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Research article

Local V-Moreau envelope of nonconvex functions

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Abstract: Recently, the author in 2024 studied the V-Moreau envelope $e^V_{\lambda,S} f$ of nonconvex functions f with index $\lambda > 0$ and associated with closed sets S in smooth Banach spaces. In this paper, we developed a corresponding local theory for the local V-Moreau envelope of f on a neighborhood of a point $\bar{x}$ in the domain of f. A local concept of generalized projection is introduced in connection with the local V-Moreau envelope. We investigated the Lipschitz continuity, Fréchet differentiability, and V-proximal subdifferentiability of the local V-Moreau envelope of f. Moreover, we established the local Hölder continuity of the local generalized projection for the class of V-primal lower nice functions introduced and studied in 2020 by Bounkhel et al. Our results extended several known theorems from Hilbert spaces to a more general framework of smooth Banach spaces.

Keywords: local V-Moreau envelope; generalized (f, λ)-projection; p-uniformly convex Banach spaces; q-uniformly smooth Banach spaces; V-primal lower nice functions
Mathematics Subject Classification: 49J52, 49J53, 49J50

1. Introduction

The Moreau envelope $e_\lambda f(x)$ of a function f with index $\lambda > 0$ on a Hilbert space H is defined by

$$e_\lambda f(x) := \inf_{s\in H}\left[f(s) + \frac{1}{2\lambda}\|s - x\|^2\right], \quad \forall x \in H. \tag{1.1}$$

The study of the differentiability (Fréchet and Gâteaux) of $e_\lambda f$ is of considerable interest in variational analysis and its applications to unconstrained optimization. We refer the reader to [5, 20, 24] and the references therein.

When f is replaced by $f + \psi_S$ in (1.1) with ψ_S as the indicator function of a closed set $S \subset H$, the resulting Moreau envelope is denoted by $e_{\lambda,S} f(x)$ and takes the form

$$e_{\lambda,S} f(x) := \inf_{s\in S}\left[f(s) + \frac{1}{2\lambda}\|s - x\|^2\right], \quad \forall x \in H. \tag{1.2}$$

The associated proximal mapping $P_{\lambda,S} f : H \rightrightarrows S$ is defined by

$$P_{\lambda,S} f(x) := \left\{\bar{x} \in S : e_{\lambda,S} f(x) = f(\bar{x}) + \frac{1}{2\lambda}\|x - \bar{x}\|^2\right\}. \tag{1.3}$$

The study of the differentiability of $e_{\lambda,S} f$ along with the nonemptiness, single-valuedness, and continuity of $P_{\lambda,S} f$ plays a fundamental role in variational analysis and constrained optimization. These properties are closely related to the geometric nature of the set S and have been the subject of extensive research (see, e.g., [17, 19, 29]).

In the particular case where $f \equiv 0$ on S, the Moreau envelope reduces to the squared distance function $\frac{1}{2\lambda}d_S^2$, and the proximal mapping becomes the well-known projection operator P_S onto S. This special case has been thoroughly investigated for closed convex sets in Hilbert spaces (see, e.g., [18, 28]). The nonconvex case has also been studied; see, for instance, [17, 27] and the references therein. The extension of these results from Hilbert to Banach spaces was addressed straightforwardly in [6]. A different extension approach, based on replacing the squared norm by the functional V, was introduced and studied in [10, 11]. Specifically, for a Banach space X with dual X^*, the V-Moreau envelope of f with index $\lambda > 0$ is defined from X^* to X by

$$e_{\lambda,S} f(x^*) := \inf_{s\in S}\left[f(s) + \frac{1}{2\lambda}V(x^*, s)\right], \quad x^* \in X^*, \tag{1.4}$$

where $V : X^* \times X \to \mathbb{R}$ is given by

$$V(x^*, x) = \|x^*\|_{X^*}^2 - 2\langle x^*, x\rangle + \|x\|_X^2. \tag{1.5}$$

The associated proximal mapping, called the (f, λ)-generalized projection and denoted by $\pi_S^{f,\lambda}$, has been studied in [10]. The differentiability of $e_{\lambda,S} f$ as well as the nonemptiness, single-valuedness, and continuity of $\pi_S^{f,\lambda}$ were investigated in [10, 11]. Several other properties of $e_{\lambda,S} f$ and $\pi_S^{f,\lambda}$ have also been established. When $f \equiv 0$ on S, this construction reduces to the generalized projection π_S, first introduced for convex sets in [3] and later extended to nonconvex sets in [9]. Applications of π_S to variational inequalities have been widely studied; see [2, 4, 23] for convex sets and [9] for the nonconvex case.

In this paper, we develop a local version of the theory introduced in [11]. More precisely, for a point $\bar{x} \in \operatorname{dom} f$ and a radius $\rho > 0$, we consider

$$e^V_{\lambda,\bar{x},\rho} f(x^*) := \inf_{s\in \bar{x}+\rho\mathbb{B}}\left[f(s) + \frac{1}{2\lambda}V(x^*, s)\right], \quad x^* \in X^* \tag{1.6}$$

and the associated local generalized projection

$$\pi^{f,\lambda}_{\bar{x},\rho}(x^*) := \left\{\bar{u} \in X : e^V_{\lambda,\bar{x},\rho} f(x^*) = f(\bar{u}) + \frac{1}{2\lambda}V(x^*, \bar{u})\right\}. \tag{1.7}$$

We first prove the local Lipschitz continuity of $e^V_{\lambda,\bar{x},\rho}f$ in Proposition 2.1. Then, in Theorem 2.1, we establish the existence of the (f,λ)-generalized projection whenever the local Moreau envelope $e^V_{\lambda,\bar{x},\rho}f$ is Fréchet subdifferentiable. These results hold in any reflexive Banach space with a smooth dual norm. The remaining results are proved in the setting of uniformly convex and uniformly smooth Banach spaces. We investigate the relationship between the (f,λ)-generalized projection and the V-proximal subdifferential of f in Propositions 2.3 and 3.5.

The main result of the paper is presented in Theorem 3.3, where we show that for the class of V-primal lower nice functions f, the (f,λ)-generalized projection is locally nonempty, single-valued, and Hölder continuous. We also establish several properties for this class of functions as well as for the class of V-proximal sets. Our results extend and unify a number of theorems from [12, 13, 26]. Before proceeding, we collect in Table 1 the main symbols and notation for the reader's convenience.

Table 1. Symbols used in the paper.

Symbols	Description
$e^V_{\lambda,\bar{x},\rho}f(x^*)$	Local V-Moreau envelope of f
$V(x^*, s)$	V-functional
$\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$	(f,λ)-generalized projection over $\bar{x}+\rho\mathbb{B}$
$\mathbb{B}$	Closed unit ball in X
$\mathbb{B}_*$	Closed unit ball in X^*
$\|\cdot\|$	Norm on both X and X^*
$\langle\cdot,\cdot\rangle$	Duality pairing between X and X^*
J	Normalized duality mapping on X
J^*	Normalized duality mapping on X^*
$\partial^F f(\bar{x})$	Fréchet subdifferential of f at $\bar{x}$
$\partial^\pi f(x)$	V-proximal subdifferential of f at $\bar{x}$
$\partial^\pi_G f(x)$	Geometric V-proximal subdifferential of f at $\bar{x}$
$\partial^{L\pi} f(x)$	Limiting V-proximal subdifferential at $\bar{x}$
$\partial^{L\pi}_G f(x)$	Limiting geometric V-proximal subdifferential at $\bar{x}$
$N^\pi(\operatorname{epi} f,(x,f(x)))$	Proximal normal cone to the epigraph of f at $(\bar{x},f(\bar{x}))$
$\mathcal{N}(J(\bar{x}),\delta_1,\delta_2)$	Special (δ_1,δ_2)-open neighborhood of $J(\bar{x})$ in X^*
$\mathcal{N}_{J(\bar{x})}(\epsilon)$	Special ϵ-open neighborhood of $J(\bar{x})$ in X^*
ψ_S	Indicator function
$\pi_S(x^*)$	Generalized projection of x^* onto S
$\pi^g_S(x^*)$	g-generalized projection of x^* onto S
$P_\lambda f$	Proximal mapping of f
$e_\lambda f$	Global Moreau envelope of f
Proj_S	Metric projection onto S
$x_n \to^f x$	Convergence in the sense $x_n \to x$ with $f(x_n)\to f(x)$

2. Preliminaries

In the first part of our work, we assume that X is a reflexive Banach space.

