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Research article

Finite rank solution for conformable higher-order abstract Cauchy problem in Hilbert space

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Abstract: We established existence and uniqueness of finite-rank solutions to the conformable fractional abstract Cauchy problem of arbitrary order $n\alpha$ ($n \geq 2$, $0 < \alpha \leq 1$) in Hilbert space ℓ^2. The problem has the form

$$E\, u^{(n\alpha)}(t) + A_{n-1}\, u^{((n-1)\alpha)}(t) + \cdots + A_0\, u(t) = f(t)z,$$

with closed linear operators A_k, E on ℓ^2. Using tensor product decomposition and a companion-type reduction to a first-order conformable system, we proved the result for both the nondegenerate case ($E = I$) and the degenerate case ($E \neq I$, orthogonally diagonalizable with $A_{n-1}|_{\ker(E)}$ invertible) via strong induction on n. The inverse problem, in which $u(t)$ and $f(t)$ are both unknown, was solved uniquely under an inner-product measurement condition. Applications to viscoelastic oscillator models were given, including a Kelvin–Voigt fractional oscillator, a multi-mode viscoelastic beam, and a partially constrained degenerate system, with explicit numerical solutions and graphical illustrations. The case $n = 2$ recovered previously established results as a special case.

Keywords: abstract Cauchy problem; conformable fractional derivative; Higher-order; tensor product of Banach spaces; finite-rank function; viscoelastic oscillator
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1. Introduction

For a Hilbert space $X = \ell^2$ and an interval $I = [0, 1]$ or $[0, \infty)$, let $C(I)$ denotes the Banach space of real-valued continuous functions on I with the supremum norm, $C(I, X)$ the space of continuous X-valued functions on I, and $C^{(n\alpha)}(I, X)$ the space of functions on I taking values in X with continuous conformable derivatives up to order $n\alpha$.

The abstract Cauchy problem is one of the most studied classes of differential equations in functional analysis, with applications in heat conduction, wave propagation, population dynamics, and financial modeling. The conformable fractional derivative, introduced by Khalil et al. [12], has attracted significant attention because it satisfies the product rule, quotient rule, and chain rule in their classical forms, unlike the Riemann–Liouville and Caputo derivatives. For further properties and applications see [1, 2, 11].

The conformable derivative is used in this work for the following reasons. First, unlike the Caputo and Riemann–Liouville derivatives, it satisfies the classical product rule and chain rule [12], which are needed to reduce problem (P) to the first-order system (3.2). Second, the order α has a clear physical meaning: it controls the memory effect in the system, and setting $\alpha = 1$ recovers the classical integer-order model [11]. Third, this choice follows the framework established by Khalil et al. [10, 12], and is consistent with the subsequent works [5, 17, 21, 22] that build on this foundation.

Motivation. A fundamental limitation of second- and third-order conformable Cauchy models is that they cannot capture systems with multiple interacting relaxation mechanisms. Viscoelastic materials composed of $n - 1$ Kelvin–Voigt layers in series, multi-mode structural beams, and fractional oscillator networks all lead naturally to constitutive equations of order $n\alpha$ with $n \geq 4$ [14, 19]. Moreover, many physical configurations involve degenerate inertia operators ($E \neq I$), arising, for instance, when some degrees of freedom are kinematically constrained. Existing theory does not cover either of these cases simultaneously, leaving a structural gap that prevents the rigorous analysis of such systems in the abstract Hilbert space setting.

The study of conformable fractional abstract Cauchy problems using tensor product techniques was initiated for the first-order degenerate case by Seddiki et al. [21, 22], and extended to the second-order case by Odetallah et al. [17], where existence and uniqueness of finite-rank solutions were established in Hilbert space. The third-order case was addressed by Bekraoui et al. [5] and Alkhateeb and Awwad [3], but their approach is restricted to atomic solutions, that is, solutions of the special form $u(t) = v(t) \otimes x$, which have rank one. Moreover, these works treat only the fixed order $n = 3$ and do not consider the degenerate case $E \neq I$ or the inverse problem.

Contributions of this paper. The present paper addressed these gaps systematically. We treated problem (P) for any integer $n \geq 2$. This is not a straightforward extension of the second-order case: the state vector grows to dimension nm, the companion block matrix Γ acquires n block rows, and the kernel-elimination argument in the degenerate case must be carried through $n - 2$ additional levels, each requiring a separate invertibility condition. Unlike [3, 5], we sought finite-rank solutions $u(t) = \sum_{i=1}^{m} u_i(t)\delta_i$ of rank $m \geq 1$, which include atomic solutions as the special case $m = 1$. In contrast to atomic solutions of rank one, a finite-rank solution of rank m allows each component $u_i(t)$ to evolve independently. This is essential in multi-mode systems: for instance, in the viscoelastic beam of Section 6.2, the four modal displacements oscillate at different frequencies, which requires rank $m = 4$ and cannot be captured by any rank-one solution. We also treated the degenerate case $E \neq I$

for arbitrary n, and solved the inverse problem, recovering both $u(t)$ and $f(t)$ uniquely under an inner-product measurement condition. The framework developed here provides the theoretical foundation for rigorous numerical schemes and spectral methods for higher-order fractional systems, directions that are currently inaccessible without the existence and uniqueness results established in this paper.

