger\right| A\right|^2 x, x\right\rangle^{1/2}\left\langle\left|V^\dagger VB^*\right|^2 y, y\right\rangle^{1/2}\\ &\leq\left(\left\langle \| V| A|^2 x, x\right\rangle + \left\langle\left|\left|V^\dagger\right| A\right|^2 x, x\right\rangle\right)^{1/2}\\ &\times\left(\left\langle\left|VV^\dagger B^*\right|^2 y, y\right\rangle + \left\langle\left|V^\dagger VB^*\right|^2 y, y\right\rangle\right)^{1/2}\\ &=\left\langle\left(\| V| A|^2 + \left|\left|V^\dagger\right| A\right|^2\right)x, x\right\rangle^{1/2}\left\langle\left(\left|VV^\dagger B^*\right|^2 + \left|V^\dagger VB^*\right|^2\right)y, y\right\rangle^{1/2} \end{aligned}$$

for all $x, y \in H$.

By the triangle inequality, we have that

$$\begin{aligned} |\langle BVAx, y\rangle| + \left|\left\langle BV^\dagger Ax, y\right\rangle\right| &\geq \left|\langle BVAx, y\rangle + \left\langle BV^\dagger Ax, y\right\rangle\right|\\ &= \left|\left\langle B\left(V + V^\dagger\right)Ax, y\right\rangle\right| = 2\left|\left\langle B\left(S_\dagger V\right) Ax, y\right\rangle\right| \end{aligned}$$

and

$$|\langle BVAx, y\rangle| + \left|\left\langle BV^\dagger Ax, y\right\rangle\right| \geq \left|\langle BVAx, y\rangle - \left\langle BV^\dagger Ax, y\right\rangle\right|$$

$$= \left|\left\langle B\left(V - V^{\dagger}\right)Ax, y\right\rangle\right| = 2\left|\left\langle B\left(D_{\dagger}V\right)Ax, y\right\rangle\right|,$$

which gives that

$$\begin{aligned} &|\langle BVAx, y\rangle| + \left|\left\langle BV^{\dagger}Ax, y\right\rangle\right| \\ &\geq 2\max\left\{\left|\left\langle B\left(S_{\dagger}V\right)Ax, y\right\rangle\right|, \left|\left\langle B\left(D_{\dagger}V\right)Ax, y\right\rangle\right|\right\} \end{aligned} \tag{3.12}$$

for all $x, y \in H$.

By utilizing (3.10), (3.11), and (3.12), we derive the desired result (3.9). □

Remark 3.2. If we take $A = B = I$ in (3.7), then we get

$$\begin{aligned} &\max\left\{\left|\left\langle\left(S_{\dagger}V\right)x, y\right\rangle\right|, \left|\left\langle\left(D_{\dagger}V\right)x, y\right\rangle\right|\right\} \\ &\leq \frac{1}{2}\left\langle\left(|V|^2 + \left|V^{\dagger}\right|^2\right)x, x\right\rangle^{1/2}\left\langle\left(VV^{\dagger} + V^{\dagger}V\right)y, y\right\rangle^{1/2} \end{aligned}$$

for all $x, y \in H$.

We can also obtain several power inequalities that provide bounds for the quantity

$$|\langle BVAx, y\rangle|^2 + \left|\left\langle BV^{\dagger}Ax, y\right\rangle\right|^2$$

with $x, y \in H$.

Proposition 3.2. *Let $V \in CR(H)$ and $A, B \in B(H)$. Then for all $x, y \in H$, we have that*

$$\begin{aligned} &|\langle BVAx, y\rangle|^2 + \left|\left\langle BV^{\dagger}Ax, y\right\rangle\right|^2 \\ &\leq \|x\|^{2/q}\|y\|^{2/p}\left\langle\left(||V|A|^{2p} + \left||V^{\dagger}|A\right|^{2p}\right)x, x\right\rangle^{1/p} \\ &\quad\times\left\langle\left(\left|VV^{\dagger}B^*\right|^{2q} + \left|V^{\dagger}VB^*\right|^{2q}\right)y, y\right\rangle^{1/q} \end{aligned} \tag{3.13}$$

for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In particular,

$$\begin{aligned} |\langle BVAx, y\rangle|^2 + \left|\left\langle BV^{\dagger}Ax, y\right\rangle\right|^2 \leq \|x\|\|y\|&\left\langle\left(||V|A|^4 + \left||V^{\dagger}|A\right|^4\right)x, x\right\rangle^{1/2} \\ &\times\left\langle\left(\left|VV^{\dagger}B^*\right|^4 + \left|V^{\dagger}VB^*\right|^4\right)y, y\right\rangle^{1/2} \end{aligned} \tag{3.14}$$

for all $x, y \in H$.

Proof. If we add (3.1) and (3.5), then we get that

$$\begin{aligned} &|\langle BVAx, y\rangle|^2 + \left|\left\langle BV^{\dagger}Ax, y\right\rangle\right|^2 \\ &\leq \left\langle||V|A|^2x, x\right\rangle\left\langle\left|VV^{\dagger}B^*\right|^2y, y\right\rangle + \left\langle\left||V^{\dagger}|A\right|^2x, x\right\rangle\left\langle\left|V^{\dagger}VB^*\right|^2y, y\right\rangle \end{aligned} \tag{3.15}$$

for all $x, y \in H$.

We use the elementary Hölder inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, which shows that

$$0 \le ac + bd \le (a^p + b^p)^{1/p} (c^q + c^q)^{1/q}, \ a, b, c, d \ge 0.$$

Therefore, for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have that

$$\begin{aligned}
&\left\langle \| V| A|^2 x, x \right\rangle \left\langle \left| VV^\dagger B^* \right|^2 y, y \right\rangle + \left\langle \left| |V^\dagger| A \right|^2 x, x \right\rangle \left\langle \left| V^\dagger VB^* \right|^2 y, y \right\rangle \\
&\le \left(\left\langle \| V| A|^2 x, x \right\rangle^p + \left\langle \left| |V^\dagger| A \right|^2 x, x \right\rangle^p \right)^{1/p} \\
&\quad \times \left(\left\langle \left| VV^\dagger B^* \right|^2 y, y \right\rangle^q + \left\langle \left| V^\dagger VB^* \right|^2 y, y \right\rangle^q \right)^{1/q}
\end{aligned} \tag{3.16}$$

for all $x, y \in H$.

Recall McCarthy's inequality, which states that $\langle Px, x \rangle^r \le \langle P^r x, x \rangle \|x\|^{2(r-1)}$ for $x \in H$ and $r \ge 1$. Utilizing this inequality, we have that

$$\begin{aligned}
\left\langle \| V| A|^2 x, x \right\rangle^p + \left\langle \left| |V^\dagger| A \right|^2 x, x \right\rangle^p &\le \left(\left\langle \| V| A|^{2p} x, x \right\rangle + \left\langle \left| |V^\dagger| A \right|^{2p} x, x \right\rangle \right) \|x\|^{2(p-1)} \\
&= \left\langle \left(\| V| A|^{2p} + \left| |V^\dagger| A \right|^{2p} \right) x, x \right\rangle \|x\|^{2(p-1)}
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle \left| VV^\dagger B^* \right|^2 y, y \right\rangle^q + \left\langle \left| V^\dagger VB^* \right|^2 y, y \right\rangle^q \\
&\le \left(\left\langle \left| VV^\dagger B^* \right|^{2q} y, y \right\rangle + \left\langle \left| V^\dagger VB^* \right|^{2q} y, y \right\rangle \right) \|y\|^{2(q-1)} \\
&= \left\langle \left(\left| VV^\dagger B^* \right|^{2q} + \left| V^\dagger VB^* \right|^{2q} \right) y, y \right\rangle \|y\|^{2(q-1)}
\end{aligned}$$