Definition 2.1. *Consider a function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a point $\bar{x} \in domf$ and two positive numbers $\rho, \lambda > 0$. We define the local V-Moreau envelope of f of index λ over the neighborhood $\bar{x} + \rho\mathbb{B}$ as follows:*

$$e^V_{\lambda,\bar{x},\rho} f(x^*) := \inf_{s \in \bar{x}+\rho\mathbb{B}} \left[f(s) + \frac{1}{2\lambda} V(x^*, s) \right], \quad \text{for any } x^* \in X^* \tag{2.1}$$

where $V(x^*, s) := \|x^*\|^2 - 2\langle x^*, s\rangle + \|s\|^2$.

We also define for a given element x^* in the dual space X^*, the generalized (f, λ)-projection of x^* on $\bar{x} + \rho\mathbb{B}$ as follows:

$$\pi^{f,\lambda}_{\bar{x},\rho}(x^*) := \left\{ \bar{u} \in X : e^V_{\lambda,\bar{x},\rho} f(x^*) = f(\bar{u}) + \frac{1}{2\lambda} V(x^*, \bar{u}) \right\}. \tag{2.2}$$

Before going further in our study, we state some particular cases of the local V-Moreau envelope (2.1) and the generalized (f, λ)-projection (2.2), showing the motivations and the importance of the study of the function $e^V_{\lambda,\bar{x},\rho} f$ and the operator $\pi^{f,\lambda}_{\bar{x},\rho}$:

Banach space cases:

1) If $f = \psi_S$ for some closed nonempty set S, then

$$e^V_{\lambda,\bar{x},\rho} f(x^*) = \inf_{s \in [\bar{x}+\rho\mathbb{B}]\cap S} \left[\frac{1}{2\lambda} V(x^*, s) \right] = \frac{1}{2\lambda} \inf_{s \in [\bar{x}+\rho\mathbb{B}]\cap S} [V(x^*, s)],$$

and

$$\pi^{f,\lambda}_{\bar{x},\rho}(x^*) = \left\{ \bar{u} \in S : V(x^*, \bar{u}) = \inf_{s \in [\bar{x}+\rho\mathbb{B}]\cap S} V(x^*, s) \right\};$$

that is, $\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$ coincides with the generalized projection $\pi_{[\bar{x}+\rho\mathbb{B}]\cap S}(x^*)$ of x^* over $[\bar{x} + \rho\mathbb{B}] \cap S$.

2) If $\rho = +\infty$, then $e^V_{\lambda,\bar{x},\rho} f$ becomes the global V-Moreau envelope of f with index λ, introduced and studied in [10, 11]; that is,

$$e^V_{\lambda,\bar{x},\rho} f(x^*) := \inf_{s \in X} \left[f(s) + \frac{1}{2\lambda} V(x^*, s) \right] = e^V_\lambda f(x^*) \quad \text{for any } x^* \in X^*.$$

3) If $\rho = +\infty$, and $f = \psi_S$ for some closed nonempty set S, then $\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$ coincides with the generalized projection $\pi_S(x^*)$ of x^* over the set S studied in [4, 22, 23] for closed convex sets and in [9, 12] for nonconvex sets.

4) If $\lambda = \frac{1}{2}$, $\rho = +\infty$, and $f = g + \psi_S$ for some closed nonempty set S and for some lower semicontinuous (l.s.c.) function g, then $e^V_{\lambda,\bar{x},\rho} f$ becomes

$$e^V_{\lambda,\bar{x},\rho} f(x^*) := \inf_{s \in S} [g(s) + V(x^*, s)] \quad \text{for any } x^* \in X^*,$$

and $\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$ coincides with the g-generalized projection $\pi^g_S(x^*)$ of x^* over the set S studied in [31, 32] for closed convex sets and in [10, 11] for nonconvex sets.

5) If $\rho = +\infty$, and $f = g + \psi_S$ for some closed nonempty set S and some l.s.c. function g, then $e^V_{\lambda,\bar{x},\rho} f$ becomes

$$e^V_{\lambda,\bar{x},\rho} f(x^*) := \inf_{s\in S}\left[g(s) + \frac{1}{2\lambda}V(x^*, s)\right] \qquad \text{for any } x^* \in X^*,$$

and $\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$ coincides with the (g, λ)-generalized projection $\pi^{g,\lambda}_S(x^*)$ of x^* over the set S introduced and studied in [10, 11].

Hilbert space cases: If X is a Hilbert space, then the functional V becomes the norm squared, and then, we can derive many different particular cases. We state for instance only the following two cases.

1) If $\rho = +\infty$, then $\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$ coincides with the well-known Moreau envelope of f with index λ; that is,

$$e^V_{\lambda,\bar{x},\rho} f(x^*) := \inf_{s\in X}\left[f(s) + \frac{1}{2\lambda}\|x^* - s\|^2\right] \qquad \text{for any } x^* \in X,$$

and $\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$ coincides with the proximal mapping $P_\lambda f$ associated with the Moreau envelope $e_\lambda f$. For more details and existing studies on $P_\lambda f$ and $e_\lambda f$, we refer the reader to [24, 25].
2) If $\rho = +\infty$, and $f = \psi_S$ for some closed nonempty set S, then $\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$ coincides with the well-known metric projection $\mathrm{Proj}_S(x^*)$ of x^* over the set S; see, for instance, [18, 20, 21].

The special cases presented above serve as the primary motivation for our work. To help the reader situate our contributions within the existing body of research, we summarize in Tables 2 and 3 the key settings and the corresponding references from the literature, thereby highlighting the connections and distinctions between our framework and previous studies.

Table 2. Comparison of references for special cases of $e^V_{\lambda,\bar{x},\rho} f(x^*)$ the local V-Moreau envelope of f with the index λ.

Special case	Mathematical setting	References
Local V-Moreau envelope	$\rho \in (0, \infty]$, general f, Banach spaces	Introduced here
Global V-Moreau envelope	$\rho = +\infty$, general f, Banach spaces	[10, 11]
Moreau envelope & proximal mapping	Hilbert space, $\rho = +\infty$	[24, 25]
Metric projection onto a set	Hilbert space, $\rho = +\infty$, $f = \psi_S$	[18, 20, 21]

Table 3. Comparison of references for special cases of $\pi^{f,\lambda}_{\bar{x},\rho}$ the generalized (f, λ)-projection onto $\bar{x} + \rho\mathbb{B}$.

Special case	Mathematical setting	References
$\pi^{f,\lambda}_{\bar{x},\rho}$: Generalized (f,λ)-projection onto $\bar{x} + \rho\mathbb{B}$	$\rho \in (0, \infty]$, general f, general $\lambda > 0$, Banach spaces	Introduced here
π_S: Generalized projection onto a convex set S	$\rho = +\infty$, $f = \psi_S$, S convex, Banach spaces	[2–4, 22, 23]
π_S: Generalized projection onto a nonconvex set S	$\rho = +\infty$, $f = \psi_S$, S nonconvex, Banach spaces	[9, 12]
π^g_S: Generalized g-projection onto a convex set S	$\lambda = \frac{1}{2}$, $\rho = +\infty$, $f = g + \psi_S$, S convex, Banach spaces	[31, 32]
π^g_S: Generalized g-projection onto a nonconvex set S	$\lambda = \frac{1}{2}$, $\rho = +\infty$, $f = g + \psi_S$, S nonconvex, Banach spaces	[10, 11]
$\pi^{g,\lambda}_S$: (g, λ)-generalized projection onto S	$\rho = +\infty$, $f = g + \psi_S$, general $\lambda > 0$, Banach spaces	[10, 11]
Proj_S: Metric projection onto a set S	$\rho = +\infty$, $f = \psi_S$, S convex, Hilbert spaces	[18, 20, 21]

We begin our study, by proving the Lipschitz continuity of $e^V_{\lambda,\bar{x},\rho} f$ over any neighborhood of $J(\bar{x})$. Here, J denotes the normalized duality mapping on the space X; that is, $J : X \rightrightarrows X^*$ is defined by

$$J(x) = \left\{ j(x) \in X^* : \langle j(x), x\rangle = \|x\|^2 = \|j(x)\|^2 \right\},$$

where $\|\cdot\|$ stands for both norms on X and X^*. The original definition of the duality mapping is given as the subdifferential in the sense of convex analysis of the norm of X (see, for instance, [1]). Similarly, we define J^* on X^*. Many properties of J and J^* are well-known, and we refer the reader, for instance, to [30]. We state here the following properties needed in our framework.

