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Research article

Approximation on univariate and bivariate modified Bernstein-Kantorovich operators

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Abstract: In this manuscript, we introduce a new ρ-th-order Kantorovich-type generalization of the modified Bernstein operators. First, we establish Voronovskaya-type theorems for the proposed operators. Then, we derive rates of convergence and establish global approximation theorems by means of the modulus of continuity, the $\mathbb{K}$-functional, the Ditzian-Totik weighted moduli of smoothness and the weighted $\mathbb{K}$-functional. Also, several numerical examples are presented to illustrate the approximation behavior using MATLAB. Finally, we construct a bivariate analog of these operators and investigate its convergence properties.

Keywords: Bernstein operators; Voronovskaya-type theorems; modulus of continuity; uniform convergence; global approximation; bivariate operators
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1. Introduction

As is well-known, the classical Bernstein operators were constructed in [1] to prove the Weierstrass approximation theorem as follows:

$$B_\lambda(\varphi;\nu)=\sum_{\mu=0}^{\lambda}\binom{\lambda}{\mu}\nu^\mu(1-\nu)^{\lambda-\mu}\varphi\left(\frac{\mu}{\lambda}\right),\ \nu\in[0,1], \tag{1.1}$$

where $\mu=0,1,\cdots,\lambda$, $\lambda\in\mathbb{N}^+:=\{1,2,\cdots\}$ and $\varphi\in C[0,1]$.

Later, many modifications and generalizations of Bernstein operators were introduced and studied extensively, including α-Bernstein operators [2–4], λ-Bernstein operators [5–8], q-Bernstein operators [9–11] and (p,q)-Bernstein operators [12–14], among others. In [15], Usta constructed the following modified Bernstein operators by using the second-order central moment of the classical Bernstein operators (1.1):

$$B_{\lambda}^{*}(\varphi;\nu)=\sum_{\mu=0}^{\lambda}p_{\lambda,\mu}(\nu)\varphi\left(\frac{\mu}{\lambda}\right),\ \nu\in[0,1], \tag{1.2}$$

where

$$p_{\lambda,\mu}(\nu)=\begin{cases}\lambda\nu(1-\nu)^{\lambda-1}, & \mu=0,\\ (\lambda\nu-1)^{2}(1-\nu)^{\lambda-2}, & \mu=1,\\ \dfrac{1}{\lambda}\dbinom{\lambda}{\mu}(\mu-\lambda\nu)^{2}\nu^{\mu-1}(1-\nu)^{\lambda-\mu-1}, & 1<\mu<\lambda-1,\\ [\lambda(1-\nu)-1]^{2}\nu^{\lambda-1}(1-\nu), & \mu=\lambda-1,\\ \lambda(1-\nu)\nu^{\lambda-1}, & \mu=\lambda.\end{cases}$$

The q-analog of the operators (1.2) was discussed in [16]. In [17], Cai et al. presented the beta-type generalization of the operators (1.2). Based on Usta's method in [15], several other operators have been constructed and studied by many authors (see [18–20]). In [21], Khan defined a ρ-th-order Kantorovich-type generalization of λ-Bernstein operators. Related studies were also considered by many authors; see, for example, [22–24]. Motivated by the above works, we define a new Kantorovich-type variant of the modified Berstein operators (1.2) on $[0,1]$ as follows:

$$K_{\lambda,\rho}^{*}(\varphi;\nu)=\sum_{\mu=0}^{\lambda}p_{\lambda,\mu}(\nu)\int_{0}^{1}\cdots\int_{0}^{1}\varphi\left(\frac{\mu+u_1+\cdots u_{\rho}}{\lambda+\rho}\right)\mathrm{d}u_1\cdots\mathrm{d}u_{\rho}. \tag{1.3}$$

2. Preliminaries

Let $e_m(u)=u^m$, $m\in\mathbb{N}:=\{0,1,2,\cdots\}$ denote the test functions, which play an important role in the operator approximation theory.

Lemma 2.1. *[16, Lemmas 1 and 2, q=1] For any $\nu\in[0,1]$, the following relations hold:*

$$B_{\lambda}(e_{m+1};\nu)=\frac{1}{\lambda}\nu(1-\nu)\left(B_{\lambda}(e_m;\nu)\right)'+\nu B_{\lambda}(e_m;\nu),$$

and

$$B_{\lambda}^{*}(e_m;\nu)=\frac{1}{\lambda}\left(\nu(1-\nu)\left(B_{\lambda}(e_m;\nu)\right)'\right)'+B_{\lambda}(e_m;\nu),$$

where $\lambda\in\mathbb{N}^{+}, m\in\mathbb{N}$.

The following lemma follows immediately.

Lemma 2.2. *For any $\nu\in[0,1]$, $\lambda\in\mathbb{N}^{+}$, we have*

$$\begin{aligned}B_{\lambda}^{*}(e_0;\nu)=&1,\\ B_{\lambda}^{*}(e_1;\nu)=&\left(\frac{\lambda-2}{\lambda}\right)\nu+\frac{1}{\lambda},\\ B_{\lambda}^{*}(e_2;\nu)=&\left(\frac{\lambda^2-7\lambda+6}{\lambda^2}\right)\nu^2+\left(\frac{5\lambda-6}{\lambda^2}\right)\nu+\frac{1}{\lambda^2},\end{aligned}$$

$$B_\lambda^*(e_3;\nu)=\left(\frac{\lambda^3-15\lambda^2+38\lambda-24}{\lambda^3}\right)\nu^3+12\left(\frac{\lambda^2-4\lambda+3}{\lambda^3}\right)\nu^2+\left(\frac{13\lambda-14}{\lambda^3}\right)\nu+\frac{1}{\lambda^3},$$
$$B_\lambda^*(e_4;\nu)=\left(\frac{\lambda^4-26\lambda^3+131\lambda^2-226\lambda+120}{\lambda^4}\right)\nu^4+\left(\frac{22\lambda^3-186\lambda^2+404\lambda-240}{\lambda^4}\right)\nu^3$$
$$+\left(\frac{61\lambda^2-211\lambda+150}{\lambda^4}\right)\nu^2+\left(\frac{29\lambda-30}{\lambda^4}\right)\nu+\frac{1}{\lambda^4}.$$

By direct calculation, we obtain the following lemma.

Lemma 2.3. *For $\nu\in[0,1]$, $\mu=\{0,1,\cdots\lambda\}$, $\lambda\in\mathbb{N}^+$,*

$$\left(\frac{\mu+u_1+\cdots u_\rho}{\lambda+\rho}\right)^m=\sum_{i_0+\cdots+i_\rho=m}\binom{m}{i_0,\cdots,i_\rho}\frac{\mu^{i_0}u_1^{i_1}\cdots u_\rho^{i_\rho}}{(\lambda+\rho)^m}.$$

Moreover,

$$\int_0^1\cdots\int_0^1\frac{\mu^{i_0}u_1^{i_1}\cdots u_\rho^{i_\rho}}{(\lambda+\rho)^m}\mathrm{d}u_1\cdots\mathrm{d}u_\rho=\frac{\mu^{i_0}}{(\lambda+\rho)^m(i_1+1)\cdots(i_\rho+1)}.$$

Using the lemma above, we can obtain the following moment relation between the operators (1.2) and (1.3).

Lemma 2.4. *For any $\nu\in[0,1]$, $\rho,\lambda\in\mathbb{N}^+$, we have*

$$K_{\lambda,\rho}^*(e_m;\nu)=\sum_{i_0+\cdots+i_\rho=m}\binom{m}{i_0,\cdots,i_\rho}\frac{\lambda^{i_0}}{(\lambda+\rho)^m(i_1+1)\cdots(i_\rho+1)}B_\lambda^*(e_{i_0};\nu). \tag{2.1}$$

Proof. By the definition of the operators $K_{\lambda,\rho}^*$, we have

$$\begin{aligned}K_{\lambda,\rho}^*(e_m;\nu)&=\sum_{\mu=0}^{\lambda}p_{\lambda,\mu}(\nu)\int_0^1\cdots\int_0^1\left(\frac{\mu+u_1+\cdots u_\rho}{\lambda+\rho}\right)^m\mathrm{d}u_1\cdots\mathrm{d}u_\rho\\&=\sum_{\mu=0}^{\lambda}p_{\lambda,\mu}(\nu)\sum_{i_0+\cdots+i_\rho=m}\binom{m}{i_0,\cdots,i_\rho}\int_0^1\cdots\int_0^1\frac{\mu^{i_0}u_1^{i_1}\cdots u_\rho^{i_\rho}}{(\lambda+\rho)^m}\mathrm{d}u_1\cdots\mathrm{d}u_\rho\\&=\sum_{\mu=0}^{\lambda}p_{\lambda,\mu}(\nu)\sum_{i_0+\cdots+i_\rho=m}\binom{m}{i_0,\cdots,i_\rho}\frac{\mu^{i_0}}{(\lambda+\rho)^m(i_1+1)\cdots(i_\rho+1)}\\&=\sum_{i_0+\cdots+i_\rho=m}\binom{m}{i_0,\cdots,i_\rho}\frac{\lambda^{i_0}}{(\lambda+\rho)^m(i_1+1)\cdots(i_\rho+1)}\sum_{\mu=0}^{\lambda}\frac{\mu^{i_0}}{\lambda^{i_0}}p_{\lambda,\mu}(\nu)\\&=\sum_{i_0+\cdots+i_\rho=m}\binom{m}{i_0,\cdots,i_\rho}\frac{\lambda^{i_0}}{(\lambda+\rho)^m(i_1+1)\cdots(i_\rho+1)}B_\lambda^*(e_{i_0};\nu).\end{aligned}$$

This proves (2.1). □

Combining Lemmas 2.2 and 2.4, we immediately obtain the following three lemmas.

Lemma 2.5. *For any $\nu \in [0,1], \rho, \lambda \in \mathbb{N}^+$, the following equalities and asymptotic formulas hold as $\lambda \to \infty$:*

$$
\begin{aligned}
K^*_{\lambda,\rho}(e_0;\nu) =& 1,\\
K^*_{\lambda,\rho}(e_1;\nu) =& \frac{\lambda-2}{\lambda+\rho}\nu + \frac{\rho+2}{2(\lambda+\rho)},\\
K^*_{\lambda,\rho}(e_2;\nu) =& \frac{\lambda^2-7\lambda+6}{(\lambda+\rho)^2}\nu^2 + \frac{5\lambda-6+(\lambda-2)\rho}{(\lambda+\rho)^2}\nu + \frac{3\rho^2+13\rho+12}{12(\lambda+\rho)^2},\\
K^*_{\lambda,\rho}(e_3;\nu) =& \frac{\lambda^3-15\lambda^2+38\lambda}{(\lambda+\rho)^3}\nu^3 + \left(\frac{3\rho(\lambda^2-7\lambda)}{2(\lambda+\rho)^3} + \frac{12(\lambda^2-4\lambda)}{(\lambda+\rho)^3}\right)\nu^2 + \left(\frac{3\lambda\rho^2+31\lambda\rho+52\lambda}{4(\lambda+\rho)^3}\right)\nu + o\left(\frac{1}{\lambda^2}\right),\\
K^*_{\lambda,\rho}(e_4;\nu) =& \frac{\lambda^2(\lambda^2-26\lambda+131)}{(\lambda+\rho)^4}\nu^4 + \left(\frac{\lambda^2(22\lambda-186)}{(\lambda+\rho)^4} + \frac{2\lambda^2\rho(\lambda-15)}{(\lambda+\rho)^4}\right)\nu^3\\
&+ \left(\frac{122\lambda^2+49\lambda^2\rho+3\lambda^2\rho^2}{2(\lambda+\rho)^4}\right)\nu^2 + o\left(\frac{1}{\lambda^2}\right).
\end{aligned}
$$

Lemma 2.6. *For any $\nu \in [0,1]$ and $\rho, \lambda \in \mathbb{N}^+$, we have*

$$
\begin{aligned}
\alpha_{\lambda,\rho}(\nu) :=& K^*_{\lambda,\rho}(e_1-\nu;\nu) = -\frac{\rho+2}{\lambda+\rho}\nu + \frac{\rho+2}{2(\lambda+\rho)},\\
\beta_{\lambda,\rho}(\nu) :=& K^*_{\lambda,\rho}\left((e_1-\nu)^2;\nu\right) = \frac{3\lambda-\rho^2-4\rho-6}{(\lambda+\rho)^2}\nu(1-\nu) + \frac{3\rho^2+13\rho+12}{12(\lambda+\rho)^2},
\end{aligned}
$$

and, as $\lambda \to \infty$,

$$
K^*_{\lambda,\rho}\left((e_1-\nu)^4;\nu\right) = \frac{15\lambda^2}{(\lambda+\rho)^4}\nu^2(\nu-1)^2 + o\left(\frac{1}{\lambda^2}\right).
$$