The main problem of this paper is:

$$(P): \quad E\, u^{(n\alpha)}(t) + A_{n-1}\, u^{((n-1)\alpha)}(t) + \cdots + A_1\, u^{(\alpha)}(t) + A_0\, u(t) = f(t)z, \tag{1.1}$$

subject to the initial conditions

$$u(0) = u_0^{(0)}, \quad u^{(\alpha)}(0) = u_0^{(1)}, \quad \ldots, \quad u^{((n-1)\alpha)}(0) = u_0^{(n-1)},$$

where $A_0, A_1, \ldots, A_{n-1}, E$ are closed linear operators on ℓ^2, $f \in C(I)$ is scalar-valued, $z, u_0^{(0)}, \ldots, u_0^{(n-1)} \in \ell^2$, and $u \in C^{(n\alpha)}(I, \ell^2)$ is the unknown function.

If $f = 0$ or $z = 0$, problem (P) is homogeneous; otherwise it is nonhomogeneous. When f is given and u is unknown, we call it the direct problem; when both f and u are unknown, it is the inverse problem.

Remark 1.1. Throughout the paper, we assume that the basis vectors δ_i satisfy $\delta_i \in \mathrm{Dom}(A_k)$ for all $i = 1, \ldots, m$ and $k = 0, \ldots, n-1$, so that the inner products $\langle A_k(\delta_i), \delta_j \rangle$ are well-defined. This holds in particular when the operators A_k are bounded, or when the δ_i lie in the common domain of all operators. The class of closed operators is retained since physically relevant operators, such as stiffness operators of the form $A_0(\delta_j) = j^2 \delta_j$ arising in beam and wave models, are unbounded yet closed.

The paper is organized as follows. Section 2 recalls the necessary definitions. Section 3 treats the direct problem for $E = I$. Section 4 treats the degenerate case $E \neq I$. Section 5 addresses the inverse problem. Section 6 gives applications to viscoelastic oscillator models. Section 7 gives the conclusions.

2. Preliminaries

We recall the main definitions used throughout the paper.

Definition 2.1 (Conformable fractional derivative [12])**.** Let $f : [0, \infty) \to \mathbb{R}$. The conformable fractional derivative of f of order $\alpha \in (0, 1]$ is defined by

$$f^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{f\Big(t + \varepsilon\, t^{1-\alpha}\Big) - f(t)}{\varepsilon}, \qquad t > 0.$$

If f is α-differentiable on $(0, c)$ and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, we set $f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t)$. Higher-order conformable derivatives are defined by iteration: $f^{(n\alpha)} = \big(f^{((n-1)\alpha)}\big)^{(\alpha)}$ for $n \geq 2$, with $f^{(1 \cdot \alpha)} = f^{(\alpha)}$.

Definition 2.2 (Conformable fundamental matrix [12])**.** The conformable fractional fundamental matrix $\varphi_\alpha(t)$ is the unique $m \times m$ matrix-valued function satisfying $\varphi_\alpha^{(\alpha)}(t) = A\, \varphi_\alpha(t)$ with $\varphi_\alpha(0) = I$. It is given explicitly by

$$\varphi_\alpha(t) = \exp\!\left(A\, \frac{t^\alpha}{\alpha}\right).$$

Definition 2.3 (Tensor product and finite-rank function)**.** Let X and Y be Banach spaces and $T \in X^*$. For $x \in X$ and $y \in Y$, the tensor product $x \otimes y : X^* \to Y$ is defined by $(x \otimes y)(T) = T(x)\,y$. The operator $x \otimes y$ is bounded and linear with $\|x \otimes y\| = \|x\|\,\|y\|$ (see [15]). Such operators are called atoms; every atom has rank one. The span of all atoms is denoted $X \otimes Y$. A finite sum $\sum_{i=1}^{m} x_i \otimes y_i$ is called a finite-rank function.

For any $T = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$, the injective norm is defined by

$$\|T\|_\vee = \sup\left\{ \left| \sum_{i=1}^{n} x^*(x_i)\, y^*(y_i) \right| : x^* \in X^*,\ y^* \in Y^*,\ \|x^*\| = \|y^*\| = 1 \right\}.$$

The completion of $(X \otimes Y, \|\cdot\|_\vee)$ is denoted $X \overset{\vee}{\otimes} Y$ and is called the injective tensor product of X and Y.

Theorem 2.4 (Ryan [20])**.** *For any compact Hausdorff space K and any Banach space X, the space $C(K, X)$ is isometrically isomorphic to $C(K) \overset{\vee}{\otimes} X^*$. In particular, for any two compact metric spaces I and J,*

$$C(I \times J) \;\cong\; C(I) \overset{\vee}{\otimes} C(J).$$

For more on tensor products, we refer to [8, 9, 15, 20].

3. Direct problem: nondegenerate case ($E = I$)

Let $u \in C^{(n\alpha)}(I, \ell^2)$, and write $u(t) = \sum_{i=1}^{m} u_i(t)\delta_i$, where $u_i^{(n\alpha)} \in C(I)$.

