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Theory article

Fractional calculus for the one-dimensional Laplacian with Robin boundary condition

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Abstract: We study fractional powers of the one-dimensional Robin Laplacian $A_h = -d^2/dx^2$ on the half-line, subject to the boundary condition $u'(0) = hu(0)$, in the non-negative regime $h \geq 0$. The analysis is based on a Robin-deformed cosine transform with kernel

$$\cos_h(x, \lambda) = \cos(\lambda x) + \frac{h}{\lambda}\sin(\lambda x),$$

which diagonalizes A_h. We prove a product formula for these eigenfunctions and use it to define positive generalized translations and a commutative Robin convolution. We then derive an explicit Robin heat kernel and Gaussian bounds. Finally, we define the fractional powers $A_h^{s/2}$, $0 < s < 2$, by spectral calculus and obtain their Bochner representation and singular-integral formula in terms of the Robin translations and an associated Lévy kernel. The Caffarelli–Silvestre extension is formulated for an extension exponent $0 < \sigma < 1$; when it is applied to the operator $A_h^{s/2}$, one takes $\sigma = s/2$. The case $h = 0$ recovers the fractional Neumann Laplacian on the half-line.

Keywords: Robin Laplacian; fractional powers of operators; Robin-deformed cosine transform; Bochner subordination; Caffarelli–Silvestre extension; maximum principle; spectral positivity estimate; Lévy kernel; heat semigroup
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1. Introduction

Boundary conditions play a decisive role in spectral theory. They determine the self-adjoint realization of a differential operator and, consequently, the associated functional calculus. In this paper we study the one-dimensional Robin Laplacian on the half-line,

$$A_h u = -u'', \qquad \mathrm{Dom}(A_h) = \{u \in H^2(0,\infty) : u'(0) = hu(0)\}, \qquad h \in \mathbb{R}. \tag{1.1}$$

The parameter h interpolates between the Neumann condition ($h = 0$) and the Dirichlet condition in the limiting regime $h \to +\infty$. For $h \geq 0$, the operator A_h is non-negative, while for $h < 0$ it has one negative bound state. In applications, the Robin parameter describes, for example, heat exchange at the boundary, a delta-type boundary interaction in quantum mechanics, or partial reflection of Brownian motion at the origin; see [1, 2].

The aim of the paper is to develop a fractional calculus for A_h in the non-negative regime $h \geq 0$. The main point is that the nonlocal operator $A_h^{s/2}$, $0 < s < 2$, is not obtained by simply restricting the usual fractional Laplacian to the half-line. The boundary condition changes the underlying spectral transform, the translation structure, and the associated Lévy kernel. In particular, the zero-energy eigenfunction is

$$\cos_h(x, 0) = 1 + hx,$$

rather than the constant function 1. This feature is reflected in the singular-integral formula for $A_h^{s/2}$, where the usual difference $f(x) - f(x+y)$ is replaced by a Robin-adapted difference involving the factor $1 + hy$. The analysis is based on the Robin-deformed cosine transform

$$(C_h f)(\lambda) = \int_0^\infty f(x) \cos_h(x, \lambda)\, dx, \qquad \lambda > 0, \tag{1.2}$$

where

$$\cos_h(x, \lambda) = \cos(\lambda x) + \frac{h}{\lambda} \sin(\lambda x), \qquad \cos_h(x, 0) = 1 + hx.$$

This transform diagonalizes the Robin Laplacian. The corresponding spectral measure is

$$d\mu_h(\lambda) = \frac{2}{\pi} \frac{\lambda^2}{h^2 + \lambda^2}\, d\lambda,$$

which incorporates the Robin normalization; see [3–5]. When $h = 0$, the transform reduces to the usual Fourier cosine transform.

Fractional powers of self-adjoint operators and boundary value problems have been widely studied. The Caffarelli–Silvestre extension [6] realizes the fractional Laplacian as a Dirichlet-to-Neumann operator, and the abstract spectral version was developed by Stinga and Torrea [7]. Related developments include fractional operators on domains with boundary conditions [8–10], fractional Bessel-type operators [11, 12], and ground-state representations for half-line Dirichlet fractional Laplacians [13]. Recent applied works also show the continuing role of fractional, stochastic, and numerical techniques in boundary-sensitive and nonlocal models; see, for example, [14–16]. The present work focuses on the Robin half-line case and gives an explicit harmonic-analytic construction adapted to the boundary condition.

The first part of the paper develops the spectral and convolution structure associated with C_h. We prove inversion, Plancherel and diagonalization formulas, and then establish a product formula for $\cos_h$ when $h \geq 0$. This product formula leads to positive generalized translations and to a commutative Robin convolution. We also show that the Robin transform can be factored through the classical cosine transform by a Volterra transmutation operator.

The second part studies the Robin heat semigroup. We obtain an explicit closed formula for the heat kernel,

$$p_{h,t}(y) = \frac{1}{\sqrt{\pi t}} e^{-y^2/(4t)} - h e^{hy+h^2 t} \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + h\sqrt{t}\right), \qquad h \geq 0,$$

where erfc is the complementary error function. We prove positivity and Gaussian bounds for this kernel. These estimates are then used to define and analyze the Robin Lévy kernel.

The third part is devoted to the fractional powers $A_h^{s/2}$, $0 < s < 2$. We define them by spectral calculus, prove a Bochner subordination formula, and derive the singular-integral representation

$$A_h^{s/2} f(x) = \int_0^\infty [(1 + hy) f(x) - (\tau_x^h f)(y)] L_{h,s}(y)\, dy.$$

Here, τ_x^h is the Robin translation and $L_{h,s}$ is the associated Lévy kernel. We compute the leading asymptotics of $L_{h,s}$: Near the origin it has the same singularity as the classical fractional Laplacian,

$$L_{h,s}(y) \sim c_s y^{-1-s}, \qquad y \downarrow 0,$$

whereas for $h > 0$ it decays faster at infinity,

$$L_{h,s}(y) \sim K_{h,s} y^{-2-s}, \qquad y \to +\infty.$$

Thus, the Robin boundary condition does not change the local order of the fractional operator, but it does modify its global tail.

The final part connects the construction with standard tools of nonlocal analysis. For $0 < s < 1$, we prove a Caffarelli–Silvestre type extension: If $a = 1 - 2s$, then the extension solves

$$\partial_x^2 U + \partial_t^2 U + \frac{a}{t} \partial_t U = 0, \qquad x > 0,\ t > 0,$$

with the Robin boundary condition propagated to the spatial boundary,

$$\partial_x U(0, t) = hU(0, t).$$

The conormal derivative at $t = 0$ recovers A_h^s up to the constant

$$\kappa_s = \frac{\Gamma(1 - s)}{2^{2s-1}\Gamma(s)}.$$

We also prove a pointwise maximum principle and a spectral positivity inequality for powers of A_h. We also include a concrete example with $f(x) = e^{-x}$, illustrating how the Robin parameter enters the transform and the fractional spectral multiplier. Finally, we compare the Robin fractional operator with the classical Riesz fractional derivative on $\mathbb{R}$ and with the Neumann and Dirichlet half-line limits.

Notation. Throughout, $h \in \mathbb{R}$ denotes the Robin parameter, while the main positivity and convolution results are stated for $h \geq 0$. The fractional exponent is denoted by $s \in (0, 2)$, with $s \in (0, 1)$ in the extension problem. We write $L^p(0, \infty)$ for the Lebesgue space with respect to dx, and $d\mu_h$ for the Robin spectral measure. The notation $A \lesssim B$ means that $A \leq CB$, where the constant C may depend on the fixed parameters.

All normalization constants appearing below, such as c_s, $C_{1,s}$, $K_{h,s}$, κ_s, and $\kappa_{s/2}$, are introduced explicitly at the point where they enter the corresponding kernel estimate, asymptotic formula, or extension identity.

2. The Robin Laplacian and its spectral transform

The self-adjoint Robin Laplacian A_h is defined by (1.1). Its regular eigenfunctions, normalized by $u(0) = 1$, are

$$\cos_h(x, \lambda) = \cos(\lambda x) + \frac{h}{\lambda} \sin(\lambda x), \qquad \lambda > 0, \tag{2.1}$$

extended continuously to $\cos_h(x, 0) = 1+hx$. They satisfy $A_h \cos_h(\cdot, \lambda) = \lambda^2 \cos_h(\cdot, \lambda)$ in the generalized sense, with $\cos_h(0, \lambda) = 1$, $\partial_x \cos_h(0, \lambda) = h$. The Robin-deformed cosine transform is

$$(C_h f)(\lambda) = \int_0^\infty f(x) \cos_h(x, \lambda)\, dx, \qquad f \in C_c^\infty(0, \infty),\ \lambda > 0. \tag{2.2}$$

The Weyl–Titchmarsh theory of half-line Sturm–Liouville operators [3–5, 17] gives the absolutely continuous spectral density relative to the regular solution: with $E = \lambda^2$, $\lambda > 0$,

$$d\mu_h(\lambda) = \frac{2}{\pi} \frac{\lambda^2}{h^2 + \lambda^2}\, d\lambda. \tag{2.3}$$

For $h \geq 0$, $\operatorname{spec}(A_h) = [0, \infty)$ is purely absolutely continuous. For $h < 0$, A_h has the additional simple eigenvalue

$$E_h = -h^2, \qquad \phi_h(x) = \sqrt{-2h}\, e^{hx}, \qquad \|\phi_h\|_{L^2} = 1. \tag{2.4}$$

We now prove the Plancherel and inversion formulas for C_h, treating uniformly the case where a bound state is present.

Theorem 2.1. *The transform C_h admits a unique extension to a partial isometry on $L^2(0, \infty)$. Specifically:*

(a) *If $h \geq 0$, $C_h\colon L^2(0, \infty) \to L^2((0, \infty), d\mu_h)$ is unitary, and for every $f \in L^2(0, \infty)$,*

$$f(x) = \int_0^\infty (C_h f)(\lambda) \cos_h(x, \lambda) d\mu_h(\lambda), \tag{2.5}$$

$$\|f\|^2_{L^2(0,\infty)} = \int_0^\infty |(C_h f)(\lambda)|^2\, d\mu_h(\lambda); \tag{2.6}$$

(b) *If $h < 0$, the same identities hold with the bound-state contribution:*

$$f(x) = \langle f, \phi_h \rangle \phi_h(x) + \int_0^\infty (C_h f)(\lambda) \cos_h(x, \lambda) d\mu_h(\lambda), \tag{2.7}$$

$$\|f\|^2_{L^2} = |\langle f, \phi_h \rangle|^2 + \int_0^\infty |(C_h f)(\lambda)|^2\, d\mu_h(\lambda). \tag{2.8}$$

For every $f \in \operatorname{Dom}(A_h)$ belonging to the absolutely continuous subspace,

$$C_h(A_h f)(\lambda) = \lambda^2 (C_h f)(\lambda), \quad \text{a.e. } \lambda > 0. \tag{2.9}$$

If, in addition, f is chosen so that both sides are continuous functions of λ on $(0, \infty)$, then the identity holds pointwise for every $\lambda > 0$. Thus, the earlier “a.e.” wording was only a spectral-measure statement and is not needed in the continuous-kernel formulation.

Proof. Let

$$\tau u = -u'', \qquad x > 0,$$

with the Robin boundary condition

$$u'(0) = hu(0).$$

Thus, A_h is the self-adjoint realization of τ in $L^2(0, \infty)$ with this boundary condition.

For $z \in \mathbb{C} \setminus [0, \infty)$, choose the branch

$$k(z) = \sqrt{z}, \qquad \Im k(z) > 0.$$

The regular solution normalized by

$$u(0) = 1, \qquad u'(0) = h$$

is

$$c_h(x, z) := \cos(k(z)x) + \frac{h}{k(z)} \sin(k(z)x).$$

Hence, for $z = \lambda^2 > 0$, one has

$$c_h(x, \lambda^2) = \cos_h(x, \lambda),$$

where $\cos_h$ is the Robin cosine introduced in (2.1). Moreover,

$$c_h(0, z) = 1, \qquad c_h'(0, z) = h, \qquad -c_h''(x, z) = z\, c_h(x, z).$$

The Weyl solution at $+\infty$ is the unique solution of

$$-u'' = zu$$

which belongs to $L^2(0, \infty)$. With our choice of branch, it is

$$\psi(x, z) = e^{ik(z)x}.$$

Its Wronskian with c_h is

$$\begin{aligned} W(c_h, \psi) &= c_h(x, z)\psi'(x, z) - c_h'(x, z)\psi(x, z) \\ &= ik(z) - h. \end{aligned}$$

Consequently, the Green kernel of the resolvent $(A_h - z)^{-1}$ is

$$G_h(x, y; z) = \frac{c_h(\min(x, y), z)\psi(\max(x, y), z)}{h - ik(z)}.$$

Indeed, $G_h(\cdot, y; z)$ satisfies the Robin boundary condition at 0, is square-integrable at $+\infty$, solves

$$\left(-\frac{d^2}{dx^2} - z\right) G_h(x, y; z) = 0 \qquad x \neq y,$$

and has the correct jump of derivative at $x = y$.

The Weyl–Titchmarsh m-function in the normalization $c_h(0,z) = 1$, $c_h'(0,z) = h$, is therefore

$$m_h(z) = \frac{1}{h - i\sqrt{z}}.$$

By the Weyl–Titchmarsh theorem, m_h admits the spectral representation

$$m_h(z) = \int_{\mathbb{R}} \frac{d\rho_h(E)}{E - z}, \qquad z \in \mathbb{C} \setminus \mathbb{R},$$

where ρ_h is the spectral measure associated with the regular solution c_h.

We now compute the absolutely continuous part of this measure. Let $E = \lambda^2 > 0$. Since the boundary value of $\sqrt{E + i0}$ is λ, we have

$$m_h(E + i0) = \frac{1}{h - i\lambda} = \frac{h + i\lambda}{h^2 + \lambda^2}.$$

Hence, by the Stieltjes inversion formula,

$$\frac{d\rho_h}{dE}(E) = \frac{1}{\pi}\Im m_h(E + i0) = \frac{1}{\pi}\frac{\lambda}{h^2 + \lambda^2}.$$

Since $E = \lambda^2$ and $dE = 2\lambda\, d\lambda$, this becomes

$$d\rho_h(E) = d\mu_h(\lambda) = \frac{2}{\pi}\frac{\lambda^2}{h^2 + \lambda^2}\, d\lambda, \qquad \lambda > 0.$$

This is exactly the spectral measure (2.3).