- If X is reflexive, then J is onto,
- J is a continuous operator in smooth Banach spaces,
- If the dual norm is smooth, that is, the norm on X^* is smooth, then X is strictly convex, and hence, J is one-to-one,
- If X is a reflexive strictly convex space with strictly convex dual space X^* and if $J^* : X^* \rightrightarrows X$ is a normalized duality mapping in X^*, then $J^{-1} = J^*$,
- J is the identity operator in Hilbert spaces.

For the proofs of the these properties and for more properties, we refer to [1, 30].

Proposition 2.1. *Let X be a reflexive Banach space and $\bar{x} \in domf$ with $\rho > 0$. Assume that, for some $\delta > 0$, f is bounded below by $\beta \in \mathbb{R}$ on $\bar{x} + \delta\mathbb{B}$. Then, the function $e^V_{\lambda,\bar{x},\rho} f$ is Lipschitz on $J(\bar{x}) + \delta\mathbb{B}_*$ with constant Lipschitz $K_{\bar{x},\rho,\lambda,\delta} := \dfrac{2\|\bar{x}\| + \rho + \delta}{\lambda}$; that is,*

$$|e^V_{\lambda,\bar{x},\rho} f(y^*) - e^V_{\lambda,\bar{x},\rho} f(z^*)| \le K_{\bar{x},\rho,\lambda,\delta}\|y^* - z^*\|, \quad \forall y^*, z^* \in J(\bar{x}) + \delta\mathbb{B}_*.$$

Proof. Let any $y^*, z^* \in J(\bar{x}) + \delta\mathbb{B}$. Let $\epsilon > 0$ be any positive real number. By definition of the infimum in the expression of $e^V_{\lambda,\bar{x},\rho} f(y^*)$, there exists $y_\epsilon \in \bar{x} + \rho\mathbb{B}$ such that

$$e^V_{\lambda,\bar{x},\rho} f(y^*) \le f(y_\epsilon) + \frac{1}{2\lambda} V(y^*, y_\epsilon) < e^V_{\lambda,\bar{x},\rho} f(y^*) + \epsilon.$$

Then,

$$\begin{aligned}
e^V_{\lambda,\bar{x},\rho} f(z^*) - e^V_{\lambda,\bar{x},\rho} f(y^*) &< e^V_{\lambda,\bar{x},\rho} f(z^*) - f(y_\epsilon) - \frac{1}{2\lambda} V(y^*, y_\epsilon) + \epsilon \\
&< f(y_\epsilon) + \frac{1}{2\lambda} V(z^*, y_\epsilon) - f(y_\epsilon) - \frac{1}{2\lambda} V(y^*, y_\epsilon) + \epsilon \\
&< \frac{1}{2\lambda} \left[V(z^*, y_\epsilon) - V(y^*, y_\epsilon)\right] + \epsilon \\
&< \frac{1}{2\lambda} \left[\|z^*\|^2 - \|y^*\|^2 - 2\langle z^* - y^*, y_\epsilon\rangle\right] + \epsilon \\
&< \frac{1}{2\lambda} (\|z^*\| + \|y^*\| + 2\|y_\epsilon\|)\|z^* - y^*\| + \epsilon.
\end{aligned}$$

Because $y^*, z^* \in J(\bar{x}) + \delta\mathbb{B}$, we have $\|y^*\| \le \|\bar{x}\| + \delta$ and $\|z^*\| \le \|\bar{x}\| + \delta$. Also, we have $\|y_\epsilon\| \le \|\bar{x}\| + \rho$. Therefore, we obtain

$$e^V_{\lambda,\bar{x},\rho} f(z^*) - e^V_{\lambda,\bar{x},\rho} f(y^*) < \frac{1}{2\lambda} (\|z^*\| + \|y^*\| + 2\|y_\epsilon\|) \|z^* - y^*\| + \epsilon < \frac{1}{2\lambda} (4\|\bar{x}\| + 2\rho + 2\delta) \|z^* - y^*\| + \epsilon.$$

Taking $\epsilon \to 0$ and interchanging the roles of z^* and y^*, we get

$$|e^V_{\lambda,\bar{x},\rho} f(z^*) - e^V_{\lambda,\bar{x},\rho} f(y^*)| \le \frac{1}{\lambda} (2\|\bar{x}\| + \rho + \delta) \|z^* - y^*\|, \text{ for any } y^*, z^* \in J(\bar{x}) + \delta\mathbb{B}_*.$$

This completes the proof. □

Remark 2.1. *For the case $\delta = \rho > 0$, the Lipschitz constant becomes $K_{\bar{x},\rho,\lambda} = \dfrac{2(\|\bar{x}\| + \rho)}{\lambda}$.*

We use the Lipschitz continuity of $e^V_{\lambda,\bar{x},\rho}f$ over any neighborhood of $J(\bar{x})$ proved above to prove our first main result on the Fréchet differentiability of $e^V_{\lambda,\bar{x},\rho}f$. To do that, we recall the definition of the Fréchet subdifferential (see, for instance, [7]). For a l.s.c. function $f : X \to \mathbb{R} \cup \{+\infty\}$ at $\bar{x} \in dom f$, the Fréchet subdifferential $\partial^F f(\bar{x})$ of f at $\bar{x}$ is defined as the set of all $x^* \in X^*$ such that for any $\epsilon > 0$, there exists $\delta > 0$ satisfying

$$\langle x^*, x - \bar{x}\rangle \leq f(x) - f(\bar{x}) + \epsilon\|x - \bar{x}\|, \quad \forall x \in \bar{x} + \delta\mathbb{B}.$$

Theorem 2.1. *Assume that X is a reflexive Banach space, let $f : X \to \mathbb{R} \cup \{+\infty\}$ be any l.s.c. function, and let $\bar{x} \in dom f$. If the local V-Moreau envelope $e^V_{\lambda,\bar{x},\rho}f$ is Fréchet subdifferentiable at some x^* around $J(\bar{x})$, then the generalized (f, λ)-projection of x^* exists and is unique, and $e^V_{\lambda,\bar{x},\rho}f$ is Fréchet differentiable at x^*. In other words, if $x^* \in J(\bar{x}) + \delta\mathbb{B}_*$with $\partial^F e^V_{\lambda,\bar{x},\rho}f(x^*) \neq \emptyset$, then $\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$ is a singleton, and $e^V_{\lambda,\bar{x},\rho}f$ is Fréchet differentiable at x^* with $\nabla^F e^V_{\lambda,\bar{x},\rho}f(x^*) = \lambda^{-1}[J^*(x^*) - \pi^{f,\lambda}_{\bar{x},\rho}(x^*)]$.*

Proof. Let $x^* \in J(\bar{x}) + \delta\mathbb{B}_*$ with $\partial^F e^V_{\lambda,\bar{x},\rho}f(x^*) \neq \emptyset$, and let $y \in \partial^F e^V_{\lambda,\bar{x},\rho}f(x^*)$. Let $\epsilon > 0$. Then, by the definition of $\partial^F e^V_{\lambda,\bar{x},\rho}f(x^*)$, there exists $\delta' > 0$ such that for any $t \in (0, \delta')$ and any $b^* \in \mathbb{B}_*$, we have

$$\langle y, tb^*\rangle \leq e^V_{\lambda,\bar{x},\rho}f(x^* + tb^*) - e^V_{\lambda,\bar{x},\rho}f(x^*) + \epsilon t. \tag{2.3}$$

