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Research article

Aggregation of partial T-indistinguishability operators: an application to image recognition

Elif Güner*

Department of Mathematics, Kocaeli University, Umuttepe Campus, Kocaeli 41380, Turkey

* **Correspondence:** Email:elif.guner@kocaeli.edu.tr.

Abstract: In this paper, we introduce the notion of residual domination, which is a stronger version of the usual concept of domination. Then, we show how a family of partial T-indistinguishability operators can be aggregated into one operator using residual domination. This new idea allows the use of partial T-indistinguishability operators instead of T-indistinguishability operators, thus providing a wider variety of operators when we need to apply them to real-life problems. Additionally, we present a practical application to the image recognition problem by using T-indistinguishability operators and partial T-indistinguishability operators. Finally, we give an illustrative example to explain the proposed technique.

Keywords: t-norm; residuum operator; aggregation operator; T-indistinguishability operator; similarity; image recognition
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1. Introduction

Image recognition has emerged as a critical component in artificial intelligence (AI) with applications in healthcare, autonomous driving, security, and robotics. The primary challenge in image recognition lies in accurately distinguishing elements within an image, despite the ambiguity and uncertainty that often characterize visual data. To address these challenges, T-indistinguishability operators, which measure the degree of similarity between objects, can be taken as a useful tool. Initially introduced by Trillas [1] in 1982, the T-indistinguishability operator (or T-equivalence) aimed to fuzzify the classical concept of equivalence relations. By introducing a notion of similarity, these operators established a more flexible framework for comparing elements in datasets where data boundaries might not be well-defined, thereby enabling comparison methods. Since then, extensions and generalizations of the T-indistinguishability operator have been developed by researchers. In the literature, there are two different definitions of partial T-indistinguishability operator $E\colon X \times X \rightarrow [0,1]$. Calvo Sanchez et al. [2] considered the triangle inequality axiom as $T(E(x,z), E(z,y)) \leq T(E(x,y), E(z,z))$ where T is a t-norm. The other definition of the partial T-indistinguishability operator (given by [17]) is based on the residuum operator where the triangle inequality is considered as $E(x,z) \geq T(E(x,y),\ E(y,y) \rightarrow_T E(y,z))$. Both of these concepts generalize the T-indistinguishability operators to handle situations where similarity relations may be partial or incomplete. This has led to considerable research interest in exploring the properties and applications of partial T-indistinguishability operators, especially due to its duality relationship with metric structures [2–4]. This relationship allows T-indistinguishability operators to be viewed as tools that bridge similarity relations and metric spaces, providing a topological base in which T-indistinguishability operators can be analyzed for properties like continuity and convergence-key considerations in complex tasks such as image recognition. The modular indistinguishability operator, introduced by Miñana and Valero [5], represents another significant advancement. This operator allows similarity measurement in a relative context through a parameter. This flexibility not only broadens the application of T-indistinguishability operators but also brings them closer to the topological concepts of proximity and neighborhood, fundamental in analyzing spatial relationships within an image. More recently, Fuster-Parra et al. [6] highlighted the relevance of partial T-equivalence in distinguishing between pseudo-metric and metric structures, further refining the application of T-indistinguishability operators by associating them with topological concepts that better accommodate the complex and variable structure of image data.

Given the complex nature of image data, relying on a single T-indistinguishability operator may be insufficient for precise recognition. Therefore, aggregating multiple T-indistinguishability operators has emerged as a valuable approach. Aggregation techniques, originally used in data fusion across fields such as decision theory [7–9] and information sciences [10, 11], make a means to combine various similarity measures, particularly when inputs contain vagueness or uncertainty. In this way, aggregation functions are applied in fuzzy relations and have proven especially effective in applied sciences [12]. Through aggregation, multiple T-indistinguishability measures can be integrated into a single composite measure, preserving essential topological properties like continuity and T-transitivity. Various studies have tackled the challenge of combining T-indistinguishability operators into a unified operator [2,3,13–15] with recent approaches emphasizing the need to preserve properties such as T-transitivity. Calvo Sanchez et al. [2] presented a technique to aggregate the partial T-indistinguishability operators (in the means of [2]) by means of additive generator functions. Saminger et al. [16] demonstrated that an aggregation operator A preserves the T-transitivity of fuzzy relations if and only if A dominates the t-norm T. This result, extending previous findings on t-norms, emphasizes the importance of transitivity in fuzzy relations, ensuring that similar elements maintain consistent relationships. In a topological context, this preservation of transitivity is crucial, as it maintains the structural consistency of aggregated data, thereby supporting applications requiring accuracy and adaptability, such as image recognition.

In this paper, we define the notion of residual domination, a stronger concept than domination within the context of aggregation. Using residual domination, we present a method for combining partial T-indistinguishability operators (in the means of [17]) into a single, unified operator. One of the objectives of this study is to incorporate partial T-indistinguishability operators, which extend the class of T-indistinguishability operators with this idea, into the application. In this way, the operators we can use are also diversified. Then, we present a novel technique for image recognition by using aggregated T-indistinguishability operators and aggregated partial T-indistinguishability operators. The presented approach is advantageous for image recognition tasks, as aggregated T-indistinguishability operators can enhance both the precision and adaptability of recognition systems by ensuring the topological consistency required for detailed image analysis. We then demonstrate an application of this methodology in robotics, where aggregated operators enable robots to detect and classify elements within images effectively. Comparatively, other advanced methods, such as deep learning-based techniques, have shown great promise in image recognition by learning hierarchical feature representations. However, these approaches often require large amounts of labeled data and significant computational resources. In contrast, the presented residual domination framework provides an alternative that can be more computationally efficient, especially in scenarios where labeled data is scarce or costly to obtain.

The rest of the paper is organized as follows. In Section 2, we recall some basic definitions such as indistinguishability operator, aggregation function, domination, and associated notions. Section 3 introduces the notion of residual domination and aggregation of partial T-indistinguishability operators. Section 4 presents an application of the proposed technique and demonstrates a comparison with the T-indistinguishability operator. Section 5 gives some directions related to the future developments of this methodology, including potential improvements.

2. Preliminaries

In this section, we recall some fundamental notions such as t-norm, residuum operator, (partial) T-indistinguishability operator, aggregation function, and domination.

Definition 2.1 ([18]). A triangular norm (briefly, t-norm) is a binary operation T on $[0,1]$ if the following axioms are satisfied for all $x, y, z \in [0,1]$:

(T1) $T(x,y) = T(y,x)$, (Commutativity)

(T2) $T(x,T(y,z)) = T(T(x,y),z)$, (Associativity)

(T3) $T(x,y) \leq T(x,z)$ if $y \leq z$, (Monotonicity)

(T4) $T(x,1) = x$. (Boundary condition)

In addition, if $T\colon [0,1]\times[0,1] \to [0,1]$ is a continuous function (with respect to the usual topology), then T is called a continuous t-norm. The most commonly used t-norms are the minimum t-norm ($T_\wedge$) where

$$T_\wedge(x,y) = x \wedge y,$$

the product t-norm (T_p) where

$$T_p(x,y) = xy,$$

the Łukasiewicz t-norm ($T_\mathfrak{L}$) where

$$T_\mathfrak{L}(x,y) = \max\{0,\ x+y-1\},$$

and the drastic t-norm (T_D) where

$$T_D(x,y) = \begin{cases} \min\{x,y\}, & \text{if } x = 1 \text{ or } y = 1, \\ 0, & \text{otherwise.} \end{cases}$$

In the following, the definition of the residuum operator that is associated with the t-norm is presented:

Definition 2.2 ([18]). Let T be a t-norm on $[0,1]$. If there exists a binary operation $\rightarrow_T$ on $[0,1]$ satisfying the following condition for all $x, y, z \in [0,1]$, then $\rightarrow_T$ is called the T-residuum operator of the t-norm T:

$$T(x,y) \leq z \quad \text{if and only if} \quad x \leq y \rightarrow_T z.$$

Proposition 2.3 ([18]). *Let T be a continuous t-norm, then T-residuum operator $\rightarrow_T$ of T is uniquely determined by the formula*

$$x \rightarrow_T y = \begin{cases} 1, & \text{if } x \leq y, \\ \sup\{z \in [0,1] : T(x,z) = y\}, & \text{if } x > y. \end{cases} \tag{2.1}$$

The details about residuum operators can be found in [18]. We now recall the definitions of the T-indistinguishability operator (given by Trillas [1]) and the partial T-indistinguishability operator (given by Höhle [17]), which will be used later.