If $h \geq 0$, the denominator $h - i\sqrt{z}$ has no zero on $(-\infty, 0)$. Therefore, A_h has no negative eigenvalue, and the spectrum is purely absolutely continuous:

$$\operatorname{spec}(A_h) = [0, \infty).$$

If $h < 0$, then m_h has a simple pole at

$$E_h = -h^2.$$

Indeed,

$$\sqrt{E_h} = i(-h),$$

and therefore

$$h - i\sqrt{E_h} = 0.$$

Near $z = -h^2$, one obtains

$$m_h(z) = \frac{2h}{z + h^2} + O(1).$$

On the other hand, an atom of mass a at $E_h = -h^2$ in the spectral measure contributes

$$\frac{a}{E_h - z} = -\frac{a}{z + h^2}$$

to the Weyl function. Comparing residues gives

$$a = -2h > 0.$$

The corresponding eigenfunction is

$$c_h(x, -h^2) = e^{hx}.$$

After normalization in $L^2(0, \infty)$, this gives

$$\phi_h(x) = \sqrt{-2h}\, e^{hx},$$

which is precisely (2.4).

We now apply the Weyl–Titchmarsh spectral theorem. Initially, for $f \in C_c^\infty(0, \infty)$, define

$$\mathcal{F}_h f(E) = \int_0^\infty f(x) c_h(x, E)\ dx.$$

The spectral theorem extends this map uniquely to a unitary map from $L^2(0, \infty)$ onto $L^2(\mathbb{R}, d\rho_h)$, with the possible point mass at $E_h = -h^2$ included when $h < 0$.

On the absolutely continuous part, $E = \lambda^2 > 0$, the transform $\mathcal{F}_h$ is exactly the Robin-deformed cosine transform:

$$\mathcal{F}_h f(\lambda^2) = \int_0^\infty f(x) \cos_h(x, \lambda)\ dx = (C_h f)(\lambda).$$

Therefore, if $h \geq 0$, there is no point spectrum, and the Weyl–Titchmarsh inversion formula gives

$$f(x) = \int_0^\infty (C_h f)(\lambda) \cos_h(x, \lambda)\ d\mu_h(\lambda).$$

The corresponding Parseval identity gives

$$\|f\|^2_{L^2(0,\infty)} = \int_0^\infty |(C_h f)(\lambda)|^2\ d\mu_h(\lambda).$$

This proves (2.5) and (2.6).

If $h < 0$, the spectral measure has one additional atom at

$$E_h = -h^2.$$

Thus, the spectral expansion splits into the orthogonal sum of the bound state and the absolutely continuous part:

$$f = \langle f, \phi_h \rangle \phi_h + f_{\text{ac}}.$$

Consequently,

$$f(x) = \langle f, \phi_h \rangle \phi_h(x) + \int_0^\infty (C_h f)(\lambda) \cos_h(x, \lambda)\ d\mu_h(\lambda),$$

and Parseval's identity becomes

$$\|f\|^2_{L^2} = |\langle f, \phi_h \rangle|^2 + \int_0^\infty |(C_h f)(\lambda)|^2\ d\mu_h(\lambda).$$

This proves (2.7) and (2.8). In this case, C_h is unitary on the absolutely continuous subspace and vanishes on span$\{\phi_h\}$. Hence, C_h is a partial isometry on $L^2(0, \infty)$.

It remains to prove the diagonalization formula. For f in a smooth core of A_h, integration by parts gives

$$\begin{aligned} C_h(A_h f)(\lambda) &= \int_0^\infty (-f''(x)) \cos_h(x,\lambda)\, dx \\ &= \Big[-f'(x)\cos_h(x,\lambda) + f(x)\partial_x \cos_h(x,\lambda)\Big]_0^\infty \\ &\quad + \int_0^\infty f(x)(-\partial_x^2 \cos_h(x,\lambda))\, dx. \end{aligned}$$

The boundary term at $+\infty$ vanishes by the core approximation. The boundary term at 0 vanishes because

$$f'(0) = hf(0), \qquad \partial_x \cos_h(0,\lambda) = h\cos_h(0,\lambda) = h.$$

Moreover,

$$-\partial_x^2 \cos_h(x,\lambda) = \lambda^2 \cos_h(x,\lambda).$$

Therefore,

$$C_h(A_h f)(\lambda) = \lambda^2 (C_h f)(\lambda).$$

By density in the graph norm of A_h, equivalently by the spectral theorem, this identity extends to every $f \in \mathrm{Dom}(A_h)$ belonging to the absolutely continuous subspace. This proves (2.9). □

3. Volterra transmutation

For $h \in \mathbb{R}$, define

$$(\mathcal{U}_h f)(x) = f(x) + h\int_0^x f(t)\, dt, \qquad x \geq 0. \tag{3.1}$$

A direct computation shows $\mathcal{U}_h(\cos(\lambda\,\cdot))(x) = \cos_h(x,\lambda)$ for all $\lambda \geq 0$. The Volterra inverse is

$$(\mathcal{U}_h^{-1} f)(x) = f(x) - h\int_0^x e^{-h(x-t)} f(t)\, dt, \tag{3.2}$$

which one verifies by direct substitution. Composing these for two parameters $h_1, h_2 \in \mathbb{R}$ gives $T_{h_1,h_2} := \mathcal{U}_{h_2}\mathcal{U}_{h_1}^{-1}$, which acts as

$$(T_{h_1,h_2} f)(x) = f(x) + (h_2 - h_1)\int_0^x e^{-h_1(x-t)} f(t)\, dt. \tag{3.3}$$

The dual operator on test functions reads

$$({}^t\mathcal{U}_h g)(x) = g(x) + h\int_x^\infty g(t)\, dt, \tag{3.4}$$

and one verifies $\int_0^\infty (\mathcal{U}_h f) g\, dx = \int_0^\infty f({}^t\mathcal{U}_h g)\, dx$ for $f, g \in C_c^\infty(0,\infty)$.

The next proposition shows that the Robin-deformed cosine kernel is obtained from the classical cosine kernel by an explicit triangular integral operator, and consequently C_h factors through the cosine transform.

Proposition 3.1. *Let $h_1, h_2 \geq 0$, and assume that the transmutation operator $\mathcal{U}_h$ satisfies*

$$\mathcal{U}_h(\cos(\lambda\,\cdot)) = \cos_h(\cdot, \lambda), \qquad \lambda \geq 0.$$

Define

$$T_{h_1,h_2} := \mathcal{U}_{h_2}\mathcal{U}_{h_1}^{-1}.$$

Then, for every $\lambda \geq 0$,

$$T_{h_1,h_2}\cos_{h_1}(\cdot, \lambda) = \cos_{h_2}(\cdot, \lambda). \tag{3.5}$$

Moreover, on $C_c^\infty(0, \infty)$,

$$C_h = \mathcal{F}_c \circ {}^t\mathcal{U}_h, \tag{3.6}$$

where

$$(\mathcal{F}_c g)(\lambda) := \int_0^\infty g(x)\cos(\lambda x)\,dx$$

is the classical cosine transform, and ${}^t\mathcal{U}_h$ denotes the transpose of $\mathcal{U}_h$ with respect to the $L^2(0, \infty)$-pairing. More generally,

$$C_{h_2} = C_{h_1} \circ {}^tT_{h_1,h_2} \qquad \text{on } C_c^\infty(0, \infty). \tag{3.7}$$

Proof. By the definition of T_{h_1,h_2}, we have

$$T_{h_1,h_2} = \mathcal{U}_{h_2}\mathcal{U}_{h_1}^{-1}.$$

Since

$$\cos_{h_1}(\cdot, \lambda) = \mathcal{U}_{h_1}(\cos(\lambda\,\cdot)),$$

we get

$$\begin{aligned} T_{h_1,h_2}\cos_{h_1}(\cdot, \lambda) &= \mathcal{U}_{h_2}\mathcal{U}_{h_1}^{-1}\mathcal{U}_{h_1}(\cos(\lambda\,\cdot)) \\ &= \mathcal{U}_{h_2}(\cos(\lambda\,\cdot)) \\ &= \cos_{h_2}(\cdot, \lambda). \end{aligned}$$

This proves (3.5).

We now prove the factorization of the transform. Let $f \in C_c^\infty(0, \infty)$. By the definition of the transpose,

$$\langle {}^t\mathcal{U}_h f, \psi\rangle = \langle f, \mathcal{U}_h\psi\rangle$$

for every admissible test function ψ. Taking $\psi(x) = \cos(\lambda x)$, we obtain

$$\begin{aligned} (\mathcal{F}_c({}^t\mathcal{U}_h f))(\lambda) &= \int_0^\infty ({}^t\mathcal{U}_h f)(x)\cos(\lambda x)\,dx \\ &= \int_0^\infty f(x)\,\mathcal{U}_h(\cos(\lambda\,\cdot))(x)\,dx \\ &= \int_0^\infty f(x)\cos_h(x, \lambda)\,dx \\ &= (C_h f)(\lambda). \end{aligned}$$

Hence,

$$C_h = \mathcal{F}_c \circ {}^t\mathcal{U}_h$$

on $C_c^\infty(0,\infty)$.

Finally, using (3.5), for $f \in C_c^\infty(0,\infty)$ we have

$$\begin{aligned}(C_{h_2}f)(\lambda) &= \int_0^\infty f(x)\cos_{h_2}(x,\lambda)\,dx \\ &= \int_0^\infty f(x)\,(T_{h_1,h_2}\cos_{h_1}(\cdot,\lambda))(x)\,dx \\ &= \int_0^\infty ({}^tT_{h_1,h_2}f)(x)\cos_{h_1}(x,\lambda)\,dx \\ &= (C_{h_1}({}^tT_{h_1,h_2}f))(\lambda).\end{aligned}$$

Therefore,

$$C_{h_2} = C_{h_1} \circ {}^tT_{h_1,h_2}$$

on $C_c^\infty(0,\infty)$. The proof is complete. □

4. Generalised translation and Robin convolution

Throughout this section we assume $h \geq 0$. Positivity of the translation measures, hence of the convolution, fails for $h < 0$; this is consistent with the loss of non-negativity of A_h in that regime.
To derive a generalized translation operator, we consider the boundary value problem for the wave equation

$$\begin{cases} u_{xx} = u_{yy}, & x, y > 0, \\ u(x,0) = f(x), & \\ u_y(x,0) = hf(x), & x > 0, \end{cases} \tag{4.1}$$

where $f \in C^1([0,\infty))$. In characteristic variables $\xi = x+y, \eta = x-y$, the equation reduces to $u_{\xi\eta} = 0$, so $u(x,y) = F(x+y) + G(x-y)$ for some C^1 functions F, G. The boundary data at $y = 0$ yield $F(x) + G(x) = f(x)$ and $F'(x) - G'(x) = hf(x)$; differentiating the first relation gives $F'(x) + G'(x) = f'(x)$, and, solving, $F' = \frac{1}{2}(f' + hf)$, $G' = \frac{1}{2}(f' - hf)$. Setting $H(x) = \int_0^x f(t)dt$ and choosing constants to satisfy $F + G = f$, $F(x) = \frac{1}{2}f(x) + (h/2)H(x)$, $G(x) = \frac{1}{2}f(x) - (h/2)H(x)$. Therefore, for $x \geq y \geq 0$,

$$u(x,y) = \tfrac{1}{2}[f(x+y) + f(x-y)] + \frac{h}{2}\int_{x-y}^{x+y} f(t)\,dt.$$

By symmetry of the resulting expression in x, y, the formula extends to all $x, y \geq 0$ on replacing $x - y$ by $|x-y|$. This motivates the following definition (cf. [18] for the analogous construction in the Bessel setting, and [19] for the general theory of hypergroups).

For $h \geq 0$ and $x, y \geq 0$, we define a positive measure $\delta_x *_h \delta_y$ on $[0,\infty)$ by

$$\delta_x *_h \delta_y := \frac{1}{2}\delta_{|x-y|} + \frac{1}{2}\delta_{x+y} + \frac{h}{2}\mathbf{1}_{[|x-y|,x+y]}(t)\ dt. \tag{4.2}$$

Here, δ_a denotes the Dirac mass at the point a, and $\mathbf{1}_{[|x-y|,x+y]}(t)\ dt$ denotes the restriction of the Lebesgue measure to the interval

$$[|x-y|\,, x+y].$$

Thus, the measure $\delta_x *_h \delta_y$ consists of two atomic parts, located at $|x-y|$ and $x+y$, together with an absolutely continuous part which is uniformly distributed on the interval $[|x-y|, x+y]$ with density $h/2$.

Its total mass is

$$\begin{aligned}(\delta_x *_h \delta_y)([0,\infty)) &= \frac{1}{2} + \frac{1}{2} + \frac{h}{2}\int_{|x-y|}^{x+y} dt \\ &= 1 + \frac{h}{2}(x+y-|x-y|) \\ &= 1 + h\min\{x,y\}.\end{aligned}$$

Definition 4.1. For $h \geq 0$ and $x, y \geq 0$, the *Robin translation* τ_x^h is defined by

$$(\tau_x^h f)(y) = \int_{[0,\infty)} f(t)(\delta_x *_h \delta_y)(dt), \tag{4.3}$$

that is,

$$(\tau_x^h f)(y) = \frac{1}{2}f(|x-y|) + \frac{1}{2}f(x+y) + \frac{h}{2}\int_{|x-y|}^{x+y} f(t)\, dt. \tag{4.4}$$

The product formula below identifies $\cos_h(\cdot, \lambda)$ as a one-parameter family of multiplicative characters of the Robin hypergroup.