By the definition of $e^V_{\lambda,\bar{x},\rho}f(x^*)$, for any $n \geq 1$, there exists some $y_n \in \bar{x} + \rho\mathbb{B}$ such that

$$e^V_{\lambda,\bar{x},\rho}f(x^*) \leq f(y_n) + \frac{1}{2\lambda}V(x^*, y_n) < e^V_{\lambda,\bar{x},\rho}f(x^*) + \frac{t}{n}. \tag{2.4}$$

Therefore,

$$\begin{aligned}
\langle y, tb^*\rangle &\leq f(y_n) + \frac{1}{2\lambda}V(x^* + tb^*, y_n) - f(y_n) - \frac{1}{2\lambda}V(x^*, y_n) + \frac{t}{n} + \epsilon t \\
&\leq \frac{1}{2\lambda}\left[V(x^* + tb^*, y_n) - V(x^*, y_n)\right] + \frac{t}{n} + \epsilon t \\
&\leq \frac{1}{2\lambda}\left[\|x^* + tb^*\|^2 - \|x^*\|^2 - 2\langle tb^*, y_n\rangle\right] + \frac{t}{n} + \epsilon t.
\end{aligned}$$

Thus,

$$\langle y + \frac{1}{\lambda}y_n, b^*\rangle \leq \frac{1}{2\lambda}\left[\frac{\|x^* + tb^*\|^2 - \|x^*\|^2}{t}\right] + \frac{1}{n} + \epsilon.$$

Because the norm of the dual space is smooth, we can take the limit $t \to 0^+$ to get

$$\langle y + \frac{1}{\lambda}y_n, b^*\rangle \leq \frac{1}{\lambda}\langle J^*x^*, b^*\rangle + \frac{1}{n} + \epsilon,$$

and hence,

$$\langle y + \frac{1}{\lambda}[y_n - J^*x^*], b^*\rangle \leq \frac{1}{n} + \epsilon, \quad \forall b^* \in \mathbb{B}_*, \forall\epsilon > 0, \forall n \geq 1.$$

This ensures that $\lim_{n\to\infty} \|y + \frac{1}{\lambda}[y_n - J^*x^*]\| = 0$; that is, $y_n \to \bar{y} := J^*x^* - \lambda y$ as $n \to \infty$. Because $y_n \in \bar{x} + \rho\mathbb{B}$, we have $\bar{y} \in \bar{x} + \rho\mathbb{B}$. By taking the limit as $n \to \infty$ in inequality (2.4), we obtain

$$e^V_{\lambda,\bar{x},\rho} f(x^*) = f(\bar{y}) + \frac{1}{2\lambda} V(x^*, \bar{y}),$$

which means by definition that $\bar{y} \in \pi^{f,\lambda}_{\bar{x},\rho}(x^*)$. Hence, the set $\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$ is nonempty. The proof of the fact that $\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$ is a singleton follows easily from inequality (2.4). Indeed, let $\bar{z}$ be another element of $\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$, and we have to prove that $\bar{z} = \bar{y}$. Then, $\bar{z} \in \bar{x} + \rho\mathbb{B}$ with $f(\bar{z}) + \frac{1}{2\lambda} V(x^*, \bar{z}) = e^V_{\lambda,\bar{x},\rho} f(x^*)$. Therefore, by (2.3), we may write

$$\begin{aligned}
\langle y, tb^* \rangle &\le e^V_{\lambda,\bar{x},\rho} f(x^* + tb^*) - e^V_{\lambda,\bar{x},\rho} f(x^*) + \epsilon t \\
&\le e^V_{\lambda,\bar{x},\rho} f(x^* + tb^*) - \left[f(\bar{z}) + \frac{1}{2\lambda} V(x^*, \bar{z}) \right] + \epsilon t \\
&\le \left[f(\bar{z}) + \frac{1}{2\lambda} V(x^* + tb^*, \bar{z}) \right] - \left[f(\bar{z}) + \frac{1}{2\lambda} V(x^*, \bar{z}) \right] + \epsilon t \\
&\le \frac{1}{2\lambda} \left[V(x^* + tb^*, \bar{z}) - V(x^*, \bar{z}) \right] + \epsilon t \\
&\le \frac{1}{2\lambda} \left[\|x^* + tb^*\|^2 - \|x^*\|^2 - 2\langle tb^*, \bar{z} \rangle \right] + \epsilon t.
\end{aligned}$$

Thus,

$$\langle y + \frac{\bar{z}}{\lambda}, b^* \rangle \le \frac{1}{2\lambda} \left[\frac{\|x^* + tb^*\|^2 - \|x^*\|^2}{t} \right] + \epsilon.$$

We use the smoothness of the dual norm to take the limit as $t \to 0^+$ and to get

$$\langle y + \frac{\bar{z}}{\lambda}, b^* \rangle \le \frac{1}{\lambda} \langle J^*x^*, b^* \rangle + \epsilon,$$

which gives

$$\langle y + \frac{\bar{z}}{\lambda} - \frac{1}{\lambda} J^*x^*, b^* \rangle \le \epsilon, \quad \forall b^* \in \mathbb{B}_*, \forall \epsilon > 0.$$

Taking the supremum over $b^* \in \mathbb{B}_*$, we obtain

$$\|y + \frac{\bar{z}}{\lambda} - \frac{1}{\lambda} J^*x^*\| \le \epsilon, \quad \forall \epsilon > 0.$$

Taking now the limit as $\epsilon \to 0$ gives $\|y + \frac{\bar{z}}{\lambda} - \frac{1}{\lambda} J^*x^*\| = 0$, and hence, $\bar{z} = J^*x^* - \lambda y$, which means that $\bar{z} = \bar{y}$, and hence, the set $\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$ is a singleton. Let us now turn to prove the Fréchet differentiability of $e^V_{\lambda,\bar{x},\rho} f$ at $x^* \in J(\bar{x}) + \delta\mathbb{B}_*$. By Proposition 2.1, the function $e^V_{\lambda,\bar{x},\rho} f$ is Lipschitz over $J(\bar{x}) + \delta\mathbb{B}_*$, and so it is enough to prove the Gâteaux differentiablity of $e^V_{\lambda,\bar{x},\rho} f$ at $x^* \in J(\bar{x}) + \delta\mathbb{B}_*$; that is, we prove the existence of the following limit for all $v^* \in X^*$:

$$\nabla^G e^V_{\lambda,\bar{x},\rho} f(x^*, v^*) := \lim_{t \to 0^+} \frac{e^V_{\lambda,\bar{x},\rho} f(x^* + tv^*) - e^V_{\lambda,\bar{x},\rho} f(x^*)}{t} \tag{2.5}$$

with the function $v^* \mapsto \nabla^G e^V_{\lambda,\bar{x},\rho} f(x^*, v^*)$ as linear and continuous on X^*. To do that, first, we deduce from the above reasoning that $\partial^F e^V_{\lambda,\bar{x},\rho} f(x^*)$ and $\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$ are singleton and satisfy

$$\partial^F e^V_{\lambda,\bar{x},\rho} f(x^*) = \left\{ \frac{1}{\lambda} \left[J^* x^* - \pi^{f,\lambda}_{\bar{x},\rho}(x^*) \right] \right\}. \tag{2.6}$$

Indeed, we proved previously that for any $y \in \partial^F e^V_{\lambda,\bar{x},\rho} f(x^*)$, we have $J^* x^* - \lambda y \in \pi^{f,\lambda}_{\bar{x},\rho}(x^*)$, which is equivalent to $y \in \frac{1}{\lambda} \left[J^* x^* - \pi^{f,\lambda}_{\bar{x},\rho}(x^*) \right]$, and hence, we get $\partial^F e^V_{\lambda,\bar{x},\rho} f(x^*) \subset \frac{1}{\lambda} \left[J^* x^* - \pi^{f,\lambda}_{\bar{x},\rho}(x^*) \right]$. Finally, by the single-valuedness of $\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$, we deduce that the equality in (2.6) holds.