Definition 2.4 ([1]). Let X be a set, T a t-norm, and $E : X \times X \rightarrow [0,1]$ a fuzzy relation on X. The fuzzy relation E is called a T-indistinguishability operator if it satisfies the following conditions for all $x, y, z \in X$:
(E1) $E(x,x) = 1$, (Reflexivity)
(E2) $E(x,y) = E(y,x)$, (Symmetry)
(E3) $E(x,z) \geq T(E(x,y), E(y,z))$. (T-transitivity)

As mentioned in the paper [5], T-indistinguishability operators can be considered as the generalization of the concept of fuzzy metrics.

Example 2.5 ([5]). Let the fuzzy relation $E_t\colon \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow [0,1]$ be defined by

$$E_t(x,y) = \frac{t}{t + |x-y|}$$

for all $t > 0$ and $x, y \in \mathbb{R}^+$. Then E_t is a T-indistinguishability operator for each t-norm T.

Definition 2.6 ([17]). Let X be a set, T a t-norm, and $E : X \times X \rightarrow [0,1]$ a fuzzy relation on X. The fuzzy relation E is called a partial T-indistinguishability operator if it satisfies the following conditions for all $x, y, z \in X$:
(PE1) $E(x,x) \geq E(x,y)$, (Partial reflexivity)
(PE2) $E(x,y) = E(y,x)$, (Symmetry)
(PE3) $E(x,z) \geq T(E(x,y), E(y,y) \rightarrow_T E(y,z))$. (Partial T-transitivity)

It is clear from the above definitions that partial T-indistinguishability operators are generalizations of the T-indistinguishability operators. Also, the notion of fuzzy partial metric spaces and partial T-indistinguishability operators are related. Hence, we can take the mapping as a partial T-indistinguishability operator shown in the following example that is given as a fuzzy partial metric space in [20].

Example 2.7. Let the fuzzy relation $E_t\colon \mathbb{R}^+ \times \mathbb{R}^+ \to [0,1]$ be defined by

$$E_t(x,y) = \frac{t}{t + \max\{x,y\}}$$

for all $t > 0$ and $x, y \in \mathbb{R}^+$. Then E_t is a partial T_p-indistinguishability operator.

The aggregation function is fundamental in the fusion of different pieces of information. This function was introduced by Mesiar and Pap [19] as follows:

Definition 2.8 ([19]). A function $A : [0,1]^n \to [0,1]$ is called an aggregation function if the following conditions are satisfied for all $x, y \in [0,1]^n$:
(A1) $A(0) = 0, A(1) = 1,$ (Boundary)
(A2) $A(x) \leq A(y)$ if $x \leq y$. (Monotonicity)

Definition 2.9 ([16]). Let A be an aggregation function and $E_1, E_2, ..., E_n$ be n-fuzzy relations on a set X. Then the fuzzy relation

$$A(E_1, E_2, ..., E_n) : X^n \times X^n \to [0,1]$$

defined by

$$A(E_1, E_2, \ldots, E_n)(x,y) = A(E_1(x_1,y_1), E_2(x_2,y_2), \ldots, E_n(x_n,y_n))$$

is said to be the aggregation of $E_1, E_2, ..., E_n$ by A.

The next definition gives the concept of domination for aggregation operators that allow the preservation of T-transitivity.

Definition 2.10 ([16]). Let A be an aggregation function and T a t-norm. Then we say that A dominates T if the following inequality is satisfied for all $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in [0,1]^n$:

$$A(T(x_1,y_1), T(x_2,y_2), \ldots, T(x_n,y_n)) \geq T(A(x_1,x_2,\ldots,x_n),\ A(y_1,y_2,\ldots,y_n)).$$

The preservation of T-transitivity in aggregating fuzzy relations is closely related to the domination of an aggregation operator over the corresponding t-norm T given as in the following theorem:

Theorem 2.11 ([16]). *Let X be a set such that $|X| \geq 3$ and T a t-norm. Then $A(E_1, E_2, ...E_n)$ is a T-indistinguishability operator for every T-indistinguishability operators $E_1, E_2, ..., E_n$ if and only if A dominates T.*

Since the aggregation function $A\colon [0,1]^n \to [0,1]$ defined by

$$A(x) = x_1 x_2 ... x_n$$

for all

$$x = (x_1, x_2, \ldots, x_n) \in [0,1]^n$$

dominates the product t-norm T_p, the following corollary has been obtained:

Corollary 2.12 ([16]). *If $E_1, E_2, ..., E_n$ are T_p-indistinguishability operators, then the fuzzy relation*

$$E_1 E_2 ... E_n : X^n \times X^n \to [0,1]$$

defined by

$$(E_1 E_2 \cdots E_n)(x,y) = E_1(x_1,y_1)\, E_2(x_2,y_n) \cdots E_n(x_n,y_n)$$

is a T_p-indistinguishability operator.

3. Aggregation of partial T-indistinguishability operators

In this section, we first provide the notion of residual domination, which is a stronger concept than domination. Then, we show that the partial T-transitivity is preserved under aggregation with the condition of residual domination.

Definition 3.1. Let A be an aggregation function, T a t-norm, and $\rightarrow_T$ a T-residuum operator. Then, we say that A residual dominates T if the following condition is satisfied for all $x, y, z \in [0, 1]^n$ whenever $y \geq z$:

$$A(T(x_1, y_1 \rightarrow_T z_1), T(x_2, y_2 \rightarrow_T z_2), \ldots, T(x_n, y_n \rightarrow_T z_n)) \geq T(A(x_1, x_2, \ldots, x_n), A(y_1, y_2, \ldots, y_n) \rightarrow_T A(z_1, z_2, \ldots, z_n)).$$

The next proposition shows that the concept of residual domination is stronger than the concept of domination.

Proposition 3.2. *If an aggregation operator A residual dominates T, then A dominates T.*

Proof. Let A residual dominate T. Then, we have

$$A(T(x_1, y_1 \rightarrow_T z_1), T(x_2, y_2 \rightarrow_T z_2), \ldots, T(x_n, y_n \rightarrow_T z_n)) \geq T(A(x_1, x_2, \ldots, x_n), A(y_1, y_2, \ldots, y_n) \rightarrow_T A(z_1, z_2, \ldots, z_n)),$$

for all $x, y, z \in [0, 1]^n$ whenever $y \geq z$. If we take $y = (1, 1, ..., 1)$, then we obtain

$$\begin{aligned} A(T(x_1, z_1), T(x_2, z_2), \ldots, T(x_n, z_n)) &= A(T(x_1, 1 \rightarrow_T z_1), T(x_2, 1 \rightarrow_T z_2), \ldots, T(x_n, 1 \rightarrow_T z_n)) \\ &\geq T(A(x_1, x_2, \ldots, x_n),\ A(1, 1, \ldots, 1) \rightarrow_T A(z_1, z_2, \ldots, z_n)) \\ &= T(A(x_1, x_2, \ldots, x_n),\ A(z_1, z_2, \ldots, z_n)), \end{aligned}$$

which means that A dominates T. □

In the following, we show that aggregation of partial T-indistinguishability operators is a partial T-indistinguishability operator but in a narrow set.