Lemma 4.2 (Robin product formula)**.** *For all $x, y \geq 0$ and $\lambda \geq 0$,*

$$\tau_x^h \cos_h(\cdot, \lambda)(y) = \cos_h(x, \lambda)\cos_h(y, \lambda). \tag{4.5}$$

Proof. We give the details because the endpoint $x = y$ is important. By symmetry in x and y, it is enough to assume $x \geq y \geq 0$. Then, $|x-y| = x-y$, including the endpoint case $x = y$, and the translation formula gives

$$\tau_x^h f(y) = \frac{1}{2}[f(x-y) + f(x+y)] + \frac{h}{2}\int_{x-y}^{x+y} f(r)\, dr.$$

First, let $\lambda > 0$ and set $f(r) = \cos_h(r, \lambda)$. Since

$$\cos_h(r, \lambda) = \cos(\lambda r) + \frac{h}{\lambda}\sin(\lambda r),$$

the even part of the translation is

$$\frac{1}{2}[\cos_h(x-y, \lambda) + \cos_h(x+y, \lambda)] = \cos(\lambda x)\cos(\lambda y) + \frac{h}{\lambda}\sin(\lambda x)\cos(\lambda y).$$

Indeed, this follows from

$$\frac{1}{2}[\cos(\lambda(x-y)) + \cos(\lambda(x+y))] = \cos(\lambda x)\cos(\lambda y),$$

and

$$\frac{1}{2}[\sin(\lambda(x-y)) + \sin(\lambda(x+y))] = \sin(\lambda x)\cos(\lambda y).$$

For the integral part, direct integration gives

$$\frac{h}{2}\int_{x-y}^{x+y}\cos(\lambda r)\,dr = \frac{h}{\lambda}\cos(\lambda x)\sin(\lambda y),$$

and

$$\frac{h}{2}\int_{x-y}^{x+y}\frac{h}{\lambda}\sin(\lambda r)\,dr = \frac{h^2}{\lambda^2}\sin(\lambda x)\sin(\lambda y).$$

Adding the even part and the integral part, we obtain

$$\begin{aligned}\tau_x^h\cos_h(\cdot,\lambda)(y) &= \cos(\lambda x)\cos(\lambda y) + \frac{h}{\lambda}\sin(\lambda x)\cos(\lambda y)\\ &\quad + \frac{h}{\lambda}\cos(\lambda x)\sin(\lambda y) + \frac{h^2}{\lambda^2}\sin(\lambda x)\sin(\lambda y)\\ &= \left(\cos(\lambda x) + \frac{h}{\lambda}\sin(\lambda x)\right)\left(\cos(\lambda y) + \frac{h}{\lambda}\sin(\lambda y)\right)\\ &= \cos_h(x,\lambda)\cos_h(y,\lambda).\end{aligned}$$

The endpoint $x = y$ is included in the above computation, because the lower limit of the integral is then 0 and no singular term appears.

It remains to consider $\lambda = 0$. Since

$$\cos_h(r,0) = 1 + hr,$$

we get, again for $x \geq y$,

$$\begin{aligned}\tau_x^h\cos_h(\cdot,0)(y) &= \frac{1}{2}[1 + h(x-y)] + \frac{1}{2}[1 + h(x+y)] + \frac{h}{2}\int_{x-y}^{x+y}(1+hr)\,dr\\ &= 1 + hx + hy + h^2xy\\ &= (1+hx)(1+hy)\\ &= \cos_h(x,0)\cos_h(y,0).\end{aligned}$$

By symmetry, the same identity holds for $x < y$. This proves the product formula for all $x, y \geq 0$ and all $\lambda \geq 0$. □

Proposition 4.3. *Let $h \geq 0$. For every $x \geq 0$, the Robin translation τ_x^h, has the following properties:*

(a) $\tau_0^h f = f$, *and* $(\tau_x^h f)(y) = (\tau_y^h f)(x)$, $\qquad x, y \geq 0$;

(b) *If $f \geq 0$, then $\tau_x^h f \geq 0$;*

(c) *For every $f \in C_c(0,\infty)$ and every $\lambda \geq 0$,*

$$C_h(\tau_x^h f)(\lambda) = \cos_h(x,\lambda)(C_h f)(\lambda);$$

(d) *τ_x^h is symmetric on $L^2(0,\infty)$. Moreover, for every $1 \leq p \leq \infty$, τ_x^h extends uniquely to a bounded operator on $L^p(0,\infty)$, and its operator norm is exactly*

$$\|\tau_x^h\|_{L^p(0,\infty)\to L^p(0,\infty)} = 1 + hx.$$

Proof. The identities in (a) and the positivity statement in (b) follow immediately from the explicit formula

$$(\tau_x^h f)(y) = \frac{1}{2}f(|x-y|) + \frac{1}{2}f(x+y) + \frac{h}{2}\int_{|x-y|}^{x+y} f(t)\ dt.$$

We now prove the L^2-symmetry and the L^p-boundedness directly from this formula. This avoids any use of an involution property of the Robin transform, which is not available here.

Fix $x \geq 0$. Write

$$\tau_x^h f = M_x f + \frac{h}{2}K_x f,$$

where

$$(M_x f)(y) := \frac{1}{2}f(|x-y|) + \frac{1}{2}f(x+y),$$

and

$$(K_x f)(y) := \int_{|x-y|}^{x+y} f(t)\ dt.$$

First, we prove symmetry on $L^2(0,\infty)$. Let $f, g \in C_c(0,\infty)$. For the operator M_x, using changes of variables in the two terms, one obtains

$$\begin{aligned}\int_0^\infty (M_x f)(y)\overline{g(y)}\ dy &= \frac{1}{2}\int_0^\infty f(|x-y|)\overline{g(y)}\ dy + \frac{1}{2}\int_0^\infty f(x+y)\overline{g(y)}\ dy\\ &= \int_0^\infty f(t)\overline{(M_x g)(t)}\ dt.\end{aligned}$$

Hence, M_x is symmetric.

For the integral part, Fubini's theorem gives

$$\begin{aligned}\int_0^\infty (K_x f)(y)\overline{g(y)}\ dy &= \int_0^\infty \left(\int_{|x-y|}^{x+y} f(t)\ dt\right)\overline{g(y)}\ dy\\ &= \int_0^\infty \int_0^\infty f(t)\overline{g(y)}\mathbf{1}_{\{|x-y|\leq t\leq x+y\}}\ dt\ dy.\end{aligned}$$

The condition

$$|x-y| \leq t \leq x+y$$

is symmetric in the variables y and t. Indeed, it is equivalent to the triangle inequalities

$$|x-t| \leq y \leq x+t.$$

Therefore,

$$\begin{aligned}\int_0^\infty (K_x f)(y)\overline{g(y)}\ dy &= \int_0^\infty f(t)\overline{\left(\int_{|x-t|}^{x+t} g(y)\ dy\right)}\ dt\\ &= \int_0^\infty f(t)\overline{(K_x g)(t)}\ dt.\end{aligned}$$

Thus, K_x is symmetric. Since

$$\tau_x^h = M_x + \frac{h}{2}K_x,$$

it follows that τ_x^h is symmetric on $C_c(0,\infty)$. The boundedness proved below then extends this symmetry to $L^2(0,\infty)$.

Next we prove the L^p-estimate. The operator M_x is the usual cosine translation. It is a positive contraction on $L^1(0,\infty)$ and on $L^\infty(0,\infty)$. Hence, by interpolation,

$$\|M_x f\|_{L^p(0,\infty)} \le \|f\|_{L^p(0,\infty)}, \qquad 1 \le p \le \infty.$$

For K_x, its kernel is

$$\mathbf{1}_{\{|x-y|\le t\le x+y\}}.$$

For fixed y, the length of the interval $[|x-y|, x+y]$ is $2\min\{x,y\}$, and therefore

$$\sup_{y>0}\int_0^\infty \mathbf{1}_{\{|x-y|\le t\le x+y\}}\, dt \le 2x.$$

Similarly, by the same triangle-inequality symmetry,

$$\sup_{t>0}\int_0^\infty \mathbf{1}_{\{|x-y|\le t\le x+y\}}\, dy \le 2x.$$

Schur's test gives

$$\|K_x f\|_{L^p(0,\infty)} \le 2x\|f\|_{L^p(0,\infty)}, \qquad 1 \le p \le \infty.$$

Consequently,

$$\|\tau_x^h f\|_{L^p(0,\infty)} \le \|M_x f\|_{L^p} + \frac{h}{2}\|K_x f\|_{L^p} \le (1+hx)\|f\|_{L^p(0,\infty)}.$$

This proves the boundedness asserted in (d).

The constant $1+hx$ is sharp. Indeed, if $x=0$, then $\tau_0^h = I$, so the optimal constant is 1. Assume now $x>0$. For $p=\infty$, take $f\equiv 1$. Then,

$$(\tau_x^h 1)(y) = 1 + h\min\{x,y\},$$

and therefore

$$\|\tau_x^h 1\|_{L^\infty(0,\infty)} = 1+hx = (1+hx)\|1\|_{L^\infty(0,\infty)}.$$

Hence, the constant cannot be improved for $p=\infty$.

Let $1\le p<\infty$. For $R>x$ and $N>2x$, set

$$f_{R,N} := \mathbf{1}_{[R,R+N]}.$$

Then,

$$\|f_{R,N}\|_{L^p(0,\infty)} = N^{1/p}.$$

For every

$$y \in [R+x, R+N-x],$$

we have

$$y-x \in [R,R+N], \qquad y+x \in [R,R+N],$$

and the interval $[y - x, y + x]$ is contained in $[R, R + N]$. Consequently,

$$f_{R,N}(|x - y|) = 1, \qquad f_{R,N}(x + y) = 1,$$

and

$$\int_{|x-y|}^{x+y} f_{R,N}(t)\ dt = \int_{y-x}^{y+x} 1\ dt = 2x.$$

Thus, on the interval $[R + x, R + N - x]$,

$$(\tau_x^h f_{R,N})(y) = \frac{1}{2} + \frac{1}{2} + \frac{h}{2}(2x) = 1 + hx.$$

Therefore,

$$\|\tau_x^h f_{R,N}\|_{L^p(0,\infty)} \geq (1 + hx)(N - 2x)^{1/p}.$$

Dividing by $\|f_{R,N}\|_{L^p} = N^{1/p}$, we obtain

$$\frac{\|\tau_x^h f_{R,N}\|_{L^p}}{\|f_{R,N}\|_{L^p}} \geq (1 + hx)\left(1 - \frac{2x}{N}\right)^{1/p}.$$

Letting $N \to \infty$, we get

$$\|\tau_x^h\|_{L^p \to L^p} \geq 1 + hx.$$

Together with the upper estimate already proved,

$$\|\tau_x^h f\|_{L^p(0,\infty)} \leq (1 + hx)\|f\|_{L^p(0,\infty)},$$

this gives the exact operator norm

$$\|\tau_x^h\|_{L^p(0,\infty) \to L^p(0,\infty)} = 1 + hx, \qquad 1 \leq p \leq \infty.$$

Hence, the L^p-inequality in Proposition 4.3 is sharp.

It remains to prove the transform identity in (c). By the symmetry just proved and by the product formula of Lemma 4.2, for $f \in C_c(0, \infty)$ and $\lambda \geq 0$,

$$\begin{aligned} C_h(\tau_x^h f)(\lambda) &= \int_0^\infty (\tau_x^h f)(y) \cos_h(y, \lambda)\ dy \\ &= \int_0^\infty f(y)(\tau_x^h \cos_h(\cdot, \lambda))(y)\ dy \\ &= \int_0^\infty f(y) \cos_h(x, \lambda) \cos_h(y, \lambda)\ dy \\ &= \cos_h(x, \lambda)(C_h f)(\lambda). \end{aligned}$$

This proves (c). The identity extends to the natural L^2 classes by density and the boundedness of τ_x^h. □

Definition 4.4. For suitable f, g on $[0, \infty)$,

$$(f *_h g)(x) := \int_0^\infty (\tau_x^h f)(y)g(y)\, dy. \tag{4.6}$$

The next theorem states the convolution structure compatible with C_h.

Theorem 4.5. *Let $h \geq 0$. The Robin convolution $*_h$ is commutative, associative on compactly supported continuous functions, and satisfies*

$$C_h(f *_h g)(\lambda) = (C_h f)(\lambda)(C_h g)(\lambda), \qquad \lambda \geq 0, \tag{4.7}$$

for $f, g \in C_c(0, \infty)$.

Proof. Commutativity follows from Proposition 4.3(a). The transform identity follows from the product formula by Fubini's theorem:

$$\begin{aligned} C_h(f *_h g)(\lambda) &= \int_0^\infty \int_0^\infty (\tau_x^h f)(y) g(y) \cos_h(x, \lambda)\, dy\, dx \\ &= \int_0^\infty \int_0^\infty f(t) g(y) \Big[\int_{[0,\infty)} \cos_h(x, \lambda)(\delta_t *_h \delta_y)(\, dx) \Big]\, dt\, dy \\ &= \int_0^\infty \int_0^\infty f(t) g(y) \cos_h(t, \lambda) \cos_h(y, \lambda)\, dt\, dy \\ &= (C_h f)(\lambda)(C_h g)(\lambda). \end{aligned}$$

Associativity at the level of point masses $(\delta_x *_h \delta_y) *_h \delta_z = \delta_x *_h (\delta_y *_h \delta_z)$ follows from the product formula and the injectivity of C_h on bounded measures with compact support; see [19, Ch. 1]. □

We now prove a weighted Young-type estimate for $*_h$, reflecting the non-Markov nature of the Robin translation.

Proposition 4.6. *Let $h \geq 0$, $1 \leq p \leq \infty$, $f \in L^p(0, \infty)$, and $g \in L^1((0, \infty), (1 + hy)dy)$. Then,*

$$\|f *_h g\|_{L^p(0,\infty)} \leq \|f\|_{L^p(0,\infty)} \int_0^\infty |g(y)|\, (1 + hy)\, dy. \tag{4.8}$$

Proof. Decompose $\tau_x^h f(y) = A_y f(x) + (h/2) K_y f(x)$, where $A_y f(x) = \frac{1}{2}[f(|x - y|) + f(x + y)]$ and $K_y f(x) = \int_{|x-y|}^{x+y} f(t)dt$. The cosine-translation A_y is an L^p-contraction. For K_y, the kernel $\mathbf{1}_{\{|x-y| \leq t \leq x+y\}}$ of the Schwartz kernel of $K_y \colon f \mapsto K_y f$ satisfies, for fixed $y > 0$,

$$\sup_{x>0} \int_0^\infty \mathbf{1}_{\{|x-y| \leq t \leq x+y\}}\, dt = 2y, \qquad \sup_{t>0} \int_0^\infty \mathbf{1}_{\{|x-y| \leq t \leq x+y\}}\, dx = 2 \min\{t, y\} \leq 2y,$$

so by Schur's test $\left\|K_y\right\|_{L^p \to L^p} \leq 2y$ for every $1 \leq p \leq \infty$. Hence, $\left\|\tau_{(\cdot)}^h f(y)\right\|_{L_x^p} \leq (1 + hy)\, \|f\|_{L^p}$, and Minkowski's integral inequality gives (4.8). □

The total mass $1 + h \min\{x, y\}$ of $\delta_x *_h \delta_y$ exceeds 1 whenever $h > 0$: the Robin translation is positive but *not* Markov. This is the key analytical reason for the appearance of the weighted measure $(1 + hy)dy$ in (4.8) and of the factor $1 + hy$ in the singular integral representation (6.12) below. The associated hypergroup is non-compact and non-conservative, with modular function $1 + hx$; see [19] for the general abstract framework.