for all $x, y \in H$. These imply that

$$\begin{aligned}
&\left(\left\langle \| V| A|^2 x, x \right\rangle^p + \left\langle \left| |V^\dagger| A \right|^2 x, x \right\rangle^p \right)^{1/p} \\
&\quad \times \left(\left\langle \left| VV^\dagger B^* \right|^2 y, y \right\rangle^q + \left\langle \left| V^\dagger VB^* \right|^2 y, y \right\rangle^q \right)^{1/q} \\
&\le \left\langle \left(\| V| A|^{2p} + \left| |V^\dagger| A \right|^{2p} \right) x, x \right\rangle^{1/p} \|x\|^{2(1-1/p)} \\
&\quad \times \left\langle \left(\left| VV^\dagger B^* \right|^{2q} + \left| V^\dagger VB^* \right|^{2q} \right) y, y \right\rangle^{1/q} \|y\|^{2(1-1/q)} \\
&= \|x\|^{2/q} \|y\|^{2/p} \left\langle \left(\| V| A|^{2p} + \left| |V^\dagger| A \right|^{2p} \right) x, x \right\rangle^{1/p} \\
&\quad \times \left\langle \left(\left| VV^\dagger B^* \right|^{2q} + \left| V^\dagger VB^* \right|^{2q} \right) y, y \right\rangle^{1/q}
\end{aligned} \tag{3.17}$$

for all $x, y \in H$.

By making use of (3.15), (3.16), and (3.17), we get (3.13). □

Remark 3.3. If we take $A = B = I$ in Proposition 3.2, then we get for $V \in CR(H)$ that

$$\begin{aligned}&|\langle Vx, y\rangle|^2 + \left|\left\langle V^\dagger x, y\right\rangle\right|^2 \\ &\le \|x\|^{2/q} \|y\|^{2/p} \left\langle \left(|V|^{2p} + \left|V^\dagger\right|^{2p}\right) x, x\right\rangle^{1/p} \left\langle \left(VV^\dagger + V^\dagger V\right) y, y\right\rangle^{1/q}\end{aligned}$$

for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $x, y \in H$.

In particular,

$$|\langle Vx, y\rangle|^2 + \left|\left\langle V^\dagger x, y\right\rangle\right|^2 \le \|x\| \|y\| \left\langle \left(|V|^4 + \left|V^\dagger\right|^4\right) x, x\right\rangle^{1/2} \left\langle \left(VV^\dagger + V^\dagger V\right) y, y\right\rangle^{1/2}$$

for all $x, y \in H$.

4. Norm and numerical radius inequalities

It is natural now to find upper bounds for the norm of a product of three operators as follows:

Theorem 4.1. *Let $V \in CR(H)$ and $A, B \in B(H)$. Then*

$$\|BVA\| \le \||V| A\| \left\|BVV^\dagger\right\| \tag{4.1}$$

and

$$\left\|BV^\dagger A\right\| \le \left\|\left|V^\dagger\right| A\right\| \left\|BV^\dagger V\right\|. \tag{4.2}$$

Proof. If we take the supremum over $\|x\| = \|y\| = 1$ in (3.1), then we get

$$\begin{aligned}\|BVA\|^2 &= \sup_{\|x\|=\|y\|=1} |\langle BVAx, y\rangle|^2 \le \sup_{\|x\|=\|y\|=1} \left[\left\langle \||V| A|^2 x, x\right\rangle \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle\right] \\ &= \sup_{\|x\|=1} \left\langle \||V| A|^2 x, x\right\rangle \sup_{\|y\|=1} \left\langle \left|VV^\dagger B^*\right|^2 y, y\right\rangle = \left\|\||V| A|^2\right\| \left\|\left|VV^\dagger B^*\right|^2\right\| \\ &= \||V| A\|^2 \left\|VV^\dagger B^*\right\|^2.\end{aligned}$$

Since

$$\left\|VV^\dagger B^*\right\| = \left\|\left(VV^\dagger B^*\right)^*\right\| = \left\|B\left(VV^\dagger\right)^*\right\| = \left\|BVV^\dagger\right\|,$$

then inequality (4.1) is proved.

If we replace V with $V^\dagger$ in (4.1), then we get (4.2). □

Corollary 4.1. *Let $V \in CR(H)$ with $V \neq 0$. Then*

$$\min\left\{\left\||V| V^\dagger\right\|, \left\|\left|V^\dagger\right| V\right\|\right\} \ge 1. \tag{4.3}$$

Proof. From Theorem 4.1, we get

$$\left\|V^\dagger VV^\dagger\right\| \le \left\||V| V^\dagger\right\| \left\|V^\dagger VV^\dagger\right\|$$

and

$$\left\|VV^\dagger V\right\| \le \left\|\left|V^\dagger\right| V\right\| \left\|VV^\dagger V\right\|,$$

namely,

$$\left\|V^{\dagger}\right\| \leq \left\||V| V^{\dagger}\right\| \left\|V^{\dagger}\right\|$$

and

$$\|V\| \leq \left\|\left|V^{\dagger}\right| V\right\| \|V\|,$$

which gives (4.3). □

Further on, we can also obtain several bounds for the numerical radius of a product of three operators as follows:

Theorem 4.2. *Let $V \in CR(H)$ and $A, B \in B(H)$. Then for all $\lambda \in [0,1]$,*

$$w(BVA) \leq \min\{k_1, k_2, k_3\},$$

where

$$\begin{aligned} k_1 := \min\Big\{ & \left\|(1-\lambda)\,\||V|A|^2 + \lambda \left|VV^{\dagger}B^*\right|^2\right\|^{1/2} \||V|A\|^{\lambda} \left\|BVV^{\dagger}\right\|^{1-\lambda}, \\ & \left\|\lambda\,\||V|A|^2 + (1-\lambda)\left|VV^{\dagger}B^*\right|^2\right\|^{1/2} \||V|A\|^{1-\lambda} \left\|BVV^{\dagger}\right\|^{\lambda}\Big\}, \end{aligned}$$

$$k_2 := \left\|(1-\lambda)\,\||V|A|^2 + \lambda \left|VV^{\dagger}B^*\right|^2\right\|^{1/2} \left\|\lambda\,\||V|A|^2 + (1-\lambda)\left|VV^{\dagger}B^*\right|^2\right\|^{1/2},$$

and

$$k_3 := \left\|\frac{\||V|A|^2 + \left|VV^{\dagger}B^*\right|^2}{2}\right\|.$$

Proof. Observe that for $x \in H$ with $\|x\| = 1$, we have that

$$\begin{aligned} & \min\Big\{\left\langle\left[(1-\lambda)\,\||V|A|^2 + \lambda\left|VV^{\dagger}B^*\right|^2\right]x, x\right\rangle \\ & \times \left\langle \||V|A|^2 x, x\right\rangle^{\lambda} \left\langle \left|VV^{\dagger}B^*\right|^2 x, x\right\rangle^{1-\lambda}, \\ & \left\langle\left(\lambda\,\||V|A|^2 + (1-\lambda)\left|VV^{\dagger}B^*\right|^2\right)x, x\right\rangle \\ & \times \left\langle \||V|A|^2 x, x\right\rangle^{1-\lambda} \left\langle \left|VV^{\dagger}B^*\right|^2 x, x\right\rangle^{\lambda}\Big\} \\ & \leq \min\Big\{\left\|(1-\lambda)\,\||V|A|^2 + \lambda\left|VV^{\dagger}B^*\right|^2\right\| \||V|A\|^{2\lambda} \left\|VV^{\dagger}B^*\right\|^{2(1-\lambda)}, \\ & \left\|\lambda\,\||V|A|^2 + (1-\lambda)\left|VV^{\dagger}B^*\right|^2\right\| \||V|A\|^{2(1-\lambda)} \left\|VV^{\dagger}B^*\right\|^{2\lambda}\Big\} \end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