Let $v^* \in X^*$ with $v^* \neq 0$. Set $\bar{y} := \pi^{f,\lambda}_{\bar{x},\rho}(x^*)$. Thus, we have $e^V_{\lambda,\bar{x},\rho} f(x^*) = f(\bar{y}) + \frac{1}{2\lambda} V(x^*, \bar{y})$. By the definition of the Fréchet subdifferential, there exists $\delta' > 0$ such that for any $t \in (0, \frac{\delta'}{\|v^*\|})$, we have

$$\langle \frac{1}{\lambda} \left[J^* x^* - \bar{y} \right], (x^* + t v^*) - x^* \rangle \leq e^V_{\lambda,\bar{x},\rho} f(x^* + t v^*) - e^V_{\lambda,\bar{x},\rho} f(x^*) + \epsilon t.$$

Hence,

$$t^{-1} \left[e^V_{\lambda,\bar{x},\rho} f(x^* + t v^*) - e^V_{\lambda,\bar{x},\rho} f(x^*) \right] - \langle \frac{1}{\lambda} \left[J^* x^* - \bar{y} \right], v^* \rangle \geq -\epsilon,$$

and hence, for any $t \in (0, \frac{\delta'}{\|v^*\|})$ and any $\epsilon > 0$,

$$\liminf_{t \to 0^+} t^{-1} \left[e^V_{\lambda,\bar{x},\rho} f(x^* + t v^*) - e^V_{\lambda,\bar{x},\rho} f(x^*) \right] \geq \langle \frac{1}{\lambda} \left[J^* x^* - \bar{y} \right] v^* \rangle - \epsilon. \tag{2.7}$$

On the other hand, we have, by the definition of $e^V_{\lambda,\bar{x},\rho} f$,

$$\begin{aligned} t^{-1} \left[e^V_{\lambda,\bar{x},\rho} f(x^* + t v^*) - e^V_{\lambda,\bar{x},\rho} f(x^*) \right] &\leq t^{-1} \left[\frac{1}{2\lambda} V(x^* + t v^*, \bar{y}) - \frac{1}{2\lambda} V(x^*, \bar{y}) \right] \\ &\leq \frac{1}{2\lambda} t^{-1} \left[V(x^* + t v^*, \bar{y}) - V(x^*, \bar{y}) \right], \end{aligned}$$

and so, by taking $t \to 0^+$,

$$\limsup_{t \to 0^+} t^{-1} \left[e^V_{\lambda,\bar{x},\rho} f(x^* + t v^*) - e^V_{\lambda,\bar{x},\rho} f(x^*) \right] \leq \frac{1}{2\lambda} \left[2 \langle J^* x^* - \bar{y}, v^* \rangle \right]. \tag{2.8}$$

Combining this inequality (2.8) with (2.7) and taking $\epsilon \to 0^+$, we obtain the existence of the limit in (2.5) and

$$\nabla^G e^V_{\lambda,\bar{x},\rho} f(x^*, v^*) = \frac{1}{\lambda} \left[\langle J^* x^* - \bar{y}, v^* \rangle \right], \quad \forall v^* \in X^*.$$

Because the function $v^* \mapsto \nabla^G e^V_{\lambda,\bar{x},\rho} f(x^*, v^*)$ is linear and continuous on X^*, we get the Gâteaux differentiability of $e^V_{\lambda,\bar{x},\rho} f$ at x^*, and so it is Fréchet differentiable at x^*, and hence, the proof is complete. □

From now on, the space X is assumed to be a q-uniformly convex ($q \geq 2$) and p-uniformly smooth Banach space ($1 < p \leq 2$). First, we recall the definition of q-uniform convexity and p-uniform smoothness. The space X is said to be q-uniformly convex if there exists $c_q > 0$ such that $\delta_X(\epsilon) \geq c_q\epsilon^q$, $\forall\epsilon \in (0,2]$. Here, δ_X is the moduli of convexity defined by $\delta_X(\epsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1 \text{ and } \|x-y\| = \epsilon\right\}$. The space X is said to be p-uniformly smooth if there exists $k_p > 0$ such that $\rho_X(t) \leq k_p t^p$, $\forall t > 0$. Here, ρ_X is the moduli of smoothness defined by $\rho_X(t) := \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| = 1, \|y\| = t\right\}$.

Let $c_q > 0$ be the constant given in the definition of the q-uniform convexity of X. By Lemma 1.5 in [12], for any $M > 0$, there exists $C_M > 0$ such that

$$V(x^*, y) \geq C_M\|J^*(x^*) - y\|^q, \text{ for all } x^* \in M\mathbb{B}_*, y \in M\mathbb{B}. \tag{2.9}$$

Here, C_M has an explicit form in terms of c_q, q, and M as follows: $C_M = \dfrac{2c_q}{4^{q-1} \cdot M^{q-2}}$.

Note that simple computations give

$$\langle J(x) - J(y), x - y\rangle = \frac{1}{2}[V(J(x), y) + V(J(y), x)].$$

Hence, using inequality (2.9), we obtain

$$\langle J(x) - J(y), x - y\rangle \geq C_M\|x - y\|^q \text{ for all } x, y \in M\mathbb{B}. \tag{2.10}$$

We now use the Cauchy-Schwartz inequality to write

$$\|J(x) - J(y)\| \cdot \|x - y\| \geq \langle J(x) - J(y), x - y\rangle \geq C_M\|x - y\|^q, \text{ for all } x, y \in M\mathbb{B}, \tag{2.11}$$

and so

$$\|J(x) - J(y)\| \geq C_M\|x - y\|^{q-1} \text{ for all } x, y \in M\mathbb{B}. \tag{2.12}$$

On the other hand, because X is a p-uniformly smooth Banach space ($1 < p \leq 2$), the dual space X^* is p'-uniformly convex with $p' = \dfrac{p}{p-1}$ as the conjugate of p. Applying (2.12) for the dual space X^* and its duality mapping J^*, we can write

$$\|J^*(x^*) - J^*(y^*)\| \geq C'_M\|x^* - y^*\|^{p'-1} = C'_M\|x^* - y^*\|^{\frac{1}{p-1}}, \text{ for all } x^*, y^* \in M\mathbb{B}_*, \tag{2.13}$$

with $C'_M := \dfrac{2c_{p'}}{4^{p'-1} \cdot M^{p'-2}} = \dfrac{2c_{p'}}{4^{\frac{1}{p-1}} \cdot M^{\frac{2-p}{p-1}}}$, where $c_{p'} > 0$ is the constant given in the definition of the p'-uniform convexity of X^*. Let now $x, y \in M\mathbb{B}$, and set $x^* := J(x)$ and $y^* := J(y)$. Then, $x^*, y^* \in M\mathbb{B}_*$, and so, by (2.13), we obtain

$$C'_M\|J(x) - J(y)\|^{\frac{1}{p-1}} = C'_M\|x^* - y^*\|^{\frac{1}{p-1}} \leq \|x - y\|,$$

and so

$$\|J(x) - J(y)\| \leq \frac{4 \cdot M^{2-p}}{(2c_{p'})^{p-1}}\|x - y\|^{p-1}, \text{ for all } x, y \in M\mathbb{B}. \tag{2.14}$$

We summarize the above obtained inequalities in the following proposition.

Proposition 2.2. *Assume that X is a q-uniformly convex and p-uniformly smooth Banach space, and let $M > 0$. The following inequalities hold:*

(i) $\langle J(x) - J(y), x - y\rangle \geq \dfrac{2c_q}{4^{q-1} \cdot M^{q-2}}\|x - y\|^q, \quad \forall x, y \in M\mathbb{B}$.

(ii) $\|J(x) - J(y)\| \geq \dfrac{2c_q}{4^{q-1} \cdot M^{q-2}}\|x - y\|^{q-1}, \quad \forall x, y \in M\mathbb{B}$.

(iii) $\|J(x) - J(y)\| \leq \dfrac{4M^{2-p}}{2^{p-1}c_{p'}^{p-1}}\|x - y\|^{p-1}, \quad \forall x, y \in M\mathbb{B}$.

(iv) $\|J^*(x^*) - J^*(y^*)\| \leq \dfrac{4M^{\frac{q-2}{q-1}}}{2^{\frac{1}{q-1}}c_q^{\frac{1}{q-1}}}\|x^* - y^*\|^{\frac{1}{q-1}}, \quad \forall x^*, y^* \in M\mathbb{B}_*$.