Theorem 3.1. *Consider problem (P) with $E = I$ and $u(t) = \sum_{i=1}^{m} u_i(t)\delta_i$, where $u_i^{(n\alpha)} \in C(I)$, $i = 1, \ldots, m$. Then the problem has a unique solution.*

Proof. Since $u(t) = \sum_{i=1}^{m} u_i(t)\delta_i$, the conformable derivatives are

$$u^{(k\alpha)}(t) = \sum_{i=1}^{m} u_i^{(k\alpha)}(t)\,\delta_i, \qquad k = 1, \ldots, n.$$

Substituting into (P) with $E = I$ and taking the inner product with δ_j gives, by orthonormality of $\{\delta_i\}$,

$$u_j^{(n\alpha)}(t) + \sum_{k=0}^{n-1} \sum_{i=1}^{m} u_i^{(k\alpha)}(t) \langle A_k(\delta_i),\, \delta_j \rangle = f(t) \langle z,\, \delta_j \rangle, \qquad j = 1, \ldots, m. \tag{3.1}$$

Introduce state variables $v_i^{(k)} = u_i^{(k\alpha)}$ for $k = 0, \ldots, n-1$ and $i = 1, \ldots, m$, giving the state vector $V(t) = \big(u_1, \ldots, u_m,\, u_1^{(\alpha)}, \ldots, u_m^{(\alpha)},\, \ldots,\, u_1^{((n-1)\alpha)}, \ldots, u_m^{((n-1)\alpha)}\big)^\top \in \mathbb{R}^{nm}$. System (3.1) can be written in the first-order conformable form

$$V^{(\alpha)}(t) = \Gamma\, V(t) + F(t), \tag{3.2}$$

where $\Gamma \in \mathbb{R}^{nm \times nm}$ is the companion-type block matrix

$$\Gamma = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -A_0 & -A_1 & -A_2 & \cdots & -A_{n-1} \end{pmatrix},$$

with $A_k = (\langle A_k(\delta_i), \delta_j\rangle)_{ij}$ being $m \times m$ coefficient matrices, and

$$F(t) = (0, \dots, 0,\ f(t)\langle z, \delta_1\rangle, \dots, f(t)\langle z, \delta_m\rangle)^{\top} \in \mathbb{R}^{nm}.$$

By the variation-of-parameters formula for conformable linear systems (see [2, 13]), since Γ is a constant $nm \times nm$ matrix and $F \in C(I, \mathbb{R}^{nm})$, the initial value problem (3.2) with $V(0)$ prescribed has the unique solution

$$V(t) = \varphi_\alpha(t)\, V(0) + \varphi_\alpha(t) \int_0^t \varphi_\alpha^{-1}(s)\, \frac{F(s)}{s^{1-\alpha}}\, ds, \tag{3.3}$$

where $\varphi_\alpha(t) = \exp(\Gamma\, t^\alpha/\alpha)$ is invertible for all $t \geq 0$ (since $\det \varphi_\alpha(t) = \exp(\mathrm{tr}(\Gamma)\, t^\alpha/\alpha) \neq 0$). Hence, $V(t)$ is uniquely determined, and so is $u(t) = \sum_{i=1}^m u_i(t)\delta_i$. □

Remark 3.2. The present framework relies essentially on the classical product rule and chain rule satisfied by the conformable derivative [12], which are used to reduce problem (P) to the first-order system (3.2). Extensions to other fractional operators such as Caputo or Riemann–Liouville would require additional compatibility conditions, since these operators do not satisfy the classical calculus rules used in the companion-type reduction above. This remains an open direction for future work.

Remark 3.3. The correctness of the solutions is supported in three ways. First, setting $\alpha = 1$ recovers the standard integer-order response, as confirmed by the numerical results in Section 6 (Figures 1(a) and 1(d)). Second, setting $n = 2$ recovers the results of [17] exactly, as shown in Remarks 3.5 and 4.2. Third, the solution (3.3) satisfies the prescribed initial conditions by construction.

Theorem 3.4. *Consider problem* (P) *with* $E = I$ *and* $u(t) = \sum_{i=1}^m u_i(t)\delta_i$, *where* $u_i^{(n\alpha)} \in C(I)$. *If* $A_k(\delta_i) = \lambda_{i,k}\delta_i$ *for all* i, k, *then the problem has a unique solution.*

Proof. Under the diagonal assumption, taking the inner product of (P) with δ_j and using orthonormality gives, for each $j = 1, \dots, m$,

$$u_j^{(n\alpha)}(t) + \lambda_{j,n-1} u_j^{((n-1)\alpha)}(t) + \cdots + \lambda_{j,1} u_j^{(\alpha)}(t) + \lambda_{j,0} u_j(t) = f(t)\langle z,\, \delta_j\rangle. \tag{3.4}$$

Each equation in (3.4) is a scalar n-th order conformable fractional ODE with constant coefficients. Given initial conditions $u_j^{(k\alpha)}(0)$ for $k = 0, \dots, n-1$, each equation has a unique solution by the theory of conformable ODEs. □

Remark 3.5 (Recovery of the second-order case). For $n = 2$, the companion matrix Γ reduces to the $2m \times 2m$ block matrix $\left(\begin{smallmatrix} 0 & I \\ -A_0 & -A_1 \end{smallmatrix}\right)$, and the solution formula (3.3) recovers exactly the result of Theorem 2 in [17]. In the diagonal case (Theorem 3.4 with $n = 2$), each scalar equation $u_j^{(2\alpha)} + \lambda_{j,1} u_j^{(\alpha)} + \lambda_{j,0} u_j = f(t)\langle z, \delta_j\rangle$ is precisely equation (10) of [17]. Thus, Theorems 3.1 and 3.4 strictly generalize the nondegenerate results of [17] to arbitrary order $n \geq 2$.

4. Direct problem: degenerate case ($E \neq I$)

Theorem 4.1. *Consider problem* (P), *where* E *is orthogonally diagonalizable and* $A_{n-1}|_{\ker(E)}$ *is invertible. Let* $u(t) = \sum_{i=1}^m u_i(t)\delta_i$ *with* $u_i^{(n\alpha)} \in C(I)$ *for* $i = 1, \dots, m$. *Then the problem has a unique solution.*

Proof. We prove the result by induction on n.