By (3.2), we get that

$$\begin{aligned} & |\langle BVAx, x\rangle|^2 \\ & \leq \min\Big\{\left\|(1-\lambda)\,\||V|A|^2 + \lambda\left|VV^{\dagger}B^*\right|^2\right\| \||V|A\|^{2\lambda} \left\|VV^{\dagger}B^*\right\|^{2(1-\lambda)}, \end{aligned}$$

$$\left\|\lambda \left\||V| A\right|^2 + (1-\lambda)\left|VV^{\dagger} B^*\right|^2\right\| \left\||V| A\right\|^{2(1-\lambda)} \left\|VV^{\dagger} B^*\right\|^{2\lambda}\Bigg\}$$

for $x \in H$ with $\|x\| = 1$, which gives, by taking the supremum over $\|x\| = 1$, that

$$\begin{aligned} & w(BVA) \\ & \leq \min\left\{\left\|(1-\lambda)\left\||V| A\right|^2 + \lambda\left|VV^{\dagger} B^*\right|^2\right\|^{1/2} \left\||V| A\right\|^{\lambda} \left\|VV^{\dagger} B^*\right\|^{1-\lambda},\right. \\ & \qquad \left.\left\|\lambda \left\||V| A\right|^2 + (1-\lambda)\left|VV^{\dagger} B^*\right|^2\right\|^{1/2} \left\||V| A\right\|^{1-\lambda} \left\|VV^{\dagger} B^*\right\|^{\lambda}\right\}. \end{aligned}$$

Since

$$\left\|VV^{\dagger} B^*\right\| = \left\|\left(VV^{\dagger} B^*\right)^*\right\| = \left\|B\left(VV^{\dagger}\right)^*\right\| = \left\|BVV^{\dagger}\right\|,$$

then the first part of the theorem is proved.

Moreover,

$$\begin{aligned} & \left\langle\left[(1-\lambda)\left\||V| A\right|^2 + \lambda\left|VV^{\dagger} B^*\right|^2\right] x, x\right\rangle \\ & \times\left\langle\left[\lambda \left\||V| A\right|^2 + (1-\lambda)\left|VV^{\dagger} B^*\right|^2\right] x, x\right\rangle \\ \leq & \left\|(1-\lambda)\left\||V| A\right|^2 + \lambda\left|VV^{\dagger} B^*\right|^2\right\| \\ & \times\left\|\lambda \left\||V| A\right|^2 + (1-\lambda)\left|VV^{\dagger} B^*\right|^2\right\|, \end{aligned}$$

which implies, by (3.2), that

$$\begin{aligned} |\langle BVAx, x\rangle|^2 \leq & \left\|(1-\lambda)\left\||V| A\right|^2 + \lambda\left|VV^{\dagger} B^*\right|^2\right\| \\ & \times\left\|\lambda \left\||V| A\right|^2 + (1-\lambda)\left|VV^{\dagger} B^*\right|^2\right\|, \end{aligned}$$

for $x \in H$ with $\|x\| = 1$. This gives, by taking the supremum over $\|x\| = 1$, that

$$\begin{aligned} w(BVA) \leq & \left\|(1-\lambda)\left\||V| A\right|^2 + \lambda\left|VV^{\dagger} B^*\right|^2\right\|^{1/2} \\ & \times\left\|\lambda \left\||V| A\right|^2 + (1-\lambda)\left|VV^{\dagger} B^*\right|^2\right\|^{1/2}. \end{aligned}$$

Finally, since

$$\left\langle\left(\frac{\left\||V| A\right|^2 + \left|VV^{\dagger} B^*\right|^2}{2}\right) x, x\right\rangle^2 \leq \left\|\frac{\left\||V| A\right|^2 + \left|VV^{\dagger} B^*\right|^2}{2}\right\|^2$$

for $x \in H$ with $\|x\| = 1$, then by (3.2), we get that

$$w(BVA) \leq \left\|\frac{\left\||V| A\right|^2 + \left|VV^{\dagger} B^*\right|^2}{2}\right\|.$$

This completes the proof of the theorem. □

We observe that, by taking $\lambda = 1/2$ in k_1, we also have that

$$w(BVA) \leq \min\left\{ \left\| |V|A \right\|^{1/2} \left\| BVV^{\dagger} \right\|^{1/2}, \left\| \frac{\left||V|A\right|^2 + \left|VV^{\dagger}B^*\right|^2}{2} \right\|^{1/2} \right\} \times \left\| \frac{\left||V|A\right|^2 + \left|VV^{\dagger}B^*\right|^2}{2} \right\|^{1/2}.$$

We also observe that, if we take $A = B = I$ in Theorem 4.2, then we get, for a closed-range operator V and $\lambda \in [0, 1]$, that

$$w(V) \leq \min\{u_1, u_2, u_3\}, \tag{4.4}$$

where

$$u_1 := \min \left\| (1-\lambda)|V|^2 + \lambda VV^{\dagger} \right\|^{1/2} \|V\|^{1/2},$$

$$u_2 := \left\| (1-\lambda)|V|^2 + \lambda VV^{\dagger} \right\|^{1/2} \left\| \lambda |V|^2 + (1-\lambda) VV^{\dagger} \right\|^{1/2},$$

and

$$u_3 := \left\| \frac{|V|^2 + VV^{\dagger}}{2} \right\|.$$

We observe that the bound u_3 gives the known inequality (1.10) from the Introduction.

If we take $V = I$, then $u_1 = u_2 = u_3 = 1$ and we obtain in both terms of (4.4) the same quantity 1, which shows that (4.4) is sharp. This implies that the inequalities from Theorem 4.2 are also sharp.

For $\lambda = 1/2$, we obtain

$$w(V) \leq \min\left\{ \|V\|^{1/2}, \left\| \frac{|V|^2 + VV^{\dagger}}{2} \right\|^{1/2} \right\} \left\| \frac{|V|^2 + VV^{\dagger}}{2} \right\|^{1/2}.$$

If we replace V with $V^{\dagger}$ in Theorem 4.2, then we can state:

Corollary 4.2. *Let $V \in CR(H)$ and $A, B \in B(H)$. Then for all $\lambda \in [0, 1]$,*

$$w\left(BV^{\dagger}A\right) \leq \min\{\ell_1, \ell_2, \ell_3\},$$

where

$$\ell_1 := \min\left\{ \left\| (1-\lambda)\left|\left|V^{\dagger}\right|A\right|^2 + \lambda\left|V^{\dagger}VB^*\right|^2 \right\|^{1/2} \left\| \left|V^{\dagger}\right|A \right\|^{\lambda} \left\| BVV^{\dagger} \right\|^{1-\lambda}, \right.$$
$$\left. \left\| \lambda\left|\left|V^{\dagger}\right|A\right|^2 + (1-\lambda)\left|V^{\dagger}VB^*\right|^2 \right\|^{1/2} \left\| \left|V^{\dagger}\right|A \right\|^{1-\lambda} \left\| BVV^{\dagger} \right\|^{\lambda} \right\},$$

$$\ell_2 := \left\| (1-\lambda)\left|\left|V^{\dagger}\right|A\right|^2 + \lambda\left|V^{\dagger}VB^*\right|^2 \right\|^{1/2} \times \left\| \lambda\left|\left|V^{\dagger}\right|A\right|^2 + (1-\lambda)\left|V^{\dagger}VB^*\right|^2 \right\|^{1/2},$$

and

$$\ell_3 := \left\| \frac{\left|\left|V^{\dagger}\right|A\right|^2 + \left|V^{\dagger}VB^*\right|^2}{2} \right\|.$$