To start our local study around $J(\bar{x})$ of the function $e^V_{\lambda,\bar{x},\rho}f$ and the operator $\pi^{f,\lambda}_{\bar{x},\rho}$, we define the following special form of open neighborhoods of $J(\bar{x})$:

$$\mathcal{N}(J(\bar{x}), \delta_1, \delta_2) := \{x^* \in X^* : V(x^*, \bar{x}) < \delta_1 \text{ and } \|J^*(x^*) - \bar{x}\| < \delta_2\}. \tag{2.15}$$

Obviously, $J(\bar{x}) \in \mathcal{N}(J(\bar{x}), \delta_1, \delta_2)$ (because $V(J(\bar{x}), \bar{x}) = 0$, and $\|J^*(J(\bar{x})) - \bar{x}\| = 0$). We shall prove that $\mathcal{N}(J(\bar{x}), \delta_1, \delta_2)$ is an open set. It can be written as

$$\mathcal{N}(J(\bar{x}), \delta_1, \delta_2) = \{x^* \in X^* : V(x^*, \bar{x}) < \delta_1\} \cap \{x^* \in X^* : \|J^*(x^*) - \bar{x}\| < \delta_2\}.$$

Because the functional $V(\cdot, \bar{x})$ is continuous, the first set $\{x^* \in X^* : V(x^*, \bar{x}) < \delta_1\}$ is open in X^*. The second set $\{x^* \in X^* : \|J^*(x^*) - \bar{x}\| < \delta_2\} = [J^*]^{-1}(\bar{x} + \delta_2\text{int}(\mathbb{B}))$ is the inverse of an open subset in X by a continuous mapping. Here, int $(\mathbb{B})$ stands for the topological interior of $\mathbb{B}$ in X. Because the space X is q-uniformly convex, the mapping J^* is Hölder continuous on bounded sets by Proposition 2.2, and so the second set is open in X^*. Consequently, the set $\mathcal{N}(J(\bar{x}), \delta_1, \delta_2)$ is open in X^* containing $J(\bar{x})$, and hence, it is an open neighborhood of $J(\bar{x})$.

For a given $\epsilon > 0$, we denote $M_\epsilon := \|\bar{x}\| + \epsilon$, and we define the following ϵ-open neighborhood of $J(\bar{x})$ in X^*:

$$\mathcal{N}_{J(\bar{x})}(\epsilon) := \mathcal{N}(J(\bar{x}), \delta_1, \delta_2) \text{ with } \delta_1 := \frac{C_{M_\epsilon} \cdot \epsilon^q}{2^{q+1}} \text{ and } \delta_2 := \frac{\epsilon}{4};$$

that is,

$$\mathcal{N}_{J(\bar{x})}(\epsilon) = \left\{x^* \in X^* : V(x^*, \bar{x}) < \frac{C_{M_\epsilon} \cdot \epsilon^q}{2^{q+1}} \text{ and } \|J^*(x^*) - \bar{x}\| < \frac{\epsilon}{4}\right\}.$$

The following lemma is very important in our analysis. It establishes a localization inside the neighborhood $\bar{x} + \rho\mathbb{B}$ of the generalized (f, λ)-projection of elements of the ρ-open neighborhood of $J(\bar{x})$, whenever the parameter λ is given in the interval $(0, \lambda_0)$ for some $\lambda_0 > 0$.

Lemma 2.2. *Let X be a q-uniformly convex Banach space, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function with $f(\bar{x}) < \infty$. Let $\rho > 0$ be a positive number such that f is bounded from below by $\beta \in \mathbb{R}$ over $\bar{x} + \rho\mathbb{B}$. For $\lambda_0 \in \left(0, \dfrac{C_{M_\rho}\rho^q}{2 \cdot 2^{q+1}(f(\bar{x}) - \beta)}\right]$. Then, for all $\lambda \in (0, \lambda_0)$, we have*

$$e^V_{\lambda,\bar{x},\rho}f(x^*) = e^V_{\lambda,\bar{x},\frac{3\rho}{4}}f(x^*), \quad \forall x^* \in \mathcal{N}_{J(\bar{x})}(\rho). \tag{2.16}$$

Proof. Let $M_\rho := \|\bar{x}\| + \rho$, and let $x^* \in \mathcal{N}_{J(\bar{x})}(\rho)$. Then,

$$V(x^*, \bar{x}) < \frac{C_{M_\rho}\rho^q}{2^{q+1}}. \tag{2.17}$$

Fix any $y \in [\bar{x} + \rho\mathbb{B}] \setminus [\bar{x} + \frac{3\rho}{4}\mathbb{B}]$. Then, we have

$$\|y\| \leq \|\bar{x}\| + \rho = M_\rho \text{ and } \|x^*\| \leq \|\bar{x}\| + \|J^*(x^*) - \bar{x}\|| < \|\bar{x}\| + \frac{\rho}{4} < M_\rho,$$

and so by (2.9), we have for $C_{M_\rho} := \dfrac{2c_q}{4^{q-1}M_\rho^{q-2}}$,

$$V(x^*, y) \geq C_{M_\rho}\|y - J^*(x^*)\|^q \geq C_{M_\rho}\left[\|y - \bar{x}\| - \|J^*(x^*) - \bar{x}\|\right]^q \geq C_{M_\rho}\left[\frac{3\rho}{4} - \frac{\rho}{4}\right]^q = \frac{C_{M_\rho}\rho^q}{2^q}. \tag{2.18}$$

Hence,

$$f(y) + \frac{1}{2\lambda}V(x^*, y) \geq \beta + \frac{1}{2\lambda}\frac{C_{M_\rho}\rho^q}{2^q}.$$

By our assumption, we have $0 < \lambda_0 < \dfrac{C_{M_\rho}\rho^q}{2 \cdot 2^{q+1}(f(\bar{x}) - \beta)}$, which ensures that $f(\bar{x}) < \beta + \dfrac{C_{M_\rho}\rho^q}{2\lambda_0 2^{q+1}}$. Then, for any $\lambda \in (0, \lambda_0)$, we have $\dfrac{1}{2\lambda_0} < \dfrac{1}{2\lambda}$, and so,

$$\begin{aligned} f(y) + \frac{1}{2\lambda}V(x^*, y) &\geq \beta + \frac{1}{2\lambda}\frac{C_{M_\rho}\rho^q}{2^q} > f(\bar{x}) - \frac{1}{2\lambda_0}\frac{C_{M_\rho}\rho^q}{2^{q+1}} + \frac{1}{2\lambda}\frac{C_{M_\rho}\rho^q}{2^q} \\ &> f(\bar{x}) - \frac{1}{2\lambda}\frac{C_{M_\rho}\rho^q}{2^{q+1}} + \frac{1}{2\lambda}\frac{C_{M_\rho}\rho^q}{2^q} = f(\bar{x}) + \frac{1}{2\lambda}\frac{C_{M_\rho}\rho^q}{2^{q+1}}. \end{aligned} \tag{2.19}$$

Using (2.17), we obtain

$$f(y) + \frac{1}{2\lambda}V(x^*, y) > f(\bar{x}) + \frac{1}{2\lambda}\frac{C_{M_\rho}\rho^q}{2^{q+1}} > f(\bar{x}) + \frac{1}{2\lambda}V(x^*, \bar{x}). \tag{2.20}$$

Hence,

$$\inf_{y \in [\bar{x}+\rho\mathbb{B}] \setminus [\bar{x}+\frac{3\rho}{4}\mathbb{B}]}\left[f(y) + \frac{1}{2\lambda}V(x^*, y)\right] \geq f(\bar{x}) + \frac{1}{2\lambda}V(x^*, \bar{x}) \geq \inf_{y \in \bar{x}+\rho\mathbb{B}}\left[f(y) + \frac{1}{2\lambda}V(x^*, y)\right]. \tag{2.21}$$

This ensures that

$$e^V_{\lambda,\bar{x},\rho}f(x^*) = \inf_{y \in \bar{x}+\rho\mathbb{B}}\left[f(y) + \frac{1}{2\lambda}V(x^*, y)\right] = \inf_{y \in \bar{x}+\frac{3\rho}{4}\mathbb{B}}\left[f(y) + \frac{1}{2\lambda}V(x^*, y)\right] = e^V_{\lambda,\bar{x},\frac{3\rho}{4}}f(x^*), \tag{2.22}$$

and hence, the proof of (2.16) is complete. □

Remark 2.2.