Lemma 2.7. *For any $\nu \in [0,1]$ and $\rho \in \mathbb{N}^+$, the following limits hold:*

$$
\lim_{\lambda\to\infty}(\lambda+\rho)K^*_{\lambda,\rho}(e_1-\nu;\nu) = \left(\frac{1}{2}-\nu\right)(\rho+2), \tag{2.2}
$$

$$
\lim_{\lambda\to\infty}(\lambda+\rho)K^*_{\lambda,\rho}((e_1-\nu)^2;\nu) = 3\nu(1-\nu), \tag{2.3}
$$

$$
\lim_{\lambda\to\infty}(\lambda+\rho)^2K^*_{\lambda,\rho}\left((e_1-\nu)^4;\nu\right) = 15\nu^2(\nu-1)^2. \tag{2.4}
$$

Lemma 2.8. *Let $C[0,1]$ be endowed with the maximum norm*

$$
\|\varphi\| = \max_{\nu\in[0,1]} |\varphi(\nu)|.
$$

Then, for any $\varphi \in C[0,1]$, we have

$$
\left\|K^*_{\lambda,\rho}(\varphi)\right\| \le \|\varphi\|.
$$

Proof. By the definition of the operators (1.3) and Lemma 2.5, we have

$$
\left|K^*_{\lambda,\rho}(\varphi;\nu)\right| \le \sum_{\mu=0}^{\lambda} p_{\lambda,\mu}(\nu)\int_0^1\cdots\int_0^1\left|\varphi\left(\frac{\mu+u_1+\cdots u_\rho}{\lambda+\rho}\right)\right| \mathrm{d}u_1\cdots\mathrm{d}u_\rho \le \|\varphi\|K^*_{\lambda,\rho}(e_0;\nu) = \|\varphi\|.
$$

Taking the maximum over $\nu \in [0, 1]$, we obtain

$$\left\| K^*_{\lambda,\rho}(\varphi) \right\| \le \|\varphi\|.$$

This completes the proof. □

Lemma 2.9. *For any $\varphi \in C[0, 1]$, $\rho \in \mathbb{N}^+$, we have*

$$\lim_{\lambda\to\infty} K^*_{\lambda,\rho}(\varphi; \nu) = \varphi(\nu)$$

uniformly in $[0, 1]$.

Proof. Because

$$K^*_{\lambda,\rho}(e_0; \nu) = 1,$$

and

$$K^*_{\lambda,\rho}(e_1; \nu) \to \nu, K^*_{\lambda,\rho}(e_2; \nu) \to \nu^2, \text{ as } \lambda \to \infty$$

uniformly in $[0, 1]$, the classical Korovkin Theorem [25, Theorem 3.1] implies that $K^*_{\lambda,\rho}(\varphi; \nu)$ converges to $\varphi(\nu)$ uniformly on $[0, 1]$. This completes the proof. □

3. Main results

3.1. Voronovskaya-type theorems

Theorem 3.1. *Let $\varphi \in C[0, 1]$. If φ'' exists at a point $\nu \in [0, 1]$, then the following limit holds*

$$\lim_{\lambda\to\infty}(\lambda + \rho)\left(K^*_{\lambda,\rho}(\varphi; \nu) - \varphi(\nu)\right) = (\rho + 2)\left(\frac{1}{2} - \nu\right)\varphi'(\nu) + \frac{3}{2}\nu(1 - \nu)\varphi''(\nu).$$

Proof. By Taylor's formula with Peano's remainder to φ, we have

$$\varphi(u) = \varphi(\nu) + \varphi'(\nu)(u - \nu) + \frac{1}{2}\varphi''(\nu)(u - \nu)^2 + \mathrm{R}(u, \nu)(u - \nu)^2, \tag{3.1}$$

where

$$\mathrm{R}(u, \nu) = \begin{cases} \dfrac{\varphi(u) - \varphi(\nu) - \varphi'(\nu)(u - \nu) - \frac{1}{2}\varphi''(\nu)(u - \nu)^2}{(u - \nu)^2}, & u \ne \nu, \\ 0, & u = \nu. \end{cases}$$

Because $\varphi''(\nu)$ exists, by L'Hôspital's Rule, we have

$$\lim_{u\to\nu} \mathrm{R}(u, \nu) = \frac{1}{2} \lim_{u\to\nu} \frac{\varphi'(u) - \varphi'(\nu)}{u - \nu} - \frac{1}{2}\varphi''(\nu) = 0 = \mathrm{R}(\nu, \nu).$$

Thus, for the fixed point $\nu \in [0, 1]$, the function $R(\cdot, \nu)$ belongs to $C[0, 1]$. By Lemma 2.9, it follows that

$$\lim_{\lambda\to\infty} K^*_{\lambda,\rho}(\mathrm{R}(u, \nu); \nu) = \mathrm{R}(\nu, \nu) = 0. \tag{3.2}$$

Applying the operators $K^*_{\lambda,\rho}$ to both sides of (3.1), we obtain

$$K^*_{\lambda,\rho}(\varphi;\nu) - \varphi(\nu) = \varphi'(\nu)\alpha_{\lambda,\rho}(\nu) + \frac{1}{2}\varphi''(\nu)\beta_{\lambda,\rho}(\nu) + K^*_{\lambda,\rho}\left(\mathrm{R}(u,\nu)(u-\nu)^2;\nu\right).$$

For the last term, the Cauchy-Bunyakovsky-Schwarz inequality gives

$$\left|K^*_{\lambda,\rho}\left(\mathrm{R}(u,\nu)(u-\nu)^2;\nu\right)\right| \le \sqrt{K^*_{\lambda,\rho}\left((\mathrm{R}(u,\nu))^2;\nu\right)}\sqrt{K^*_{\lambda,\rho}\left((u-\nu)^4;\nu\right)}. \tag{3.3}$$

Therefore, by (2.4), (3.2), and (3.3), we obtain

$$\lim_{\lambda\to\infty}(\lambda+\rho)K^*_{\lambda,\rho}\left(\mathrm{R}(u,\nu)(u-\nu)^2;\nu\right) = 0. \tag{3.4}$$

Combining (2.2), (2.3), and (3.4), we arrive at the desired result. □

Corollary 3.2. *Let* $\varphi'' \in C[0,1]$ *and* $\rho \in \mathbb{N}^+$. *Then,*

$$\lim_{\lambda\to\infty}(\lambda+\rho)\left(K^*_{\lambda,\rho}(\varphi;\nu) - \varphi(\nu)\right) = (\rho+2)\left(\frac{1}{2}-\nu\right)\varphi'(\nu) + \frac{3}{2}\nu(1-\nu)\varphi''(\nu).$$

uniformly for $\nu \in [0,1]$.

Theorem 3.3. *Let* $\phi, \varphi \in C[0,1]$ *satisfy* $\phi'', \varphi'' \in C[0,1]$. *Then,*

$$\lim_{\lambda\to\infty}(\lambda+\rho)\left(K^*_{\lambda,\rho}(\phi\cdot\varphi;\nu) - K^*_{\lambda,\rho}(\phi;\nu)\cdot K^*_{\lambda,\rho}(\varphi;\nu)\right) = 3\nu(1-\nu)\phi'(\nu)\cdot\varphi'(\nu).$$

Proof. We first recall that

$$\begin{aligned}(\phi\cdot\varphi)(\nu) &= \phi(\nu)\cdot\varphi(\nu),\\ (\phi\cdot\varphi)'(\nu) &= \phi'(\nu)\cdot\varphi(\nu) + \phi(\nu)\cdot\varphi'(\nu),\end{aligned}$$

and

$$(\phi\cdot\varphi)''(\nu) = \phi''(\nu)\cdot\varphi(\nu) + 2\phi'(\nu)\cdot\varphi'(\nu) + \phi(\nu)\cdot\varphi''(\nu).$$

By direct computation, for every $\nu \in [0,1]$, we have

$$\begin{aligned}&K^*_{\lambda,\rho}(\phi\cdot\varphi;\nu) - K^*_{\lambda,\rho}(\phi;\nu)\cdot K^*_{\lambda,\rho}(\varphi;\nu)\\ =&\left(K^*_{\lambda,\rho}(\phi\cdot\varphi;\nu) - \phi\cdot\varphi(\nu) - (\phi\cdot\varphi(\nu))'\alpha_{\lambda,\rho}(\nu) - \frac{(\phi\cdot\varphi(\nu))''}{2}\beta_{\lambda,\rho}(\nu)\right)\\ &- \varphi(\nu)\left(K^*_{\lambda,\rho}(\phi;\nu) - \phi(\nu) - \phi'(\nu)\alpha_{\lambda,\rho}(\nu) - \frac{\phi''(\nu)}{2}\beta_{\lambda,\rho}(\nu)\right)\\ &- K^*_{\lambda,\rho}(\phi;\nu)\left(K^*_{\lambda,\rho}(\varphi;\nu) - \varphi(\nu) - \varphi'(\nu)\alpha_{\lambda,\rho}(\nu) - \frac{\varphi''(\nu)}{2}\beta_{\lambda,\rho}(\nu)\right)\\ &+ \frac{1}{2}\beta_{\lambda,\rho}(\nu)\left(\phi(\nu)\cdot\varphi''(\nu) + 2\phi'(\nu)\cdot\varphi'(\nu) - K^*_{\lambda,\rho}(\phi;\nu)\cdot\varphi''(\nu)\right)\\ &+ \alpha_{\lambda,\rho}\left(\phi(\nu)\cdot\varphi'(\nu) - K^*_{\lambda,\rho}(\phi;\nu)\cdot\varphi'(\nu)\right).\end{aligned}$$

Using (2.3), Lemma 2.9, and Theorem 3.1, we obtain

$$\lim_{\lambda\to\infty}(\lambda+\rho)\left(K^*_{\lambda,\rho}(\phi\cdot\varphi;\nu) - K^*_{\lambda,\rho}(\phi;\nu)\cdot K^*_{\lambda,\rho}(\varphi;\nu)\right) = 3\nu(1-\nu)\phi'(\nu)\cdot\varphi'(\nu).$$

This completes the proof. □

3.2. Rate of convergence

In this subsection, we discuss the rate of convergence for the operators (1.3) in terms of the modulus of continuity, the Peetre's $\mathbb{K}$-functional, the Steklov mean function, and the elements of Lipschitz-type function classes.

For $\varphi \in C[0,1]$, the usual modulus of continuity and the second-order modulus of smoothness of φ are defined respectively by

$$\omega(\varphi;\chi) = \sup_{0<|u|<\chi} \sup_{\nu,\nu+u\in[0,1]} |\varphi(\nu+u)-\varphi(\nu)|,$$

and

$$\omega_2(\varphi;\chi) = \sup_{0<|u|<\chi} \sup_{\nu,\nu+2u\in[0,1]} |\varphi(\nu+2u)-2\varphi(\nu+u)+\varphi(\nu)|.$$

The following two properties of the usual modulus of continuity are readily obtained:

$$|\varphi(u)-\varphi(\nu)| \leq \omega(\varphi;\chi)\left(1+\frac{|u-\nu|}{\chi}\right), \tag{3.5}$$

$$|\varphi(u)-\varphi(\nu)| \leq \omega(\varphi;\chi)\left(1+\frac{(u-\nu)^2}{\chi^2}\right). \tag{3.6}$$

The classical Peetre's $\mathbb{K}$-functional associated with the function $\varphi \in C[0,1]$ is defined by

$$\mathbb{K}(\varphi;\chi) = \inf_{\phi''\in C[0,1]} \{\|\varphi-\phi\| + \chi\|\phi''\|\},$$

where $\chi > 0$. By [25, Theorem 2.4], there exists an absolute constant $C_1 > 0$ such that

$$\mathbb{K}(\varphi;\chi) \leq C_1\omega_2(\varphi;\sqrt{\chi}). \tag{3.7}$$

We first derive four estimates for the rate of convergence of the operators $K^*_{\lambda,\rho}$ in terms of the modulus of continuity and the classical Peetre's $\mathbb{K}$-functional.