Remark 4.1. If we take $A = B = V^{\dagger}$ in Theorem 4.2 and use the properties of $V^{\dagger}$, then we get for $\lambda \in [0,1]$ that

$$w\left(V^{\dagger}\right) \leq \min\{m_1, m_2, m_3\},$$

where

$$\begin{aligned} m_1 := \min\Bigg\{ & \left\| (1-\lambda)\left| |V|\, V^{\dagger} \right|^2 + \lambda \left| V \left| \left(V^{\dagger}\right)^{*} \right|^2 \right|^2 \right\|^{1/2} \left\| |V|\, V^{\dagger} \right\|^{\lambda} \|V\|^{1-\lambda}, \\ & \left\| \lambda \left| |V|\, V^{\dagger} \right|^2 + (1-\lambda) \left| V \left| \left(V^{\dagger}\right)^{*} \right|^2 \right|^2 \right\|^{1/2} \times \left\| |V|\, V^{\dagger} \right\|^{1-\lambda} \|V\|^{\lambda} \Bigg\}, \end{aligned}$$

$$\begin{aligned} m_2 := & \left\| (1-\lambda)\left| |V|\, V^{\dagger} \right|^2 + \lambda \left| V \left| \left(V^{\dagger}\right)^{*} \right|^2 \right|^2 \right\|^{1/2} \\ & \times \left\| \lambda \left| |V|\, V^{\dagger} \right|^2 + (1-\lambda) \left| V \left| \left(V^{\dagger}\right)^{*} \right|^2 \right|^2 \right\|^{1/2}, \end{aligned}$$

and

$$m_3 := \left\| \frac{\left| |V|\, V^{\dagger} \right|^2 + \left| V \left| \left(V^{\dagger}\right)^{*} \right|^2 \right|^2}{2} \right\|.$$

If we take $A = B = V$ in Corollary 4.2, then we also get that

$$w(V) \leq \min\{n_1, n_2, n_3\},$$

where

$$\begin{aligned} n_1 := \min\Big\{ & \left\| (1-\lambda)\left| \left|V^{\dagger}\right| V \right|^2 + \lambda \left| V^{\dagger} |V^{*}|^2 \right|^2 \right\|^{1/2} \left\| \left|V^{\dagger}\right| V \right\|^{\lambda} \|V\|^{1-\lambda}, \\ & \left\| \lambda \left| \left|V^{\dagger}\right| V \right|^2 + (1-\lambda) \left| V^{\dagger} |V^{*}|^2 \right|^2 \right\|^{1/2} \left\| \left|V^{\dagger}\right| V \right\|^{1-\lambda} \|V\|^{\lambda} \Big\}, \end{aligned}$$

$$\begin{aligned} n_2 := & \left\| (1-\lambda)\left| \left|V^{\dagger}\right| V \right|^2 + \lambda \left| V^{\dagger} |V^{*}|^2 \right|^2 \right\|^{1/2} \\ & \times \left\| \lambda \left| \left|V^{\dagger}\right| V \right|^2 + (1-\lambda) \left| V^{\dagger} |V^{*}|^2 \right|^2 \right\|^{1/2}, \end{aligned}$$

and

$$n_3 := \left\| \frac{\left| \left|V^{\dagger}\right| V \right|^2 + \left| V^{\dagger} |V^{*}|^2 \right|^2}{2} \right\|.$$

If we take $\lambda = 1/2$, then we also have

$$\begin{aligned} w(V) \leq \min & \left\{ \left\| \left|V^{\dagger}\right| V \right\|^{1/2} \|V\|^{1/2}, \left\| \frac{\left| \left|V^{\dagger}\right| V \right|^2 + \left| V^{\dagger} |V^{*}|^2 \right|^2}{2} \right\|^{1/2} \right\} \\ & \times \left\| \frac{\left| \left|V^{\dagger}\right| V \right|^2 + \left| V^{\dagger} |V^{*}|^2 \right|^2}{2} \right\|^{1/2}. \end{aligned}$$

We can also state the following power inequalities for the numerical radius of a product of three operators:

Theorem 4.3. *Let $V \in CR(H)$, $A, B \in B(H)$, and $p \geq 1$. Then*

$$w^{2p}(BVA) \leq \frac{\| |V| A \|^{2p} \left\| BVV^{\dagger} \right\|^{2p} + w^{p}\left(\left| VV^{\dagger}B^{*} \right|^{2} |V| A|^{2} \right)}{2} \leq \| |V| A \|^{2p} \left\| BVV^{\dagger} \right\|^{2p} \tag{4.5}$$

and

$$w^{2p}(BVA) \leq \frac{1}{2} \left\| |V| A|^{4p} + \left| VV^{\dagger}B^{*} \right|^{4p} \right\|. \tag{4.6}$$

In particular, for $p = 1$, we get

$$w^{2}(BVA) \leq \frac{\| |V| A \|^{2} \left\| BVV^{\dagger} \right\|^{2} + w\left(\left| VV^{\dagger}B^{*} \right|^{2} |V| A|^{2} \right)}{2} \leq \| |V| A \|^{2} \left\| BVV^{\dagger} \right\|^{2} \tag{4.7}$$

and

$$w^{2}(BVA) \leq \frac{1}{2} \left\| |V| A|^{4} + \left| VV^{\dagger}B^{*} \right|^{4} \right\|. \tag{4.8}$$

Proof. First, recall Buzano's inequality [10], which states that

$$|\langle u, e \rangle \langle e, v \rangle| \leq \frac{\|u\| \|v\| + |\langle u, v \rangle|}{2} \tag{4.9}$$

for $x, y, e \in H$ with $\|e\| = 1$.

If we take $u = |V| A|^{2} x$, $v = \left| VV^{\dagger}B^{*} \right|^{2} x$, and $e = x$ in (4.9) with $\|x\| = 1$, then we get

$$\begin{aligned}
&\left\langle |V| A|^{2} x, x \right\rangle \left\langle \left| VV^{\dagger}B^{*} \right|^{2} x, x \right\rangle \\
&\leq \frac{\left\| |V| A|^{2} x \right\| \left\| \left| VV^{\dagger}B^{*} \right|^{2} x \right\| + \left| \left\langle |V| A|^{2} x, \left| VV^{\dagger}B^{*} \right|^{2} x \right\rangle \right|}{2} \\
&= \frac{\left\| |V| A|^{2} x \right\| \left\| \left| VV^{\dagger}B^{*} \right|^{2} x \right\| + \left| \left\langle \left| VV^{\dagger}B^{*} \right|^{2} |V| A|^{2} x, x \right\rangle \right|}{2}.
\end{aligned}$$

Taking the power $p \geq 1$, then we get, by the convexity of the power function, that

$$\begin{aligned}
&\left\langle |V| A|^{2} x, x \right\rangle^{p} \left\langle \left| VV^{\dagger}B^{*} \right|^{2} x, x \right\rangle^{p} \\
&\leq \left(\frac{\left\| |V| A|^{2} x \right\| \left\| \left| VV^{\dagger}B^{*} \right|^{2} x \right\| + \left| \left\langle \left| VV^{\dagger}B^{*} \right|^{2} |V| A|^{2} x, x \right\rangle \right|}{2} \right)^{p}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\left\| \| V | A|^2 x\right\|^p \left\| \left|VV^\dagger B^*\right|^2 x\right\|^p + \left|\left\langle \left|VV^\dagger B^*\right|^2 \| V | A|^2 x, x\right\rangle\right|^p}{2} \\
&\leq \frac{\left\| \| V | A|^2\right\|^p \left\| \left|VV^\dagger B^*\right|^2\right\|^p + w^p\left(\left|VV^\dagger B^*\right|^2 \| V | A|^2\right)}{2} \\
&= \frac{\| | V | A \|^{2p} \left\|VV^\dagger B^*\right\|^{2p} + w^p\left(\left|VV^\dagger B^*\right|^2 \| V | A|^2\right)}{2}
\end{aligned}
$$

for $x \in H$ with $\|x\| = 1$.

From (3.2), we then get, by taking the power p, that

$$
|\langle BVAx, x\rangle|^{2p} \leq \frac{\| | V | A \|^{2p} \left\|VV^\dagger B^*\right\|^{2p} + w^p\left(\left|VV^\dagger B^*\right|^2 \| V | A|^2\right)}{2}
$$

for $x \in H$ with $\|x\| = 1$, which gives, by taking the supremum over $x \in H$ with $\|x\| = 1$, the desired inequality (4.5).