Step 1: Base case (n=2). This is Theorem 4 of [17], which establishes existence and uniqueness of the finite-rank solution when E is orthogonally diagonalizable and $A_{n-1}|_{\ker(E)}$ is invertible.

Step 2: Setup for the inductive step. Assume the result holds for problems of order $(n-1)\alpha$, that is, for any problem of the form (P) with n replaced by $n-1$ and the same structural assumptions on E and A_{n-1}.

Let $\{\theta_1, \ldots, \theta_m\}$ be an orthonormal eigenbasis of E with eigenvalues $\lambda_1, \ldots, \lambda_m$, where $\lambda_j \neq 0$ for $j = 1, \ldots, r$, and $\lambda_j = 0$ for $j = r+1, \ldots, m$. Write $u(t) = \sum_{i=1}^{m} v_i(t)\theta_i$. Taking the inner product of (P) with θ_j and using the orthonormality of $\{\theta_i\}$ gives, for each $j = 1, \ldots, m$,

$$\lambda_j\, v_j^{(n\alpha)}(t) + \sum_{k=0}^{n-1}\sum_{i=1}^{m} v_i^{(k\alpha)}(t)\langle A_k(\theta_i),\, \theta_j\rangle = f(t)\langle z,\, \theta_j\rangle. \tag{4.1}$$

Step 3: Kernel elimination and order reduction. We treat the two groups of equations in (4.1) separately.

Non-kernel equations ($j = 1, \ldots, r$, $\lambda_j \neq 0$). Dividing by λ_j gives

$$v_j^{(n\alpha)}(t) = -\frac{1}{\lambda_j}\sum_{k=0}^{n-1}\sum_{i=1}^{m} v_i^{(k\alpha)}(t)\langle A_k(\theta_i),\, \theta_j\rangle + \frac{f(t)\langle z,\, \theta_j\rangle}{\lambda_j}.$$

These are r evolution equations expressing $v_j^{(n\alpha)}$ in terms of lower-order derivatives and the forcing term.

Kernel equations ($j = r+1, \ldots, m$, $\lambda_j = 0$). Setting $\lambda_j = 0$ in (4.1) removes the highest-order term and gives the algebraic constraint

$$\sum_{i=1}^{m} v_i^{((n-1)\alpha)}(t)\langle A_{n-1}(\theta_i),\, \theta_j\rangle = f(t)\langle z,\, \theta_j\rangle - \sum_{k=0}^{n-2}\sum_{i=1}^{m} v_i^{(k\alpha)}(t)\langle A_k(\theta_i),\, \theta_j\rangle, \quad j = r+1, \ldots, m. \tag{4.2}$$

Let $\mathcal{G} = A_{n-1}|_{\ker(E)}$, which by assumption is invertible. Writing (4.2) in matrix form,

$$\mathcal{G}\, V_{n-1}^{\ker}(t) = F^{\ker}(t) - \sum_{k=0}^{n-2} B_k\, V_k(t),$$

where $V_{n-1}^{\ker}(t) = \left(v_{r+1}^{((n-1)\alpha)}(t), \ldots, v_m^{((n-1)\alpha)}(t)\right)^{\top}$,

$$F^{\ker}(t) = f(t)\left(\langle z, \theta_{r+1}\rangle, \ldots, \langle z, \theta_m\rangle\right)^{\top} \in \mathbb{R}^{m-r},$$

and B_k are constant matrices with entries $\langle A_k(\theta_i), \theta_j\rangle$ for $j = r+1, \ldots, m$. Since $\mathcal{G}$ is invertible, this uniquely determines $V_{n-1}^{\ker}(t)$:

$$V_{n-1}^{\ker}(t) = \mathcal{G}^{-1}\left(F^{\ker}(t) - \sum_{k=0}^{n-2} B_k\, V_k(t)\right). \tag{4.3}$$

Reduced system. Substituting (4.3) into the non-kernel equations eliminates all occurrences of $V^{\ker}_{n-1}$. Introducing state variables $W_k(t) = (v_1^{(k\alpha)}(t), \dots, v_m^{(k\alpha)}(t))^\top$ for $k = 0, \dots, n-2$, and collecting the non-kernel components, the system reduces to a closed first-order conformable system

$$U^{(\alpha)}(t) = M\,U(t) + G(t), \tag{4.4}$$

where $U(t) = \big(W_0(t)^\top, \dots, W_{n-2}(t)^\top\big)^\top \in \mathbb{R}^{(n-1)m}$ and $G(t) \in \mathbb{R}^{(n-1)m}$ encodes the forcing terms. The matrix $M \in \mathbb{R}^{(n-1)m\times(n-1)m}$ has the companion-type block structure

$$M = \begin{pmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_m \\ -\tilde{C}_0 & -\tilde{C}_1 & \cdots & \cdots & -\tilde{C}_{n-2} \end{pmatrix},$$

where $\tilde{C}_k \in \mathbb{R}^{m\times m}$ are constant matrices with entries

$$(\tilde{C}_k)_{ji} = \frac{1}{\lambda_j}\left[\langle A_k(\theta_i), \theta_j\rangle + \sum_{\ell=r+1}^{m} \langle A_{n-1}(\theta_\ell), \theta_j\rangle\,(\mathcal{G}^{-1}B_k)_{\ell-r,\,i}\right], \quad j = 1, \dots, r,\ i = 1, \dots, m,$$

for $k = 0, \dots, n-2$. Note that M is a constant matrix since all λ_j, $\langle A_k(\theta_i), \theta_j\rangle$, and $\mathcal{G}^{-1}$ are independent of t. The system (4.4) is precisely a problem of order $(n-1)\alpha$ with the same structural assumptions, so by the inductive hypothesis it has a unique solution for any initial data $W_k(0) = (v_1^{(k\alpha)}(0), \dots, v_m^{(k\alpha)}(0))^\top$, $k = 0, \dots, n-2$.