Proposition 3.3. *Let X be a set such that $|X| \leq 2$, A be an aggregation function, and T a t-norm. If $E_1, E_2, ..., E_n$ are n partial T-indistinguishability operators, then $A(E_1, E_2, ..., E_n)$ is a partial T-indistinguishability operator.*

Proof. Let $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in X^n$.
(PE1) Since $E_i(x_i, x_i) \geq E_i(x_i, y_i)$ for all $i = 1, 2, ..., n$ and A is a monotone function, we have

$$\begin{aligned} A(E_1, E_2, \ldots, E_n)(x, x) &= A(E_1(x_1, x_1), E_2(x_2, x_2), \ldots, E_n(x_n, x_n)) \\ &\geq A(E_1(x_1, y_1), E_2(x_2, y_2), \ldots, E_n(x_n, y_n)) \\ &= A(E_1, E_2, \ldots, E_n)(x, y). \end{aligned}$$

(PE2) Since $E_i(x_i, y_i) = E_i(y_i, x_i)$ for all $i = 1, 2, ..., n$, we obtain

$$\begin{aligned} A(E_1, E_2, \ldots, E_n)(x, y) &= A(E_1(x_1, y_1), E_2(x_2, y_2), \ldots, E_n(x_n, y_n)) \\ &= A(E_1(y_1, x_1), E_2(y_2, x_2), \ldots, E_n(y_n, x_n)) \\ &= A(E_1, E_2, \ldots, E_n)(y, x). \end{aligned}$$

(PE3) For all $x, y \in X$, we have

$$\begin{aligned}
&T\Big(A(E_1, E_2, \ldots, E_n)(x, y),\ A(E_1, E_2, \ldots, E_n)(y, y) \to_T A(E_1, E_2, \ldots, E_n)(y, x)\Big) \\
&\le T\Big(A(E_1, E_2, \ldots, E_n)(y, y),\ A(E_1, E_2, \ldots, E_n)(y, y) \to_T A(E_1, E_2, \ldots, E_n)(y, x)\Big) \\
&\le A(E_1, E_2, \ldots, E_n)(y, x).
\end{aligned}$$

□

The following example shows that the previous proposition is not satisfied when $|X| \ge 3$.

Example 3.4. Let $X = \{x, y, z\}$ and $a, b, c \in [0, 1]$ satisfy $c < b < a < 1$. Define the fuzzy relation $E : X \times X \to [0, 1]$ as follows:

$$E(x, y) = E(y, x) = b, \quad E(x, z) = E(z, x) = c, \quad E(y, z) = E(z, y) = b, \quad E(x, x) = E(y, y) = E(z, z) = a.$$

(PE1) Since

$$\begin{aligned}
E(x, x) = a > b = E(x, y), \quad & E(x, x) = a > c = E(x, z), \\
E(y, y) = a > b = E(y, x), \quad & E(y, y) = a > b = E(y, z), \\
E(z, z) = a > c = E(z, x), \quad & E(z, z) = a > b = E(z, y),
\end{aligned}$$

we obtain that

$$E(x, x) \le E(x, y) \quad \text{for all } x, y \in X.$$

It is clear from the definition of E that the axiom (PE2) is satisfied. Also, we have

$$\begin{aligned}
&0 = T_D(b,\ a \to_{T_D} b) = T_D(E(x, y),\ E(y, y) \to_{T_D} E(y, z)) \le c = E(x, z), \\
&T_D(E(x, z),\ E(z, z) \to_{T_D} E(z, y)) = T_D(c,\ a \to_{T_D} b) \le c < b = E(x, y), \\
&T_D(E(y, x),\ E(x, x) \to_{T_D} E(x, z)) = T_D(b,\ a \to_{T_D} c) \le b = E(y, z)
\end{aligned}$$

for all $x, y, z \in X$, which shows that (PE3) holds. Then, E is a partial T_D-indistinguishability operator. However, if we take an aggregation function $A : [0, 1] \times [0, 1] \to [0, 1]$ satisfying $A(a, a) = A(b, b) = 1$ and $A(c, c) < 1$, then we obtain

$$\begin{aligned}
T_D(A(E, E)(x, y),\ A(E, E)(y, y) \to_{T_D} A(E, E)(y, z)) &= T_D(A(b, b),\ A(a, a) \to_{T_D} A(b, b)) \\
&= 1 > A(E, E)(x, z),
\end{aligned}$$

which implies that $A(E, E)$ does not satisfy the partial T-transitivity. Hence, $A(E, E)$ is not a partial T_D-indistinguishability operator.

The last example shows that the aggregation of two partial T-indistinguishability operators may not be a partial T-indistinguishability operator when $|X| \ge 3$ even for self-aggregation. We now show when the partial T-transitivity is preserved under the aggregation function:

Theorem 3.5. *Let X be a set such that $|X| \ge 3$ and T a t-norm. Then $A(E_1, E_2, ...E_n)$ is a partial T- indistinguishability operator for every partial T-indistinguishability operators $E_1, E_2, ..., E_n$ if and only if A residual dominates T.*

Proof. ($\Rightarrow$) Assume that $A(E_1, E_2, ..., E_n)$ is a partial T-indistinguishability operator for any partial T-indistinguishability operators $E_1, E_2, ..., E_n$. We have to show that

$$A(T(x_1, y_1 \to_T z_1), T(x_2, y_2 \to_T z_2), \dots, T(x_n, y_n \to_T z_n)) \geq T(A(x_1, x_2, \dots, x_n), A(y_1, y_2, \dots, y_n) \to_T A(z_1, z_2, \dots, z_n)).$$

Let

$$x = (x_1, x_2, \dots, x_n), \quad y = (y_1, y_2, \dots, y_n), \quad z = (z_1, z_2, \dots, z_n) \in [0, 1]^n$$

satisfying $y \geq z$. Since X contains at least three points a_i, b_i, c_i (say), there exists a fuzzy relation $R_i : X \times X \to [0, 1]$ such as

$$\begin{aligned}
&R_i(a_i, a_i) = R_i(b_i, b_i) = R_i(c_i, c_i) = x_i,\\
&R_i(a_i, b_i) = R(b_i, a_i) = y_i,\\
&R_i(b_i, c_i) = R(c_i, b_i) = z_i,\\
&R_i(a_i, c_i) = T(y_i, x_i \to_T z_i),\\
&R_i(x, y) = 0 \; for \; all \; x \neq y \; and \; x \in X \setminus \{a_i, b_i, c_i\} \; and \; y \in X \setminus \{a_i, b_i, c_i\}
\end{aligned}$$

for any $x_i, y_i, z_i \in [0, 1]$ whenever $x_i \geq y_i$ and $x_i \geq z_i$. We now claim that R_i is a partial T-indistinguishability operator for all $i = 1, 2, ..., n$. So, we need to show that

$$T\big(R_i(x, y),\ R_i(y, y) \to_T R_i(y, z)\big) \leq R_i(x, z)$$

for all $x, y, z \in X$. If at least one argument of R_i is from $X \setminus \{a_i, b_i, c_i\}$, then this inequality is satisfied directly. For arguments $x, y, z \in \{a_i, b_i, c_i\}$, we have the following:

$$\begin{aligned}
T(R_i(a_i, b_i), R_i(b_i, b_i) \to_T R_i(b_i, c_i)) &= T(y_i, x_i \to_T z_i) = R_i(a_i, c_i),\\
T(R_i(a_i, c_i), R_i(c_i, c_i) \to_T R_i(c_i, b_i)) &= T(T(y_i, x_i \to_T z_i), x_i \to_T z_i) le T(y_i, x_i \to_T z_i) \leq y_i = R_i(a_i, b_i),\\
T(R_i(b_i, c_i), R_i(c_i, c_i) \to_T R_i(c_i, a_i)) &= T(z_i, x_i \to_T T(y_i, x_i \to_T z_i))\\
&\leq T(z_i, x_i \to_T y_i) \leq T(x_i, x_i \to_T y_i) \leq y_i = R_i(a_i, b_i).
\end{aligned}$$

Thus, we deduce that R_i is a partial T-indistinguishability operator. Since $A(R_1, R_2, ..., R_n)$ is a partial T-indistinguishability operator, we obtain

$$\begin{aligned}
&T\Big(A(y_1, y_2, \dots, y_n),\ A(x_1, x_2, \dots, x_n) \to_T A(z_1, z_2, \dots, z_n)\Big)\\
&\quad = T\Big(A(R_1(a_1, b_1), R_2(a_2, b_2), \dots, R_n(a_n, b_n)),\\
&\qquad A(R_1(b_1, b_1), R_2(b_2, b_2), \dots, R_n(b_n, b_n)) \to_T A(R_1(b_1, c_1), R_2(b_2, c_2), \dots, R_n(b_n, c_n))\Big)\\
&\quad = T\Big(A(R_1, R_2, \dots, R_n)(a, b),\ A(R_1, R_2, \dots, R_n)(b, b) \to_T A(R_1, R_2, \dots, R_n)(b, c)\Big)\\
&\quad \leq A(R_1, R_2, \dots, R_n)(a, c)\\
&\quad = A(R_1(a_1, c_1), R_2(a_2, c_2), \dots, R_n(a_n, c_n))\\
&\quad = A(T(y_1, x_1, \to_T z_1), T(y_2, x_2, \to_T z_2), \dots, T(y_n, x_n, \to_T z_n)),
\end{aligned}$$

which means that A residual dominates T.