Theorem 3.4. *Let $\varphi \in C[0,1]$ and $\rho \in \mathbb{N}^+$. Then, for every $\nu \in [0,1]$, we have*

$$\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right| \leq 2\omega\left(\varphi;\sqrt{\beta_{\lambda,\rho}(\nu)}\right).$$

Proof. By the positivity and linearity of $K^*_{\lambda,\rho}$ together with (3.6), we obtain

$$\begin{aligned}\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right| &\leq K^*_{\lambda,\rho}\left(|\varphi(u)-\varphi(\nu)|;\nu\right)\\ &\leq \omega(\varphi;\chi)\left(1+\frac{K^*_{\lambda,\rho}\left((e_1-\nu)^2;\nu\right)}{\chi^2}\right)\\ &= \omega(\varphi;\chi)\left(1+\frac{\beta_{\lambda,\rho}(\nu)}{\chi^2}\right).\end{aligned}$$

Choosing $\chi = \sqrt{\beta_{\lambda,\rho}(\nu)}$, we obtain

$$\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right| \leq 2\omega\left(\varphi;\sqrt{\beta_{\lambda,\rho}(\nu)}\right).$$

This completes the proof. □

Theorem 3.5. *Let $\varphi \in C[0,1]$ and $\rho \in \mathbb{N}^+$. Then, for every $\nu \in [0,1]$, we have*

$$\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right| \leq \omega\left(\varphi;\ \sqrt{\beta_{\lambda,\rho}(\nu)}\right)+\frac{3}{2}\omega_2\left(\varphi;\ \sqrt{\beta_{\lambda,\rho}(\nu)}\right).$$

Proof. By a result of R. Păltănea [26], for any positive linear operator L, the following inequality holds:

$$|L(\varphi;\nu)-\varphi(\nu)| \leq |L(e_0;\nu)-1|\cdot|\varphi(\nu)|+\frac{1}{\chi}\,|L(e_1-\nu;\nu)|\,\omega(\varphi;\chi)+\left(L(e_0;\nu)+\frac{1}{2\chi^2}L\left((e_1-\nu)^2;\nu\right)\right)\omega_2(\varphi;\chi).$$

Taking $L := K^*_{\lambda,\rho}$ and using Lemma 2.5, we have

$$\left|K^*_{\lambda,\rho}(e_1-\nu;\nu)\right| \leq K^*_{\lambda,\rho}\left(|e_1-\nu|;\nu\right) \leq \sqrt{K^*_{\lambda,\rho}\left((e_1-\nu)^2;\nu\right)},$$

and then we have

$$\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right| \leq \frac{\sqrt{K^*_{\lambda,\rho}\left((e_1-\nu)^2;\nu\right)}}{\chi}\omega(\varphi;\chi)+\left(1+\frac{K^*_{\lambda,\rho}\left((e_1-\nu)^2;\nu\right)}{\chi^2}\right)\omega_2(\varphi;\chi).$$

Taking $\chi = \sqrt{\beta_{\lambda,\rho}(\nu)} = \sqrt{K^*_{\lambda,\rho}\left((e_1-\nu)^2;\nu\right)}$, we complete the proof of Theorem 3.5. □

Theorem 3.6. *Let $\varphi \in C^1[0,1]$, $\rho \in \mathbb{N}^+$. Then, for every $\nu \in [0,1]$, we have*

$$\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right| \leq \left|\alpha_{\lambda,\rho}(\nu)\right||\varphi'(\nu)|+2\sqrt{\beta_{\lambda,\rho}(\nu)}\,\omega\left(\varphi';\ \sqrt{\beta_{\lambda,\rho}(\nu)}\right).$$

Proof. For $u,\nu \in [0,1]$, we write

$$\begin{aligned}\varphi(u) &= \varphi(\nu)+\varphi'(\nu)(u-\nu)+\varphi(u)-\varphi(\nu)-\varphi'(\nu)(u-\nu)\\ &= \varphi(\nu)+\varphi'(\nu)(u-\nu)+\int_\nu^u(\varphi'(\sigma)-\varphi'(\nu))\mathrm{d}\sigma.\end{aligned}$$

Applying $K^*_{\lambda,\rho}$ on both sides of this equality and using (3.5), we obtain

$$\begin{aligned}\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right| &\leq |\varphi'(\nu)|\left|K^*_{\lambda,\rho}\left(e_1-\nu;\nu\right)\right|+\left|K^*_{\lambda,\rho}\left(\int_\nu^u(\varphi'(\sigma)-\varphi'(\nu))\mathrm{d}\sigma;\nu\right)\right|\\ &= |\varphi'(\nu)|\left|\alpha_{\lambda,\rho}(\nu)\right|+\left|K^*_{\lambda,\rho}\left(\int_\nu^u(\varphi'(\sigma)-\varphi'(\nu))\mathrm{d}\sigma;\nu\right)\right|.\end{aligned}$$

By the definition of the modulus of continuity, for any $\chi > 0$,

$$\left|\int_\nu^u(\varphi'(\sigma)-\varphi'(\nu))\mathrm{d}\sigma\right| \leq \omega(\varphi';\chi)\left(|u-\nu|+\frac{(u-\nu)^2}{\chi}\right).$$

Therefore, by the positivity of $K^*_{\lambda,\rho}$ and the Cauchy-Bunyakovsky inequality,

$$\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right| \leq |\varphi'(\nu)|\left|\alpha_{\lambda,\rho}(\nu)\right|+\omega(\varphi';\chi)K^*_{\lambda,\rho}\left(|u-\nu|+\frac{(u-\nu)^2}{\chi};\nu\right)$$

$$\leq |\varphi'(\nu)| \left|\alpha_{\lambda,\rho}(\nu)\right| + \omega\left(\varphi';\chi\right) \sqrt{K^*_{\lambda,\rho}\left((e_1-\nu)^2;\nu\right)} \left(1 + \frac{\sqrt{K^*_{\lambda,\rho}\left((e_1-\nu)^2;\nu\right)}}{\chi}\right).$$

Choosing

$$\chi = \sqrt{\beta_{\lambda,\rho}(\nu)} = \sqrt{K^*_{\lambda,\rho}\left((e_1-\nu)^2;\nu\right)},$$

we finally get

$$\left|K^*_{\lambda,\rho}(\varphi;\nu) - \varphi(\nu)\right| \leq \left|\alpha_{\lambda,\rho}(\nu)\right| |\varphi'(\nu)| + 2\sqrt{\beta_{\lambda,\rho}(\nu)}\,\omega\left(\varphi';\ \sqrt{\beta_{\lambda,\rho}(\nu)}\right).$$

This completes the proof. □

Theorem 3.7. *For any $\varphi \in C[0,1]$ and $\rho \in \mathbb{N}^+$, we have*

$$\left|K^*_{\lambda,\rho}(\varphi;\nu) - \varphi(\nu)\right| \leq 4C_1\omega_2\left(\varphi;\ \sqrt{\left(\alpha_{\lambda,\rho}(\nu)\right)^2 + \beta_{\lambda,\rho}(\nu)}\right) + \omega\left(\varphi; \left|\alpha_{\lambda,\rho}(\nu)\right|\right).$$

Proof. For each $\nu \in [0,1]$, define the following auxiliary operators:

$$K^{**}_{\lambda,\rho}(\varphi;\nu) = K^*_{\lambda,\rho}(\varphi;\nu) + \varphi(\nu) - \varphi\left(\alpha_{\lambda,\rho}(\nu) + \nu\right), \tag{3.8}$$

By Lemmas 2.5 and 2.8, we obtain

$$K^{**}_{\lambda,\rho}(e_0;\nu) = 1,\ K^{**}_{\lambda,\rho}(e_1;\nu) = \nu, \tag{3.9}$$

and

$$\|K^{**}_{\lambda,\rho}(\varphi)\| \leq 3\|\varphi\|. \tag{3.10}$$

For any ϕ with $\phi'' \in C[0,1]$, Taylor's formula with integral remainder gives, for $u, \nu \in [0,1]$,

$$\phi(u) = \phi(\nu) + \phi'(\nu)(u-\nu) + \int_\nu^u (\sigma-\nu)\phi''(\sigma)\mathrm{d}\sigma. \tag{3.11}$$

Applying $K^{**}_{\lambda,\rho}$ to both sides of (3.11) and using (3.8) and (3.9), we obtain

$$\begin{aligned}
K^{**}_{\lambda,\rho}(\phi;\nu) - \phi(\nu) &= K^{**}_{\lambda,\rho}\left((e_1-\nu)\phi'(\nu);\nu\right) + K^{**}_{\lambda,\rho}\left(\int_\nu^u (\sigma-\nu)\phi''(\sigma)\mathrm{d}\sigma;\nu\right) \\
&= K^{**}_{\lambda,\rho}\left(\int_\nu^u (\sigma-\nu)\phi''(\sigma)\mathrm{d}\sigma;\nu\right) \\
&= K^{*}_{\lambda,\rho}\left(\int_\nu^u (\sigma-\nu)\phi''(\sigma)\mathrm{d}\sigma;\nu\right) + \int_\nu^{\alpha_{\lambda,\rho}(\nu)+\nu} (\sigma-\nu)\phi''(\sigma)\mathrm{d}\sigma.
\end{aligned}$$

Furthermore,

$$\left|\int_\nu^u (\sigma-\nu)\phi''(\sigma)\mathrm{d}\sigma\right| \leq \left|\int_\nu^u |\sigma-\nu|\phi''(\sigma)\mathrm{d}\sigma\right| \leq (u-\nu)^2\|\phi''\|,$$

and

$$\left|\int_\nu^{\alpha_{\lambda,\rho}(\nu)+\nu} (\sigma-\nu)\phi''(\sigma)\mathrm{d}\sigma\right| \leq \left|\int_\nu^{\alpha_{\lambda,\rho}(\nu)+\nu} |\sigma-\nu|\,\phi''(\sigma)\mathrm{d}\sigma\right| \leq \left(\alpha_{\lambda,\rho}(\nu)\right)^2\|\phi''\|.$$

Hence,

$$\left|K^{**}_{\lambda,\rho}(\phi;\nu)-\phi(\nu)\right|\le\left(\left(\alpha_{\lambda,\rho}(\nu)\right)^2+\beta_{\lambda,\rho}(\nu)\right)\|\phi''\|.$$

For $\varphi\in C[0,1]$ and any ϕ with$\phi''\in C[0,1]$, using (3.8) and (3.10), we have

$$\begin{aligned}\left|K^{*}_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right|&\le\left|K^{**}_{\lambda,\rho}(\varphi;\nu)+\varphi\left(\alpha_{\lambda,\rho}(\nu)+\nu\right)-2\varphi(\nu)\right|\\&\le\left|K^{**}_{\lambda,\rho}(\varphi-\phi;\nu)\right|+|\varphi(\nu)-\phi(\nu)|+\left|K^{**}_{\lambda,\rho}(\phi;\nu)-\phi(\nu)\right|+\left|\varphi\left(\alpha_{\lambda,\rho}(\nu)+\nu\right)-\varphi(\nu)\right|\\&\le 4\|\varphi-\phi\|+\left(\left(\alpha_{\lambda,\rho}(\nu)\right)^2+\beta_{\lambda,\rho}(\nu)\right)\|\phi''\|+\omega\left(\varphi;\left|\alpha_{\lambda,\rho}(\nu)\right|\right).\end{aligned}$$

Taking the infimum over all such ϕ and using (3.7), we obtain

$$\begin{aligned}\left|K^{*}_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right|&\le 4\mathbb{K}(\varphi;\left(\alpha_{\lambda,\rho}(\nu)\right)^2+\beta_{\lambda,\rho}(\nu))+\omega\left(\varphi;\left|\alpha_{\lambda,\rho}(\nu)\right|\right)\\&\le 4C_1\omega_2\left(\varphi;\sqrt{\left(\alpha_{\lambda,\rho}(\nu)\right)^2+\beta_{\lambda,\rho}(\nu)}\right)+\omega\left(\varphi;\left|\alpha_{\lambda,\rho}(\nu)\right|\right).\end{aligned}$$

This completes the proof of Theorem 3.7. □

For $\varphi\in C[0,1]$ and $\iota\in(0,1]$, the Steklov mean function is defined by

$$\varphi_\iota(\nu)=\frac{4}{\iota^2}\int_0^{\frac{\iota}{2}}\int_0^{\frac{\iota}{2}}\left(2\varphi(\nu+\sigma+\varsigma)-\varphi\left(\nu+2(\sigma+\varsigma)\right)\right)\mathrm{d}\sigma\mathrm{d}\varsigma.$$