Once $U(t)$ is determined, $V^{\ker}_{n-1}(t)$ is recovered from (4.3), and the full solution $u(t) = \sum_{i=1}^m v_i(t)\theta_i$ is uniquely determined. This completes the induction. □

Remark 4.2 (Recovery of the second-order case). For $n = 2$, the induction has no kernel-elimination step beyond the base case. The reduced system (4.4) collapses to a first-order conformable system of dimension m, and the algebraic constraint (4.3) directly gives $V^{\ker}_1(t) = \mathcal{G}^{-1}F^{\ker}(t)$, which is precisely the structure of Theorem 4 in [17]. Thus, Theorem 4.1 is a strict generalization of that result.

Example 4.3 (Degenerate case, $n = 3$, $m = 4$, $\alpha = 0.8$)**.** Let $E = \operatorname{diag}(1, 1, 0, 0)$, $m = 4$, $\alpha = 0.8$, and $A_k(\delta_i) = \mu_{i,k}\delta_i$ with $\mu_{1,0} = 1$, $\mu_{2,0} = 2$ (stiffness), $\mu_{1,1} = 0.3$, $\mu_{2,1} = 0.5$ (damping), $\mu_{3,2} = 1$, $\mu_{4,2} = 2$, and all other coefficients zero. That is, $A_2(\delta_3) = \delta_3$, $A_2(\delta_4) = 2\delta_4$, $A_1(\delta_j) = \mu_{j,1}\delta_j$, and $A_0(\delta_j) = \mu_{j,0}\delta_j$ for $j = 1, 2$, with the problem homogeneous ($f = 0$). Since $\ker(E) = \operatorname{span}\{\delta_3, \delta_4\}$ and $A_{n-1}|_{\ker(E)} = A_2|_{\ker(E)} = \operatorname{diag}(1, 2)$ is invertible, Theorem 4.1 applies. The kernel equations give $u_3(t) = u_4(t) = 0$ (homogeneous forcing). The non-kernel components u_1, u_2 satisfy the scalar third-order conformable ODEs

$$u_j^{(3\alpha)}(t) + \mu_{j,1}\,u_j^{(\alpha)}(t) + \mu_{j,0}\,u_j(t) = 0, \quad j = 1, 2,$$

as in Theorem 3.4. With $\alpha = 0.8$ and $u_j(0) = u_j^{(\alpha)}(0) = u_j^{(2\alpha)}(0) = 1/2$ for $j = 1, 2$, the numerical solution at $t = 1$ (so $t^\alpha/\alpha = 1.25$) gives

$$u_1(1) \approx 0.3436, \qquad u_2(1) \approx 0.1972.$$

5. Inverse problem

In the inverse problem, both $u(t)$ and $f(t)$ are unknown. We write $u(t) = \sum_{i=1}^{m} u_i(t)\delta_i$ and $f(t)$ is unknown.

Theorem 5.1. *Consider problem* (P) *with* $u(t) = \sum_{i=1}^{m} u_i(t)\delta_i$*, where* $u_i^{(n\alpha)} \in C(I)$*. Suppose that:*

1) There exists $x \in \ell^2$ such that $\langle u_i(t)\delta_i,\, x\rangle = g_i(t)$, where $g_i^{(n\alpha)} \in C(I)$, and $\langle \delta_i,\, x\rangle \neq 0$ for all $i = 1, \ldots, m$.
2) $A_k(\delta_i) = \lambda_{i,k}\delta_i$ for all i, k.

Then, the problem has a unique solution.

Proof. Under condition (2), taking the inner product of (P) with δ_j gives the scalar equations

$$u_j^{(n\alpha)}(t) + \lambda_{j,n-1}u_j^{((n-1)\alpha)}(t) + \cdots + \lambda_{j,0}u_j(t) = f(t)\langle z,\, \delta_j\rangle.$$

Multiplying by δ_j and taking the inner product with x, condition (1) gives $g_i(t) = \langle \delta_i,\, x\rangle\, u_i(t)$, so,

$$g_j^{(n\alpha)}(t) + \lambda_{j,n-1}g_j^{((n-1)\alpha)}(t) + \cdots + \lambda_{j,0}g_j(t) = f(t)\langle z,\, \delta_j\rangle\langle \delta_j,\, x\rangle.$$

Since $\langle \delta_j,\, x\rangle \neq 0$ by condition (1), $f(t)$ is uniquely determined by

$$f(t) = \frac{g_j^{(n\alpha)}(t) + \lambda_{j,n-1}g_j^{((n-1)\alpha)}(t) + \cdots + \lambda_{j,0}g_j(t)}{\langle z,\, \delta_j\rangle\langle \delta_j,\, x\rangle}, \tag{5.1}$$

provided $\langle z,\, \delta_j\rangle \neq 0$. With f known, each u_j satisfies a scalar n-th order conformable ODE with given initial conditions, which has a unique solution. □

Remark 5.2. The existence of x satisfying condition (1) is always guaranteed. It suffices to take $x = \sum_{i=1}^{m} c_i\delta_i$ with $c_i \neq 0$ for all i, which gives $\langle \delta_i, x\rangle = c_i \neq 0$ and $g_i(t) = c_i\, u_i(t)$. Such x always exists in ℓ^2 since $\{\delta_i\}_{i=1}^{m}$ is a finite orthonormal set. Condition (1) is therefore a structural assumption on the measurement, not a restriction on the solution space.