($\Leftarrow$) Now suppose that A residual dominates T. Take $x, y, z \in X^n$. First, we need to show that the axiom (PE1) holds. Since $E_1, E_2, \ldots, E_n$ are partial T-indistinguishability operators, we have

$$E_i(x_i, y_i) \leq E_i(x_i, x_i) \quad \text{for all } i = 1, 2, \ldots, n.$$

This follows that

$$\begin{aligned} A(E_1, E_2, \ldots, E_n)(x, y) &= A(E_1(x_1, y_1), E_2(x_2, y_2), \ldots, E_n(x_n, y_n)) \\ &\leq A(E_1(x_1, x_1), E_2(x_2, x_2), \ldots, E_n(x_n, x_n)) \\ &= A(E_1, E_2, \ldots, E_n)(x, x), \end{aligned}$$

and so this shows that (PE1) holds. It is clear that the axiom (PE2) is satisfied. Let us show that the axiom (PE3) is satisfied. Then, we obtain

$$\begin{aligned} & T\Big(A(E_1, E_2, \ldots, E_n)(x, y),\ A(E_1, E_2, \ldots, E_n)(y, y) \to_T A(E_1, E_2, \ldots, E_n)(y, z)\Big) \\ &= T\Big(A(E_1(x_1, y_1), E_2(x_2, y_2), \ldots, E_n(x_n, y_n)), \\ &\qquad A(E_1(y_1, y_1), E_2(y_2, y_2), \ldots, E_n(y_n, y_n)) \to_T A(E_1(y_1, z_1), E_2(y_2, z_2), \ldots, E_n(y_n, z_n))\Big) \\ &\leq A\Big(T(E_1(x_1, y_1), E_1(y_1, y_1) \to_T E_1(y_1, z_1)), T(E_2(x_2, y_2), E_2(y_2, y_2) \to_T E_2(y_2, z_2)), \\ &\qquad T(E_n(x_n, y_n), E_n(y_n, y_n) \to_T E_n(y_n, z_n))\Big) \\ &\leq A(E_1(x_1, z_1), E_2(x_2, z_2), \ldots, E_n(x_n, z_n)) \\ &= A(E_1, E_2, \ldots, E_n)(x, z), \end{aligned}$$

since A residual dominates T and A is a monotone function. Hence, $A(E_1, E_2, ..., E_n)$ is a partial T-indistinguishability operator. □

The following gives a proposition related to the residual domination of the product t-norm:

Proposition 3.6. *Let $a, b, c, d \in [0, 1]$ such that $a \geq b$ and $c \geq d$. Then, the following equality holds:*

$$T_p(a \to_{T_p} b,\ c \to_{T_p} d) = T_p(a, c) \to_{T_p} T_p(b, d).$$

Proof. If $a = b$ or $c = d$, then the equality is satisfied obviously.

Let $a > b$ and $c > d$. Then we have

$$\begin{aligned} T_p(a \to_{T_p} b,\ c \to_{T_p} d) &= T_p\left(\frac{b}{a},\ \frac{d}{c}\right) = \frac{bd}{ac}, \\ T_p(a, c) \to_{T_p} T_p(b, d) &= ac \to_{T_p} bd = \frac{bd}{ac}, \end{aligned}$$

which means that the equality is satisfied. □

By considering the above proposition, we obtain that the aggregation function $A : [0, 1]^n \to [0, 1]$ defined by $A(x) = x_1 x_2 ... x_n$ for all $x = (x_1, x_2, ..., x_n) \in [0, 1]^n$ residual dominates the product t-norm T_p, and therefore, we have the following corollary:

Corollary 3.7. *If $E_1, E_2, ..., E_n$ are partial T_p-indistinguishability operators, then the fuzzy relation $E_1 E_2 ... E_n : X^n \times X^n \to [0, 1]$ defined by*

$$(E_1 E_2 \cdots E_n)(x, y) = E_1(x_1, y_1)\, E_2(x_2, y_2) \cdots E_n(x_n, y_n)$$

is a partial T_p-indistinguishability operator.

4. An application to image recognition problem

Image recognition is the ability of robots to detect, classify, and recognize objects. Recent developments in this field have brought the technology closer to human-level image processing capabilities. One of the best examples of image recognition solutions is facial recognition, which we use daily to unlock our smartphones. In this process, the system must first detect a face, classify it as a human face, and then determine if it belongs to the smartphone's owner.

4.1. Method

In this section, we present a technique for image recognition in a type of robot equipped with a single eye capable of capturing and storing visual information. In this method, as an initial step, the robot is shown an image (named Image-X). Then, the robot memorizes the coordinates of the cluster points in Image-X by retrieving the colors at specific coordinates within its field of vision. The set of cluster points is shown by CL. To filter out noise, coordinates are selected in clusters of four. Subsequently, the robot is presented with various images and tasked with determining whether each was similar to Image-X based on the stored cluster information. To perform this comparison, the robot considers the hue, saturation, and value (HSV) values of the selected four-point pixels. The HSV color model represents how colors appear under different lighting conditions and aligns with the way human vision perceives color attributes. In this model, the hues are arranged in a radial slice around a central axis of neutral colors, ranging from black at the bottom to white at the top. Here, the hue is defined as the "attribute of a visual sensation according to which an area appears similar to one of the perceived colors: red, yellow, green, and blue, or to a combination of two." Saturation is the "colorfulness of a stimulus relative to its brightness", while brightness is the "attribute of a visual sensation according to which an area appears to emit more or less light". HSV is often used in computer vision and image analysis for feature detection or image segmentation. In most cases, computer vision algorithms for color images are straightforward extensions of algorithms designed for grayscale images, such as k-means or fuzzy clustering of pixel colors, or Canny edge detection. Typically, each color component is processed separately through the same algorithm. Therefore, the features of interest must be distinguishable by the chosen color dimensions. Because the R, G, and B components of an object's color in a digital image are all correlated with the amount of light hitting the object and therefore with each other, describing images in terms of these components can make object discrimination challenging. Descriptions based on hue, lightness, and chroma or hue, lightness, and saturation are often more effective. Therefore, the robot requires the HSV values of the images to initiate the detection process. Then, all HSV values of the images are shown in tables. Next, the HSV values of Image-X with the other images are compared using T-indistinguishability operators (or partial T-indistinguishability operators). Then, these data are merged by applying the presented aggregation process. Finally, the similarity indices between the data obtained from this process are calculated by using the following formula:

$$d = \frac{1}{nt} \sum_{s=1}^{n} \sum_{k(s) \in \mathrm{CL}} \max\{k(s)(i, j) : i, j \in \{1, 2, \ldots, n\}\}, \tag{4.1}$$

where CL is the set of cluster points, $t = |CL|$, and n denotes the number of taken four pixels locations. By taking the maximum value in each cluster, this formula emphasizes the strongest similarity relationship among the corresponding image features.