5. Robin heat semigroup

For $h \geq 0$, the operator A_h is non-negative self-adjoint and hence generates the heat semigroup $P_t^h = e^{-tA_h}$. On the *Robin-deformed cosine transform* side,

$$C_h(P_t^h f)(\lambda) = e^{-t\lambda^2}(C_h f)(\lambda),$$

which by Theorem 4.5 translates to a convolution:

$$P_t^h f = f *_h p_{h,t},$$

where the heat kernel is

$$p_{h,t}(y) = \int_0^\infty e^{-t\lambda^2} \cos_h(y, \lambda) d\mu_h(\lambda). \tag{5.1}$$

We now compute the closed form of $p_{h,t}$. We shall use the classical error function and complementary error function; see [20]. For $z \in \mathbb{C}$, the error function is defined by

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2}\, du,$$

where the integral is taken along any path joining 0 to z, since e^{-u^2} is entire. The complementary error function is defined by

$$\operatorname{erfc}(z) := 1 - \operatorname{erf}(z).$$

In particular, for real x,

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-u^2}\, du.$$

Thus, erfc represents the normalized Gaussian tail. It naturally appears when Gaussian heat kernels are integrated over a half-line. More precisely, for $a > 0$ and $b \in \mathbb{R}$,

$$\int_b^{+\infty} e^{-au^2}\, du = \frac{\sqrt{\pi}}{2\sqrt{a}} \operatorname{erfc}(\sqrt{a}\, b).$$

This elementary identity is the mechanism through which the complementary error function enters the explicit heat-kernel formula below.

Theorem 5.1. *Let $h \geq 0$, $t > 0$, and $y \geq 0$. Then,*

$$p_{h,t}(y) = \frac{1}{\sqrt{\pi t}} e^{-y^2/(4t)} - h\, e^{hy+h^2 t} \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + h\sqrt{t}\right). \tag{5.2}$$

Proof. By the spectral representation (2.1) together with the measure formula (2.3), we may write

$$p_{h,t}(y) = G(t, y) + C_h(t, y),$$

where

$$G(t, y) := \frac{2}{\pi} \int_0^\infty e^{-t\lambda^2} \cos(\lambda y)\, d\lambda$$

and

$$C_h(t,y) := \frac{2}{\pi}\int_0^\infty e^{-t\lambda^2}\left[-\frac{h^2}{h^2+\lambda^2}\cos(\lambda y) + \frac{h\lambda}{h^2+\lambda^2}\sin(\lambda y)\right]d\lambda.$$

The first integral is the standard Gaussian Fourier integral:

$$G(t,y) = \frac{1}{\sqrt{\pi t}}\, e^{-y^2/(4t)}.$$

It remains to compute $C_h(t,y)$. If $h=0$, then $C_0(t,y)=0$, and the formula follows immediately. Hence, assume $h>0$. Define

$$I_h(t,y) := \Re\int_0^\infty e^{-t\lambda^2}\frac{e^{-i\lambda y}}{h+i\lambda}\,d\lambda.$$

Since

$$\Re\left(\frac{e^{-i\lambda y}}{h+i\lambda}\right) = \frac{h\cos(\lambda y) - \lambda\sin(\lambda y)}{h^2+\lambda^2},$$

we obtain

$$C_h(t,y) = -\frac{2h}{\pi}\, I_h(t,y).$$

We now derive an ordinary differential equation for I_h as a function of y. The differentiation under the integral sign is justified because, for $t>0$,

$$e^{-t\lambda^2} \quad \text{and} \quad \lambda e^{-t\lambda^2}(h^2+\lambda^2)^{-1/2}$$

are integrable on $(0,\infty)$. Therefore,

$$\begin{aligned}(\partial_y - h)I_h(t,y) &= \Re\int_0^\infty e^{-t\lambda^2}\left(\frac{-i\lambda - h}{h+i\lambda}\right)e^{-i\lambda y}\,d\lambda\\ &= -\Re\int_0^\infty e^{-t\lambda^2}e^{-i\lambda y}\,d\lambda\\ &= -\int_0^\infty e^{-t\lambda^2}\cos(\lambda y)\,d\lambda\\ &= -\frac{\sqrt{\pi}}{2\sqrt{t}}\, e^{-y^2/(4t)}.\end{aligned}$$

Moreover,

$$I_h(t,y) \longrightarrow 0 \qquad \text{as } y\to+\infty,$$

by the Riemann–Lebesgue lemma, since

$$\lambda \mapsto e^{-t\lambda^2}(h+i\lambda)^{-1}$$

belongs to $L^1(0,\infty)$. Solving the first-order linear equation

$$(\partial_y - h)I_h(t,y) = -\frac{\sqrt{\pi}}{2\sqrt{t}}\, e^{-y^2/(4t)}$$

with the boundary condition at $+\infty$, we get

$$I_h(t,y) = \frac{\sqrt{\pi}}{2\sqrt{t}}\, e^{hy} \int_y^{+\infty} e^{-hs} e^{-s^2/(4t)}\, ds.$$

Indeed, multiplying the equation by e^{-hy} gives

$$\frac{d}{dy}\left(e^{-hy} I_h(t,y)\right) = -\frac{\sqrt{\pi}}{2\sqrt{t}}\, e^{-hy} e^{-y^2/(4t)},$$

and integration from y to $+\infty$ yields the displayed formula. It remains to evaluate the Gaussian tail integral. Completing the square,

$$hs + \frac{s^2}{4t} = \frac{(s+2ht)^2}{4t} - h^2 t.$$

Hence,

$$\int_y^{+\infty} e^{-hs-s^2/(4t)}\, ds = e^{h^2 t} \int_y^{+\infty} \exp\left(-\frac{(s+2ht)^2}{4t}\right) ds.$$

With the change of variable

$$r = \frac{s}{2\sqrt{t}} + h\sqrt{t}, \qquad ds = 2\sqrt{t}\, dr,$$

we obtain

$$\begin{aligned}\int_y^{+\infty} e^{-hs-s^2/(4t)}\, ds &= 2\sqrt{t}\, e^{h^2 t} \int_{\frac{y}{2\sqrt{t}}+h\sqrt{t}}^{+\infty} e^{-r^2}\, dr \\ &= \sqrt{\pi t}\, e^{h^2 t}\, \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + h\sqrt{t}\right).\end{aligned}$$

Consequently,

$$I_h(t,y) = \frac{\pi}{2}\, e^{hy+h^2 t}\, \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + h\sqrt{t}\right).$$

Since $C_h(t,y) = -(2h/\pi) I_h(t,y)$, we arrive at

$$C_h(t,y) = -h\, e^{hy+h^2 t}\, \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + h\sqrt{t}\right).$$

Combining this with the expression for $G(t,y)$ gives

$$p_{h,t}(y) = \frac{1}{\sqrt{\pi t}}\, e^{-y^2/(4t)} - h\, e^{hy+h^2 t}\, \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + h\sqrt{t}\right),$$

which proves (5.2). The case $h = 0$ is included, since the second term vanishes and the formula reduces to the Gaussian part $G(t,y)$. This formula is consistent with the classical heat kernel for Brownian motion on the half-line with Robin, or elastic, boundary behavior as presented by Itô and McKean [1]. □

The next theorem gives a Gaussian upper estimate. We no longer call this bound "sharp" in the two-sided sense, since no matching lower Gaussian estimate is claimed here.

Theorem 5.2. *Let $h \geq 0$, $t > 0$, and $y \geq 0$. Then,*

$$0 \leq p_{h,t}(y) \leq \frac{1}{\sqrt{\pi t}}\, e^{-y^2/(4t)}. \tag{5.3}$$

Proof. We use the closed formula obtained in Theorem 5.1:

$$p_{h,t}(y) = G(t,y) - R_h(t,y),$$

where

$$G(t,y) := \frac{1}{\sqrt{\pi t}}\, e^{-y^2/(4t)}$$

and

$$R_h(t,y) := h\, e^{hy+h^2 t}\ \mathrm{erfc}\left(\frac{y}{2\sqrt{t}} + h\sqrt{t}\right).$$

Since $h \geq 0$ and $\mathrm{erfc}(x) \geq 0$ for $x \geq 0$, we have

$$R_h(t,y) \geq 0.$$

Therefore,

$$p_{h,t}(y) = G(t,y) - R_h(t,y) \leq G(t,y),$$

which gives the desired Gaussian upper bound. It remains to prove the non-negativity. The case $h = 0$ is immediate, since then $R_0(t,y) = 0$ and $p_{0,t}(y) = G(t,y)$. Assume now that $h > 0$. Set

$$z = \frac{y}{2\sqrt{t}} + h\sqrt{t}.$$

Then, $z > 0$, and we use the classical Gaussian-tail estimate

$$\mathrm{erfc}(z) \leq \frac{e^{-z^2}}{\sqrt{\pi}\, z}, \qquad z > 0.$$

For completeness, recall that this follows from

$$\int_z^{+\infty} e^{-r^2}\, dr \leq \int_z^{+\infty} \frac{r}{z} e^{-r^2}\, dr = \frac{e^{-z^2}}{2z}.$$

Multiplying by $2/\sqrt{\pi}$ gives the estimate for erfc.

Since

$$z^2 = \frac{y^2}{4t} + hy + h^2 t,$$

we obtain

$$\begin{aligned} R_h(t,y) &= h\, e^{hy+h^2 t}\, \mathrm{erfc}(z) \\ &\leq h\, e^{hy+h^2 t} \frac{e^{-z^2}}{\sqrt{\pi}\, z} \\ &= \frac{h}{\sqrt{\pi}\, z} e^{-y^2/(4t)}. \end{aligned}$$

Moreover,

$$z = \frac{y}{2\sqrt{t}} + h\sqrt{t} \geq h\sqrt{t},$$

and therefore

$$\frac{h}{z} \leq \frac{1}{\sqrt{t}}.$$

Consequently,

$$R_h(t,y) \leq \frac{1}{\sqrt{\pi t}}\, e^{-y^2/(4t)} = G(t,y).$$

Hence,

$$p_{h,t}(y) = G(t,y) - R_h(t,y) \geq 0.$$

Combining this with the already proved inequality $p_{h,t}(y) \leq G(t,y)$ gives

$$0 \leq p_{h,t}(y) \leq \frac{1}{\sqrt{\pi t}}\, e^{-y^2/(4t)}.$$

□

6. Fractional powers of the Robin Laplacian

Throughout this section, $h \geq 0$ and $0 < s < 2$. The fractional power $A_h^{s/2}$ is defined on its spectral domain

$$\mathrm{Dom}(A_h^{s/2}) = \Big\{ f \in L^2(0,\infty) : \int_0^\infty \lambda^{2s}\, |(C_h f)(\lambda)|^2\, d\mu_h(\lambda) < \infty \Big\}, \tag{6.1}$$

by

$$C_h(A_h^{s/2} f)(\lambda) = \lambda^s (C_h f)(\lambda), \qquad \lambda > 0. \tag{6.2}$$

Recall the elementary identity, for $a \geq 0$ and $0 < s < 2$,

$$a^{s/2} = \frac{1}{|\Gamma(-s/2)|} \int_0^\infty (1 - e^{-ta})\, \frac{dt}{t^{1+s/2}}. \tag{6.3}$$

The next proposition states the Bochner subordination representation of $A_h^{s/2}$ (cf. [7, 10] for abstract spectral analogues).

Proposition 6.1. *Let $0 < s < 2$, and set $\alpha = s/2 \in (0,1)$. If $f \in \mathrm{Dom}(A_h^{s/2})$, then*

$$A_h^{s/2} f = \frac{1}{|\Gamma(-s/2)|} \int_0^\infty (f - P_t^h f)\, \frac{dt}{t^{1+s/2}}, \tag{6.4}$$

where the integral is understood as a Bochner integral in $L^2(0,\infty)$, that is, as the limit

$$\lim_{\varepsilon \downarrow 0,\, R \to +\infty} \frac{1}{|\Gamma(-s/2)|} \int_\varepsilon^R (f - P_t^h f)\, \frac{dt}{t^{1+s/2}}$$

in $L^2(0,\infty)$.