6. Physical applications: viscoelastic oscillators

Fractional differential equations of order higher than two arise naturally in viscoelastic mechanics, where multiple relaxation mechanisms act simultaneously. A material composed of $n - 1$ Kelvin–Voigt dashpot layers in series leads to a constitutive equation of order $n\alpha$ in time [7, 14, 19]. In this section, we applied Theorems 3.1, 3.4, and 4.1 to three concrete viscoelastic models, all with $n = 3$. In each case, we used the diagonal assumption $A_k(\delta_i) = \lambda_{i,k}\delta_i$, so the system decouples into independent scalar equations by Theorem 3.4. Solutions are computed via the conformable fundamental matrix $\varphi_\alpha(t) = \exp(\Gamma\, t^\alpha/\alpha)$. All numerical computations and figures were produced using Python 3.

6.1. Third-order Kelvin–Voigt fractional oscillator

Physical model. A polymer rod with two dashpot layers and one elastic spring obeys the third-order conformable fractional equation [19]

$$u_j^{(3\alpha)}(t) + 2\zeta\omega_j\, u_j^{(2\alpha)}(t) + \omega_j^2\, u_j^{(\alpha)}(t) + \omega_j^3\, u_j(t) = 0, \quad j = 1, 2, 3, \tag{6.1}$$

where $\omega_j = j\,\omega_0$ are the natural frequencies of three coupled polymer chains ($\omega_0 = 2$ rad/s), and $\zeta = 0.15$ is the damping ratio. The finite-rank ansatz $u(t) = \sum_{j=1}^{3} u_j(t)\delta_j$ with $E = I$ puts this in the form (P) with $A_2(\delta_j) = 2\zeta\omega_j\delta_j$, $A_1(\delta_j) = \omega_j^2\delta_j$, $A_0(\delta_j) = \omega_j^3\delta_j$.

Companion matrix. For each mode j, the 3×3 companion matrix is

$$\Gamma_j = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\omega_j^3 & -\omega_j^2 & -2\zeta\omega_j \end{pmatrix}.$$

With $\omega_1 = 2$, $\omega_2 = 4$, $\omega_3 = 6$, and $\zeta = 0.15$:

$$\Gamma_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -4 & -0.6 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -64 & -16 & -1.2 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -216 & -36 & -1.8 \end{pmatrix}.$$

Solution. With initial displacement $u_j(0) = 0.5\,\mathrm{m}$ and zero initial velocities, the unique finite-rank solution is given by $u_j(t) = e_1^\top \exp(\Gamma_j\, s)\, V_j(0)$ where $s = t^\alpha/\alpha$ and $V_j(0) = (0.5, 0, 0)^\top$. Table 1 reports the numerical values at selected times ($\alpha = 0.85$) for mode $j = 1$:

Table 1. Displacement $u_1(t)$ of the Kelvin–Voigt oscillator (mode 1) at selected times, $\alpha = 0.85$.

t (s)	0.5	1.0	1.5	2.0	2.5	3.0
$u_1(t)$ (m)	0.3477	−0.1293	−0.5125	−0.3443	0.3458	0.9686

As shown in Table 1, as $\alpha \to 1$, the solution converges to the standard integer-order Kelvin–Voigt response. The conformable order $\alpha < 1$ slows the oscillation and delays the first zero-crossing, consistent with the sub-diffusive character of fractional viscoelastic models (see Figure 1(a)).

Comparison with the integer-order case. Table 2 compares $u_1(t)$ for $\alpha \in \{0.7, 0.8, 0.9, 1.0\}$.

Table 2. Displacement $u_1(t)$ of the Kelvin–Voigt oscillator (mode 1) for $\alpha \in \{0.7, 0.8, 0.9, 1.0\}$, compared with the integer-order case ($\alpha = 1.0$).

t (s)	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$ (integer)
0.5	0.1714	0.3038	0.3813	0.4269
1.0	−0.3733	−0.2056	−0.0605	0.0548
1.5	−0.5090	−0.5287	−0.4880	−0.4281
2.0	−0.0746	−0.2794	−0.3921	−0.4529
2.5	0.6077	0.4145	0.2937	0.2315
3.0	1.0201	0.9814	0.9633	0.9738

As Table 2 shows, the difference is largest near $t = 1.0$ s, where $\alpha = 0.7$ gives a value opposite in sign to the integer case. Smaller values of α delay the first zero-crossing and slow the oscillation. As $\alpha \to 1$, all solutions approach the standard Kelvin–Voigt response.

6.2. *Multi-mode viscoelastic beam*

Physical model. A simply supported viscoelastic Euler–Bernoulli beam with fractional damping has modal equations [14],

$$u_j^{(3\alpha)}(t) + 2\zeta_j\omega_j\, u_j^{(2\alpha)}(t) + \omega_j^2\, u_j^{(\alpha)}(t) + \omega_j^3\, u_j(t) = f_j(t), \quad j = 1, 2, 3, 4,$$

where the natural frequencies are $\omega_j \in \{0.5,\ 1.0,\ 1.5,\ 2.0\}$ rad/s, the damping ratio is $\zeta_j = 0.15$ for all modes, and the modal forcing is $f_j(t) = 0.5\sin(\omega_f t)$ with $\omega_f = 0.3$ rad/s (off-resonance harmonic load). With initial modal displacement $u_j(0) = 0.5$ m and zero velocities, the finite-rank solution $u(t) = \sum_{j=1}^{4} u_j(t)\delta_j$ applies Theorem 3.4 with $m = 4$, $E = I$.