By utilizing McCarthy's inequality, we then obtain that

$$
\begin{aligned}
\left\langle \| V | A|^2 x, x\right\rangle^p \left\langle \left|VV^\dagger B^*\right|^2 x, x\right\rangle^p &\leq \left\langle \| V | A|^{2p} x, x\right\rangle \left\langle \left|VV^\dagger B^*\right|^{2p} x, x\right\rangle \\
&\leq \frac{\left\langle \| V | A|^{2p} x, x\right\rangle^2 + \left\langle \left|VV^\dagger B^*\right|^{2p} x, x\right\rangle^2}{2} \\
&\leq \frac{\left\langle \| V | A|^{4p} x, x\right\rangle + \left\langle \left|VV^\dagger B^*\right|^{4p} x, x\right\rangle}{2} \\
&= \frac{1}{2}\left\langle \left(\| V | A|^{4p} + \left|VV^\dagger B^*\right|^{4p}\right) x, x\right\rangle \\
&\leq \frac{1}{2}\left\| \| V | A|^{4p} + \left|VV^\dagger B^*\right|^{4p}\right\|
\end{aligned}
$$

for $x \in H$ with $\|x\| = 1$.

From (3.2) we then get, by taking the power p, that

$$
|\langle BVAx, x\rangle|^{2p} \leq \frac{1}{2}\left\| \| V | A|^{4p} + \left|VV^\dagger B^*\right|^{4p}\right\|
$$

for $x \in H$ with $\|x\| = 1$, which gives, by taking the supremum over $x \in H$ with $\|x\| = 1$, the desired inequality (4.5). □

We also observe that, if we take $A = B = I$ in Theorem 4.3, then we get for $V \in CR(H)$ and $p \geq 1$ that

$$
w^{2p}(V) \leq \frac{\|V\|^{2p} + w^p\left(VV^\dagger |V|^2\right)}{2}
$$

and

$$
w^{2p}(V) \leq \frac{1}{2}\left\| |V|^{4p} + VV^\dagger\right\|.
$$

If we take $V = I$ in these two inequalities, we obtain the same quantity 1 on both sides. This shows that the inequalities in Theorem 4.3 are sharp.

We notice that this inequality can also be obtained from (1.11) for $\lambda = 1$.

Corollary 4.3. *With the assumptions of Theorem 4.3, we have that*

$$\begin{aligned} w^{2p}\left(BV^{\dagger}A\right) &\leq \frac{1}{2}\left[\left\|\left|V^{\dagger}\right|A\right\|^{2p}\left\|BV^{\dagger}V\right\|^{2p} + w^{p}\left(\left|V^{\dagger}VB^{*}\right|^{2}\left\|V^{\dagger}\right|A\right|^{2}\right)\right] \\ &\leq \left\|\left|V^{\dagger}\right|A\right\|^{2p}\left\|BV^{\dagger}V\right\|^{2p} \end{aligned} \tag{4.10}$$

and

$$w^{2p}\left(BV^{\dagger}A\right) \leq \frac{1}{2}\left\|\left|\left|V^{\dagger}\right|A\right|^{4p} + \left|V^{\dagger}VB^{*}\right|^{4p}\right\|. \tag{4.11}$$

In particular, for $p = 1$, we obtain

$$\begin{aligned} w^{2}\left(BV^{\dagger}A\right) &\leq \frac{1}{2}\left[\left\|\left|V^{\dagger}\right|A\right\|^{2}\left\|BV^{\dagger}V\right\|^{2} + w\left(\left|V^{\dagger}VB^{*}\right|^{2}\left\|V^{\dagger}\right|A\right|^{2}\right)\right] \\ &\leq \left\|\left|V^{\dagger}\right|A\right\|^{2}\left\|BV^{\dagger}V\right\|^{2} \end{aligned}$$

and

$$w^{2}\left(BV^{\dagger}A\right) \leq \frac{1}{2}\left\|\left|\left|V^{\dagger}\right|A\right|^{4} + \left|V^{\dagger}VB^{*}\right|^{4}\right\|.$$

Remark 4.2. If we select $A = B = V^{\dagger}$ in Theorem 4.2, then for $V \in CR(H)$, $A, B \in B(H)$ and $p \geq 1$, we get that

$$w^{2p}\left(V^{\dagger}\right) \leq \frac{\left\|\left|V\right|V^{\dagger}\right\|^{2p}\left\|V\right\|^{2p} + w^{p}\left(\left|V\left|\left(V^{\dagger}\right)^{*}\right|^{2}\right|^{2}\left\|V\right|V^{\dagger}\right|^{2}\right)}{2} \tag{4.12}$$

and

$$w^{2p}\left(V^{\dagger}\right) \leq \frac{1}{2}\left\|\left|\left|V\right|V^{\dagger}\right|^{4p} + \left|V\left|\left(V^{\dagger}\right)^{*}\right|^{2}\right|^{4p}\right\|. \tag{4.13}$$

Observe that

$$\begin{aligned} \left|V\left|\left(V^{\dagger}\right)^{*}\right|^{2}\right|^{2} &= \left(V\left|\left(V^{\dagger}\right)^{*}\right|^{2}\right)^{*}V\left|\left(V^{\dagger}\right)^{*}\right|^{2} = \left(VV^{\dagger}\left(V^{\dagger}\right)^{*}\right)^{*}VV^{\dagger}\left(V^{\dagger}\right)^{*} \\ &= V^{\dagger}VV^{\dagger}VV^{\dagger}\left(V^{\dagger}\right)^{*} = V^{\dagger}VV^{\dagger}\left(V^{\dagger}\right)^{*} = V^{\dagger}\left(V^{\dagger}\right)^{*} = \left|\left(V^{\dagger}\right)^{*}\right|^{2} \end{aligned}$$

and

$$\left|\left|V\right|V^{\dagger}\right|^{2} = \left(\left|V\right|V^{\dagger}\right)^{*}\left|V\right|V^{\dagger} = \left(V^{\dagger}\right)^{*}\left|V\right|\left|V\right|V^{\dagger} = \left(V^{\dagger}\right)^{*}\left|V\right|^{2}V^{\dagger},$$

then

$$\left|V\left|\left(V^{\dagger}\right)^{*}\right|^{2}\right|^{2}\left|\left|V\right|V^{\dagger}\right|^{2} = V^{\dagger}\left(V^{\dagger}\right)^{*}\left(V^{\dagger}\right)^{*}\left|V\right|^{2}V^{\dagger} = V^{\dagger}\left(\left(V^{\dagger}\right)^{*}\right)^{2}\left|V\right|^{2}V^{\dagger}.$$

By (4.12), we then get

$$w^{2p}\left(V^{\dagger}\right) \leq \frac{\left\|\left|V\right|V^{\dagger}\right\|^{2p}\left\|V\right\|^{2p} + w^{p}\left(V^{\dagger}\left(\left(V^{\dagger}\right)^{*}\right)^{2}\left|V\right|^{2}V^{\dagger}\right)}{2},$$

while, by (4.13), we obtain

$$w^{2p}\left(V^{\dagger}\right) \le \frac{1}{2}\left\|\left||V|\,V^{\dagger}\right|^{4p} + \left|\left(V^{\dagger}\right)^{*}\right|^{4p}\right\|.$$

The case $p = 1$ gives that

$$w^{2}\left(V^{\dagger}\right) \le \frac{\left\||V|\,V^{\dagger}\right\|^{2}\|V\|^{2} + w\left(V^{\dagger}\left(\left(V^{\dagger}\right)^{*}\right)^{2}|V|^{2}\,V^{\dagger}\right)}{2}$$

and

$$w^{2}\left(V^{\dagger}\right) \le \frac{1}{2}\left\|\left||V|\,V^{\dagger}\right|^{4} + \left|\left(V^{\dagger}\right)^{*}\right|^{4}\right\|.$$