4.2. Application of the proposed method

We now provide an example to explain the proposed approach in detail. Initially, the robot is shown an image (referred to as Image-O), and the image captured by the robot's eye at that moment is shown in Figure 1.

Figure 1. Original image (Image-O).

Then, the robot memorizes the coordinates of the cluster points in Image-O by retrieving the colors at specific coordinates within its field of vision. These coordinates are selected from visually distinct regions of the image, such as the eyes, ears, and body, in order to capture the characteristic color distribution of the object. These clusters are shown in Figure 2.

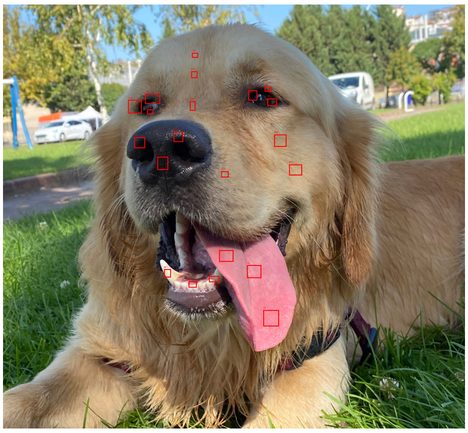

Figure 2. The clusters of Image-O.

Let the set of cluster points for Image-O be represented as

$$\{LE_i, RE_i, LF_i, CF_i, RF_i, T_i, M_i \mid i = 1, 2, 3\}.$$

Also, the coordinates of these clusters for Image-O are provided in Table 1.

Next, we present three images (named Image-A, Image-B, and Image-C), shown in Figure 3 to the robot.

(a) Image-A

(b) Image-B

(c) Image-C

Figure 3. Image-A, Image-B, and Image-C.

The robot's task is to recognize the image that most closely resembles Image-O. To do this, the HSV values of Image-O, Image-A, Image-B, and Image-C are required. Let the HSV values at the selected cluster points of each image be denoted as follows:

$$O_k(i, j) = (O^H_k(i, j), O^S_k(i, j), O^V_k(i, j))$$

for all

$$k \in \{LE_i, RE_i, LF_i, CF_i, RF_i, T_i, M_i \mid i = 1, 2, 3\}$$

and $i, j \in \{1, 2, 3\}$ for Image-O. i.e.,

$$O_{LE_1}(1, 1) = (O^H_{LE_1}(1, 1), O^S_{LE_1}(1, 1), O^V_{LE_1}(1, 1))$$

denotes the HSV values of Image-O of the pixel given in (1, 1)-place for $n = 1$ in Table 1.

Table 1. The coordinates of the clusters of Image-O.

	First position ($n = 1$)	Second position ($n = 2$)	Third position ($n = 3$)
LE_n	(552, 420), (612, 420)	(627, 393), (693, 393)	(636, 462), (663, 462)
	(552, 480), (612, 480)	(627, 438), (693, 438)	(636, 483), (663, 483)
RE_n	(1083, 378), (1122, 378)	(1155, 363), (1185, 363)	(1167, 414), (1209, 414)
	(1083, 423), (1122, 423)	(1155, 384), (1185, 384)	(1167, 447), (1209, 447)
LF_n	(579, 585), (627, 585)	(753, 561), (795, 561)	(681, 671), (729, 671)
	(579, 633), (627, 633)	(753, 606), (795, 606)	(681, 729), (729, 729)
CF_n	(834, 213), (861, 213)	(831, 294), (858, 294)	(825, 426), (849, 426)
	(834, 234), (861, 234)	(831, 324), (858, 324)	(825, 468), (849, 468)
RF_n	(963, 735), (996, 735)	(1197, 573), (1251, 573)	(1263, 705), (1320, 705)
	(693, 762), (996, 762)	(1197, 627), (1251, 627)	(1263, 753), (1320, 753)
T_n	(954, 1083), (1017, 1083)	(1077, 1149), (1140, 1149)	(1149, 1347), (1218, 1347)
	(954, 1134), (1017, 1134)	(1077, 1206), (1140, 1206)	(1149, 1419), (1218, 1419)
M_n	(711, 1167), (741, 1167)	(819, 1221), (855, 1221)	(909, 1197), (954, 1197)
	(711, 1203), (741, 1203)	(819, 1248), (855, 1248)	(909, 1224), (954, 1224)

Similarly, we denote

$$
\begin{aligned}
A_k(i,j) &= (A_k^H(i,j), A_k^S(i,j), A_k^V(i,j)) \quad \text{for Image-A;}\\
B_k(i,j) &= (B_k^H(i,j), B_k^S(i,j), B_k^V(i,j)) \quad \text{for Image-B;}\\
C_k(i,j) &= (C_k^H(i,j), C_k^S(i,j), C_k^V(i,j)) \quad \text{for Image-C,}
\end{aligned}
$$

where

$$k \in \{LE_i, RE_i, LF_i, CF_i, RF_i, T_i, M_i \mid i = 1,2,3\} \quad \text{and} \quad i,j \in \{1,2,3\}.$$

These HSV values corresponding to the selected pixel coordinates were directly taken using image-processing software and are shown in Tables 2–5.

Table 2. The HSV values of the original Image-O.

	H value of Image-O						S value of Image-O						V value of Image-O					
	$n=1$		$n=2$		$n=3$		$n=1$		$n=2$		$n=3$		$n=1$		$n=2$		$n=3$	
LE_n	68,6	35,9	24,3	40,3	203,9	219,8	6,3	49,3	8,8	24,7	27,1	24,7	55,3	22,3	47,5	36,6	15,5	36,2
	42	332,8	221,9	251,5	10,9	353	20,1	5,4	93,9	13,8	9,5	13,4	49,1	18,2	7	18,6	25	32,9
RE_n	352,9	228,6	13	24,2	253,8	14,1	20,5	53,7	26,2	20,4	8,7	43,3	21,5	4,9	35,7	56,4	24,9	16,1
	323,7	228,4	240	341	30,6	26,9	19,6	70,5	84,5	75,4	18	25,2	15	11	5	5,8	27,9	56,6
LF_n	60	51	40,2	37,4	30,2	27,8	6,3	12,5	16,8	23,7	23,8	20,7	55,3	52,7	53,5	49,8	50	53,1
	50,4	48,8	32,2	32,3	30,2	33,5	10,6	15	21,9	24	20,3	28,5	58,2	53,2	58,2	53,1	58,6	62,5
CF_n	204,1	219,6	203	226,4	215,1	240	8,6	40,4	19,6	16,3	13,3	12,5	48,4	22,1	16,6	13,5	27,3	27,3
	219,8	226,5	226,4	253,8	203,1	253,8	18,5	76,1	25,1	23,3	57,5	58,2	24,2	2,9	8,8	9,2	5,6	3,7
RF_n	64,3	348,4	354,1	354,8	329,1	30,6	7,7	103,5	38,5	43	44,3	22,1	89,9	4,8	76,3	50,4	21,7	45,5
	27,3	38,1	42,3	2,5	17,9	340	49,9	16,1	8,1	8,1	20,8	60,3	28,6	101	82,7	89,4	87,2	11,9
T_n	323	349,9	346,7	345,9	351,1	351,1	41,9	30,2	37,8	38,1	45,4	43,6	81,5	69	81,6	92,4	95,1	99,1
	351	351,5	351,5	349,5	350,7	350,7	41,2	33,9	35,1	38,6	44,5	44,5	93,5	87	83,8	89,9	99,2	99,2
M_n	35,3	34,7	30,4	35,3	34,9	36,3	19,1	26,8	34,4	30	30,6	29,8	68,8	69,5	69,3	63,7	72,9	79,2
	31	35,3	34,9	35,5	32,6	33,4	22	28,1	29,3	27,7	33,2	33,3	60,3	68	76,1	73,9	74,4	79,6

Table 3. The HSV values of Image-A.