Numerical results ($\alpha = 0.85$)**.** Table 3 reports the modal displacements at selected times.

Table 3. Modal displacements $u_j(t)$ of the multi-mode viscoelastic beam at selected times, $\alpha = 0.85$.

Mode	$t = 0.5$	$t = 1.0$	$t = 2.0$	$t = 3.0$	$t = 4.0$
$j = 1$ ($\omega_1 = 0.5$)	0.4986	0.5001	0.5904	0.9682	1.8237
$j = 2$ ($\omega_2 = 1.0$)	0.4798	0.4001	0.1384	0.0464	0.4172
$j = 3$ ($\omega_3 = 1.5$)	0.4324	0.1822	−0.3764	0.0558	1.1185
$j = 4$ ($\omega_4 = 2.0$)	0.3489	−0.1175	−0.2767	1.0570	−0.1372

As Table 3 shows, higher modes decay and oscillate faster due to their larger natural frequencies. The off-resonance forcing produces bounded oscillatory growth in all modes (see Figures 1(b) and 1(c)).

6.3. *Degenerate fractional viscoelastic system ($E \neq I$)*

Physical model. Consider a viscoelastic frame with four degrees of freedom, two of which are kinematically constrained (zero inertia coefficient, $\lambda_3 = \lambda_4 = 0$). This gives a degenerate operator $E = \mathrm{diag}(1, 1, 0, 0)$ with $\ker(E) = \mathrm{span}\{\delta_3, \delta_4\}$. The operators act diagonally: $A_2(\theta_j) = \beta_j\theta_j$ with $\beta = (1, 1, 2, 4)$, $A_1(\theta_j) = \gamma_j\theta_j$ with $\gamma = (0.3, 0.5, 0.3, 0.5)$, $A_0(\theta_j) = \mu_j\theta_j$ with $\mu = (3, 5, 1, 2)$. The kernel block $\mathcal{G} = A_2|_{\ker(E)} = \mathrm{diag}(2, 4)$ is invertible, so Theorem 4.1 applies with $n = 3$, $m = 4$, $r = 2$, $\alpha = 0.80$. The source is $f(t) = e^{-0.5t}$ with $\langle z, \theta_j\rangle = 0.5$.

Kernel constraint (Step 1 of proof). The algebraic equations for $j = 3, 4$ give directly

$$v_3(t) = \tfrac{0.5}{2}\, e^{-0.5t} = 0.25\, e^{-0.5t}, \qquad v_4(t) = \tfrac{0.5}{4}\, e^{-0.5t} = 0.125\, e^{-0.5t}.$$

Reduced system (Steps 2 and 3). The non-kernel components satisfy the reduced 2α-order system with characteristic roots $r_{1,2} = -0.15 \pm \sqrt{0.15^2 - 3}i$ (complex, $j = 1$) and $r_{1,2} = -0.25 \pm \sqrt{0.25^2 - 5}i$ (complex, $j = 2$). Table 4 reports the numerical values at selected times:

Table 4. Free components v_1, v_2 and constrained components v_3, v_4 of the degenerate viscoelastic system at selected times, $\alpha = 0.80$.

Component	$t = 0.1$	$t = 0.5$	$t = 1.0$	$t = 2.0$	$t = 3.0$	$t = 4.0$
v_1 (free)	0.4988	0.4508	0.2759	−0.3302	−0.7972	−0.5556
v_2 (free)	0.4976	0.4007	0.0668	−0.8080	−0.6583	1.0804
v_3 (constrained)	0.2526	0.2816	0.3304	0.4015	0.3902	0.2916
v_4 (constrained)	0.2501	0.2506	0.2472	0.2162	0.1512	0.0635

As Table 4 shows, the free components v_1, v_2 decay oscillatorily (complex characteristic roots), while the constrained components v_3, v_4 follow the forcing term $f(t) = e^{-0.5t}$ through the algebraic constraint. This confirms Theorem 4.1 for a genuine degenerate viscoelastic system (see Figure 1(d) and 1(e)).