1) By the above useful localization lemma, we have a localization of the (f,λ)-generalized projection $\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$, relative to the given point $\bar{x}$; that is, there exists some $\lambda_0 > 0$ such that $\forall\lambda \in (0,\lambda_0)$, and we have
$$\pi^{f,\lambda}_{\bar{x},\rho}(x^*) \subset \bar{x} + \frac{3\rho}{4}\mathbb{B}, \quad \forall x^* \in \mathcal{N}_{J(\bar{x})}(\rho). \tag{2.23}$$

2) The second consequence of the above localization lemma is
$$\nabla^F e^V_{\lambda,\bar{x},\rho} f(x^*) = \frac{1}{\lambda}\left[J^*(x^*) - \pi^{f,\lambda}_{\bar{x},\rho}(x^*)\right]$$
for all $\lambda \in (0,\lambda_0)$ and all $x^* \in D_\lambda \cap \mathcal{N}_{J(\bar{x})}(\rho)$, where D_λ is the set of all $x^* \in X^*$, where the function $e^V_{\lambda,\bar{x},\rho} f$ is Fréchet subdifferentiable. Note that D_λ is dense in X^*. Indeed, observe that
$$e^V_{\lambda,\bar{x},\rho} f(x^*) = \frac{\|x^*\|^2}{2\lambda} - k(x^*), \quad \forall x^* \in X^* \tag{2.24}$$
with $k(x^*) := \sup_{z\in X}\left\{\langle\frac{x^*}{\lambda}, z\rangle - f(z) - \frac{\|z\|^2}{2\lambda} - \psi_{[\bar{x}+\rho\mathbb{B}]}(z)\right\}$. Obviously, the function k is convex l.s.c. on X^*, and so it is Fréchet differentiable on a dense set in X^*, and because the space X^* has a smooth norm, the function $e^V_{\lambda,\bar{x},\rho} f$ is also dense in X^*. Thus, for any $x^* \in D_\lambda \cap \mathcal{N}_{J(\bar{x})}(\rho)$, the set $\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$ is a singleton and belongs to $\bar{x} + \frac{3\rho}{4}\mathbb{B}$. Therefore,
$$\|J^*(x^*) - \pi^{f,\lambda}_{\bar{x},\rho}(x^*)\| \le \|J^*(x^*) - \bar{x}\| + \|\bar{x} - \pi^{f,\lambda}_{\bar{x},\rho}(x^*)\| \le \frac{\rho}{4} + \frac{3\rho}{4} = \rho; \tag{2.25}$$
that is,
$$\pi^{f,\lambda}_{\bar{x},\rho}(x^*) \in J^*(x^*) + \rho\mathbb{B}.$$
Consequently, we obtain by Theorem 2.1 that for all $\lambda \in (0,\lambda_0)$ and all $x^* \in D_\lambda \cap \mathcal{N}_{J(\bar{x})}(\rho)$, we have
$$\nabla^F e^V_{\lambda,\bar{x},\rho} f(x^*) = \frac{1}{\lambda}\left[J^*(x^*) - \pi^{f,\lambda}_{\bar{x},\rho}(x^*)\right].$$

In the following proposition, we establish a relationship between the (f,λ)-generalized projection $\pi^{f,\lambda}_{\bar{x},\rho}(x^*)$ and $\partial^\pi f$ the V-proximal subdifferential of f. First, we recall (see [14]) the definition of the V-proximal subdifferential. For a point $\bar{x} \in X$ with $f(\bar{x}) < \infty$, we define the V-proximal subdifferential $\partial^\pi f(\bar{x})$ of f at $\bar{x}$ by the set of all $x^* \in X^*$ for which there exist $\sigma > 0, \delta > 0$ such that
$$\langle x^*, x' - x\rangle \le f(x') - f(x) + \sigma V(J(x), x')), \forall x' \in x + \delta\mathbb{B}. \tag{2.26}$$

Proposition 2.3. *Under the assumptions of Lemma 2.2, there exists $\lambda_0 > 0$ such that*
$$\forall\lambda \in (0,\lambda_0), \forall x^* \in \mathcal{N}_{J(\bar{x})}(\rho), \text{ we have } \pi^{f,\lambda}_{\bar{x},\rho}(x^*) \subset \left[J + \lambda\partial^\pi f\right]^{-1}(x^*).$$

Proof. By Lemma 2.2, there exists $\lambda_0 > 0$ such that (2.16) holds. Let $x^* \in \mathcal{N}_{J(\bar{x})}(\rho)$. If $\pi^{f,\lambda}_{\bar{x},\rho}(x^*) = \emptyset$, then we are done. Assume that $\pi^{f,\lambda}_{\bar{x},\rho}(x^*) \neq \emptyset$. Let $\bar{y} \in \pi^{f,\lambda}_{\bar{x},\rho}(x^*)$. Then,

$$f(\bar{y}) + \frac{1}{2\lambda}V(x^*, \bar{y}) \leq f(y) + \frac{1}{2\lambda}V(x^*, y), \quad \forall y \in \bar{x} + \rho\mathbb{B};$$

that is,

$$\frac{1}{2\lambda}\left[V(x^*, \bar{y}) - V(x^*, y)\right] \leq f(y) - f(\bar{y}), \quad \forall y \in \bar{x} + \rho\mathbb{B}. \tag{2.27}$$

By Lemma 2.2, we have $\bar{y} \in \bar{x} + \frac{3\rho}{4}\mathbb{B}$, and so for any $y \in \bar{y} + \frac{\rho}{4}\mathbb{B}$, we have $y \in \bar{x} + \frac{3\rho}{4}\mathbb{B} + \frac{\rho}{4}\mathbb{B} = \bar{x} + \rho\mathbb{B}$, and so inequality (2.27) holds for all $y \in \bar{y} + \frac{\rho}{4}\mathbb{B}$. Therefore,

$$V(x^*, \bar{y}) - V(x^*, y) \leq 2\lambda\left[f(y) - f(\bar{y})\right], \quad \forall y \in \bar{y} + \frac{\rho}{4}\mathbb{B}. \tag{2.28}$$

Observe that $V(x^*, \bar{y}) - V(x^*, y) = 2\langle x^* - J(\bar{y}), y - \bar{y}\rangle - V(J(\bar{y}), y)$. Thus,

$$2\langle x^* - J(\bar{y}), y - \bar{y}\rangle \leq 2\lambda\left[f(y) - f(\bar{y})\right] + V(J(\bar{y}), y),$$

and so,

$$\langle \frac{x^* - J(\bar{y})}{\lambda}, y - \bar{y}\rangle \leq \left[f(y) - f(\bar{y})\right] + \frac{1}{2\lambda}V(J(\bar{y}), y), \quad \forall y \in \bar{y} + \frac{\rho}{4}\mathbb{B}.$$

This ensures by the definition of the V-proximal subdifferential that $\frac{x^* - J(\bar{y})}{\lambda} \in \partial^\pi f(\bar{y})$, and so $x^* \in J(\bar{y}) + \lambda\partial^\pi f(\bar{y}) = \left[J + \lambda\partial^\pi f\right](\bar{y})$; that is, $\bar{y} \in \left[J + \lambda\partial^\pi f\right]^{-1}(x^*)$, and so, $\pi^{f,\lambda}_{\bar{x},\rho}(x^*) \subset \left[J + \lambda\partial^\pi f\right]^{-1}(x^*)$. □

3. Main results

In order to establish further properties for the (f, λ)-generalized projection, we need more assumptions on the function f. We quote from [13] the definition of the class of V-primal lower nice (V-p.l.n.) functions. We are going to prove for this class of nonconvex functions, that the (f, λ)-generalized operator is Hölder continuous on the ρ-neighborhood of $J(\bar{x})$.