By direct computation, we have
(A) $\|\varphi_\iota-\varphi\|\le\omega_2\left(\varphi;\iota\right)$,
(B) $\varphi'_\iota,\varphi''_\iota\in C[0,1]$ and

$$\|\varphi'_\iota\|\le\frac{5}{\iota}\omega\left(\varphi;\iota\right),\ \|\varphi''_\iota\|\le\frac{9}{\iota^2}\omega_2\left(\varphi;\iota\right).$$

Theorem 3.8. *Let $\varphi\in C[0,1]$ and $\rho\in\mathbb{N}^+$. Then, for every $\nu\in[0,1]$, we have*

$$\left|K^{*}_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right|\le 5\sqrt{\lambda}\left|\alpha_{\lambda,\rho}(\nu)\right|\omega\left(\varphi;\frac{1}{\sqrt{\lambda}}\right)+\left(\frac{9\lambda}{2}\beta_{\lambda,\rho}(\nu)+2\right)\omega_2\left(\varphi;\frac{1}{\sqrt{\lambda}}\right).$$

Proof. For $\iota\in(0,1],\nu,\nu+2\iota\in[0,1]$, applying the Steklov mean function φ_ι, we obtain

$$\left|K^{*}_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right|\le K^{*}_{\lambda,\rho}\left(|\varphi-\varphi_\iota|;\nu\right)+\left|K^{*}_{\lambda,\rho}\left(\varphi_\iota-\varphi_\iota(\nu);\nu\right)\right|+\left|\varphi_\iota(\nu)-\varphi(\nu)\right|.$$

By property (A) of the Steklov mean function and Lemma 2.8, we have

$$K^{*}_{\lambda,\rho}\left(|\varphi-\varphi_\iota|;\nu\right)\le\left\|K^{*}_{\lambda,\rho}\left(|\varphi-\varphi_\iota|\right)\right\|\le\|\varphi-\varphi_\iota\|\le\omega_2\left(\varphi;\iota\right).$$

By Taylor's expansion formula applied to φ_ι, we have

$$\varphi_\iota(u)=\varphi_\iota(\nu)+\varphi'_\iota(\nu)(u-\nu)+\int_\nu^u(u-\sigma)\varphi''_\iota(\sigma)\mathrm{d}\sigma.\tag{3.12}$$

Applying $K^*_{\lambda,\rho}$ to both sides of (3.12) and using property (B) of the Steklov mean function together with Lemma 2.6, we obtain

$$\begin{aligned}\left|K^*_{\lambda,\rho}\left(\varphi_\iota-\varphi_\iota(\nu);\nu\right)\right| &\leq |\varphi_\iota'(\nu)|\left|\alpha_{\lambda,\rho}(\nu)\right|+\frac{1}{2}\|\varphi_\iota''(\nu)\|\beta_{\lambda,\rho}(\nu)\\ &\leq \|\varphi_\iota'(\nu)\|\left|\alpha_{\lambda,\rho}(\nu)\right|+\frac{1}{2}\|\varphi_\iota''(\nu)\|\beta_{\lambda,\rho}(\nu)\\ &\leq \frac{5}{\lambda}\left|\alpha_{\lambda,\rho}(\nu)\right|\omega\left(\varphi;\iota\right)+\frac{9}{2\iota^2}\beta_{\lambda,\rho}(\nu)\omega_2\left(\varphi;\iota\right).\end{aligned}$$

Combing the above estimate with property (B), we get

$$\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right|\leq\frac{5}{\iota}\left|\alpha_{\lambda,\rho}(\nu)\right|\omega\left(\varphi;\iota\right)+\left(\frac{9}{2\iota^2}\beta_{\lambda,\rho}(\nu)+2\right)\omega_2\left(\varphi;\iota\right).$$

Taking $\iota=\frac{1}{\sqrt{\lambda}}$, we arrive at

$$\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right|\leq 5\sqrt{\lambda}\left|\alpha_{\lambda,\rho}(\nu)\right|\omega\left(\varphi;\frac{1}{\sqrt{\lambda}}\right)+\left(\frac{9\lambda}{2}\beta_{\lambda,\rho}(\nu)+2\right)\omega_2\left(\varphi;\frac{1}{\sqrt{\lambda}}\right).$$

This completes the proof. □

For $\gamma\in(0,1]$, a function φ is said to belong to $\varphi\in\mathrm{Lip}_M^\gamma$ if

$$|\varphi(u)-\varphi(\nu)|\leq M\,|u-\nu|^\gamma,\ u,\nu\in[0,1],$$

where $M>0$ is a constant depending only on φ and γ.

Theorem 3.9. *Let $\varphi\in\mathrm{Lip}_M^\gamma$, $\gamma\in(0,1]$, and $\rho\in\mathbb{N}^+$. Then, for every $\nu\in[0,1]$,*

$$\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right|\leq M\left(\beta_{\lambda,\rho}(\nu)\right)^{\frac{\gamma}{2}}.$$

Proof. By the monotonicity and the linearity of the operators $K^*_{\lambda,\rho}$, we have

$$\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right|\leq K^*_{\lambda,\rho}\left(|\varphi(u)-\varphi(\nu)|\,;\nu\right)\leq MK^*_{\lambda,\rho}\left(|u-\nu|^\gamma\,;\nu\right).$$

Applying the well-known Hölder inequality with $p=\dfrac{2}{\gamma}$, $q=\dfrac{2}{2-\gamma}$, we obtain

$$K^*_{\lambda,\rho}\left(|u-\nu|^\gamma\,;\nu\right)\leq\left(K^*_{\lambda,\rho}\left((u-\nu)^{\gamma p};u\right)\right)^{\frac{1}{p}}\left(K^*_{\lambda,\rho}(1^q;\nu)\right)^{\frac{1}{q}}.$$

Because $\gamma p=2$, Lemma 2.6 gives

$$K^*_{\lambda,\rho}\left(|u-\nu|^\gamma\,;\nu\right)\leq\left(\beta_{\lambda,\rho}(\nu)\right)^{\frac{\gamma}{2}}.$$

Therefore,

$$\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right|\leq M\left(\beta_{\lambda,\rho}(\nu)\right)^{\frac{\gamma}{2}}.$$

This completes the proof. □

In [27], B. Lenze introduced the following Lipschitz-type maximal function of order γ:

$$\widetilde{\omega}_\gamma(\varphi;\nu)=\sup_{u\neq\nu,u,\nu\in[0,1]}\frac{|\varphi(u)-\varphi(\nu)|}{|u-\nu|^\gamma},\ \gamma\in\in(0,1]. \tag{3.13}$$

Theorem 3.10. *Let* $C[0,1]$, $\gamma\in(0,1]$, *and* $\rho\in\mathbb{N}^+$. *Then, for every* $\nu\in[0,1]$, *we have*

$$\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right|\leq\widetilde{\omega}_\gamma(\varphi;\nu)\left(\beta_{\lambda,\rho}(\nu)\right)^{\frac{\gamma}{2}}.$$

Proof. By the monotonicity and the linearity of the operators $K^*_{\lambda,\rho}$ together with Lemma 2.6 and the definition of $\widetilde{\omega}_\gamma(\varphi;\nu)$, we obtain

$$\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right|\leq K^*_{\lambda,\rho}\left(|\varphi(u)-\varphi(\nu)|;\nu\right)\ \leq\widetilde{\omega}_\gamma(\varphi;\nu)K^*_{\lambda,\rho}\left(|u-\nu|^\gamma;\nu\right).$$

Applying the well-known Hölder with $p=\dfrac{2}{\gamma}$, $q=\dfrac{2}{2-\gamma}$, Lemma 2.6, and (3.13), we obtain

$$\begin{aligned}\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right|&\leq K^*_{\lambda,\rho}\left(|\varphi(u)-\varphi(\nu)|;\nu\right)\\&\leq\widetilde{\omega}_\gamma(\varphi;\nu)K^*_{\lambda,\rho}\left(|u-\nu|^\gamma;\nu\right)\\&\leq\widetilde{\omega}_\gamma(\varphi;\nu)\left(K^*_{\lambda,\rho}\left((e_1-u)^{\gamma p};u\right)\right)^{\frac{1}{p}}\left(K^*_{\lambda,\rho}(1^q;\nu)\right)^{\frac{1}{q}}\\&=\widetilde{\omega}_\gamma(\varphi;\nu)\left(\beta_{\lambda,\rho}(\nu)\right)^{\frac{\gamma}{2}}.\end{aligned}$$

□

In [28], M. A. Özarslan and H. Aktuğlu introduced the following Lipschitz-type function space:

$$\mathrm{Lip}_M^{(\kappa_1,\kappa_2)}(\gamma):=\left\{\varphi\in C[0,1]:|\varphi(u)-\varphi(\nu)|\leq M\frac{|u-\nu|^\gamma}{(u+\kappa_1\nu^2+\kappa_2\nu)^{\frac{\gamma}{2}}},\ u,\nu\in(0,1]\right\},$$

where $\gamma\in(0,1]$, $\kappa_1\geq0$, $\kappa_2>0$, M is a positive constant depending only on φ, γ, $\kappa_1\geq0$, and $\kappa_2>0$.

Theorem 3.11. *Let* $\varphi\in\mathrm{Lip}_M^{(\kappa_1,\kappa_2)}(\gamma)$, *where* $\gamma\in(0,1]$, $\kappa_1\geq0$, $\kappa_2>0$, *and* $\rho\in\mathbb{N}^+$. *Then, for every* $\nu\in(0,1]$, *we have*

$$\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right|\leq M\left(\frac{\beta_{\lambda,\rho}(\nu)}{\kappa_1\nu^2+\kappa_2\nu}\right)^{\frac{\gamma}{2}}.$$

Proof. By the well-known Hölder with $p=\frac{2}{\gamma}$, $q=\frac{2}{2-\gamma}$ together with the monotonicity and the linearity of the operators $K^*_{\lambda,\rho}$ and Lemma 2.6, we have

$$\begin{aligned}&\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right|\\&\leq\sum_{\mu=0}^{\lambda}p_{\lambda,\mu}(\nu)\int_0^1\cdots\int_0^1\left|\varphi\left(\frac{\mu+u_1+\cdots u_\rho}{\lambda+\rho}\right)-\varphi(\nu)\right|\mathrm{d}u_1\cdots\mathrm{d}u_\rho\\&\leq\left(\sum_{\mu=0}^{\lambda}p_{\lambda,\mu}(\nu)\int_0^1\cdots\int_0^1\left|\varphi\left(\frac{\mu+u_1+\cdots u_\rho}{\lambda+\rho}\right)-\varphi(\nu)\right|^p\mathrm{d}u_1\cdots\mathrm{d}u_\rho\right)^{\frac{1}{p}}\cdot\left(\sum_{\mu=0}^{\lambda}p_{\lambda,\mu}(\nu)\int_0^1\cdots\int_0^1 1^q\mathrm{d}u_1\cdots\mathrm{d}u_\rho\right)^{\frac{1}{q}}.\end{aligned}$$

Because $\varphi \in \mathrm{Lip}_M^{(\kappa_1,\kappa_2)}(\gamma)$, we further have

$$\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right| \le M\left(\sum_{\mu=0}^{\lambda} p_{\lambda,\mu}(\nu)\int_0^1\cdots\int_0^1\left|\frac{\left|\frac{\mu+u_1+\cdots u_\rho}{\lambda+\rho}-\nu\right|^{\gamma}}{\left(\frac{\mu+u_1+\cdots u_\rho}{\lambda+\rho}+\kappa_1\nu^2+\kappa_2\nu\right)^{\frac{\gamma}{2}}}\right|^{p}\mathrm{d}u_1\cdots\mathrm{d}u_\rho\right)^{\frac{1}{p}}$$
$$\le \frac{M}{(\kappa_1\nu^2+\kappa_2\nu)^{\frac{\gamma}{2}}}\left(\sum_{\mu=0}^{\lambda} p_{\lambda,\mu}(\nu)\int_0^1\cdots\int_0^1\left(\frac{\mu+u_1+\cdots u_\rho}{\lambda+\rho}-\nu\right)^{\gamma p}\mathrm{d}u_1\cdots\mathrm{d}u_\rho\right)^{\frac{1}{p}}.$$