If in Corollary 4.3 we take $A = B = V$, then we get

$$w^{2p}(V) \le \frac{1}{2}\left[\left\|\left|V^{\dagger}\right|V\right\|^{2p}\|V\|^{2p} + w^{p}\left(\left|V^{\dagger}\,|V^{*}|^{2}\right|^{2}\left|\left|V^{\dagger}\right|V\right|^{2}\right)\right] \tag{4.14}$$

and

$$w^{2p}(V) \le \frac{1}{2}\left\|\left|\left|V^{\dagger}\right|V\right|^{4p} + \left|V^{\dagger}\,|V^{*}|^{2}\right|^{4p}\right\|. \tag{4.15}$$

Now, observe that

$$\begin{aligned}\left|V^{\dagger}\,|V^{*}|^{2}\right|^{2} &= \left(V^{\dagger}\,|V^{*}|^{2}\right)^{*}V^{\dagger}\,|V^{*}|^{2} = \left(V^{\dagger}VV^{*}\right)^{*}V^{\dagger}VV^{*}\\ &= VV^{\dagger}VV^{\dagger}VV^{*} = VV^{\dagger}VV^{*} = VV^{*} = |V^{*}|^{2}\end{aligned}$$

and

$$\left|\left|V^{\dagger}\right|V\right|^{2} = \left(\left|V^{\dagger}\right|V\right)^{*}\left|V^{\dagger}\right|V = V^{*}\left|V^{\dagger}\right|\left|V^{\dagger}\right|V = V^{*}\left|V^{\dagger}\right|^{2}V$$

and then

$$\left|V^{\dagger}\,|V^{*}|^{2}\right|^{2}\left|\left|V^{\dagger}\right|V\right|^{2} = VV^{*}V^{*}\left|V^{\dagger}\right|^{2}V = V(V^{*})^{2}\left|V^{\dagger}\right|^{2}V.$$

By (4.14), we then obtain

$$w^{2p}(V) \le \frac{1}{2}\left[\left\|\left|V^{\dagger}\right|V\right\|^{2p}\|V\|^{2p} + w^{p}\left(V(V^{*})^{2}\left|V^{\dagger}\right|^{2}V\right)\right]$$

while, from (4.15),

$$w^{2p}(V) \le \frac{1}{2}\left\|\left|\left|V^{\dagger}\right|V\right|^{4p} + |V^{*}|^{4p}\right\|.$$

The case $p = 1$ gives

$$w^{2}(V) \le \frac{1}{2}\left[\left\|\left|V^{\dagger}\right|V\right\|^{2}\|V\|^{2} + w\left(V(V^{*})^{2}\left|V^{\dagger}\right|^{2}V\right)\right]$$

and

$$w^{2}(V) \le \frac{1}{2}\left\|\left|\left|V^{\dagger}\right|V\right|^{4} + |V^{*}|^{4}\right\|.$$

Let (C, D) be a pair of bounded linear operators on H. The Euclidean operator radius is defined by:

$$w_e(C, D) := \sup_{\|x\|=1} \left(|\langle Cx, x\rangle|^2 + |\langle Dx, x\rangle|^2 \right)^{1/2}. \tag{4.16}$$

As pointed out in [11], $w_e : \mathcal{B}^2(H) \to [0, \infty)$ is a norm and the following inequality holds:

$$\frac{\sqrt{2}}{4} \left\| |C|^2 + |D|^2 \right\|^{1/2} \leq w_e(C, D) \leq \left\| |C|^2 + |D|^2 \right\|^{1/2}, \tag{4.17}$$

for $(C, D) \in \mathcal{B}^2(H)$, where the constants $\frac{\sqrt{2}}{4}$ and 1 are the best possible in (4.17).

The diamond numerical radius, see, for instance, [12], is defined by

$$w(C, D) := \sup_{\|x\|=1} \left(|\langle Cx, x\rangle| + |\langle Dx, x\rangle| \right).$$

For $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = |\beta| = 1$ we have that

$$|\langle (\alpha C + \beta D) x, x\rangle| \leq |\langle Cx, x\rangle| + |\langle Dx, x\rangle| \leq \sqrt{2} \left(|\langle Cx, x\rangle|^2 + |\langle Dx, x\rangle|^2 \right)^{1/2}$$

and by taking the supremum over $\|x\| = 1$, we obtain that

$$w(\alpha C + \beta D) \leq w(C, D) \leq \sqrt{2} w_e(C, D),$$

or, equivalently,

$$\frac{1}{2} w^2(\alpha C + \beta D) \leq \frac{1}{2} w^2(C, D) \leq w_e^2(C, D).$$

Proposition 4.1. *Let $V \in CR(H)$ and $A, B \in B(H)$. Then*

$$\begin{aligned} &2 \max\left\{ w\left(B\left(S_{\dagger} V\right) A\right), w\left(B\left(D_{\dagger} V\right) A\right) \right\} \\ &\leq w\left(BVA, BV^{\dagger} A\right) \leq 2 \min\{b_1, b_2\}, \end{aligned} \tag{4.18}$$

where

$$b_1 := \frac{1}{2} \left\| \left| |V| A \right|^2 + \left| \left| V^{\dagger} \right| A \right|^2 \right\|^{1/2} \left\| \left| V V^{\dagger} B^* \right|^2 + \left| V^{\dagger} V B^* \right|^2 \right\|^{1/2}$$

and

$$b_2 := \left\| \frac{\left| |V| A \right|^2 + \left| \left| V^{\dagger} \right| A \right|^2 + \left| V V^{\dagger} B^* \right|^2 + \left| V^{\dagger} V B^* \right|^2}{4} \right\|.$$

Proof. From (3.9), we have for $y = x$ that

$$\begin{aligned} &\max\left\{ \left| \left\langle B\left(S_{\dagger} V\right) A x, x \right\rangle \right|, \left| \left\langle B\left(D_{\dagger} V\right) A x, x \right\rangle \right| \right\} \\ &\leq \frac{1}{2} \left(|\langle BVAx, x\rangle| + \left| \left\langle BV^{\dagger} A x, x \right\rangle \right| \right) \\ &\leq \frac{1}{2} \left\langle \left(\left| |V| A \right|^2 + \left| \left| V^{\dagger} \right| A \right|^2 \right) x, x \right\rangle^{1/2} \left\langle \left(\left| V V^{\dagger} B^* \right|^2 + \left| V^{\dagger} V B^* \right|^2 \right) x, x \right\rangle^{1/2} \end{aligned} \tag{4.19}$$

for all $x \in H$.

If we take the supremum over $\|x\| = 1$ in (4.19), then we get

$$\begin{aligned}
&\max\left\{w\left(B\left(S_{\dagger}V\right)A\right), w\left(B\left(D_{\dagger}V\right)A\right)\right\} \\
&= \sup_{\|x\|=1}\left[\max\left\{\left|\left\langle B\left(S_{\dagger}V\right)Ax, x\right\rangle\right|, \left|\left\langle B\left(D_{\dagger}V\right)Ax, x\right\rangle\right|\right\}\right] \\
&\le \frac{1}{2}\sup_{\|x\|=1}\left(\left|\langle BVAx, x\rangle\right| + \left|\left\langle BV^{\dagger}Ax, x\right\rangle\right|\right) = \frac{1}{2}w\left(BVA, BV^{\dagger}A\right) \\
&\le \frac{1}{2}\sup_{\|x\|=1}\left[\left\langle\left(\left||V|A\right|^2 + \left|\left|V^{\dagger}\right|A\right|^2\right)x, x\right\rangle^{1/2}\left\langle\left(\left|VV^{\dagger}B^*\right|^2 + \left|V^{\dagger}VB^*\right|^2\right)x, x\right\rangle^{1/2}\right] \\
&\le \frac{1}{2}\sup_{\|x\|=1}\left\langle\left(\left||V|A\right|^2 + \left|\left|V^{\dagger}\right|A\right|^2\right)x, x\right\rangle^{1/2}\sup_{\|x\|=1}\left\langle\left(\left|VV^{\dagger}B^*\right|^2 + \left|V^{\dagger}VB^*\right|^2\right)x, x\right\rangle^{1/2} \\
&= \frac{1}{2}\left\|\left||V|A\right|^2 + \left|\left|V^{\dagger}\right|A\right|^2\right\|^{1/2}\left\|\left|VV^{\dagger}B^*\right|^2 + \left|V^{\dagger}VB^*\right|^2\right\|^{1/2},
\end{aligned}$$

which proves the first part of (4.18).