Proof. Set $\alpha = s/2 \in (0,1)$. We use the spectral diagonalization of A_h by the transform C_h. Namely,

$$C_h(A_h f)(\lambda) = \lambda^2 C_h f(\lambda),$$

and therefore

$$C_h(A_h^{s/2} f)(\lambda) = \lambda^s C_h f(\lambda).$$

Moreover, the heat semigroup satisfies

$$C_h(P_t^h f)(\lambda) = e^{-t\lambda^2} C_h f(\lambda).$$

For $0 < \varepsilon < R < +\infty$, define the truncated Bochner integral

$$T_{\varepsilon,R} f := \frac{1}{|\Gamma(-\alpha)|} \int_\varepsilon^R (f - P_t^h f) \frac{dt}{t^{1+\alpha}}.$$

Since the integral is over a finite interval and P_t^h is bounded on $L^2(0,\infty)$, this is a well-defined L^2-valued integral. Applying C_h, we obtain

$$C_h(T_{\varepsilon,R} f)(\lambda) = m_{\varepsilon,R}(\lambda)\, C_h f(\lambda),$$

where

$$m_{\varepsilon,R}(\lambda) := \frac{1}{|\Gamma(-\alpha)|} \int_\varepsilon^R (1 - e^{-t\lambda^2}) \frac{dt}{t^{1+\alpha}}.$$

We now let $\varepsilon \downarrow 0$ and $R \to +\infty$. For every $\lambda \geq 0$, the scalar Bochner identity gives

$$\frac{1}{|\Gamma(-\alpha)|} \int_0^\infty (1 - e^{-t\lambda^2}) \frac{dt}{t^{1+\alpha}} = (\lambda^2)^\alpha = \lambda^s.$$

Indeed, after the change of variable $u = t\lambda^2$,

$$\int_0^\infty (1 - e^{-t\lambda^2}) \frac{dt}{t^{1+\alpha}} = \lambda^{2\alpha} \int_0^\infty (1 - e^{-u}) \frac{du}{u^{1+\alpha}} = |\Gamma(-\alpha)|\, \lambda^{2\alpha}.$$

Hence,

$$m_{\varepsilon,R}(\lambda) \longrightarrow \lambda^s \qquad \text{for every } \lambda \geq 0.$$

Furthermore, since the integrand is non-negative,

$$0 \leq m_{\varepsilon,R}(\lambda) \leq \lambda^s, \qquad \lambda \geq 0.$$

Therefore,

$$\left| m_{\varepsilon,R}(\lambda) - \lambda^s \right|^2 |C_h f(\lambda)|^2 \leq \lambda^{2s} |C_h f(\lambda)|^2 .$$

The right-hand side is integrable with respect to the spectral measure because $f \in \mathrm{Dom}(A_h^{s/2})$. By dominated convergence and Plancherel's theorem,

$$\left\| T_{\varepsilon,R} f - A_h^{s/2} f \right\|_2^2 = \int_0^\infty \left| m_{\varepsilon,R}(\lambda) - \lambda^s \right|^2 |C_h f(\lambda)|^2 \, d\mu_h(\lambda).$$

Thus, $T_{\varepsilon,R} f$ converges in $L^2(0,\infty)$ to $A_h^{s/2} f$, which proves (6.4). □

For $y > 0$, define

$$L_{h,s}(y) := \frac{1}{|\Gamma(-s/2)|} \int_0^\infty p_{h,t}(y) \frac{dt}{t^{1+s/2}}. \tag{6.5}$$

By Theorem 5.2, the kernel $p_{h,t}(y)$ is non-negative and satisfies

$$0 \le p_{h,t}(y) \le \frac{1}{\sqrt{\pi t}} e^{-y^2/(4t)}.$$

Therefore, $L_{h,s}(y)$ is well-defined and finite for every $y > 0$, since

$$\int_0^\infty \frac{1}{\sqrt{\pi t}} e^{-y^2/(4t)} \frac{dt}{t^{1+s/2}} < +\infty.$$

Substituting the closed formula from Theorem 5.1,

$$p_{h,t}(y) = \frac{1}{\sqrt{\pi t}} e^{-y^2/(4t)} - h e^{hy+h^2 t} \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + h\sqrt{t}\right),$$

we obtain the decomposition

$$L_{h,s}(y) = c_s y^{-1-s} - \frac{h e^{hy}}{|\Gamma(-s/2)|} \int_0^\infty e^{h^2 t} \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + h\sqrt{t}\right) \frac{dt}{t^{1+s/2}}, \tag{6.6}$$

where

$$c_s := \frac{2^{1+s}\Gamma\left(\frac{1+s}{2}\right)}{\sqrt{\pi}\, |\Gamma(-s/2)|}. \tag{6.7}$$

Indeed,

$$\frac{1}{|\Gamma(-s/2)|} \int_0^\infty \frac{1}{\sqrt{\pi t}} e^{-y^2/(4t)} \frac{dt}{t^{1+s/2}} = c_s y^{-1-s}.$$

Theorem 6.2. *Let $0 < s < 2$ and $h \ge 0$. Then,*

$$L_{h,s}(y) > 0, \qquad y > 0,$$

and

$$0 < L_{h,s}(y) \le c_s y^{-1-s}, \qquad y > 0. \tag{6.8}$$

Moreover, as $y \downarrow 0$,

$$L_{h,s}(y) = c_s y^{-1-s} + O_{h,s}(y^{-s}). \tag{6.9}$$

If $h > 0$, then, as $y \to +\infty$,

$$L_{h,s}(y) \sim K_{h,s} y^{-2-s}, \qquad K_{h,s} := \frac{2^{s+2}\Gamma\left(\frac{s+3}{2}\right)}{h\sqrt{\pi}\, |\Gamma(-s/2)|}. \tag{6.10}$$

For $h = 0$, one has the exact identity

$$L_{0,s}(y) = c_s y^{-1-s}, \qquad y > 0.$$

Proof. The upper bound follows directly from the Gaussian estimate for the heat kernel. Namely,

$$0 \le p_{h,t}(y) \le \frac{1}{\sqrt{\pi t}} e^{-y^2/(4t)}.$$

Integrating this inequality against $|\Gamma(-s/2)|^{-1}\, t^{-1-s/2}\, dt$, we obtain

$$0 \le L_{h,s}(y) \le c_s y^{-1-s}.$$

We now prove strict positivity. If $h = 0$, then

$$p_{0,t}(y) = \frac{1}{\sqrt{\pi t}} e^{-y^2/(4t)} > 0, \qquad t > 0,\ y > 0,$$

and hence $L_{0,s}(y) > 0$. Assume $h > 0$. From the closed formula, write

$$p_{h,t}(y) = G(t, y) - R_h(t, y),$$

where

$$G(t, y) = \frac{1}{\sqrt{\pi t}} e^{-y^2/(4t)}$$

and

$$R_h(t, y) = h e^{hy+h^2 t}\, \mathrm{erfc}\left(\frac{y}{2\sqrt{t}} + h\sqrt{t}\right).$$

Set

$$z = \frac{y}{2\sqrt{t}} + h\sqrt{t}.$$

Since $z > h\sqrt{t}$, the classical Gaussian-tail estimate

$$\mathrm{erfc}(z) \le \frac{e^{-z^2}}{\sqrt{\pi} z}, \qquad z > 0,$$

gives

$$R_h(t, y) \le \frac{h}{\sqrt{\pi} z} e^{-y^2/(4t)} < \frac{1}{\sqrt{\pi t}} e^{-y^2/(4t)} = G(t, y).$$

Thus, $p_{h,t}(y) > 0$ for every $t > 0$ and $y > 0$, and therefore $L_{h,s}(y) > 0$. It remains to study the asymptotics. The case $h = 0$ is immediate from the closed formula, since the Robin correction vanishes identically. Hence, assume first that $h > 0$. For the small-y behavior, denote the correction term in (6.6) by

$$D_{h,s}(y) := \frac{h e^{hy}}{|\Gamma(-s/2)|} \int_0^\infty e^{h^2 t}\, \mathrm{erfc}\left(\frac{y}{2\sqrt{t}} + h\sqrt{t}\right) \frac{dt}{t^{1+s/2}}.$$

Then,

$$L_{h,s}(y) = c_s y^{-1-s} - D_{h,s}(y).$$

Make the change of variables $t = y^2 u$. We get

$$D_{h,s}(y) = \frac{h e^{hy}}{|\Gamma(-s/2)|}\, y^{-s} \int_0^\infty e^{h^2 y^2 u}\, \mathrm{erfc}\left(\frac{1}{2\sqrt{u}} + hy\sqrt{u}\right) \frac{du}{u^{1+s/2}}.$$

For $a, b \geq 0$, one has

$$e^{a^2}\operatorname{erfc}(a+b) \leq \operatorname{erfc}(b).$$

Indeed,

$$e^{a^2}\operatorname{erfc}(a+b) = \frac{2}{\sqrt{\pi}}\int_b^{+\infty} e^{-r^2-2ar}\,dr \leq \frac{2}{\sqrt{\pi}}\int_b^{+\infty} e^{-r^2}\,dr = \operatorname{erfc}(b).$$

Applying this with

$$a = hy\sqrt{u}, \qquad b = \frac{1}{2\sqrt{u}},$$

we obtain

$$e^{h^2y^2u}\operatorname{erfc}\left(\frac{1}{2\sqrt{u}} + hy\sqrt{u}\right) \leq \operatorname{erfc}\left(\frac{1}{2\sqrt{u}}\right).$$

The function

$$u \mapsto \operatorname{erfc}\left(\frac{1}{2\sqrt{u}}\right)u^{-1-s/2}$$

is integrable on $(0, \infty)$: near 0 it has exponential decay, and near $+\infty$ it behaves like $u^{-1-s/2}$. Therefore,

$$D_{h,s}(y) = O_{h,s}(y^{-s}) \qquad \text{as } y \downarrow 0.$$

Consequently,

$$L_{h,s}(y) = c_s y^{-1-s} + O_{h,s}(y^{-s}), \qquad y \downarrow 0.$$

We now prove the large-y asymptotic for $h > 0$. Set $t = y^2u$. Then,

$$L_{h,s}(y) = \frac{y^{-s}}{|\Gamma(-s/2)|}\int_0^{\infty} p_{h,y^2u}(y)\frac{du}{u^{1+s/2}}.$$

For fixed $u > 0$, set

$$a = hy\sqrt{u}, \qquad b = \frac{1}{2\sqrt{u}}, \qquad z = a + b.$$

Using

$$G(y^2u, y) = \frac{1}{y\sqrt{\pi u}}e^{-1/(4u)}$$

and the identity

$$e^{hy+h^2y^2u} = e^{z^2}e^{-1/(4u)},$$

we can write

$$p_{h,y^2u}(y) = \frac{e^{-1/(4u)}}{\sqrt{\pi}y\sqrt{u}}\left[1 - a\,M(z)\right],$$

where

$$M(z) := \sqrt{\pi}e^{z^2}\operatorname{erfc}(z).$$

The classical asymptotic expansion of the Mills ratio gives

$$M(z) = \frac{1}{z} + O(z^{-3}), \qquad z \to +\infty.$$

Since $z = a + b$ and $a \to +\infty$ as $y \to +\infty$, we obtain

$$1 - aM(z) = 1 - \frac{a}{a+b} + O\left(\frac{a}{(a+b)^3}\right) = \frac{b}{a+b} + O(y^{-2}).$$

Thus,

$$1 - aM(z) \sim \frac{b}{a} = \frac{1}{2hyu}.$$

It follows that, for every fixed $u > 0$,

$$y^2 p_{h,y^2u}(y) \longrightarrow \frac{1}{2h\sqrt{\pi}}\, u^{-3/2} e^{-1/(4u)}.$$

We may pass to the limit under the integral sign. Indeed, the two-sided Mills estimate

$$\frac{1}{z} - \frac{1}{2z^3} \le M(z) \le \frac{1}{z}, \qquad z > 0,$$

implies

$$0 \le 1 - aM(z) \le \frac{b}{a+b} + \frac{a}{2(a+b)^3} \le \frac{C_h}{yu}, \qquad y \ge 1,\ u > 0.$$

Therefore,

$$0 \le y^2 p_{h,y^2u}(y) \le C_h\, u^{-3/2} e^{-1/(4u)}.$$

After multiplication by $u^{-1-s/2}$, the dominating function becomes

$$C_h\, u^{-(s+5)/2} e^{-1/(4u)},$$

which is integrable on $(0, \infty)$. Hence, dominated convergence gives

$$\begin{aligned}\lim_{y\to+\infty} y^{s+2} L_{h,s}(y) &= \frac{1}{|\Gamma(-s/2)|} \int_0^\infty \frac{1}{2h\sqrt{\pi}}\, u^{-3/2} e^{-1/(4u)} \frac{du}{u^{1+s/2}} \\ &= \frac{1}{2h\sqrt{\pi}\,|\Gamma(-s/2)|} \int_0^\infty u^{-(s+5)/2} e^{-1/(4u)}\, du.\end{aligned}$$

Finally, with the change of variables $v = 1/(4u)$,

$$\int_0^\infty u^{-(s+5)/2} e^{-1/(4u)}\, du = 2^{s+3}\Gamma\left(\frac{s+3}{2}\right).$$

Therefore,

$$\lim_{y\to+\infty} y^{s+2} L_{h,s}(y) = \frac{2^{s+2}\Gamma\left(\frac{s+3}{2}\right)}{h\sqrt{\pi}\,|\Gamma(-s/2)|} = K_{h,s}.$$

This proves

$$L_{h,s}(y) \sim K_{h,s} y^{-2-s}, \qquad y \to +\infty.$$

The proof is complete. □

The next lemma is the key symbol identity.

Lemma 6.3. *Let* $0 < s < 2$*,* $h \geq 0$*, and* $\lambda \geq 0$*. Then,*

$$\lambda^s = \int_0^\infty \Big[(1 + hy) - \cos_h(y, \lambda)\Big] L_{h,s}(y)\, dy. \tag{6.11}$$

The integral is finite. More precisely, the integrand is non-negative and belongs to $L^1(0, \infty)$*.*

Proof. Recall that

$$\cos_h(y, \lambda) = \cos(\lambda y) + \frac{h}{\lambda} \sin(\lambda y), \qquad \lambda > 0,$$

with the natural convention

$$\cos_h(y, 0) = 1 + hy.$$

First, observe that, for $\lambda > 0$,

$$(1 + hy) - \cos_h(y, \lambda) = (1 - \cos(\lambda y)) + h\left(y - \frac{\sin(\lambda y)}{\lambda}\right).$$

Both terms on the right-hand side are non-negative for $y \geq 0$, since

$$1 - \cos r \geq 0, \qquad \sin r \leq r, \qquad r \geq 0.$$

Hence,

$$(1 + hy) - \cos_h(y, \lambda) \geq 0, \qquad y \geq 0,\ \lambda \geq 0.$$

For $\lambda = 0$, the expression is identically zero, and (6.11) reduces to $0 = 0$. We therefore assume below that $\lambda > 0$. By the definition of the Robin Lévy kernel,

$$L_{h,s}(y) = \frac{1}{|\Gamma(-s/2)|} \int_0^\infty p_{h,t}(y)\, \frac{dt}{t^{1+s/2}}.$$

Since both $p_{h,t}(y)$ and $(1 + hy) - \cos_h(y, \lambda)$ are non-negative, Tonelli's theorem gives

$$\begin{aligned}
&\int_0^\infty \Big[(1 + hy) - \cos_h(y, \lambda)\Big] L_{h,s}(y)\, dy \\
&\quad = \frac{1}{|\Gamma(-s/2)|} \int_0^\infty \left[\int_0^\infty \Big[(1 + hy) - \cos_h(y, \lambda)\Big] p_{h,t}(y)\, dy\right] \frac{dt}{t^{1+s/2}}.
\end{aligned}$$

We now compute the inner integral. From the spectral identity (5.1),

$$\int_0^\infty \cos_h(y, \lambda) p_{h,t}(y)\, dy = e^{-t\lambda^2}.$$

Taking $\lambda = 0$ in the same identity and using $\cos_h(y, 0) = 1 + hy$, we also have

$$\int_0^\infty (1 + hy) p_{h,t}(y)\, dy = 1.$$

Subtracting the two identities yields

$$\int_0^\infty \Big[(1 + hy) - \cos_h(y, \lambda)\Big] p_{h,t}(y)\, dy = 1 - e^{-t\lambda^2}.$$

Therefore,

$$\int_0^\infty \Big[(1+hy) - \cos_h(y,\lambda)\Big] L_{h,s}(y)\,dy = \frac{1}{|\Gamma(-s/2)|}\int_0^\infty (1-e^{-t\lambda^2})\frac{dt}{t^{1+s/2}}.$$

Finally, by the scalar Bochner identity,

$$\frac{1}{|\Gamma(-s/2)|}\int_0^\infty (1-e^{-at})\frac{dt}{t^{1+s/2}} = a^{s/2}, \qquad a \ge 0.$$

Taking $a = \lambda^2$, we obtain

$$\frac{1}{|\Gamma(-s/2)|}\int_0^\infty (1-e^{-t\lambda^2})\frac{dt}{t^{1+s/2}} = (\lambda^2)^{s/2} = \lambda^s.$$

This proves (6.11). Since the right-hand side is finite, the non-negative integrand in (6.11) is integrable on $(0,\infty)$. □

The hypersingular integral representation of $A_h^{s/2}$ is the main result of this section.