Because $p=\dfrac{2}{\gamma}$, we have $\gamma p = 2$. Therefore,

$$\left|K^*_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right| \le \frac{M}{(\kappa_1\nu^2+\kappa_2\nu)^{\frac{\gamma}{2}}}\left(\sum_{\mu=0}^{\lambda} p_{\lambda,\mu}(\nu)\int_0^1\cdots\int_0^1\left(\frac{\mu+u_1+\cdots u_\rho}{\lambda+\rho}-\nu\right)^{\gamma p}\mathrm{d}u_1\cdots\mathrm{d}u_\rho\right)^{\frac{1}{p}}$$
$$= M\left(\frac{\beta_{\lambda,\rho}(\nu)}{\kappa_1\nu^2+\kappa_2\nu}\right)^{\frac{\gamma}{2}}.$$

This completes the proof. □

3.3. Global approximation

For $\varphi \in C[0,1]$, the first- and second-order Ditzian-Totik weighted moduli of smoothness are defined, respectively, by

$$\omega^{\Psi}(\varphi;\chi) = \sup_{|\sigma|\le\chi}\ \sup_{u,u+\sigma\Psi(u)\in[0,1]} \left|\varphi\left(u+\sigma\Psi(u)\right)-\varphi\left(u\right)\right|,$$

and

$$\omega_2^{\Phi}(\varphi;\chi) = \sup_{|\sigma|\le\chi}\ \sup_{u\pm\sigma\Phi(u)\in[0,1]} \left|\varphi\left(u-\sigma\Phi(u)\right)-2\varphi\left(u\right)+\varphi\left(u+\sigma\Phi(u)\right)\right|.$$

The corresponding second-order weighted $\mathbb{K}$-functional is defined by

$$\mathbb{K}(\varphi;\chi) = \inf_{\phi\in\mathbb{W}^2(\Phi)}\left\{\|\varphi-\phi\|+\chi\|\Phi^2\phi''\|+\chi^2\|\phi''\|\right\},\ \chi>0,$$

where

$$\mathbb{W}^2(\Phi) = \{\phi\in C[0,1] : \phi' \in \mathrm{AC}_{\mathrm{loc}}\}, \Phi^2\phi'' \in C[0,1].$$

Here, $\phi' \in \mathrm{AC}_{\mathrm{loc}}$ means that ϕ is differentiable, and ϕ' is absolutely continuous on every closed subinterval $I \subset [0,1]$. By [29], there exists an absolute constant $M > 0$, such that

$$\mathbb{K}(\varphi;\chi) \le M\omega_2^{\Phi}(\varphi;\sqrt{\chi}), \tag{3.14}$$

where the weighted functions are given by

$$\Phi(\nu) = \sqrt{\nu(1-\nu)},\ \Psi(\nu) = (\rho+2)\left|1-\frac{\nu}{2}\right|,\ \nu\in[0,1].$$

Theorem 3.12. *For any $\varphi \in C[0,1]$ and $\rho \in \mathbb{N}^+$, we have*

$$\left\|K^*_{\lambda,\rho}(\varphi) - \varphi\right\| \le M_1 \omega_2^{\Phi}\left(\varphi; \frac{1}{\sqrt{\lambda+\rho}}\right) + \omega^{\Psi}\left(\varphi; \frac{1}{\lambda+\rho}\right),$$

where M_1 is a positive constant depending only on ρ and φ.

Proof. Let $K^{**}_{\lambda,\rho}$ be the operator defined in the proof of Theorem 3.7. For any $\phi \in \mathbb{W}^2(\Phi)$, Taylor's formula for ϕ, together with the argument used in the proof Theorem 3.7, gives

$$\left|K^{**}_{\lambda,\rho}(\phi;\nu) - \phi(\nu)\right| \le K^{**}_{\lambda,\rho}\left(\left|\int_\nu^u |u-\sigma||\phi''(\sigma)|\mathrm{d}\sigma; \nu\right|\right) + \left|\int_\nu^{\alpha_{\lambda,\rho}(\nu)+\nu} \left|\alpha_{\lambda,\rho}(\nu)+\nu-\sigma\right| |\phi''(\sigma)|\mathrm{d}\sigma\right|.$$

Set

$$\Theta^2(\nu) = \Phi^2(\nu) + \frac{1}{\lambda+\rho}.$$

For $u \le \sigma \le \nu$, write $\sigma = \varrho\nu + (1-\varrho)u$ with $\varrho \in [0,1]$. By the concavity of Θ, we obtain

$$\frac{|u-\sigma|}{\Theta(\sigma)} = \frac{\varrho|u-\nu|}{\Theta(\varrho\nu+(1-\varrho)u)} \le \frac{\varrho|u-\nu|}{\varrho\Theta(\nu)+(1-\varrho)\Theta(u)} \le \frac{|u-\nu|}{\Theta(\nu)}.$$

Therefore, by Lemma 2.6,

$$\begin{aligned}
\left|K^{**}_{\lambda,\rho}(\phi;\nu) - \phi(\nu)\right| &\le K^{**}_{\lambda,\rho}\left(\left|\int_\nu^u \frac{|u-\sigma|}{\Theta^2(\sigma)}\mathrm{d}\sigma; \nu\right|\right)\|\Theta^2\phi''\| + \left|\int_\nu^{\alpha_{\lambda,\rho}(\nu)+\nu} \frac{\left|\alpha_{\lambda,\rho}(\nu)+\nu-\sigma\right|}{\Theta^2(\sigma)}\mathrm{d}\sigma\right| \|\Theta^2\phi''\| \\
&\le \frac{1}{\Theta^2(\nu)}\|\Theta^2\phi''\|\left(\beta_{\lambda,\rho}(\nu) + \left(\alpha_{\lambda,\rho}(\nu)\right)^2\right) \\
&\le \frac{C_2}{\lambda+\rho}\|\Theta^2\phi''\| \le \frac{C_2}{\lambda+\rho}\left(\|\Phi^2\phi''\| + \frac{1}{\lambda+\rho}\|\phi''\|\right),
\end{aligned}$$

where $C_2 > 0$ is a constant depending only on ρ. For any $\varphi \in C[0,1]$ and $\nu \in [0,1]$, Lemma 2.8 yields

$$\begin{aligned}
\left|K^*_{\lambda,\rho}(\varphi;\nu) - \varphi(\nu)\right| &\le \left|K^{**}_{\lambda,\rho}(\varphi-\phi;\nu)\right| + \left|K^{**}_{\lambda,\rho}(\phi;\nu) - \phi(\nu)\right| + |\varphi(\nu)-\phi(\nu)| + \left|\varphi\left(\alpha_{\lambda,\rho}(\nu)+\nu\right) - \varphi(\nu)\right| \\
&\le 4\|\varphi-\phi\| + \frac{C_2}{\lambda+\rho}\|\Phi^2\phi''\| + \frac{C_2}{(\lambda+\rho)^2}\|\phi''\| + \left|\varphi\left(\alpha_{\lambda,\rho}(\nu)+\nu\right) - \varphi(\nu)\right|.
\end{aligned}$$

Taking the infimum over all $\phi \in \mathbb{W}^2(\Phi)$ on the right-hand side, we get

$$\left|K^*_{\lambda,\rho}(\varphi;\nu) - \varphi(\nu)\right| \le M_2\mathbb{K}\left(\varphi; \frac{1}{\lambda+\rho}\right) + \left|\varphi\left(\alpha_{\lambda,\rho}(\nu)+\nu\right) - \varphi(\nu)\right|. \tag{3.15}$$

Furthermore,

$$\begin{aligned}
\left|\varphi\left(\alpha_{\lambda,\rho}(\nu)+\nu\right) - \varphi(\nu)\right| &= \left|\varphi\left(\nu + \Psi(\nu)\frac{\alpha_{\lambda,\rho}(\nu)}{\Psi(\nu)}\right) - \varphi(\nu)\right| \\
&\le \sup_{\nu\in[0,1]}\left|\varphi\left(\nu + \Psi(\nu)\frac{\frac{(\rho+2)(1-\frac{\nu}{2})}{\lambda+\rho}}{\Psi(\nu)}\right) - \varphi(\nu)\right| \qquad (3.16)
\end{aligned}$$

$$\leq \omega^{\Psi}\left(\varphi; \frac{(\rho+2)\left|1-\frac{\nu}{2}\right|}{(\lambda+\rho)\Psi(\nu)}\right) \leq \omega^{\Psi}\left(\varphi; \frac{1}{\lambda+\rho}\right).$$

Combining (3.14)–(3.16) and then taking the supremum over $\nu \in [0,1]$, we obtain

$$\left\|K^{*}_{\lambda,\rho}(\varphi)-\varphi\right\| \leq M_1\omega_2^{\Phi}\left(\varphi; \frac{1}{\sqrt{\lambda+\rho}}\right)+\omega^{\Psi}\left(\varphi; \frac{1}{\lambda+\rho}\right).$$

This completes the proof. □

3.4. *Numerical examples about $K^{*}_{\lambda,\rho}$*

Let $E_{\lambda,\rho}(\varphi;\nu) = \left|K^{*}_{\lambda,\rho}(\varphi;\nu)-\varphi(\nu)\right|$ denote the error function for the approximation of φ by $K^{*}_{\lambda,\rho}$.

Example 3.13 (Approximation error)**.** *Let $\varphi(\nu) = (\nu-0.5)\sin(2\pi\nu)$, $\nu \in [0,1]$. Figures 1 and 2 show the convergence of $K^{*}_{\lambda,\rho}$ to the function φ for $\lambda = 10, 60, 120, 240$, $\rho = 3$.*

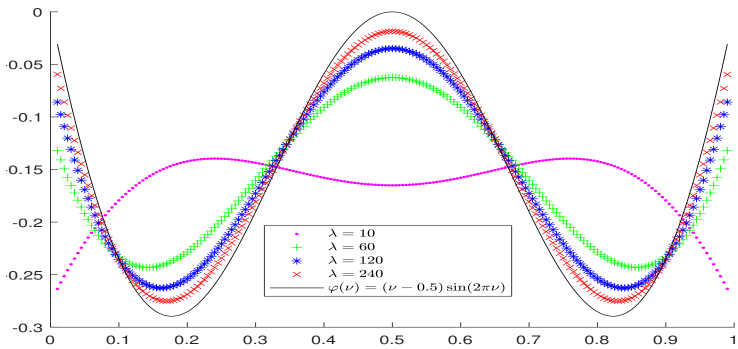

Figure 1. Approximation process by $K^{*}_{\lambda,\rho}$, $\rho = 3$.

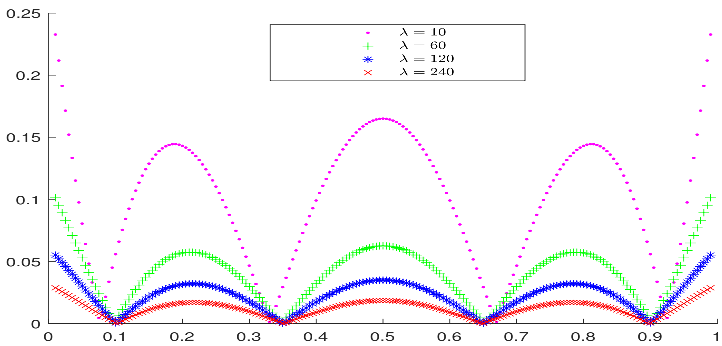

Figure 2. Approximation error function $E_{\lambda,\rho}$, $\rho = 3$.

Figure 1 shows the convergence of the operators $K^{*}_{\lambda,\rho}(\varphi;\nu)$ to the function $\varphi(\nu)$ for different values of λ. We can clearly see that as λ increases, the graph of the operators $K^{*}_{\lambda,\rho}(\varphi;\nu)$ approach the graph of the function $\varphi(\nu)$ for the given $\rho = 3$. Table 1 shows the approximation errors for $E_{10,3}(\varphi;\nu), E_{60,3}(\varphi;\nu), E_{120,3}(\varphi;\nu), E_{240,3}(\varphi;\nu)$ at the points $\{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$. Together with Figure 2, we can clearly see that when the values of λ are increasing, the approximation error $\varphi(\nu)$ decreases.

Table 1. Approximation error function $E_{\lambda,\rho}$ for $\lambda = 10, 60, 120, 240$ and $\rho = 3$.

ν	$E_{10,3}(\varphi;\nu)$	$E_{60,3}(\varphi;\nu)$	$E_{120,3}(\varphi;\nu)$	$E_{240,3}(\varphi;\nu)$
0.2	0.1430991	0.056834085	0.031458536	0.016563449
0.3	0.046598108	0.029716887	0.017472748	0.0094964475
0.4	0.099049985	0.032052316	0.017406873	0.0090821258
0.5	0.16502246	0.062536785	0.034916348	0.018497244
0.6	0.099049985	0.032052316	0.017406873	0.0090821258
0.7	0.046598108	0.029716887	0.017472748	0.0094964475
0.8	0.1430991	0.056834085	0.031458536	0.016563449

4. Construction of bivariate operators

In this section, we discuss the convergence rates of the bivariate operators $K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\varphi;\nu_1,\nu_2)$ to the function $\varphi(\nu_1,\nu_2)$ in terms of the modulus of continuity. Further results related to bivariate operators can be found in [30–32].