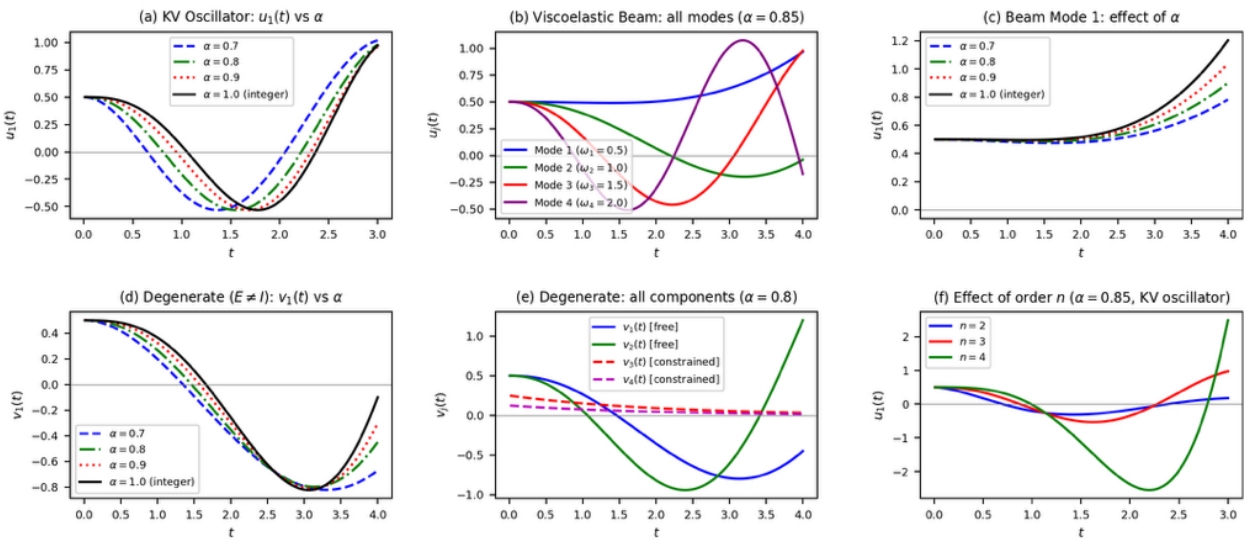

Figure 1. Finite-rank solutions to the higher-order conformable Cauchy problem for viscoelastic oscillators. (a) Kelvin–Voigt oscillator (§6.1): mode 1 displacement $u_1(t)$ for $\alpha = 0.7, 0.8, 0.9, 1.0$; smaller α delays the oscillation; (b) Viscoelastic beam (§6.2): all four modal displacements at $\alpha = 0.85$, showing frequency-dependent decay and forced oscillation; (c) Beam mode 1: effect of varying α; (d) Degenerate system (§6.3): free component $v_1(t)$ for varying α; the fractional order controls the rate of oscillatory decay; (e) Degenerate system at $\alpha = 0.80$: free components decay oscillatorily while constrained components follow the exponential forcing term; (f) Effect of order $n = 2, 3, 4$ on the Kelvin–Voigt oscillator at $\alpha = 0.85$; higher n introduces additional oscillatory modes.

7. Conclusions

This paper established a complete existence and uniqueness theory for finite-rank solutions to the n-th order conformable fractional abstract Cauchy problem in Hilbert space, valid for any integer $n \geq 2$ and any fractional order $\alpha \in (0, 1]$.

Conceptual contributions. The contributions of this paper are not a simple extension of existing results. First, as n increases, the companion matrix Γ grows from dimension $2m$ to nm, and its block structure changes at each level—this requires a separate solvability argument for each order, not a direct scaling of the second-order case. Second, the degenerate case $E \neq I$ is handled by strong induction: at each step, the kernel equations must be solved algebraically before the order can be reduced, adding $n-2$ elimination levels that do not appear in [17]. Third, to our knowledge, no previous work treats arbitrary order n, degenerate operator $E \neq I$, and the inverse problem together in a single unified framework.

Summary of main results. Theorems 3.1 and 3.4 settle the nondegenerate case $E = I$ by reducing problem (P) to a companion-type first-order conformable system of dimension nm, whose unique solvability follows from the variation-of-parameters formula. The key structural insight is that the companion block matrix Γ encodes the entire higher-order dynamics into a single first-order system, making the dimension growth from m to nm both explicit and tractable. Theorem 4.1 treats the degenerate case $E \neq I$ by strong induction on n: at each inductive step, the kernel equations algebraically determine the highest-order kernel components via the invertibility of $A_{n-1}|_{\ker(E)}$, reducing the problem to one of order $(n-1)\alpha$. This induction carries through $n-2$ additional elimination levels beyond the base case of [17], constituting the principal technical contribution of the paper. Theorem 5.1 addresses the inverse problem, showing that $f(t)$ is uniquely recovered from an inner-product measurement condition when both $u(t)$ and $f(t)$ are unknown. The case $n = 2$ recovers all results of [17] as a special case (Remarks 3.5 and 4.2).

Physical significance. The framework is validated on three viscoelastic models with $n = 3$: a Kelvin–Voigt polymer oscillator, a multi-mode viscoelastic beam, and a partially constrained degenerate frame. The numerical experiments confirm that the fractional order α controls the rate of oscillatory decay, with $\alpha \to 1$ recovering the standard integer-order response. The degenerate example demonstrates that constrained degrees of freedom follow the forcing term algebraically, while free components decay in an oscillatory manner—a qualitative behavior that the present theory now rigorously justifies for arbitrary n.

Open directions. The results of this paper open several new research directions. First, the companion structure of Γ provides a natural starting point for developing convergent numerical schemes for higher-order conformable systems, including Runge–Kutta-type methods adapted to the conformable derivative. Second, extending the theory to infinite-rank solutions via spectral theory of the operators A_k would substantially broaden the class of admissible solutions. Third, the kernel-elimination argument of Theorem 4.1 may be adaptable to non-diagonal operator coefficients, which would cover a wider class of coupled systems arising in structural mechanics. Finally, the abstract framework developed here is a prerequisite for treating higher-order conformable fractional partial differential equations in two spatial variables, where tensor product methods are expected to play an analogous role.

Author contributions

Huda Odetallah: Conceptualization, Methodology, Formal analysis, Investigation, Writing–original draft, Writing–review & editing, Supervision, Project administration; Mayada Abualhomos: Validation, Writing–review & editing; Husam Miqdad: Validation, Writing–review & editing; Tala

Sasa: Software, Investigation, Data curation, Writing–review & editing; Lubaba Shaikh: Software, Validation, Investigation, Writing–review & editing; Raja'a Al-Naimi: Formal analysis, Writing–review & editing. All authors read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used any Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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