	H value of Image-A						S value of Image-A						V value of Image-A					
	$n=1$		$n=2$		$n=3$		$n=1$		$n=2$		$n=3$		$n=1$		$n=2$		$n=3$	
LE_n	50,4	358,9	153	217,5	43,4	40,3	11,5	12,1	4,5	37,4	16,2	26,1	53,5	24,4	22,4	21,6	44,3	34,6
	32,3	206,9	33,2	71,5	210,8	189,9	28,2	45,8	18	4,3	62	12	45,3	24,4	42,9	60,4	31,4	20,1
RE_n	26,3	2,4	219,7	38	27,8	28,8	23,2	28,9	83,6	10,4	15,5	27,9	37,7	24,6	14,3	13	59,3	57,4
	14	216	26,1	31,4	210,3	30	26,6	78,3	33,6	27,4	28,6	85,4	26,4	15	64,3	75	46,5	19,2
LF_n	60	42,3	42	35,5	32,3	30,2	6,6	13,4	17,2	18,9	23,3	21,9	52,5	50,1	57,4	57,6	54,7	54,3
	42,3	40,2	32,2	30,2	29,8	32,4	11,7	15,7	22,8	22,2	12,8	32,5	57,6	57,4	55,9	53,5	53,5	59,1
CF_n	216,6	222,9	223	267,5	215,1	217,4	40,6	36	26,5	14	16,9	40,8	28,8	19,6	26,7	15,5	21,4	39,7
	226,4	226,5	240	240	226,4	240	14,6	67	9,4	8,4	15	100	15,1	3,3	9,1	10,3	14,7	3
RF_n	144,4	230,2	0,5	353,3	321,9	323,4	3,1	18,5	42,8	46,5	34,6	25,3	95,8	33	61,2	33,1	18,2	34,7
	31,2	38,2	6,6	5,6	31,8	17,9	37,4	25,3	20,8	28,8	14,3	22,6	35,5	96,8	81,6	80	95,6	80,1
T_n	350,3	352,7	349	346,4	350,7	350,6	45,8	32,6	37,6	36	44,2	45,1	86	81,5	94,8	95,1	100	98
	351,5	351,5	349,4	345,9	351,9	350,7	38,4	33,3	40	37,7	35	43,9	96,6	88,6	93,8	98,5	100,7	100,8
M_n	28,7	29,5	27,1	30,6	39,6	35,2	39,5	49,4	40,6	37,8	30,1	25,3	61,6	52,2	65,8	76,4	72,2	74,4
	32,6	27,1	29,3	25,3	32,6	34,9	31,1	39,6	36,4	51,1	32,8	31,5	69,1	67,4	63,1	70,6	75,2	71

Table 4. The HSV values of Image-B.

	H value of Image-B						S value of Image-B						V value of Image-B					
	$n = 1$		$n = 2$		$n = 3$		$n = 1$		$n = 2$		$n = 3$		$n = 1$		$n = 2$		$n = 3$	
LE_n	117,7	52,2	33,1	12	43,4	44,1	42,8	65,3	56,9	55,6	64,6	70	70,6	97,3	20	7,1	94,1	99,2
	103,2	33,5	40	33,1	43,6	44,3	65,7	55,7	52,2	59,2	63,6	66,7	66,3	23,9	27,1	19,2	94,9	94,1
RE_n	12	17,4	45	46,3	22,5	42,9	13,5	56,7	50	46,6	15,4	41,2	14,5	26,3	97,3	99,2	20,4	33,3
	29,1	50,6	46,2	56,8	35	12	61,4	56	42,9	59,2	48	11,1	44,7	98,8	99,6	100	29,4	17,6
LF_n	104,8	101,8	96,2	95,5	97,4	94,6	69,2	63,7	98,4	94	95,5	90,2	56,1	67,1	50,2	52,2	43,5	40
	93,8	91,9	97,1	96,3	0	0,8	74	75,2	95,9	94,5	100	100	67,8	64,7	48,2	43,1	92,2	89,4
CF_n	35,7	96,7	54,7	44,1	42,9	38,5	85,4	78,3	54,8	53,1	34,5	34,2	94,1	18	24,3	50,2	79,6	89,4
	25,6	67,5	132	43,7	330	42,4	100	33,3	9,4	45,4	4,9	54	29,4	9,4	20,8	98,4	16,1	98,8
RF_n	48	228	0	160	170,3	230	63	17,2	0	12	31,8	17,1	93,3	11,4	6,7	29,4	83,9	13,7
	50	30,9	22	31,9	66,7	83,1	23,1	94,1	95,2	88,6	67,5	56,5	10,2	99,6	97,6	100	94,1	97,3
T_n	3	5,9	6,1	3,9	1,8	2	100	99	100	100	100	93,8	77,6	81,2	46,3	36,5	40	25,1
	4,1	6	5,5	5,5	0,7	6	97,4	98,6	100	98,6	97,8	94,6	29,8	27,8	51,8	56,1	36,5	29
M_n	38	359,2	0	2,1	86,9	82,5	99,2	92,3	100	99,2	63,6	65,8	99,2	30,6	92,5	46,3	30,2	28,6
	1,7	5,1	3,1	0	75,7	69,4	99,1	100	99,4	98,3	66,7	55,2	42,7	73,3	69,8	68,6	24,7	22,7

Table 5. The HSV values of Image-C.

	H value of Image-C						S value of Image-C						V value of Image-C					
	$n = 1$		$n = 2$		$n = 3$		$n = 1$		$n = 2$		$n = 3$		$n = 1$		$n = 2$		$n = 3$	
LE_n	42,9	44,6	32,3	48	16	7,3	89,5	53,2	53,1	59,5	6,5	14,6	78,4	74,5	19,2	16,5	91	88,6
	28,9	38,7	54,8	37,2	37,2	29,3	92,4	90,7	53,5	61,7	13,7	21,1	67,1	75,7	16,9	18,4	83,1	80
RE_n	6,1	6,1	7,4	6,1	6,1	6,1	100	98,3	96,6	100	100	100	23,1	23,5	23,1	23,1	23,1	23,1
	5	6,1	6,1	6,1	7,4	3,9	100	100	100	100	100	100	23,5	23,1	23,1	23,1	22,4	24,3
LF_n	0	0	0	0	0	0	96,6	96,6	96,6	96,6	96,6	96,6	69,8	69,8	69,8	69,8	69,8	69,8
	0	0	0	0	0	0	96,6	96,6	96,6	96,6	96,6	96,6	69,8	69,8	69,8	69,8	69,8	69,8
CF_n	42	42	330	60	16	5,8	4	4,3	0,8	1,6	6,5	13,8	98	92,2	100	100	90,6	88,2
	44	45	50	70,9	30	33	6,8	7,8	2,4	4,6	15,2	19,8	86,3	80,4	99,6	94,5	82,4	79,2
RF_n	24,9	28,3	19,8	18,9	20,5	22,4	68,5	64	80,8	80,5	95,2	93,4	51	54,5	47,1	46,3	49,4	41,6
	24,5	29,6	29,1	25,7	22,9	24,9	82,7	54,3	40	71,7	94,4	88,7	54,5	54,1	62,7	49,8	49	41,6
T_n	40,1	45,8	44,5	45,9	36,9	33,4	83,9	83	86,6	83,4	85,3	81,4	80,4	76,1	78,8	78	80	76,1
	39,7	40,9	44,9	46,5	37,8	31,2	88,5	69,7	93,4	97,8	92,7	90,5	89	72,5	83,5	89,4	75,3	65,9
M_n	32,3	37,1	41,7	39,2	32,7	39,2	51	46,7	56,1	57,8	48,9	60,5	20	17,6	16,1	17,6	17,6	16,9
	34,3	34,3	39,2	34,3	39,2	39,2	59,6	58,3	59,1	56	56,5	59,1	18,4	18,8	17,3	19,6	18	17,3

Now, we now compare the HSV values of Image-O with those of Image-A, Image-B, and Image-C by using T-indistinguishability operator $E : \mathbb{R}^+ \times \mathbb{R}^+ \to [0,1]$ defined by $E(x,y) = \frac{1}{1+|x-y|}$ for all $x, y \in \mathbb{R}^+$. We calculate the aggregated similarity between Image-O and Image-A by using the following fuzzy relation, which is a T-indistinguishability operator:

$$E^*(O_{k_n}, A_{k_n})(i,j) = \prod_{l \in \{H,S,V\}} E(O_k^l(i,j), A_k^l(i,j)). \tag{4.2}$$

We show the results under this T-indistinguishability operator in Table 6.