Let $I^2 := I\times I$, where $I = [0,1]$. We denote by$C(I^2)$ the space of all real-valued continuous functions on I^2, endowed with the max-norm

$$\|\varphi\|_{C(I^2)} = \max_{(\nu_1,\nu_2)\in I^2} |\varphi(\nu_1,\nu_2)|.$$

Let $C^2(I^2)$ be the subspace of all functions $\varphi \in C(I^2)$ such that

$$\frac{\partial^m\varphi}{\partial\nu_1^m}, \frac{\partial^m\varphi}{\partial\nu_2^m},\ m = 1,2.$$

The norm on $C^2(I^2)$ is defined by

$$\|\varphi\|_{C^2(I^2)} = \|\varphi\|_{C(I^2)} + \sum_{m=1}^{2}\left(\left\|\frac{\partial^m\varphi}{\partial\nu_1^m}\right\|_{C(I^2)} + \left\|\frac{\partial^m\varphi}{\partial\nu_2^m}\right\|_{C(I^2)}\right) + \left\|\frac{\partial^2\varphi}{\partial\nu_1\partial\nu_2}\right\|_{C(I^2)}.$$

For $\varphi \in C(I^2)$, Peetre's $\mathbb{K}$-functional is defined by

$$\mathbb{K}(\varphi;\chi) := \inf_{\phi\in C^2(I^2)}\left\{\|\varphi-\phi\|_{C(I^2)} + \chi\|\phi\|_{C^2(I^2)}\right\},\ \chi > 0.$$

By [33, p. 192], there exists an absolute constant $M_3 > 0$ such that

$$\mathbb{K}(\varphi;\chi) \le M_3\left(\overline{\omega}_2(\varphi;\sqrt{\chi}) + \min(1,\chi)\|\varphi\|_{C(I^2)}\right),\ \chi > 0. \tag{4.1}$$

Here, $\overline{\omega}_2\left(\varphi;\sqrt{\chi}\right)$ denotes the following second-order complete modulus of continuity:

$$\overline{\omega}_2(\varphi;\sqrt{\chi}) = \sup_{0<|\mathbf{u}|<\sqrt{\chi}}\ \sup_{\mathbf{v},\mathbf{v}+2\mathbf{u}\in I^2} \|\varphi(\mathbf{v}) + \varphi(\mathbf{v}+2\mathbf{u}) - 2\varphi(\mathbf{v}+\mathbf{u})\|_{C(I^2)},$$

where $\mathbf{v} = (\nu_1, \nu_2)$, and $\mathbf{u} = (u_1, u_2) \in I^2$. Similarly, the first-order complete modulus of continuity is defined by

$$\overline{\omega}(\varphi;\chi) = \sup_{0<|\mathbf{u}|<\chi} \sup_{\mathbf{v},\mathbf{v}+\mathbf{u}\in I^2} \|\varphi(\mathbf{v}+\mathbf{u}) - \varphi(\mathbf{v})\|_{C(I^2)}.$$

Further, the partial moduli of continuity with respect to ν_1 and ν_2, respectively, are defined by

$$\omega^{(1)}(\varphi;\chi) = \sup_{0<|u_1|<\chi} \sup_{\nu_1,\nu_1+u_1,\nu_2\in I} |\varphi(\nu_1+u_1,\nu_2) - \varphi(\nu_1,\nu_2)|,$$

and

$$\omega^{(2)}(\varphi;\chi) = \sup_{0<|u_2|<\chi} \sup_{\nu_2,\nu_2+u_2,\nu_1\in I} |\varphi(\nu_1,\nu_2+u_2) - \varphi(\nu_1,\nu_2)|.$$

For any $\varphi \in C(I^2)$ and $\lambda_1, \lambda_2, \rho_1, \rho_2 \in \mathbb{N}^+$, the bivariate generalization of the operators (1.3) is constructed as follows:

$$\begin{aligned} K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\varphi;\nu_1,\nu_2) = \sum_{\mu_1=0}^{\lambda_1}\sum_{\mu_2=0}^{\lambda_2} p_{\lambda_1,\mu_1}(\nu_1)p_{\lambda_2,\mu_2}(\nu_2)\int_0^1\cdots\int_0^1\cdot\int_0^1\cdots\int_0^1 \\ \varphi\left(\frac{\mu_1+u_1+\cdots u_{\rho_1}}{\lambda_1+\rho_1}, \frac{\mu_2+w_1+\cdots w_{\rho_2}}{\lambda_2+\rho_2}\right)\mathrm{d}u_1\cdots\mathrm{d}u_{\rho_1}\cdot\mathrm{d}w_1\cdots\mathrm{d}w_{\rho_2}. \end{aligned} \tag{4.2}$$

Let $e_{n,m} := e_{n,m}(u_1,u_2) = u_1^n u_2^m$, $(n,m) \in \mathbb{N}\times\mathbb{N}$ denote the test functions in the bivariate case. By Lemmas 2.6 and 2.7, we obtain the following two lemmas.

Lemma 4.1. *For $(\nu_1,\nu_2) \in I^2$ and $\lambda_1,\lambda_2,\rho_1,\rho_2 \in \mathbb{N}^+$, the following equalities hold:*

$$\begin{aligned}
&K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(e_{0,0};\nu_1,\nu_2) = 1,\\
&K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(e_{1,0};\nu_1,\nu_2) = \frac{\lambda_1-2}{\lambda_1+\rho_1}\nu_1 + \frac{\rho_1+2}{2(\lambda_1+\rho_1)},\\
&K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(e_{0,1};\nu_1,\nu_2) = \frac{\lambda_2-2}{\lambda_2+\rho_2}\nu_2 + \frac{\rho_2+2}{2(\lambda_2+\rho_2)},\\
&K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(e_{2,0};\nu_1,\nu_2) = \frac{\lambda_1^2-7\lambda_1+6}{(\lambda_1+\rho_1)^2}\nu_1^2 + \frac{5\lambda_1-6+(\lambda_1-2)\rho_1}{(\lambda_1+\rho_1)^2}\nu_1 + \frac{3\rho_1^2+13\rho_1+12}{12(\lambda_1+\rho_1)^2},\\
&K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(e_{0,2};\nu_1,\nu_2) = \frac{\lambda_2^2-7\lambda_2+6}{(\lambda_2+\rho_2)^2}\nu_2^2 + \frac{5\lambda_2-6+(\lambda_2-2)\rho_2}{(\lambda_2+\rho_2)^2}\nu_2 + \frac{3\rho_2^2+13\rho_2+12}{12(\lambda_2+\rho_2)^2}.
\end{aligned}$$

Lemma 4.2. *For $(\nu_1,\nu_2) \in I^2$ and $\lambda_1,\lambda_2,\rho_1,\rho_2 \in \mathbb{N}^+$, the following equalities hold:*

$$\begin{aligned}
&K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(e_{1,0}-\nu_1;\nu_1,\nu_2) = -\frac{\rho_1+2}{\lambda_1+\rho_1}\nu_1 + \frac{\rho_1+2}{2(\lambda_1+\rho_1)},\\
&K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(e_{0,1}-\nu_2;\nu_1,\nu_2) = -\frac{\rho_2+2}{\lambda_2+\rho_2}\nu_2 + \frac{\rho_2+2}{2(\lambda_2+\rho_2)},\\
&K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}\left((e_{1,0}-\nu_1)^2;\nu_1,\nu_2\right) = \frac{3\lambda_1-\rho_1^2-4\rho_1-6}{(\lambda_1+\rho_1)^2}\nu_1(1-\nu_1) + \frac{3\rho_1^2+13\rho_1+12}{12(\lambda_1+\rho_1)^2},\\
&K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}\left((e_{0,1}-\nu_2)^2;\nu_1,\nu_2\right) = \frac{3\lambda_2-\rho_2^2-4\rho_2-6}{(\lambda_2+\rho_2)^2}\nu_2(1-\nu_2) + \frac{3\rho_2^2+13\rho_2+12}{12(\lambda_2+\rho_2)^2}.
\end{aligned}$$

Lemma 4.3. *For any $\varphi \in C(I^2)$ and $\lambda_1, \lambda_2, \rho_1, \rho_2 \in \mathbb{N}^+$, we have*

$$\|K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(\varphi)\|_{C(I^2)} \leq \|\varphi\|_{C(I^2)}.$$

Theorem 4.4. *For any $\varphi \in C(I^2)$ and fixed $\rho_1, \rho_2 \in \mathbb{N}^+$, the bivariate operators $\{K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(\varphi)\}$ converge uniformly to φ on I^2 as $\lambda_1, \lambda_2 \to \infty$.*

Proof. We first obtain the following limit equalities:

$$\lim_{\lambda_1,\lambda_2\to\infty} K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(e_{n,m}; \nu_1, \nu_2) = \nu_1^n \nu_2^m, \ (n,m) \in \{(0,0),(0,1),(1,0)\},$$

and

$$\lim_{\lambda_1,\lambda_2\to\infty} K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(e_{2,0} + e_{0,2}; \nu_1, \nu_2) = \nu_1^2 + \nu_2^2$$

uniformly on I^2. Hence, by [34, Theorem 2.1], the desired conclusion follows. □

Theorem 4.5. *For any $\varphi \in C(I^2)$ and $\lambda_1, \lambda_2, \rho_1, \rho_2 \in \mathbb{N}^+$, we have*

$$\left|K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(\varphi; \nu_1, \nu_2) - \varphi(\nu_1, \nu_2)\right| \leq 2\left(\omega^{(1)}\left(\varphi; \sqrt{\beta_{\lambda_1,\rho_1}(\nu_1)}\right) + \omega^{(2)}\left(\varphi; \sqrt{\beta_{\lambda_2,\rho_2}(\nu_2)}\right)\right).$$

Proof. Applying the definition of the partial moduli of continuity, together the monotonicity and the linearity of the bivariate operators $K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}$ and using the Cauchy-Bunyakovsky-Schwarz inequality, we obtain

$$\begin{aligned}
&\left|K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(\varphi; \nu_1, \nu_2) - \varphi(\nu_1, \nu_2)\right| \leq K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(|\varphi(u_1, u_2) - \varphi(\nu_1, \nu_2)|; \nu_1, \nu_2)\\
\leq& K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(|\varphi(u_1, \nu_2) - \varphi(\nu_1, \nu_2)|; \nu_1, \nu_2) + K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(|\varphi(u_1, u_2) - \varphi(u_1, \nu_2)|; \nu_1, \nu_2)\\
\leq& K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}\left(\omega^{(1)}(\varphi; |u_1 - \nu_1|); \nu_1, \nu_2\right) + K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}\left(\omega^{(2)}(\varphi; |u_2 - \nu_2|); \nu_1, \nu_2\right).
\end{aligned}$$

Then, for arbitray $\chi_1, \chi_2 > 0$, the standard estimate for the partial moduli of continuity yields

$$\begin{aligned}
&\left|K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(\varphi; \nu_1, \nu_2) - \varphi(\nu_1, \nu_2)\right|\\
\leq&\omega^{(1)}(\varphi; \chi_1)\left(1 + \frac{1}{\chi_1} K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(|u_1 - \nu_1|; \nu_1, \nu_2)\right) + \omega^{(2)}(\varphi; \chi_2)\left(1 + \frac{1}{\chi_2} K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(|u_2 - \nu_2|; \nu_1, \nu_2)\right)\\
\leq&\omega^{(1)}(\varphi; \chi_1)\left(1 + \frac{1}{\chi_1}\sqrt{K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}((u_1 - \nu_1)^2; \nu_1, \nu_2)}\right) + \omega^{(2)}(\varphi; \chi_2)\left(1 + \frac{1}{\chi_2}\sqrt{K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}((u_2 - \nu_2)^2; \nu_1, \nu_2)}\right)\\
=&\omega^{(1)}(\varphi; \chi_1)\left(1 + \frac{1}{\chi_1}\sqrt{\beta_{\lambda_1,\rho_1}(\nu_1)}\right) + \omega^{(2)}(\varphi; \chi_2)\left(1 + \frac{1}{\chi_2}\sqrt{\beta_{\lambda_2,\rho_2}(\nu_2)}\right).
\end{aligned}$$