Theorem 6.4. *Let $0 < s < 2$, $h \ge 0$, and $f \in C_c^\infty(0,\infty)$. Then, for every $x > 0$,*

$$A_h^{s/2} f(x) = \int_0^\infty \Big[(1+hy)f(x) - (\tau_x^h f)(y)\Big] L_{h,s}(y)\,dy, \tag{6.12}$$

where the integral is absolutely convergent. Equivalently, the right-hand side defines a pointwise representative of $A_h^{s/2} f$.

In the Neumann case $h = 0$, one has

$$L_{0,s}(y) = c_s y^{-1-s}$$

and

$$(\tau_x^0 f)(y) = \frac{1}{2}[f(|x-y|) + f(x+y)].$$

Consequently,

$$A_0^{s/2} f(x) = c_s \int_0^\infty \frac{f(x) - \frac{1}{2}[f(|x-y|) + f(x+y)]}{y^{1+s}}\,dy, \qquad x > 0. \tag{6.13}$$

This is the usual singular-integral formula for the fractional Neumann Laplacian on the half-line.

Proof. For $0 < \varepsilon < R < +\infty$, define the truncated integral

$$T_{\varepsilon,R} f(x) := \int_\varepsilon^R \Big[(1+hy)f(x) - (\tau_x^h f)(y)\Big] L_{h,s}(y)\,dy.$$

Since the interval $[\varepsilon, R]$ is finite, this integral is well defined. By the product formula for the Robin translation, Proposition 4.3(c), one has

$$C_h\Big[x \mapsto (\tau_x^h f)(y)\Big](\lambda) = \cos_h(y,\lambda)\,(C_h f)(\lambda).$$

Therefore,

$$C_h(T_{\varepsilon,R}f)(\lambda) = m_{\varepsilon,R}(\lambda)(C_hf)(\lambda),$$

where

$$m_{\varepsilon,R}(\lambda) := \int_\varepsilon^R \Big[(1+hy) - \cos_h(y,\lambda)\Big] L_{h,s}(y)\,dy.$$

By Lemma 6.3,

$$m_{\varepsilon,R}(\lambda) \longrightarrow \lambda^s \qquad \text{as } \varepsilon \downarrow 0,\ R \to +\infty.$$

Moreover, since

$$(1+hy) - \cos_h(y,\lambda) \geq 0,$$

we have

$$0 \leq m_{\varepsilon,R}(\lambda) \leq \lambda^s.$$

Hence,

$$\left|m_{\varepsilon,R}(\lambda) - \lambda^s\right|^2 |C_hf(\lambda)|^2 \leq \lambda^{2s} |C_hf(\lambda)|^2.$$

Since $f \in C_c^\infty(0,\infty) \subset \mathrm{Dom}(A_h^{s/2})$, the function on the right-hand side is integrable with respect to the spectral measure $d\mu_h(\lambda)$. By dominated convergence and Plancherel's theorem,

$$\left\|T_{\varepsilon,R}f - A_h^{s/2}f\right\|_2^2 = \int_0^\infty \left|m_{\varepsilon,R}(\lambda) - \lambda^s\right|^2 |C_hf(\lambda)|^2 \, d\mu_h(\lambda).$$

Thus, the truncated singular integrals converge to $A_h^{s/2}f$ in $L^2(0,\infty)$.

We next prove that the integral in (6.12) is absolutely convergent for every fixed $x > 0$. Using the explicit formula for the Robin translation,

$$(\tau_x^hf)(y) = \frac{1}{2}\big[f(|x-y|) + f(x+y)\big] + \frac{h}{2}\int_{|x-y|}^{x+y} f(r)\,dr,$$

we examine the behavior near $y = 0$. For $0 < y < x$, Taylor expansion at x gives

$$\frac{1}{2}\big[f(x-y) + f(x+y)\big] = f(x) + \frac{y^2}{2}f''(x) + O_x(y^4),$$

and

$$\frac{h}{2}\int_{x-y}^{x+y} f(r)\,dr = hyf(x) + O_x(y^3).$$

Therefore,

$$(\tau_x^hf)(y) = (1+hy)f(x) + O_x(y^2), \qquad y \downarrow 0.$$

It follows that

$$\left|(1+hy)f(x) - (\tau_x^hf)(y)\right| = O_x(y^2), \qquad y \downarrow 0.$$

Since Theorem 6.2 gives

$$L_{h,s}(y) = O_{h,s}(y^{-1-s}), \qquad y \downarrow 0,$$

the integrand is

$$O_{x,h,s}(y^{1-s}), \qquad y \downarrow 0.$$

This is integrable near 0, because $s < 2$. It remains to control the integral at infinity. Choose $R_f > 0$ such that

$$\operatorname{supp} f \subset [0, R_f].$$

For fixed $x > 0$, if $y > x + R_f$, then

$$|x - y| = y - x > R_f \qquad \text{and} \qquad x + y > R_f.$$

Hence,

$$f(|x-y|) = 0, \qquad f(x+y) = 0, \qquad \int_{|x-y|}^{x+y} f(r)\, dr = 0.$$

Thus,

$$(\tau_x^h f)(y) = 0, \qquad y > x + R_f.$$

Consequently, for large y,

$$\Big|(1+hy)f(x) - (\tau_x^h f)(y)\Big| = |(1+hy)f(x)| \le C_x(1+hy).$$

If $h = 0$, then $L_{0,s}(y) = c_s y^{-1-s}$, and hence

$$\int^{\infty} L_{0,s}(y)\, dy < +\infty.$$

If $h > 0$, Theorem 6.2 gives

$$L_{h,s}(y) = O_{h,s}(y^{-2-s}), \qquad y \to +\infty.$$

Therefore,

$$(1+y)L_{h,s}(y) = O_{h,s}(y^{-1-s}), \qquad y \to +\infty,$$

which is integrable at infinity. Hence, the integral in (6.12) is absolutely convergent for every fixed $x > 0$. By absolute convergence, the truncated integrals $T_{\varepsilon,R} f(x)$ converge pointwise, for every $x > 0$, to

$$\int_0^{\infty} \Big[(1+hy)f(x) - (\tau_x^h f)(y)\Big] L_{h,s}(y)\, dy.$$

On the other hand, the same truncated integrals converge in $L^2(0,\infty)$ to $A_h^{s/2} f$. Therefore, the absolutely convergent integral above defines the L^2-limit, and hence gives a pointwise representative of $A_h^{s/2} f$. This proves (6.12).

Finally, if $h = 0$, then the Robin translation reduces to the even Neumann translation

$$(\tau_x^0 f)(y) = \frac{1}{2}[f(|x-y|) + f(x+y)],$$

and the Lévy kernel reduces to

$$L_{0,s}(y) = c_s y^{-1-s}.$$

Substituting these two identities into (6.12) gives (6.13). □

7. Maximum principle and spectral positivity

The singular-integral representation obtained in Theorem 6.4, together with the positivity of the Robin Lévy kernel, yields a pointwise maximum principle for the fractional Robin operator. The result is analogous to the standard maximum principles for nonlocal fractional operators; see, for instance, [13, 21, 22].

Theorem 7.1. *Let $0 < s < 2$, $h \geq 0$, and let $f \in C_c^\infty(0, \infty)$ be real-valued. Assume that $x_0 > 0$ is a global maximum point of f, and that*

$$f(x_0) > 0.$$

Then,

$$A_h^{s/2} f(x_0) > 0. \tag{7.1}$$

Proof. Set

$$M := f(x_0) > 0.$$

Since x_0 is a global maximum point, we have

$$f(r) \leq M, \qquad r \geq 0.$$

Using the explicit formula for the Robin translation,

$$(\tau_{x_0}^h f)(y) = \frac{1}{2}[f(|x_0 - y|) + f(x_0 + y)] + \frac{h}{2}\int_{|x_0-y|}^{x_0+y} f(r)\,dr,$$

we estimate each term separately. The first term satisfies

$$\frac{1}{2}[f(|x_0 - y|) + f(x_0 + y)] \leq M.$$

Moreover, the length of the interval

$$[\,|x_0 - y|, x_0 + y\,]$$

is

$$x_0 + y - |x_0 - y| = 2\min\{x_0, y\} \leq 2y.$$

Therefore,

$$\frac{h}{2}\int_{|x_0-y|}^{x_0+y} f(r)\,dr \leq \frac{h}{2} \cdot 2y\,M = hyM.$$

Consequently,

$$(\tau_{x_0}^h f)(y) \leq (1 + hy)M = (1 + hy)f(x_0), \qquad y \geq 0.$$

Hence, the integrand in the singular representation satisfies

$$(1 + hy)f(x_0) - (\tau_{x_0}^h f)(y) \geq 0, \qquad y \geq 0.$$

By Theorem 6.4,

$$A_h^{s/2} f(x_0) = \int_0^\infty \Big[(1 + hy)f(x_0) - (\tau_{x_0}^h f)(y)\Big] L_{h,s}(y)\,dy.$$

Since $L_{h,s}(y) > 0$ for every $y > 0$, this already gives

$$A_h^{s/2} f(x_0) \geq 0.$$

It remains to prove strict positivity. Choose $R_f > 0$ such that

$$\operatorname{supp} f \subset [0, R_f].$$

If $y > x_0 + R_f$, then

$$|x_0 - y| = y - x_0 > R_f, \qquad x_0 + y > R_f.$$

Thus,

$$f(|x_0 - y|) = 0, \qquad f(x_0 + y) = 0,$$

and the interval $[\,|x_0 - y|, x_0 + y\,]$ lies entirely to the right of $\operatorname{supp} f$. Hence,

$$\int_{|x_0-y|}^{x_0+y} f(r)\, dr = 0,$$

and therefore

$$(\tau_{x_0}^h f)(y) = 0, \qquad y > x_0 + R_f.$$

For such y,

$$(1 + hy) f(x_0) - (\tau_{x_0}^h f)(y) = (1 + hy) M > 0.$$

Since $L_{h,s}(y) > 0$ on $(0, \infty)$, the integral over $(x_0 + R_f, \infty)$ is strictly positive. Hence,

$$A_h^{s/2} f(x_0) > 0.$$

□

Next, we record the basic positivity estimate for the quadratic form of powers of the Robin operator. This is the spectral positivity underlying the fractional calculus developed above. In contrast with a classical Hardy inequality, no positive weight appears here; the sharp constant is zero because the spectrum of A_h begins at 0.

Theorem 7.2. *Let $s > 0$ and $h \geq 0$. For every $f \in \operatorname{Dom}(A_h^s)$,*

$$\int_0^\infty \overline{f(x)}\, A_h^s f(x)\, dx \geq 0. \tag{7.2}$$

The constant 0 *is sharp. Moreover, if $C_h f$ is supported in $[1, \infty)$, then*

$$\int_0^\infty \overline{f(x)}\, A_h^s f(x)\, dx \geq \|f\|_{L^2(0,\infty)}^2. \tag{7.3}$$

More generally, if $C_h f$ is supported in $[\rho, \infty)$ for some $\rho > 0$, then

$$\int_0^\infty \overline{f(x)}\, A_h^s f(x)\, dx \geq \rho^{2s} \|f\|_{L^2(0,\infty)}^2. \tag{7.4}$$

Proof. By Plancherel's theorem and the spectral diagonalization of A_h, we have

$$C_h(A_h^s f)(\lambda) = \lambda^{2s} C_h f(\lambda).$$

Therefore,

$$\int_0^\infty \overline{f(x)}\, A_h^s f(x)\, dx = \int_0^\infty \lambda^{2s} \left|C_h f(\lambda)\right|^2 \, d\mu_h(\lambda)$$
$$\geq 0.$$

This proves (7.2).

The constant 0 is sharp because the spectrum starts at 0. More explicitly, for any $\varepsilon > 0$, one may choose $C_h f$ supported in $[0, \varepsilon]$. Then,

$$\int_0^\infty \lambda^{2s} \left|C_h f(\lambda)\right|^2 \, d\mu_h(\lambda) \leq \varepsilon^{2s} \int_0^\infty \left|C_h f(\lambda)\right|^2 \, d\mu_h(\lambda).$$

By Plancherel,

$$\int_0^\infty \left|C_h f(\lambda)\right|^2 \, d\mu_h(\lambda) = \|f\|^2_{L^2(0,\infty)}.$$

Thus, no strictly positive constant can replace 0 in (7.2) for all $f \in \mathrm{Dom}(A_h^s)$.

If $C_h f$ is supported in $[1, \infty)$, then

$$\lambda^{2s} \geq 1 \qquad \text{on } \operatorname{supp} C_h f.$$

Consequently,

$$\int_0^\infty \lambda^{2s} \left|C_h f(\lambda)\right|^2 \, d\mu_h(\lambda) \geq \int_0^\infty \left|C_h f(\lambda)\right|^2 \, d\mu_h(\lambda) = \|f\|^2_{L^2(0,\infty)},$$

which proves (7.3). The proof of (7.4) is identical, using $\lambda^{2s} \geq \rho^{2s}$ on $[\rho, \infty)$. □

8. Caffarelli–Silvestre type extension

We now characterize the fractional Robin operator A_h^s by means of a degenerate elliptic extension on the upper half-plane, in the spirit of Caffarelli–Silvestre and of the spectral extension theorem of Stinga–Torrea; see [6, 7]. Throughout this section, we fix

$$0 < s < 1, \qquad h \geq 0,$$

and we set

$$a := 1 - 2s \in (-1, 1). \tag{8.1}$$

Thus, the weight t^a belongs to the Muckenhoupt class A_2.