By the arithmetic mean-geometric mean inequality, we also have that

$$\begin{aligned}
&\left\langle\left(\left||V|A\right|^2 + \left|\left|V^{\dagger}\right|A\right|^2\right)x, x\right\rangle^{1/2}\left\langle\left(\left|VV^{\dagger}B^*\right|^2 + \left|V^{\dagger}VB^*\right|^2\right)x, x\right\rangle^{1/2} \\
&\le \frac{1}{2}\left[\left\langle\left(\left||V|A\right|^2 + \left|\left|V^{\dagger}\right|A\right|^2\right)x, x\right\rangle + \left\langle\left(\left|VV^{\dagger}B^*\right|^2 + \left|V^{\dagger}VB^*\right|^2\right)x, x\right\rangle\right] \\
&= \frac{1}{2}\left\langle\left(\left||V|A\right|^2 + \left|\left|V^{\dagger}\right|A\right|^2 + \left|VV^{\dagger}B^*\right|^2 + \left|V^{\dagger}VB^*\right|^2\right)x, x\right\rangle
\end{aligned}$$

for all $x \in H$.

By (4.19), we get

$$\left|\langle BVAx, x\rangle\right| + \left|\left\langle BV^{\dagger}Ax, x\right\rangle\right| \le \frac{1}{2}\left\langle\left(\left||V|A\right|^2 + \left|\left|V^{\dagger}\right|A\right|^2 + \left|VV^{\dagger}B^*\right|^2 + \left|V^{\dagger}VB^*\right|^2\right)x, x\right\rangle$$

for all $x \in H$. By taking the supremum over $\|x\| = 1$ we derive the second part of (4.18). □

Remark 4.3. If we take $A = B = I$ in Proposition 4.1, then we get

$$2\max\left\{w\left(S_{\dagger}V\right), w\left(D_{\dagger}V\right)\right\} \le w\left(V, V^{\dagger}\right) \le 2\min\left\{c_1, c_2\right\},$$

where

$$c_1 := \frac{1}{2}\left\||V|^2 + \left|V^{\dagger}\right|^2\right\|^{1/2}\left\|VV^{\dagger} + V^{\dagger}V\right\|^{1/2}$$

and

$$c_2 := \left\|\frac{|V|^2 + \left|V^{\dagger}\right|^2 + VV^{\dagger} + V^{\dagger}V}{4}\right\|.$$

For the Euclidean numerical radius, we have:

Proposition 4.2. *Let $V \in CR(H)$ and $A, B \in B(H)$. Then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = |\beta| = 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have that*

$$\frac{1}{2}w^2\left(B\left(\alpha V + \beta V^{\dagger}\right)A\right) \le \frac{1}{2}w^2\left(BVA, BV^{\dagger}A\right) \tag{4.20}$$

$$\leq \quad w_e^2\left(BVA, BV^{\dagger}A\right) \leq \min\{d_1, d_2\},$$

where

$$d_1 := \left\| \left\| V \right| A \right|^{2p} + \left| \left| V^{\dagger} \right| A \right|^{2p} \Big\|^{1/p} \left\| \left| VV^{\dagger}B^* \right|^{2q} + \left| V^{\dagger}VB^* \right|^{2q} \right\|^{1/q}$$

and

$$d_2 := \left\| \frac{1}{p}\left(\left\| V \right| A \right|^{2p} + \left| \left| V^{\dagger} \right| A \right|^{2p} \right) + \frac{1}{q}\left(\left| VV^{\dagger}B^* \right|^{2q} + \left| V^{\dagger}VB^* \right|^{2q} \right) \right\|.$$

In particular, we have the upper bounds

$$w_e^2\left(BVA, BV^{\dagger}A\right) \leq \min\{e_1, e_2\},$$

where

$$e_1 := \left\| \left\| V \right| A \right|^4 + \left| \left| V^{\dagger} \right| A \right|^4 \Big\|^{1/2} \left\| \left| VV^{\dagger}B^* \right|^4 + \left| V^{\dagger}VB^* \right|^4 \right\|^{1/2}$$

and

$$e_2 := \frac{1}{2}\left\| \left\| V \right| A \right|^4 + \left| \left| V^{\dagger} \right| A \right|^4 + \left| VV^{\dagger}B^* \right|^4 + \left| V^{\dagger}VB^* \right|^4 \Big\|.$$

Proof. From (3.13), we get for $y = x$ that

$$\begin{aligned} &|\langle BVAx, x\rangle|^2 + \left|\left\langle BV^{\dagger}Ax, x\right\rangle\right|^2 \\ &\leq \left\langle \left(\left\| V \right| A \right|^{2p} + \left| \left| V^{\dagger} \right| A \right|^{2p} \right) x, x \right\rangle^{1/p} \left\langle \left(\left| VV^{\dagger}B^* \right|^{2q} + \left| V^{\dagger}VB^* \right|^{2q} \right) x, x \right\rangle^{1/q} \end{aligned} \tag{4.21}$$

for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $x \in H$ with $\|x\| = 1$.

If we take the supremum over $\|x\| = 1$, then we get

$$\begin{aligned} &w_e^2\left(BVA, BV^{\dagger}A\right) \\ &= \sup_{\|x\|=1}\left(|\langle BVAx, x\rangle|^2 + \left|\left\langle BV^{\dagger}Ax, x\right\rangle\right|^2 \right) \\ &\leq \sup_{\|x\|=1} \left\langle \left(\left\| V \right| A \right|^{2p} + \left| \left| V^{\dagger} \right| A \right|^{2p} \right) x, x \right\rangle^{1/p} \\ &\quad \times \sup_{\|x\|=1} \left\langle \left(\left| VV^{\dagger}B^* \right|^{2q} + \left| V^{\dagger}VB^* \right|^{2q} \right) x, x \right\rangle^{1/q} \\ &= \left\| \left\| V \right| A \right|^{2p} + \left| \left| V^{\dagger} \right| A \right|^{2p} \Big\|^{1/p} \left\| \left| VV^{\dagger}B^* \right|^{2q} + \left| V^{\dagger}VB^* \right|^{2q} \right\|^{1/q}, \end{aligned}$$

which proves the first upper bound.

If we use Young's inequality

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q, \; a, b \geq 0,$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we also have

$$\left\langle \left(\left\| V \right| A \right|^{2p} + \left| \left| V^{\dagger} \right| A \right|^{2p} \right) x, x \right\rangle^{1/p} \left\langle \left(\left| VV^{\dagger}B^* \right|^{2q} + \left| V^{\dagger}VB^* \right|^{2q} \right) x, x \right\rangle^{1/q}$$

$$\leq \frac{1}{p}\left\langle \left(\||V|A|^{2p} + \left\| V^{\dagger} \right| A \right|^{2p} \right) x, x \right\rangle + \frac{1}{q}\left\langle \left(\left| VV^{\dagger}B^{*} \right|^{2q} + \left| V^{\dagger}VB^{*} \right|^{2q} \right) x, x \right\rangle$$
$$= \left\langle \left[\frac{1}{p}\left(\||V|A|^{2p} + \left\| V^{\dagger} \right| A \right|^{2p} \right) + \frac{1}{q}\left(\left| VV^{\dagger}B^{*} \right|^{2q} + \left| V^{\dagger}VB^{*} \right|^{2q} \right) \right] x, x \right\rangle$$

for $x \in H$ with $\|x\| = 1$.