Choosing

$$\chi_1 = \sqrt{\beta_{\lambda_1,\rho_1}(\nu_1)}, \chi_2 = \sqrt{\beta_{\lambda_2,\rho_2}(\nu_2)},$$

we obtain the desired estimate. □

Theorem 4.6. *For any $\varphi \in C^2(I^2)$ and $\rho_1, \rho_2 \in \mathbb{N}^+$, we have*

$$\begin{aligned}
&\lim_{\lambda\to\infty} \lambda\left(K_{\lambda,\lambda}^{\rho_1,\rho_2}(\varphi; \nu_1, \nu_2) - \varphi(\nu_1, \nu_2)\right)\\
=&(\rho_1 + 2)\left(\frac{1}{2} - \nu_1\right)\frac{\partial\varphi}{\partial\nu_1} + (\rho_2 + 2)\left(\frac{1}{2} - \nu_2\right)\frac{\partial\varphi}{\partial\nu_2} + \frac{3}{2}\nu_1(1 - \nu_1)\frac{\partial^2\varphi}{\partial\nu_1^2} + \frac{3}{2}\nu_2(1 - \nu_2)\frac{\partial^2\varphi}{\partial\nu_2^2}.
\end{aligned}$$

Proof. Let $(\nu_1,\nu_2)\in I^2$ be arbitrary but fixed. By Taylor's formula for functions of two variables with Peano's form of reminder, we have

$$\begin{aligned}\varphi(u_1,u_2)=&\varphi(\nu_1,\nu_2)+\frac{\partial\varphi}{\partial\nu_1}(u_1-\nu_1)+\frac{\partial\varphi}{\partial\nu_2}(u_2-\nu_2)\\&+\frac{1}{2}\left(\frac{\partial^2\varphi}{\partial\nu_1^2}(u_1-\nu_1)^2+2\frac{\partial^2\varphi}{\partial\nu_1\partial\nu_2}(u_1-\nu_1)(u_2-\nu_2)+\frac{\partial^2\varphi}{\partial\nu_2^2}(u_2-\nu_2)^2\right)\\&+\mathrm{R}(u_1,u_2,\nu_1,\nu_2)\sqrt{(u_1-\nu_1)^4+(u_2-\nu_2)^4},\end{aligned}\tag{4.3}$$

where $\mathrm{R}(u_1,u_2,\nu_1,\nu_2)\in C(I^2)$, and

$$\mathrm{R}(u_1,u_2,\nu_1,\nu_2)\to 0 \text{ as } (u_1,u_2)\to(\nu_1,\nu_2).$$

Applying the operator $K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}$ to both sides of (4.3), we obtain

$$\begin{aligned}&K^{\rho_1,\rho_2}_{\lambda,\lambda}(\varphi;\nu_1,\nu_2)=\varphi(\nu_1,\nu_2)+\frac{\partial\varphi}{\partial\nu_1}K^{\rho_1,\rho_2}_{\lambda,\lambda}(e_{1,0}-\nu_1;\nu_1,\nu_2)+\frac{\partial\varphi}{\partial\nu_2}K^{\rho_1,\rho_2}_{\lambda,\lambda}(e_{0,1}-\nu_2;\nu_1,\nu_2)\\=&\frac{1}{2}\left(\frac{\partial^2\varphi}{\partial\nu_1^2}K^{\rho_1,\rho_2}_{\lambda,\lambda}\left((e_{1,0}-\nu_1)^2;\nu_1,\nu_2\right)+2\frac{\partial^2\varphi}{\partial\nu_1\partial\nu_2}K^{\rho_1,\rho_2}_{\lambda,\lambda}\left((e_{1,0}-\nu_1)(e_{0,1}-\nu_2);\nu_1,\nu_2\right)\right.\\&\left.+\frac{\partial^2\varphi}{\partial\nu_2^2}K^{\rho_1,\rho_2}_{\lambda,\lambda}\left((e_{0,1}-\nu_2)^2;\nu_1,\nu_2\right)\right)+K^{\rho_1,\rho_2}_{\lambda,\lambda}\left(\mathrm{R}(u_1,u_2,\nu_1,\nu_2)\sqrt{(u_1-\nu_1)^4+(u_2-\nu_2)^4};\nu_1,\nu_2\right).\end{aligned}\tag{4.4}$$

By the Cauchy-Bunyakovsky-Schwarz inequality, we get

$$\begin{aligned}&\left|K^{\rho_1,\rho_2}_{\lambda,\lambda}\left(\mathrm{R}(u_1,u_2,\nu_1,\nu_2)\sqrt{(u_1-\nu_1)^4+(u_2-\nu_2)^4};\nu_1,\nu_2\right)\right|\\\le&\sqrt{K^{\rho_1,\rho_2}_{\lambda,\lambda}\left(\mathrm{R}^2(u_1,u_2,\nu_1,\nu_2);\nu_1,\nu_2\right)}\cdot\sqrt{K^{\rho_1,\rho_2}_{\lambda,\lambda}\left((u_1-\nu_1)^4+(u_2-\nu_2)^4;\nu_1,\nu_2\right)}\\=&\sqrt{K^{\rho_1,\rho_2}_{\lambda,\lambda}\left(\mathrm{R}^2(u_1,u_2,\nu_1,\nu_2);\nu_1,\nu_2\right)}\cdot\sqrt{K^{*}_{\lambda,\rho_1}\left((u_1-\nu_1)^2;\nu_1\right)+K^{*}_{\lambda,\rho_2}\left((u_2-\nu_2)^2;\nu_2\right)}.\end{aligned}$$

Combining Theorem 4.4 and the corresponding convergence result with Lemma 2.7, we obtain

$$\lim_{\lambda\to\infty}\lambda K^{\rho_1,\rho_2}_{\lambda,\lambda}\left(\mathrm{R}(u_1,u_2,\nu_1,\nu_2)\sqrt{(u_1-\nu_1)^4+(u_2-\nu_2)^4};\nu_1,\nu_2\right)=0.$$

Therefore, using the moment estimates and passing to the limit in (4.4), the desired Voronovskaya-type formula follows. □

Theorem 4.7. *For every $\varphi\in C(I^2)$ and $\lambda_1,\lambda_2,\rho_1,\rho_2\in\mathbb{N}^+$, we have*

$$\begin{aligned}&\left|K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\varphi;\nu_1,\nu_2)-\varphi(\nu_1,\nu_2)\right|\\\le&4M_3\left(\overline{\omega}_2\left(\varphi;\sqrt{T^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\nu_1,\nu_2)}\right)+\min(1,T^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\nu_1,\nu_2))\|\varphi\|_{C(I^2)}\right)+\overline{\omega}\left(\varphi;\sqrt{\left(\alpha_{\lambda_1,\rho_1}(\nu_1)\right)^2+\left(\alpha_{\lambda_2,\rho_2}(\nu_2)\right)^2}\right),\end{aligned}$$

where $T^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\nu_1,\nu_2)=\left(\alpha_{\lambda_1,\rho_1}(\nu_1)\right)^2+\left(\alpha_{\lambda_2,\rho_2}(\nu_2)\right)^2+\beta_{\lambda_1,\rho_1}(\nu_1)+\beta_{\lambda_2,\rho_2}(\nu_2).$

Proof. For $(v_1, v_2) \in I^2$, define the following auxiliary bivariate operators:

$$K^{*,\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\varphi; v_1, v_2) = K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\varphi; v_1, v_2) - \varphi\left(\alpha_{\lambda_1,\rho_1}(v_1) + v_1, \alpha_{\lambda_2,\rho_2}(v_2) + v_2\right) + \varphi(v_1, v_2). \tag{4.5}$$

By Lemma 4.3, we obtain

$$\begin{aligned} &K^{*,\rho_1,\rho_2}_{\lambda_1,\lambda_2}(e_{0,0}; v_1, v_2) = 1, K^{*,\rho_1,\rho_2}_{\lambda_1,\lambda_2}(e_{1,0}; v_1, v_2) = v_1, K^{*,\rho_1,\rho_2}_{\lambda_1,\lambda_2}(e_{0,1}; v_1, v_2) = v_2, \\ &\|K^{*,\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\varphi)\|_{C(I^2)} \leq 3\|\varphi\|_{C(I^2)}. \end{aligned} \tag{4.6}$$

For any $\phi \in C^2(I^2)$ and $(u_1, u_2) \in I^2$, $(v_1, v_2) \in I^2$, Taylor's formula with an integral remainder gives

$$\begin{aligned} \phi(u_1, u_2) - \phi(v_1, v_2) =& \phi(u_1, v_2) - \phi(v_1, v_2) + \phi(u_1, u_2) - \phi(u_1, v_2) \\ =& \frac{\partial\phi(v_1, v_2)}{\partial v_1}(u_1 - v_1) + \int_{v_1}^{u_1} (u_1 - \sigma_1)\frac{\partial^2\phi(\sigma_1, v_2)}{\partial\sigma_1^2}\mathrm{d}\sigma_1 + \frac{\partial\phi(v_1, v_2)}{\partial v_2}(u_2 - v_2) \\ &+ \int_{v_2}^{u_2} (u_2 - \sigma_2)\frac{\partial^2\phi(v_1, \sigma_2)}{\partial\sigma_2^2}\mathrm{d}\sigma_2 + \int_{v_2}^{u_2}\int_{v_1}^{u_1} \frac{\partial^2\phi(\sigma_1, \sigma_2)}{\partial\sigma_1\partial\sigma_2}\mathrm{d}\sigma_1\mathrm{d}\sigma_2. \end{aligned} \tag{4.7}$$

Applying $K^{*,\rho_1,\rho_2}_{\lambda_1,\lambda_2}$ to both sides of (4.7) and using (4.5) and (4.6), we obtain

$$\begin{aligned} &K^{*,\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\phi; v_1, v_2) - \phi(v_1, v_2) \\ =& K^{*,\rho_1,\rho_2}_{\lambda_1,\lambda_2}\left(\phi(u_1, u_2) - \phi(v_1, v_2); v_1, v_2\right) \\ =& K^{*,\rho_1,\rho_2}_{\lambda_1,\lambda_2}\left(\int_{v_1}^{u_1} (u_1 - \sigma_1)\frac{\partial^2\phi(\sigma_1, v_2)}{\partial\sigma_1^2}\mathrm{d}\sigma_1; v_1, v_2\right) + K^{*,\rho_1,\rho_2}_{\lambda_1,\lambda_2}\left(\int_{v_2}^{u_2} (u_2 - \sigma_2)\frac{\partial^2\phi(v_1, \sigma_2)}{\partial\sigma_2^2}\mathrm{d}\sigma_2; v_1, v_2\right) \\ &+ K^{*,\rho_1,\rho_2}_{\lambda_1,\lambda_2}\left(\int_{v_2}^{u_2}\int_{v_1}^{u_1} \frac{\partial^2\phi(\sigma_1, \sigma_2)}{\partial\sigma_1\partial\sigma_2}\mathrm{d}\sigma_1\mathrm{d}\sigma_2; v_1, v_2\right) \\ =& K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}\left(\int_{v_1}^{u_1} (u_1 - \sigma_1)\frac{\partial^2\phi(\sigma_1, v_2)}{\partial\sigma_1^2}\mathrm{d}\sigma_1; v_1, v_2\right) - \int_{v_1}^{\alpha_{\lambda_1,\rho_1}(v_1)+v_1} (\alpha_{\lambda_1,\rho_1}(v_1) + v_1 - \sigma_1)\frac{\partial^2\phi(\sigma_1, v_2)}{\partial\sigma_1^2}\mathrm{d}\sigma_1 \\ &+ K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}\left(\int_{v_2}^{u_2} (u_2 - \sigma_2)\frac{\partial^2\phi(v_1, \sigma_2)}{\partial\sigma_2^2}\mathrm{d}\sigma_2; v_1, v_2\right) - \int_{v_2}^{\alpha_{\lambda_2,\rho_2}(v_2)+v_2} (\alpha_{\lambda_2,\rho_2}(v_2) + v_2 - \sigma_2)\frac{\partial^2\phi(v_1, \sigma_2)}{\partial\sigma_2^2}\mathrm{d}\sigma_2 \\ =& K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}\left(\int_{v_2}^{u_2}\int_{v_1}^{u_1} \frac{\partial^2\phi(\sigma_1, \sigma_2)}{\partial\sigma_1\partial\sigma_2}\mathrm{d}\sigma_1\mathrm{d}\sigma_2; v_1, v_2\right) - \int_{v_2}^{\alpha_{\lambda_2,\rho_2}(v_2)+v_2}\int_{v_1}^{\alpha_{\lambda_1,\rho_1}(v_1)+v_1} \frac{\partial^2\phi(\sigma_1, \sigma_2)}{\partial\sigma_1\partial\sigma_2}\mathrm{d}\sigma_1\mathrm{d}\sigma_2. \end{aligned}$$