For $r > 0$, define

$$\Phi_s(r) := \frac{r^s}{2^{s-1}\Gamma(s)} K_s(r),$$

where K_s denotes the modified Bessel function of the second kind. We extend Φ_s continuously to $r = 0$ by setting

$$\Phi_s(0) = 1.$$

Equivalently, for $t \geq 0$ and $\lambda \geq 0$, we write

$$\Phi_s(t, \lambda) := \Phi_s(t\lambda),$$

that is,

$$\Phi_s(t, \lambda) = \frac{(t\lambda)^s}{2^{s-1}\Gamma(s)} K_s(t\lambda), \qquad t\lambda > 0, \tag{8.2}$$

with the limiting convention $\Phi_s(0, \lambda) = 1$.

Theorem 8.1. *Let $0 < s < 1$, $h \geq 0$, and $f \in \mathrm{Dom}(A_h^s)$. Define $U(\cdot, t)$ spectrally by*

$$U(\cdot, t) := C_h^{-1} \left[\Phi_s(t, \lambda)(C_h f)(\lambda)\right], \qquad t \geq 0. \tag{8.3}$$

Then, the following hold.

(i) *$U(\cdot, t) \to f$ in $L^2(0, \infty)$ as $t \downarrow 0$.*

(ii) *For every $t > 0$, $U(\cdot, t) \in \mathrm{Dom}(A_h^m)$ for all $m \geq 0$. Moreover, U is smooth in $t > 0$ with values in $L^2(0, \infty)$, and it solves*

$$\partial_t^2 U + \frac{a}{t}\partial_t U - A_h U = 0, \qquad t > 0, \tag{8.4}$$

in $L^2(0, \infty)$.

(iii) *In the spatial variable, U satisfies the Robin boundary condition*

$$\partial_x U(0, t) = hU(0, t), \qquad t > 0,$$

in the sense of the domain of A_h. Hence, whenever U is represented pointwise, it solves

$$\begin{cases} \partial_x^2 U + \partial_t^2 U + \dfrac{a}{t}\partial_t U = 0, & x > 0,\ t > 0, \\ \partial_x U(0, t) = hU(0, t), & t > 0, \\ U(x, 0) = f(x), & x > 0. \end{cases} \tag{8.5}$$

(iv) *The conormal derivative recovers the fractional power A_h^s:*

$$-\lim_{t\downarrow 0} t^a \partial_t U(\cdot, t) = \kappa_s A_h^s f \qquad \text{in } L^2(0, \infty), \tag{8.6}$$

where

$$\kappa_s := \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}. \tag{8.7}$$

If, in addition, $f \in C_c^\infty(0, \infty)$, then the formula also holds pointwise for every $x > 0$.

The solution U is unique in the natural finite-energy class associated with the Robin quadratic form.

We prove the theorem through a sequence of elementary spectral facts.

Lemma 8.2. *For every $\lambda > 0$, the function*

$$\varphi_\lambda(t) := \Phi_s(t, \lambda)$$

belongs to $C^\infty(0, \infty)$ and satisfies

$$\varphi_\lambda''(t) + \frac{a}{t}\varphi_\lambda'(t) - \lambda^2\varphi_\lambda(t) = 0, \qquad t > 0. \tag{8.8}$$

Moreover,

$$\lim_{t\downarrow 0} \varphi_\lambda(t) = 1. \tag{8.9}$$

Proof. Set

$$u(r) := r^s K_s(r).$$

The modified Bessel equation

$$r^2 K_s''(r) + rK_s'(r) - (r^2 + s^2)K_s(r) = 0$$

implies, by direct differentiation, that

$$u''(r) + \frac{1 - 2s}{r}u'(r) - u(r) = 0, \qquad r > 0.$$

Since $a = 1 - 2s$ and

$$\varphi_\lambda(t) = \frac{u(t\lambda)}{2^{s-1}\Gamma(s)},$$

we obtain

$$\varphi_\lambda''(t) + \frac{a}{t}\varphi_\lambda'(t) - \lambda^2\varphi_\lambda(t) = 0.$$

It remains to check the boundary value at $t = 0$. For $0 < s < 1$, the small-argument expansion of K_s gives

$$K_s(r) = 2^{s-1}\Gamma(s)r^{-s} + 2^{-s-1}\Gamma(-s)r^s + O(r^{2-s}), \qquad r \downarrow 0.$$

Therefore,

$$\Phi_s(r) = 1 + \frac{\Gamma(-s)}{2^{2s}\Gamma(s)}r^{2s} + O(r^2), \qquad r \downarrow 0.$$

In particular,

$$\lim_{r\downarrow 0} \Phi_s(r) = 1.$$

Taking $r = t\lambda$ proves (8.9). □

Lemma 8.3. *The function U defined by* (8.3) *satisfies*

$$\partial_t^2 U + \frac{a}{t}\partial_t U - A_h U = 0, \qquad t > 0,$$

in $L^2(0, \infty)$. If $f \in C_c^\infty(0, \infty)$, then U is smooth on $(0, \infty) \times (0, \infty)$ *and satisfies*

$$\partial_x^2 U + \partial_t^2 U + \frac{a}{t}\partial_t U = 0$$

pointwise.

Proof. Applying the Robin cosine transform to U, we have

$$C_h U(\lambda, t) = \varphi_\lambda(t)(C_h f)(\lambda).$$

For $t > 0$, the exponential decay of $K_s(t\lambda)$ as $\lambda \to +\infty$ implies that

$$\lambda^{2m}\varphi_\lambda(t)(C_h f)(\lambda) \in L^2(d\mu_h)$$

for every $m \geq 0$. Hence, $U(\cdot, t) \in \mathrm{Dom}(A_h^m)$ for all $m \geq 0$. The same exponential decay justifies differentiation in t in the L^2-sense. Therefore,

$$C_h\left(\partial_t^2 U + \frac{a}{t}\partial_t U - A_h U\right)(\lambda) = \left[\varphi_\lambda''(t) + \frac{a}{t}\varphi_\lambda'(t) - \lambda^2\varphi_\lambda(t)\right](C_h f)(\lambda).$$

By Lemma 8.2, the bracket vanishes. Hence,

$$\partial_t^2 U + \frac{a}{t}\partial_t U - A_h U = 0$$

in $L^2(0, \infty)$.

Since A_h is the realization of $-\partial_x^2$ with Robin boundary condition, this abstract equation is the same as

$$\partial_x^2 U + \partial_t^2 U + \frac{a}{t}\partial_t U = 0$$

whenever pointwise differentiation is justified. In particular, for $f \in C_c^\infty(0, \infty)$, the pointwise spectral integral is smooth in $(x, t) \in (0, \infty) \times (0, \infty)$, and the equation holds pointwise. □

Lemma 8.4. *The function U satisfies*

$$U(\cdot, t) \to f \qquad \textit{in } L^2(0, \infty)$$

as $t \downarrow 0$. Moreover, for every $t > 0$,

$$\partial_x U(0, t) = hU(0, t)$$

in the sense of the domain of the Robin operator. If $f \in C_c^\infty(0, \infty)$, then this boundary condition holds pointwise.

Proof. By the Plancherel theorem,

$$\|U(\cdot, t) - f\|_2^2 = \int_0^\infty |\Phi_s(t\lambda) - 1|^2 |(C_h f)(\lambda)|^2 \, d\mu_h(\lambda).$$

Since $\Phi_s(t\lambda) \to 1$ for every $\lambda \geq 0$ and Φ_s is bounded on $[0, \infty)$, dominated convergence gives

$$U(\cdot, t) \to f \qquad \text{in } L^2(0, \infty).$$

For $t > 0$, Lemma 8.3 gives

$$U(\cdot, t) \in \mathrm{Dom}(A_h).$$

By the definition of the Robin realization, every function in $\mathrm{Dom}(A_h)$ satisfies

$$\partial_x u(0) = hu(0)$$

in the trace sense. Hence,

$$\partial_x U(0,t) = hU(0,t).$$

If $f \in C_c^\infty(0,\infty)$, this can also be checked directly from the pointwise spectral representation, using

$$\partial_x \cos_h(0,\lambda) = h\cos_h(0,\lambda), \qquad \cos_h(0,\lambda) = 1.$$

□

Lemma 8.5. *Let $f \in \mathrm{Dom}(A_h^s)$. Then,*

$$-\lim_{t\downarrow 0} t^a \partial_t U(\cdot,t) = \kappa_s A_h^s f \qquad \textit{in } L^2(0,\infty),$$

where

$$\kappa_s = \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}.$$

If $f \in C_c^\infty(0,\infty)$, the convergence also holds pointwise for every $x > 0$.

Proof. We use the identity

$$\frac{d}{dr}(r^s K_s(r)) = -r^s K_{1-s}(r),$$

where $K_{s-1} = K_{1-s}$. Hence, for $r = t\lambda$,

$$\partial_t \Phi_s(t,\lambda) = -\frac{\lambda}{2^{s-1}\Gamma(s)}(t\lambda)^s K_{1-s}(t\lambda).$$

Multiplying by $t^a = t^{1-2s}$, we get

$$t^a \partial_t \Phi_s(t,\lambda) = -\frac{\lambda^{2s}}{2^{s-1}\Gamma(s)}(t\lambda)^{1-s} K_{1-s}(t\lambda).$$

Since

$$K_{1-s}(r) \sim 2^{-s}\Gamma(1-s) r^{s-1}, \qquad r \downarrow 0,$$

we obtain

$$\lim_{t\downarrow 0} t^a \partial_t \Phi_s(t,\lambda) = -\frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}\lambda^{2s} = -\kappa_s \lambda^{2s}.$$

Furthermore, the function

$$r \mapsto r^{1-s} K_{1-s}(r)$$

is bounded on $(0,\infty)$. Therefore,

$$|t^a \partial_t \Phi_s(t,\lambda)| \le C_s \lambda^{2s}, \qquad t > 0, \ \lambda \ge 0.$$

Because $f \in \mathrm{Dom}(A_h^s)$, we have

$$\lambda^{2s}(C_h f)(\lambda) \in L^2(d\mu_h).$$

Thus, dominated convergence in $L^2(d\mu_h)$ gives

$$-t^a \partial_t \Phi_s(t,\lambda)(C_h f)(\lambda) \longrightarrow \kappa_s \lambda^{2s}(C_h f)(\lambda)$$

in $L^2(d\mu_h)$. Applying the inverse Robin cosine transform, we obtain

$$-\lim_{t\downarrow 0} t^a \partial_t U(\cdot,t) = \kappa_s A_h^s f \qquad \text{in } L^2(0,\infty).$$

For $f \in C_c^\infty(0,\infty)$, the same argument, together with the rapid decay of $C_h f$, justifies the inverse transform pointwise and gives the pointwise identity. □

8.1. Proof of Theorem 8.1

Proof of Theorem 8.1. The existence statements follow from the preceding lemmas. Lemma 8.4 gives the trace condition $U(\cdot, t) \to f$ in L^2 and the Robin boundary condition at $x = 0$. Lemma 8.3 proves the extension equation, and Lemma 8.5 identifies the conormal derivative with $\kappa_s A_h^s f$. It remains only to prove uniqueness.

We prove uniqueness in the finite-energy class associated with the Robin quadratic form. For a sufficiently regular function V, set

$$\mathcal{E}_h[V] := \int_0^\infty t^a \left(\|\partial_t V(\cdot, t)\|^2_{L^2(0,\infty)} + \mathfrak{a}_h(V(\cdot, t), V(\cdot, t)) \right) dt,$$

where

$$\mathfrak{a}_h(g, g) := \int_0^\infty |g'(x)|^2 \, dx + h|g(0)|^2.$$

For $h \geq 0$, this form is non-negative and is the quadratic form of the non-negative Robin operator A_h.

Let V be a finite-energy solution of the homogeneous problem

$$\begin{cases} \partial_x^2 V + \partial_t^2 V + \dfrac{a}{t} \partial_t V = 0, & x > 0, \ t > 0, \\ \partial_x V(0, t) = hV(0, t), & t > 0, \\ V(x, 0) = 0, & x > 0, \end{cases}$$

and assume

$$\mathcal{E}_h[V] < +\infty.$$

We show that $V \equiv 0$.

Apply the Robin cosine transform in the x-variable and write

$$v(\lambda, t) := (C_h V(\cdot, t))(\lambda).$$

Since A_h is diagonalized by C_h, the equation becomes, for almost every $\lambda \geq 0$,

$$\partial_t^2 v(\lambda, t) + \frac{a}{t} \partial_t v(\lambda, t) - \lambda^2 v(\lambda, t) = 0, \qquad t > 0. \tag{8.10}$$

The zero trace condition gives

$$v(\lambda, 0) = 0$$

in the spectral trace sense, and the finite-energy assumption gives

$$\int_0^\infty t^a \left(|\partial_t v(\lambda, t)|^2 + \lambda^2 |v(\lambda, t)|^2 \right) dt < +\infty \tag{8.11}$$

for almost every $\lambda \geq 0$.

Assume first that $\lambda > 0$. The two independent solutions of (8.10) are

$$(t\lambda)^s K_s(t\lambda) \qquad \text{and} \qquad (t\lambda)^s I_s(t\lambda).$$

The K_s-branch has a finite non-zero trace at $t = 0$, since

$$(t\lambda)^s K_s(t\lambda) \longrightarrow 2^{s-1}\Gamma(s) \qquad \text{as } t \downarrow 0.$$

Thus, the zero trace condition eliminates this branch. The I_s-branch grows exponentially as $t \to +\infty$, and therefore it cannot satisfy (8.11). Hence,

$$v(\lambda, t) = 0 \qquad \text{for a.e. } \lambda > 0,\ t > 0.$$

For $\lambda = 0$, the equation reduces to

$$v''(t) + \frac{a}{t} v'(t) = 0.$$

Its solutions are

$$v(t) = C_1 + C_2 t^{2s}.$$

The zero trace condition gives $C_1 = 0$. If $C_2 \neq 0$, then

$$v'(t) = 2sC_2 t^{2s-1},$$

and hence

$$\int^{\infty} t^a |v'(t)|^2 \, dt = 4s^2 |C_2|^2 \int^{\infty} t^{2s-1} \, dt = +\infty.$$

Therefore, $C_2 = 0$. Hence, $v(0, t) = 0$ as well.