Table 6. Indistinguishability of Image-O with Image-A.

	$n = 1$		$n = 2$		$n = 3$	
LE_n	0,0030	0,00003	0,00006	0,00003	0,00002	0,0089
	0,00214	0,00003	0,0000	0,00001	0,00001	0,0018
RE_n	0,00005	0,00001	0,00000	0,00014	0,00002	0,00009
	0,00003	0,00170	0,0000	0,0000	0,00002	0,0010
LF_n	0,20243	0,01507	0,05206	0,00676	0,03773	0,06077
	0,03271	0,01178	0,15949	0,08229	0,01378	0,2165
CF_n	0,00011	0,01230	0,00054	0,00240	0,03151	0,00011
	0,00266	0,07072	0,00315	0,00202	0,00009	0,00093
RF_n	0,00032	0,0000	0,00003	0,00486	0,00253	0,00007
	0,00191	0,01714	0,00095	0,00108	0,00095	0,0000
T_n	0,00131	0,00573	0,01778	0,05812	0,05503	0,12698
	0,04279	0,24038	0,00497	0,01192	0,01732	0,24038
M_n	0,00075	0,00037	0,00718	0,00146	0,06880	0,01493
	0,00389	0,00543	0,00134	0,00085	0,39683	0,01488

Similarly, we compute the aggregated similarities between Image-O and Image-B and between Image-O and Image-C by considering the HSV values and by using

$$E^*(O_{k_n}, B_{k_n})(i, j) = \prod_{l \in \{H,S,V\}} E(O^l_k(i, j), B^l_k(i, j)). \tag{4.3}$$

$$E^*(O_{k_n}, C_{k_n})(i, j) = \prod_{l \in \{H,S,V\}} E(O^l_k(i, j), C^l_k(i, j)). \tag{4.4}$$

We show the results obtained with these T-indistinguishability operators in Tables 7 and 8.

Table 7. Indistinguishability of Image-O with Image-B.

$E^*(O, B)$	$n = 1$		$n = 2$		$n = 3$	
LE_n	0.00003	0.00004	0.00007	0.00004	0.0000	0.0000
	0.00002	0.00001	0.00001	0.00006	0.00001	0.0000
RE_n	0.00005	0.00005	0.00002	0.00004	0.0010	0.00059
	0.0000	0.0000	0.0000	0.0000	0.00239	0.00010
LF_n	0.00019	0.00002	0.00005	0.00007	0.00003	0.00001
	0.00003	0.00003	0.00002	0.00002	0.00001	0.00001
CF_n	0.0000	0.00004	0.00002	0.0000	0.0000	0.0000
	0.00001	0.00002	0.00005	0.0000	0.00001	0.00001
RF_n	0.00023	0.00001	0.0000	0.00001	0.00001	0.00003
	0.00008	0.00064	0.00003	0.00003	0.00005	0.00001
T_n	0.00001	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
M_n	0.00011	0.0000	0.00002	0.00002	0.00001	0.00001
	0.00002	0.00007	0.00006	0.00006	0.00001	0.00002

Table 8. Indistinguishability of Image-O with Image-C.

$E^*(O,C)$	$n=1$		$n=2$		$n=3$	
LE_n	0.00002	0.00040	0.00008	0.00015	0.0000	0.00001
	0.00005	0.0000	0.00001	0.00008	0.00012	0.00001
RE_n	0.00001	0.00001	0.00016	0.00002	0.00002	0.00024
	0.0000	0.00001	0.00001	0.00001	0.00008	0.00002
LF_n	0.00001	0.00001	0.00002	0.00002	0.00002	0.00003
	0.00002	0.00001	0.00003	0.00002	0.00003	0.00005
CF_n	0.00002	0.0000	0.0000	0.0000	0.00001	0.00003
	0.00001	0.0000	0.0000	0.0000	0.0000	0.0000
RF_n	0.00001	0.0000	0.0000	0.00002	0.0000	0.00031
	0.00029	0.00006	0.00010	0.00002	0.00006	0.0000
T_n	0.00004	0.00001	0.00002	0.0000	0.0000	0.0000
	0.00001	0.00001	0.00004	0.00004	0.0000	0.0000
M_n	0.00015	0.00027	0.00007	0.00015	0.00029	0.00013
	0.00014	0.00032	0.00010	0.00028	0.00009	0.00009

As a final step, we calculate the similarity index (d) by taking $t = 7$ and $n = 3$ in Eq 4.1:

$$d = \frac{1}{21}\sum_{s=1}^{3}\sum_{k(s)\in \mathrm{CL}} \max\{k(s)(i,j) : i,j \in \{1,2,3\}\}.$$

We show the similarity results of Image-O between the images Image-A, Image-B, and Image-C in Table 9.

Table 9. Similarity of Image-O with Image-A, Image-B, and Image-C under the T-indistinguishability operator E.

Similarities with Image-O		
Image-A	Image-B	Image-C
0.07175	0.0019	0.0014

As seen in Table 9, Image-A is more similar to Image-O as expected. Therefore, we can conclude that the T-indistinguishability operators can be used to calculate in the image detection process.

We now consider partial T-indistinguishability operators instead of T-indistinguishability operators in the image detection process. Let us take the partial T-indistinguishability operator

$$E\colon \mathbb{R}^+ \times \mathbb{R}^+ \to [0,1]$$

defined by

$$E(x,y) = \frac{10}{10+\max\{x,y\}}, \quad \forall x,y \in \mathbb{R}^+.$$

We calculate the aggregated similarity between Image-O and Image-A using the following fuzzy relation, which is a partial T-indistinguishability operator:

$$E^{**}(O_{k_n}, A_{k_n})(i,j) = \prod_{l\in\{H,S,V\}} E(O_k^l(i,j),\, A_k^l(i,j)). \tag{4.5}$$

Then, we show the results under this partial T-indistinguishability operator in Table 10.

Table 10. Partial indistinguishability of Image-O with Image-A.

$E^{**}(O, A)$	$n = 1$		$n = 2$		$n = 3$	
LE_n	0.0091	0.0013	0.0057	0.0020	0.0023	0.0026
	0.0085	0.0015	0.0008	0.0023	0.0015	0.0027
RE_n	0.0017	0.0019	0.0010	0.0103	0.0021	0.0072
	0.0022	0.0019	0.0006	0.0004	0.0021	0.0039
LF_n	0.0132	0.0112	0.0105	0.0093	0.0108	0.0121
	0.0112	0.0098	0.0106	0.0109	0.0120	0.0075
CF_n	0.0015	0.0027	0.0032	0.0054	0.0044	0.0016
	0.0043	0.0037	0.0060	0.0056	0.0025	0.0025
RF_n	0.0035	0.0006	0.0006	0.0008	0.0017	0.0015
	0.0089	0.0053	0.0067	0.0166	0.0074	0.0005
T_n	0.0005	0.0007	0.0006	0.0006	0.0005	0.0005
	0.0005	0.0006	0.0005	0.0005	0.0005	0.0005
M_n	0.0057	0.0047	0.0062	0.0053	0.0060	0.0061
	0.0072	0.0057	0.0056	0.0043	0.0064	0.0057

Similarly, we compute the aggregated similarities between Image-O and Image-B and between Image-O and Image-C by considering the HSV values and by using

$$E^{**}(O_{k_n}, B_{k_n})(i, j) = \prod_{l \in \{H,S,V\}} E(O_k^l(i, j),\ B_k^l(i, j)), \tag{4.6}$$

$$E^{**}(O_{k_n}, C_{k_n})(i, j) = \prod_{l \in \{H,S,V\}} E(O_k^l(i, j),\ C_k^l(i, j)). \tag{4.7}$$

Then, we show the results obtained by using this partial T-indistinguishability operator in Tables 11 and 12.