Utilizing (4.21), we then get

$$|\langle BVAx, x \rangle|^{2} + \left| \left\langle BV^{\dagger}Ax, x \right\rangle \right|^{2}$$
$$\leq \left\langle \left[\frac{1}{p}\left(\||V|A|^{2p} + \left\| V^{\dagger} \right| A \right|^{2p} \right) + \frac{1}{q}\left(\left| VV^{\dagger}B^{*} \right|^{2q} + \left| V^{\dagger}VB^{*} \right|^{2q} \right) \right] x, x \right\rangle$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$ we obtain the second upper bound as well. □

Remark 4.4. If we take $A = B = I$ in Proposition 4.2, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = |\beta| = 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\frac{1}{2}w^{2}\left(\alpha V + \beta V^{\dagger}\right) \leq \frac{1}{2}w^{2}\left(V, V^{\dagger}\right) \leq w_{e}^{2}\left(V, BV^{\dagger}A\right) \leq \min\{f_{1}, f_{2}\},$$

where

$$f_{1} := \left\| |V|^{2p} + \left| V^{\dagger} \right|^{2p} \right\|^{1/p} \left\| VV^{\dagger} + V^{\dagger}V \right\|^{1/q}$$

and

$$f_{2} := \left\| \frac{1}{p}\left(|V|^{2p} + \left| V^{\dagger} \right|^{2p} \right) + \frac{1}{q}\left(VV^{\dagger} + V^{\dagger}V \right) \right\|.$$

In particular, we have the upper bounds

$$w_{e}^{2}\left(V, V^{\dagger}\right) \leq \min\{g_{1}, g_{2}\},$$

where

$$g_{1} := \left\| |V|^{4} + \left| V^{\dagger} \right|^{4} \right\|^{1/2} \left\| VV^{\dagger} + V^{\dagger}V \right\|^{1/2}$$

and

$$g_{2} := \frac{1}{2}\left\| |V|^{4} + \left| V^{\dagger} \right|^{4} + VV^{\dagger} + V^{\dagger}V \right\|.$$

5. The case of invertible operators

When the operator $V \in B(H)$ is invertible, we have $V^{\dagger} = V^{-1}$ and we can state the following inequalities for this class of operators. The obtained results are simpler in this case.

Let $A, B \in B(H)$. Then for all $\lambda \in [0, 1]$, we get from Theorem 4.2 that

$$w(BVA) \leq \min\left\{\widehat{k_{1}}, \widehat{k_{2}}, \widehat{k_{3}}\right\},$$

where

$$\widehat{k_{1}} := \min\left\{ \left\| (1-\lambda)\, \||V|A|^{2} + \lambda\, |B^{*}|^{2} \right\|^{1/2} \||V|A\|^{\lambda}\, \|B\|^{1-\lambda}, \right.$$

$$\left\|\lambda \left||V|A\right|^2 + (1-\lambda)\left|B^*\right|^2\right\|^{1/2} \left\||V|A\right\|^{1-\lambda} \|B\|^{\lambda}\Big\},$$

$$\widehat{k_2} := \left\|(1-\lambda)\left||V|A\right|^2 + \lambda\left|B^*\right|^2\right\|^{1/2} \left\|\lambda\left||V|A\right|^2 + (1-\lambda)\left|B^*\right|^2\right\|^{1/2},$$

and

$$\widehat{k_3} := \left\|\frac{\left||V|A\right|^2 + \left|B^*\right|^2}{2}\right\|.$$

We observe that, by taking $\lambda = 1/2$ in k_1, we also have that

$$w(BVA) \le \min\left\{\left\||V|A\right\|^{1/2} \|B\|^{1/2}, \left\|\frac{\left||V|A\right|^2 + \left|B^*\right|^2}{2}\right\|^{1/2}\right\} \times \left\|\frac{\left||V|A\right|^2 + \left|B^*\right|^2}{2}\right\|^{1/2}.$$

From Theorem 4.3, we get for $A, B \in B(H)$ and $p \ge 1$ that

$$w^{2p}(BVA) \le \frac{\left\||V|A\right\|^{2p} \|B\|^{2p} + w^p\left(\left|B^*\right|^2 \left||V|A\right|^2\right)}{2} \le \left\||V|A\right\|^{2p} \|B\|^{2p}$$

and

$$w^{2p}(BVA) \le \frac{1}{2}\left\|\left||V|A\right|^{4p} + \left|B^*\right|^{4p}\right\|.$$

In particular, for $p = 1$, we get

$$w^2(BVA) \le \frac{\left\||V|A\right\|^2 \|B\|^2 + w\left(\left|B^*\right|^2 \left||V|A\right|^2\right)}{2} \le \left\||V|A\right\|^2 \|B\|^2$$

and

$$w^2(BVA) \le \frac{1}{2}\left\|\left||V|A\right|^4 + \left|B^*\right|^4\right\|.$$

Finally, if we take $A = B = I$ in the inequalities above, then we have the following bounds for the invertible operators V:

$$w(V) \le \min\{\widehat{s_1}, \widehat{s_2}, \widehat{s_3}\},$$

where

$$\widehat{s_1} := \min\left\{\left\|(1-\lambda)|V|^2 + \lambda I\right\|^{1/2} \|V\|^{\lambda}, \left\|\lambda|V|^2 + (1-\lambda)I\right\|^{1/2} \|V\|^{1-\lambda}\right\},$$

$$\widehat{s_2} := \left\|(1-\lambda)|V|^2 + \lambda I\right\|^{1/2} \left\|\lambda|V|^2 + (1-\lambda)I\right\|^{1/2},$$

and

$$\widehat{s_3} := \left\|\frac{|V|^2 + I}{2}\right\|.$$

We observe that, by taking $\lambda = 1/2$, we also have that

$$w(V) \le \min\left\{\|V\|^{1/2}, \left\|\frac{|V|^2 + I}{2}\right\|^{1/2}\right\} \left\|\frac{|V|^2 + I}{2}\right\|^{1/2}.$$

Also for $p \ge 1$, we get that

$$w^{2p}(V) \le \frac{1}{2}\left\||V|^{4p} + I\right\|.$$

Remark 5.1. The following example gives a numerical validation of the inequality

$$w^2(V) \le \min\left\{\|V\|, \left\|\frac{|V|^2+I}{2}\right\|\right\}\left\|\frac{|V|^2+I}{2}\right\| \tag{5.1}$$

that holds for any invertible operator $V \in B(H)$.

We consider the 2×2 invertible matrix

$$V = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

After some simple numerical calculations, we get that $w(V) = 2$, $\|V\| = 1 + \sqrt{2}$, and

$$\left\|\frac{|V|^2+I}{2}\right\| = 2 + \sqrt{2}.$$

If we replace these values in (5.1), then we get

$$4 \le \min\left\{1+\sqrt{2}, 2+\sqrt{2}\right\}\left(2+\sqrt{2}\right) = \left(1+\sqrt{2}\right)\left(2+\sqrt{2}\right) = 4 + 3\sqrt{2},$$

which numerically validates (5.1).

6. Conclusions

In this paper, we obtained some new norm and numerical radius inequalities for the product of three operators in terms of the Moore–Penrose inverse. Several upper bounds for powers of the numerical radius of operators $V \in CR(H)$, which complement some recent results stated in the Introduction, were also provided.

Author contributions

Najla Altwaijry: Funding acquisition, resources, writing–review and editing, project administration. Silvestru Sever Dragomir: Conceptualization, formal analysis, investigation, validation, visualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

Prof. Silvestru Sever Dragomir is the Guest Editor of special issue "Mathematical Inequalities and Applications" for AIMS Mathematics. Prof. Silvestru Sever Dragomir was not involved in the editorial review and the decision to publish this article.

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