We conclude that

$$v(\lambda, t) = 0 \qquad \text{for a.e. } \lambda \geq 0,\ t > 0.$$

By the injectivity of the Robin cosine transform,

$$V(\cdot, t) = 0 \qquad \text{in } L^2(0, \infty)$$

for every $t > 0$. Thus, $V \equiv 0$. This proves uniqueness and completes the proof of Theorem 8.1. □

9. Example

We include a simple computation to illustrate how the Robin parameter enters the spectral representation. Let

$$f(x) = e^{-x}, \qquad x \geq 0.$$

Then, $f \in L^2(0, \infty)$, and its Robin transform is explicit. Using

$$\int_0^{\infty} e^{-x} \cos(\lambda x) \, dx = \frac{1}{1 + \lambda^2}, \qquad \int_0^{\infty} e^{-x} \sin(\lambda x) \, dx = \frac{\lambda}{1 + \lambda^2},$$

we obtain, for $\lambda \geq 0$,

$$\begin{aligned} (C_h f)(\lambda) &= \int_0^{\infty} e^{-x} \left[\cos(\lambda x) + \frac{h}{\lambda} \sin(\lambda x) \right] dx \\ &= \frac{1}{1 + \lambda^2} + \frac{h}{\lambda} \frac{\lambda}{1 + \lambda^2} = \frac{1 + h}{1 + \lambda^2}. \end{aligned}$$

Thus, the fractional Robin operator acts on this example through the spectral formula

$$C_h(A_h^{s/2} f)(\lambda) = \lambda^s \frac{1+h}{1+\lambda^2}.$$

Equivalently,

$$A_h^{s/2} f(x) = (1+h) \int_0^\infty \frac{\lambda^s}{1+\lambda^2} \cos_h(x,\lambda)\, d\mu_h(\lambda),$$

whenever the inverse transform is interpreted in the spectral sense. In the Neumann case $h = 0$, this reduces to the usual cosine-transform computation. For $h > 0$, the factor $1 + h$ and the Robin spectral measure both record the influence of the boundary condition.

Similarly, the heat semigroup applied to f is given by

$$P_t^h f(x) = (1+h) \int_0^\infty \frac{e^{-t\lambda^2}}{1+\lambda^2} \cos_h(x,\lambda)\, d\mu_h(\lambda).$$

This elementary example shows that the Robin parameter does not merely change a boundary value; it modifies the whole spectral representation through both the eigenfunctions and the spectral measure.

10. Comparison with the Riesz fractional derivative and limiting boundary regimes

We close by placing the fractional Robin operator within the broader class of nonlocal boundary value problems. The comparison is made with the classical Riesz fractional derivative on the line, and with the two limiting boundary regimes $h = 0$ and $h \to +\infty$. Throughout this section, $h \geq 0$, so that A_h is non-negative and its fractional powers are defined by the spectral calculus developed above.

For $0 < s < 2$, the Riesz fractional derivative on $\mathbb{R}$ is the Fourier multiplier

$$\widehat{(-d^2/dx^2)^{s/2} f}(\xi) = |\xi|^s \widehat{f}(\xi), \qquad f \in \mathcal{S}(\mathbb{R}). \tag{10.1}$$

Equivalently,

$$(-d^2/dx^2)^{s/2} f(x) = C_{1,s}\ \mathrm{P.V.} \int_{\mathbb{R}} \frac{f(x) - f(y)}{|x-y|^{1+s}}\, dy, \qquad C_{1,s} = \frac{2^s \Gamma\left(\frac{1+s}{2}\right)}{\sqrt{\pi}\, |\Gamma(-s/2)|}. \tag{10.2}$$

This operator is the generator of the symmetric s-stable process on $\mathbb{R}$, and its natural diagonalizing transform is the ordinary Fourier transform.

The half-line operator $A_h^{s/2}$ has the same spectral homogeneity but a different harmonic analysis. It is diagonalized not by the Fourier transform, but by the Robin-deformed cosine transform C_h. Since the eigenvalue of A_h corresponding to the spectral parameter λ is λ^2, the multiplier of $A_h^{s/2}$ is again λ^s. Thus, the Robin boundary condition does not change the fractional order of the operator; it changes the eigenfunctions, the translation structure, and consequently the global form of the singular integral.

Indeed, the hypersingular representation obtained in Theorem 6.4 reads

$$A_h^{s/2} f(x) = \int_0^\infty \Big[(1+hy) f(x) - \tau_x^h f(y)\Big] L_{h,s}(y)\, dy. \tag{10.3}$$

Near the singular point $y = 0$, Theorem 6.2 gives

$$L_{h,s}(y) \sim c_s y^{-1-s}, \qquad c_s = \frac{2^{1+s}\Gamma\left(\frac{1+s}{2}\right)}{\sqrt{\pi}\,|\Gamma(-s/2)|} = 2C_{1,s}. \tag{10.4}$$

Thus, the leading singularity has exactly the same fractional order as the Riesz kernel. The factor 2 reflects the folding of the line onto the half-line. In the Neumann case $h = 0$, the operator is obtained from the Riesz operator by even reflection, and the two line contributions $|x-y|^{-1-s}$ and $(x+y)^{-1-s}$ both appear in the unfolded half-line formula.

The Robin parameter therefore leaves the local singular order unchanged. Its effect appears in the lower-order and global terms, most visibly in the correction factor $(1+hy)f(x)$, in the Robin translation τ_x^h, and in the far-field behavior of the kernel. In contrast with the Riesz derivative, whose kernel $|x-y|^{-1-s}$ controls both short and long jumps, the Robin kernel has faster decay at infinity. For $h > 0$, Theorem 6.2 yields

$$L_{h,s}(y) \sim K_{h,s} y^{-2-s}, \qquad K_{h,s} = \frac{2^{s+2}\Gamma\left(\frac{s+3}{2}\right)}{h\sqrt{\pi}\,|\Gamma(-s/2)|}, \qquad y \to \infty. \tag{10.5}$$

Thus, the Robin kernel decays one power faster than the Riesz kernel. This extra decay is a genuine boundary effect: the partially reflecting boundary reduces the influence of long jumps as measured through the Robin translation. When $h = 0$, this effect disappears and

$$L_{0,s}(y) = c_s y^{-1-s}$$

exactly.

The same distinction is visible in the extension problem. The Riesz operator is realized through the Caffarelli–Silvestre extension on $\mathbb{R} \times (0, \infty)$. For the Robin operator, the extension is posed on the quarter-plane and preserves the spatial boundary condition. Writing the extension for $A_h^{s/2}$, one obtains

$$\begin{cases} \partial_x^2 U + \partial_t^2 U + \dfrac{1-s}{t}\partial_t U = 0, & x > 0,\ t > 0, \\ \partial_x U(0,t) = hU(0,t), & t > 0, \\ U(x,0) = f(x), & x > 0. \end{cases} \tag{10.6}$$

Moreover,

$$-\lim_{t\downarrow 0} t^{1-s}\partial_t U(x,t) = \kappa_{s/2} A_h^{s/2} f(x), \qquad \kappa_{s/2} = \frac{\Gamma\left(1-\frac{s}{2}\right)}{2^{s-1}\Gamma\left(\frac{s}{2}\right)}. \tag{10.7}$$

The notation $\kappa_{s/2}$ is used here because the extension exponent is $s/2$ when the recovered operator is $A_h^{s/2}$. This corrects the former constant mismatch in this displayed formula. Thus, the fractional boundary value problem has two distinct boundaries: $t = 0$, where the nonlocal operator is recovered, and $x = 0$, where the Robin condition is retained. This separation is important: The extension procedure does not mix the spatial boundary condition with the auxiliary extension variable.

This spectral viewpoint is complementary to recent numerical and stochastic fractional modeling approaches in applied systems, including predator–prey reaction–diffusion models and computer-virus propagation models [14–16]. These references are used here only for contextual background and methodological comparison; the proofs in the present paper are based on the Robin spectral transform and the associated generalized translation.

The maximum principle also reflects the same boundary-adapted structure. For the Riesz derivative, positivity at a maximum follows directly from the symmetric difference $f(x) - f(y)$. For the Robin operator, the correct positive quantity is not $f(x) - \tau_x^h f(y)$, because the Robin translation does not have unit mass. The relevant expression is instead

$$(1 + hy)f(x) - \tau_x^h f(y).$$

At a positive global maximum of f, this quantity is non-negative, which is the mechanism behind Theorem 7.1. Similarly, since A_h is non-negative for $h \geq 0$, the quadratic form of A_h^s is non-negative, leading to the spectral positivity estimate proved in Section 7. The constant zero is sharp because $\inf \sigma(A_h) = 0$.

Finally, the Robin family interpolates between the classical Neumann and Dirichlet boundary conditions. At the Neumann endpoint $h = 0$, one has

$$\cos_h(x, \lambda) = \cos(\lambda x),$$

the translation τ_x^0 is the usual even translation on the half-line, and $A_0^{s/2}$ is the fractional Neumann Laplacian. More precisely, if $E : L^2(0, \infty) \to L^2_{\text{even}}(\mathbb{R})$ denotes the even reflection isometry, $(Ef)(x) = f(|x|)$. Then $A_0^{s/2}$ is unitarily equivalent to $(-d^2/dx^2)^{s/2}$ restricted to even functions. Formula (6.13) is exactly the unfolded Riesz formula in the even sector.

At the opposite endpoint, $h \to +\infty$, the Robin Laplacian converges to the Dirichlet Laplacian

$$A_\infty u = -u'', \qquad \mathrm{Dom}(A_\infty) = \{u \in H^2(0, \infty) : u(0) = 0\}.$$

This convergence is most naturally expressed through quadratic forms. Let

$$\mathfrak{a}_h[u] = \int_0^\infty |u'(x)|^2 \, dx + h|u(0)|^2, \qquad \mathrm{Dom}(\mathfrak{a}_h) = H^1(0, \infty),$$

and

$$\mathfrak{a}_\infty[u] = \int_0^\infty |u'(x)|^2 \, dx, \qquad \mathrm{Dom}(\mathfrak{a}_\infty) = H_0^1(0, \infty).$$

As $h \to +\infty$, the forms $\mathfrak{a}_h$ increase to $\mathfrak{a}_\infty$ in the sense of monotone convergence of closed forms [23, Sec. VIII.3]. Consequently,

$$(A_h + 1)^{-1} \longrightarrow (A_\infty + 1)^{-1} \quad \text{strongly in } L^2(0, \infty). \tag{10.8}$$

By functional calculus, this implies, for $0 < s < 2$,

$$A_h^{s/2} \longrightarrow A_\infty^{s/2} \quad \text{in the strong resolvent sense.} \tag{10.9}$$

Equivalently, the associated heat semigroups converge strongly:

$$e^{-tA_h} f \longrightarrow e^{-tA_\infty} f \quad \text{in } L^2(0, \infty), \qquad t > 0, \quad f \in L^2(0, \infty). \tag{10.10}$$

At the level of heat kernels, this corresponds to the classical limit

$$H_h(t;x,y) \longrightarrow H_\infty(t;x,y) = \frac{1}{\sqrt{4\pi t}}\left(e^{-(x-y)^2/(4t)} - e^{-(x+y)^2/(4t)}\right), \tag{10.11}$$

where H_∞ is the Dirichlet heat kernel on the half-line.

The same limiting picture appears in the eigenfunction expansion. For fixed $x \geq 0$ and $\lambda > 0$,

$$\frac{1}{h}\cos_h(x,\lambda) = \frac{1}{h}\cos(\lambda x) + \frac{\sin(\lambda x)}{\lambda} \longrightarrow \frac{\sin(\lambda x)}{\lambda}. \tag{10.12}$$

Together with

$$h^2\, d\mu_h(\lambda) = \frac{2}{\pi}\frac{h^2\lambda^2}{h^2+\lambda^2}\, d\lambda \longrightarrow \frac{2}{\pi}\lambda^2\, d\lambda,$$

this recovers the sine-transform resolution of the Dirichlet Laplacian.

Hence, $A_h^{s/2}$ is not merely the Riesz fractional derivative restricted to a half-line. It is a boundary-adapted nonlocal operator whose local singularity is Riesz-type, but whose global behavior is shaped by the Robin boundary condition. The parameter h changes the translation structure, modifies the Lévy tail, propagates into the extension boundary condition, and provides a continuous passage from the fractional Neumann Laplacian to the fractional Dirichlet Laplacian. This makes the operator especially suited to fractional boundary value problems on the half-line.

11. Conclusions

In this paper we developed a fractional calculus for the one-dimensional Laplacian on the half-line with Robin boundary condition. The construction is based on the Robin cosine transform, which diagonalizes the self-adjoint Robin realization of the Laplacian and provides the natural spectral framework for defining fractional powers. For $h \geq 0$, we proved a product formula for the Robin eigenfunctions and used it to introduce positive generalized translations and a commutative Robin convolution.

A central part of the analysis was the derivation of the Robin heat kernel and its Gaussian bounds. These estimates allowed us to construct the Robin Lévy kernel $L_{h,s}$ and to obtain the singular-integral representation of the fractional operator $A_h^{s/2}$. The resulting formula shows explicitly how the Robin boundary condition modifies the classical fractional-difference structure: the factor $1 + hy$, corresponding to the zero-energy Robin eigenfunction, replaces the constant ground state of the Neumann case. We also computed the asymptotic behavior of $L_{h,s}$, showing that the local singularity agrees with the classical fractional Laplacian, whereas for $h > 0$ the kernel has a faster decay at infinity.

We further established a Caffarelli-Silvestre-type extension theorem for the Robin fractional operator. In this extension problem, the Robin condition is propagated to the spatial boundary $x = 0$, while the fractional power is recovered as a conormal derivative at the extension boundary $t = 0$. We also proved a pointwise maximum principle and a spectral positivity estimate, which confirm that the Robin fractional calculus fits naturally into the standard framework of nonlocal elliptic analysis.

The results show that the fractional Robin Laplacian forms a natural interpolating family between the fractional Neumann and fractional Dirichlet half-line operators. They also indicate several possible directions for future work, including L^p-mapping properties, sharp weighted Hardy inequalities, probabilistic interpretations in terms of partially reflected stable processes, and extensions to higher-dimensional domains with Robin boundary conditions.

Use of Generative-AI tools declaration

During the preparation of this article, the author used ChatGPT (OpenAI) solely for language refinement and proofreading. The author subsequently reviewed and edited all generated output and takes full responsibility for the content of the publication.

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Conflict of interest

The author declares no conflict of interest.

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