Table 11. Partial indistinguishability of Image-O with Image-B.

$E^{**}(O, B)$	$n = 1$		$n = 2$		$n = 3$	
LE_n	0.0018	0.0020	0.0060	0.0065	0.0006	0.0005
	0.0015	0.0013	0.0011	0.0019	0.0024	0.0003
RE_n	0.0029	0.0017	0.0028	0.0029	0.0043	0.0082
	0.0008	0.0005	0.0004	0.0003	0.0097	0.0116
LF_n	0.0017	0.0016	0.0014	0.0015	0.0015	0.0015
	0.0015	0.0015	0.0013	0.0014	0.0022	0.0021
CF_n	0.0005	0.0015	0.0021	0.0011	0.0011	0.0009
	0.0010	0.0025	0.0039	0.0006	0.0017	0.0005
RF_n	0.0018	0.0011	0.0007	0.0009	0.0006	0.0023
	0.0072	0.0018	0.0017	0.0022	0.0016	0.0004
T_n	0.0003	0.0003	0.0003	0.0002	0.0002	0.0002
	0.0002	0.0003	0.0003	0.0003	0.0002	0.0002
M_n	0.0017	0.0003	0.0022	0.0027	0.0017	0.0016
	0.0032	0.0024	0.0024	0.0024	0.0018	0.0022

Table 12. Partial indistinguishability of Image-O with Image-C.

$E^{**}(O,C)$	$n=1$		$n=2$		$n=3$	
LE_n	0.0014	0.0034	0.0065	0.0053	0.0012	0.0013
	0.0024	0.0003	0.0015	0.0019	0.0096	0.0010
RE_n	0.0008	0.0012	0.0089	0.0040	0.0010	0.0114
	0.0008	0.0012	0.0011	0.0008	0.0059	0.0037
LF_n	0.0017	0.0019	0.0023	0.0025	0.0029	0.0031
	0.0019	0.0020	0.0028	0.0028	0.0029	0.0027
CF_n	0.0023	0.0008	0.0009	0.0015	0.0019	0.0017
	0.0016	0.0005	0.0011	0.0011	0.0008	0.0006
RF_n	0.0017	0.0004	0.0004	0.0005	0.0005	0.0043
	0.0045	0.0029	0.0041	0.0034	0.0030	0.0006
T_n	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
	0.0003	0.0004	0.0003	0.0003	0.0002	0.0003
M_n	0.0046	0.0047	0.0037	0.0041	0.0046	0.0032
	0.0046	0.0041	0.0034	0.0040	0.0036	0.0033

With a similar idea to the above process, finally, we calculate the similarity index by taking $t = 7$ and $n = 3$ in Eq (4.1):

$$d = \frac{1}{21} \sum_{s=1}^{3} \sum_{k(s) \in \mathrm{CL}} \max \{k(s)(i, j) : i, j \in \{1, 2, 3\}\}.$$

We show the similarity results of Image-O between the images Image-A, Image-B, and Image-C under the partial T-indistinguishability operator in Table 13.

Table 13. Similarity of Image-O with Image-A, Image-B, and Image-C under the partial T-indistinguishability operator E'.

Similarities with Image-O		
Image-A	Image-B	Image-C
0.0068	0.0030	0.0039

As demonstrated in Table 13, we find that Image-A gives the highest similarity to Image-O. This finding shows that partial T-indistinguishability operators can also be used in the image detection process.

4.3. Comparative study and discussion

In the literature, various techniques have been proposed for image recognition and image similarity assessment. Some of the most commonly used approaches are histogram-based methods [21], such as histogram intersection and cosine similarity, which compare the color distributions of images. Structural methods, including the structural similarity index (SSIM), evaluate similarities in terms of luminance, contrast, and structural information. More recently, deep learning-based approaches have gained considerable attention, where feature vectors extracted by convolutional neural networks are compared using similarity measures such as cosine similarity. In the present study, HSV color information is employed in conjunction with (partial) T-indistinguishability operators to assess the similarity between images. This choice enables the applicability of the proposed operators to be investigated within an image recognition framework. To show the robustness of the proposed method, histogram-based methods are selected for comparison since, similarly to the proposed approach, it relies on color information and compares the distributions of image features. So, it provides a suitable comparison for evaluating the effectiveness of the proposed (partial) T-indistinguishability-based image recognition method. The images considered in the application section (Image-O, Image-A, Image-B, and Image-C) were compared using histogram-based similarity measures (histogram intersection and cosine similarity). The obtained results are shown in Table 14.

Table 14. Similarity results obtained by histogram-based methods and the proposed methods.

Image Pair	Histogram Int.	Cosine Sim.	T-ind. op.	Partial T-ind. op.
Image-O vs. Image-A	0.7416	0.8680	0.0718	0.0068
Image-O vs. Image-B	0.1080	0.0207	0.0019	0.0030
Image-O vs. Image-C	0.0603	0.0062	0.0014	0.0039

The results (presented in Table 14) show that all methods identify Image-A as the image most similar to the Image-O. This observation is particularly important since Image-A is indeed visually the closest image to the reference image. While minor differences can be observed in the ordering of Image-B and Image-C, all methods clearly distinguish these images from Image-A by assigning substantially lower similarity values. Therefore, the results validate that the proposed approaches are consistent with those obtained by well-established histogram-based methods, supporting their applicability to image recognition problems. In addition to this, a comparison between the T-indistinguishability operator and the partial T-indistinguishability operator also shows some important observations. Both operators correctly identify Image-A as the most similar image to the reference image. The T-indistinguishability operator provides a stronger separation between the similarity values and ranks the images as Image-A, Image-B, and Image-C. The partial T-indistinguishability operator, while providing a slightly different ordering for Image-B and Image-C, still clearly distinguishes the most similar image from the remaining alternatives. Furthermore, due to its weaker structural requirements, the partial T-indistinguishability operator offers greater flexibility and enlarges the class of admissible operators that can be employed in image recognition problems.

5. Conclusions

In this paper, we introduce a new approach for constructing partial T-indistinguishability operators by applying an aggregation function to a set of finite partial T-indistinguishability operators. This method enables the development of a single, aggregated operator that captures similarity relations from multiple perspectives, thus enhancing the adaptability and accuracy of indistinguishability measurements in practical applications. We then apply these theoretical results to image recognition problems by establishing a method for robots tasked with image detection. The proposed technique allows robots to detect and distinguish objects in visual data, accounting for the partial similarities and variabilities often present in real-world environments. As a consequence, we can comment that the proposed method remains stable when the selected coordinate clusters adequately represent the characteristic regions of the images and when the compared images are obtained under similar illumination and color conditions. Under such circumstances, the extracted HSV values provide reliable information for evaluating image similarity through (partial) T-indistinguishability operators. However, the method may lose practical effectiveness when the selected clusters are not representative of the image content or when significant illumination changes, color distortions, or image degradations occur. Since the proposed method primarily relies on HSV color information, its performance may decrease in applications where shape, texture, or high-level semantic features play a dominant role.

Future research could explore optimizing the aggregation process to reduce computational overhead, making it more efficient for real-time applications. Another important direction is expanding the scope of residual domination beyond robotic vision to other areas such as data classification or decision-making in complex environments. Investigating the applicability of this method to dynamic and uncertain data environments would also be an exciting avenue for further study. Additionally, exploring the integration of residual domination with different metric structures might lead to more robust solutions, especially in domains where distance and similarity measures play a critical role. Exploring hybrid models that combine residual domination with other advanced techniques could yield even more robust results in a wider range of applications.

Declarations

Ethical approval: The author declares that there is no ethical approval required for this study, as there was no direct contact with humans and animals.

Use of Generative-AI tools declaration

The author declares that no Artificial Intelligence (AI) tools were used in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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