Moreover,

$$\left|\int_{v_1}^{u_1} (u_1 - \sigma_1)\frac{\partial^2\phi(\sigma_1, v_2)}{\partial\sigma_1^2}\mathrm{d}\sigma_1\right| \leq \left|\int_{v_1}^{u_1} |u_1 - \sigma_1|\frac{\partial^2\phi(\sigma_1, v_2)}{\partial\sigma_1^2}\mathrm{d}\sigma_1\right| \leq (u_1 - v_1)^2\left\|\frac{\partial^2\phi}{\partial v_1^2}\right\|_{C(I^2)},$$

and

$$\left|\int_{v_1}^{\alpha_{\lambda_1,\rho_1}(v_1)+v_1} (\alpha_{\lambda_1,\rho_1}(v_1) + v_1 - \sigma_1)\frac{\partial^2\phi(\sigma_1, v_2)}{\partial\sigma_1^2}\mathrm{d}\sigma_1\right| \leq \alpha^2_{\lambda_1,\rho_1}(v_1)\left\|\frac{\partial^2\phi}{\partial v_1^2}\right\|_{C(I^2)}.$$

Similarly,

$$\left|\int_{v_2}^{u_2} (u_2 - \sigma_2)\frac{\partial^2\phi(v_1, \sigma_2)}{\partial\sigma_2^2}\mathrm{d}\sigma_2\right| \leq (u_2 - v_2)^2\left\|\frac{\partial^2\phi}{\partial v_2^2}\right\|_{C(I^2)},$$

$$\left|\int_{\nu_2}^{\alpha_{\lambda_2,\rho_2}(\nu_2)+\nu_2}(\alpha_{\lambda_2,\rho_2}(\nu_2)+\nu_2-\sigma_2)\frac{\partial^2\phi(\nu_1,\sigma_2)}{\partial\sigma_2^2}\mathrm{d}\sigma_2\right|\leq\alpha^2_{\lambda_2,\rho_2}(\nu_2)\left\|\frac{\partial^2\phi}{\partial\nu_2^2}\right\|_{C(I^2)},$$

Using the Cauchy-Bunyakovsky-Schwarz inequality and the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}
&\left|K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}\left(\int_{\nu_2}^{u_2}\int_{\nu_1}^{u_1}\frac{\partial^2\phi(\sigma_1,\sigma_2)}{\partial\sigma_1\partial\sigma_2}\mathrm{d}\sigma_1\mathrm{d}\sigma_2;\nu_1,\nu_2\right)\right|\\
\leq&K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}\left(\int_{\nu_2}^{u_2}\int_{\nu_1}^{u_1}\left|\frac{\partial^2\phi(\sigma_1,\sigma_2)}{\partial\sigma_1\partial\sigma_2}\right|\mathrm{d}\sigma_1\mathrm{d}\sigma_2;\nu_1,\nu_2\right)\\
\leq&K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(|u_1-\nu_1|\cdot|u_2-\nu_2|;\nu_1,\nu_2)\left\|\frac{\partial^2\phi}{\partial\nu_1\partial\nu_2}\right\|_{C(I^2)}\\
\leq&\sqrt{K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}((u_1-\nu_1)^2);\nu_1,\nu_2)\cdot K^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}((u_2-\nu_2)^2);\nu_1,\nu_2)}\left\|\frac{\partial^2\phi}{\partial\nu_1\partial\nu_2}\right\|_{C(I^2)}\\
=&\sqrt{\beta_{\lambda_1,\rho_1}(\nu_1)\cdot\beta_{\lambda_2,\rho_2}(\nu_2)}\left\|\frac{\partial^2\phi}{\partial\nu_1\partial\nu_2}\right\|_{C(I^2)}\leq\left(\beta^2_{\lambda_1,\rho_1}(\nu_1)+\beta^2_{\lambda_2,\rho_2}(\nu_2)\right)\left\|\frac{\partial^2\phi}{\partial\nu_1\partial\nu_2}\right\|_{C(I^2)},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{\nu_2}^{\alpha_{\lambda_2,\rho_2}(\nu_2)+\nu_2}\int_{\nu_1}^{\alpha_{\lambda_1,\rho_1}(\nu_1)+\nu_1}\frac{\partial^2\phi(\sigma_1,\sigma_2)}{\partial\sigma_1\partial\sigma_2}\mathrm{d}\sigma_1\mathrm{d}\sigma_2\right|\\
\leq&|\alpha_{\lambda_1,\rho_1}(\nu_1)|\cdot|\alpha_{\lambda_2,\rho_2}(\nu_2)|\left\|\frac{\partial^2\phi}{\partial\nu_1\partial\nu_2}\right\|_{C(I^2)}\left(\left(\alpha_{\lambda_1,\rho_1}(\nu_1)\right)^2+\left(\alpha_{\lambda_2,\rho_2}(\nu_2)\right)^2\right)\left\|\frac{\partial^2\phi}{\partial\nu_1\partial\nu_2}\right\|_{C(I^2)}.
\end{aligned}$$

Hence,

$$\left|K^{*,\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\phi;\nu_1,\nu_2)-\phi(\nu_1,\nu_2)\right|\leq T^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\nu_1,\nu_2)\|\phi\|_{C(I^2)}.$$

For $\varphi\in C(I^2)$ and $\phi''\in C(I^2)$, using (4.5) and (4.6), we have

$$\begin{aligned}
&\left|K^{*,\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\varphi;\nu_1,\nu_2)-\varphi(\nu_1,\nu_2)\right|=\left|K^{\rho_1,\rho_2}_{*,\lambda_1,\lambda_2}(\varphi;\nu_1,\nu_2)+\phi\left(\alpha_{\lambda_1,\rho_1}(\nu_1)+\nu_1,\alpha_{\lambda_2,\rho_2}(\nu_2)+\nu_2\right)-2\varphi(\nu_1,\nu_2)\right|\\
\leq&\left|K^{*,\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\varphi-\phi;\nu_1,\nu_2)\right|+|\varphi(\nu_1,\nu_2)-\phi(\nu_1,\nu_2)|+\left|K^{*,\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\phi;\nu_1,\nu_2)-\phi(\nu_1,\nu_2)\right|\\
&+\left|\phi\left(\alpha_{\lambda_1,\rho_1}(\nu_1)+\nu_1,\alpha_{\lambda_2,\rho_2}(\nu_2)\right)-\varphi(\nu_1,\nu_2)\right|\\
\leq&4\|\varphi-\phi\|+T^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\nu_1,\nu_2)\|\phi\|_{C(I^2)}+\overline{\omega}\left(\varphi;\sqrt{\left(\alpha_{\lambda_1,\rho_1}(\nu_1)\right)^2+\left(\alpha_{\lambda_2,\rho_2}(\nu_2)\right)^2}\right).
\end{aligned}$$

Now, taking the infimum on the right-hand side for all $\phi''\in C(I^2)$ and using (4.1), we obtain

$$\begin{aligned}
&\left|K^{*,\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\varphi;\nu_1,\nu_2)-\varphi(\nu_1,\nu_2)\right|\\
\leq&4\mathbb{K}\left(\varphi;T^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\nu_1,\nu_2)\right)+\overline{\omega}\left(\varphi;\sqrt{\left(\alpha_{\lambda_1,\rho_1}(\nu_1)\right)^2+\left(\alpha_{\lambda_2,\rho_2}(\nu_2)\right)^2}\right)\\
\leq&4M_3\left(\overline{\omega}_2\left(\varphi;\sqrt{T^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\nu_1,\nu_2)}\right)+\min(1,T^{\rho_1,\rho_2}_{\lambda_1,\lambda_2}(\nu_1,\nu_2))\|\varphi\|_{C(I^2)}\right)+\overline{\omega}\left(\varphi;\sqrt{\left(\alpha_{\lambda_1,\rho_1}(\nu_1)\right)^2+\left(\alpha_{\lambda_2,\rho_2}(\nu_2)\right)^2}\right),
\end{aligned}$$

This completes the proof of Theorem 4.7. □

4.1. Numerical examples about $K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(\varphi;\nu_1,\nu_2)$

Let $E_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(\varphi;\nu_1,\nu_2)=\left|K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(\varphi;\nu_1,\nu_2)-\varphi(\nu_1,\nu_2)\right|$ denote the error function for the approximation of φ by $K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}$.

Example 4.8. *Let* $\varphi(\nu_1,\nu_2)=\nu_1+\nu_1\nu_2-6\nu_2^2$, $(\nu_1,\nu_2)\in[0,1]\times[0,1]$. *Here, we take the value of* $\lambda_1=\lambda_2=\lambda\in\{20,40\}$, $\rho_1=4$, $\rho_2=5$.

Figure 3 shows the convergence of the operators $K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(\varphi;\nu_1,\nu_2)$ to function $\varphi(\nu_1,\nu_2)$ for increasing values of $\lambda_1=\lambda_2=\lambda$ and fixed ρ_1 and ρ_2. Table 2 shows the approximation errors for $E_{20,20}^{4,5}(\varphi;\nu_1,\nu_2)$, $E_{40,40}^{4,5}(\varphi;\nu_1,\nu_2)$ at the points $\{(\nu_1,\nu_2):\nu_1=0.5,\nu_2=0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0\}$. We can clearly see that when the values of $\lambda_1=\lambda_2=\lambda$ are increasing, the approximation error $E_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(\varphi;\nu_1,\nu_2)$ decreases.

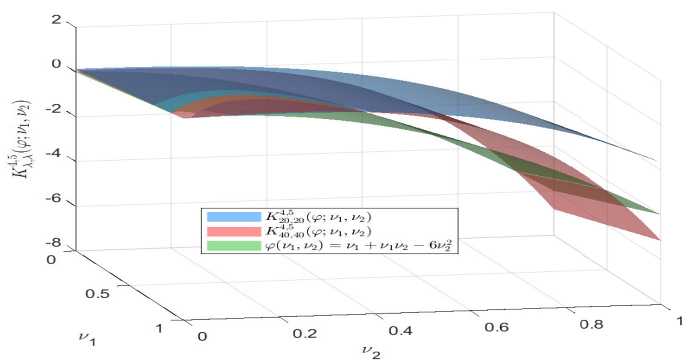

Figure 3. Approximation process by $K_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}(\varphi;\nu_1,\nu_2)$ for $\rho_1=4$, $\rho_2=5$ and $\lambda_1=\lambda_2=20,40$.

Table 2. Approximation error function $E_{\lambda_1,\lambda_2}^{\rho_1,\rho_2}$ for $\rho_1=4$, $\rho_2=5$ and $\lambda_1=\lambda_2=20,40$.

ν_1	ν_2	$E_{20,20}^{4,5}(\varphi;\nu_1,\nu_2)$	$E_{40,40}^{4,5}(\varphi;\nu_1,\nu_2)$
0.5	0.1	1.1694145	0.89364941
0.5	0.2	1.1316281	0.87305929
0.5	0.3	1.0938418	0.85246917
0.5	0.4	1.0560555	0.83187906
0.5	0.5	1.0182691	0.81128894
0.5	0.6	0.98048278	0.79069883
0.5	0.7	0.94269644	0.77010871
0.5	0.8	0.9049101	0.74951859
0.5	0.9	0.86712377	0.72892848
0.5	1.0	0.82933743	0.70833836

5. Conclusions

In this paper, we construct a new ρ-th-order Kantorovich-type generalization of the modified Bernstein operators. First, the moments and the central moments computation formulas and their quantitative properties are computed. Then, the Voronovskaya-type theorems for the proposed operators are established. Also, we derive rates of convergence and establish global approximation theorems by means of the modulus of continuity, the $\mathbb{K}$-functional, the Ditzian-Totik weighted moduli of smoothness, and the weighted $\mathbb{K}$-functional. Additionally, a bivariate analog of these operators is also constructed and investigated. Finally, some graphs and numerical examples are shown using Matlab.

Author contributions

Wentao Cheng: Methodology, formal analysis, writing–original draft and software; Yujie Liu: writing–review and editing; Weiping Zhou: writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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