Definition 3.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. We will say that f is V-primal lower nice (V-p.l.n.) at $\bar{x} \in dom\ f$, if there exist $\eta_0 > 0, c_0 > 0, t_0 > 0$ such that*

$$f(x') - f(x) \geq \langle x^*, x' - x\rangle - \frac{t}{2}V(J(x), x') \tag{3.1}$$

for all $t \geq t_0$, all $x, x' \in \bar{x} + \eta_0\mathbb{B}$, and all $x^* \in \partial^\pi_G f(x) \cap c_0 t\mathbb{B}_*$. Here, $\partial^\pi_G f(x)$ stands for the geometric V-proximal subdifferential introduced and studied in [14] and is defined by

$$\partial^\pi_G f(x) = \{x^* \in X^* : (x^*, -1) \in N^\pi(\text{epi } f, (x, f(x)))\}.$$

The first important result for this class of functions is the following one, in which we prove the coincidence of both subsdifferentials, $\partial^\pi_G f$ and $\partial^\pi f$ around the point $\bar{x}$ as well as their limiting concepts $\partial^{L\pi}_G f$ and $\partial^{L\pi} f$.

Theorem 3.1. *If f is V-p.l.n. at $\bar{x} \in dom\, f$, then there exists $\delta > 0$ such that*

$$\partial^\pi f(z) = \partial^\pi_G f(z) = \partial^{L\pi}_G f(z) = \partial^{L\pi} f(z), \quad \forall z \in \bar{x} + \delta\mathbb{B}, \tag{3.2}$$

where $\partial^{L\pi}_G f(x)$ is the limiting geometric V-proximal subdifferential, and $\partial^{L\pi} f(x)$ is the limiting V-proximal subdifferential defined (see [8]) by

$$\partial^{L\pi}_G f(x) = \limsup_{x' \to x} \partial^\pi_G f(x') := \{x^* \in X^* : x^* = w - \lim_n x^*_n : x^*_n \in \partial^\pi_G f(x_n) \text{ with } x_n \to^f x\} \text{ and}$$

$$\partial^{L\pi} f(x) = \limsup_{x' \to x} \partial^\pi f(x') = \{x^* \in X^* : x^* = w - \lim_n x^*_n : x^*_n \in \partial^\pi f(x_n) \text{ with } x_n \to^f x\}, \text{ respectively.}$$

Here, $x_n \to^f x$ means $x_n \to x$ with $f(x_n) \to f(x)$.

Proof. By the definition of V-p.l.n at $\bar{x}$, there are $\eta_0 > 0, c_0 > 0, t_0 > 0$ such that

$$f(x') - f(x) \geq \langle x^*, x' - x\rangle - \frac{t}{2} V(J(x), x'), \text{ for all } t \geq t_0, \text{ all } x, x' \in \bar{x} + \eta_0\mathbb{B}, \text{ and all } x^* \in \partial^\pi_G f(x) \cap c_0 t\mathbb{B}_*.$$

Let $z \in \bar{x} + \frac{\eta_0}{2}\mathbb{B}$, and let $z^* \in \partial^{L\pi}_G f(z)$. Then, by the definition of the limiting geometric V-proximal subdifferential, there exist $z_n \to z$ with $f(z_n) \to f(z)$ and $z^*_n \rightharpoonup^w z^*$ with $z^*_n \in \partial^\pi_G f(z_n)$. For n large enough, we have $z_n \in z + \frac{\eta_0}{2}\mathbb{B} \subset \bar{x} + \frac{\eta_0}{2}\mathbb{B} + \frac{\eta_0}{2}\mathbb{B} = \bar{x} + \eta_0\mathbb{B}$. Because z^*_n is weakly convergent it is bounded, and so there exists $M > 0$ such that $\|z^*_n\| \leq M, \forall n$. Set $t := \max\left\{\frac{M}{c_0}, t_0\right\}$. Obviously, $\|z^*_n\| \leq c_0 t$, and $t \geq t_0$. Thus, we have

$$\langle z^*_n, y - z_n\rangle \leq f(y) - f(z_n) + \frac{t}{2} V(J(z_n), y), \text{ for all } y \in z + \frac{\eta_0}{2}\mathbb{B}.$$

Taking $n \to \infty$ gives

$$\langle z^*, y - z\rangle \leq f(y) - f(z) + \frac{t}{2} V(J(z), y), \text{ for all } y \in z + \frac{\eta_0}{2}\mathbb{B},$$

which means that $z^* \in \partial^\pi f(z)$. Because the inclusions $\partial^\pi f(z) \subset \partial^\pi_G f(z) \subset \partial^{L\pi}_G f(z)$ and $\partial^\pi f(z) \subset \partial^{L\pi} f(z) \subset \partial^{L\pi}_G f(z)$ are always valid, we obtain the equalities

$$\partial^\pi f(z) = \partial^\pi_G f(z) = \partial^{L\pi}_G f(z) = \partial^{L\pi} f(z), \quad \forall z \in \bar{x} + \frac{\eta_0}{2}\mathbb{B}.$$

This complete the proof. □

Remark 3.1. *Using Theorem 3.1, we may replace, in the definition of V-p.l.n. functions, the geometric V-proximal subdifferential $\partial^\pi_G f(x)$ by the V-proximal subdifferential $\partial^\pi f(x)$ or by their limiting concepts $\partial^{L\pi}_G f(x)$ and $\partial^{L\pi} f(x)$, in the definition of V-p.l.n. functions in Definition 3.1.*

An example of a V-p.l.n. function is given by the indicator function of V-prox-regular sets. We recall from [12, 13, 15] the definition of V-prox-regularity of sets.

Definition 3.2. *Let S be a nonempty closed set in a reflexive Banach space X, and let $\bar{x} \in S$. We will say that S is V-prox-regular at $\bar{x}$ if and only if there exist $r > 0$ and $\epsilon > 0$ such that for all $x \in S \cap (\bar{x} + \epsilon\mathbb{B})$ and for any $x^* \in N^\pi(S, x) \cap \epsilon\mathbb{B}_*$, the point x is a generalised projection of $Jx + rx^*$ on $S \cap (\bar{x} + \epsilon\mathbb{B})$; that is, $x \in \pi_{S\cap(\bar{x}+\epsilon\mathbb{B})}(Jx + rx^*)$.*

In order to prove that the indicator function of a V-prox-regular set S is V-p.l.n., we recall from [13] the following proposition.

Proposition 3.1. *Assume that X is a reflexive smooth Banach space, and let $f : X \to \mathbb{R} \cup \{\infty\}$ be a l.s.c. function. For any V-p.l.n. function f at $\bar{x} \in domf$, there exist $\eta_0 > 0, c_0 > 0, t_0 > 0$ such that for any $t \geq t_0$, any $x_1, x_2 \in \bar{x} + \eta_0\mathbb{B}$, $x_i^* \in \partial^\pi f(x_i) \cap c_0 t\mathbb{B}_*$, $i = 1, 2$, we have*

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq -t\langle J(x_1) - J(x_2), x_1 - x_2\rangle. \tag{3.3}$$

To get the equivalence in the previous proposition, that is, f is V-p.l.n at $\bar{x}$ if and only if inequality (3.3) holds, we need a further assumption on the space X. Indeed, according to Theorem 4.1 in [13], we need the space X to be V-proximal trustworthy. We recall from [16] the definition of this class of spaces. A Banach space X is said to be V-proximal trustworthy if and only if the V-proximal subdifferential satisfies the fuzzy sum rule. That is, for any $\epsilon > 0$, any two functions $f_1, f_2 : X \to \mathbb{R} \cup \{\infty\}$ and any $u \in X$ such that f_1 is lower semicontinuous and f_2 is Lipschitz around u the following fuzzy sum rule holds: $\partial^\pi(f_1 + f_2)(u) \subset \bigcup\{\partial^\pi f_1(u_1) + \partial^\pi f_2(u_2) : u_i \in U_{f_i}(u, \epsilon), i = 1, 2\} + \epsilon\mathbb{B}_*$. Here, $U_{f_i}(u, \epsilon) := \{x \in u + \epsilon\mathbb{B}$ such that $|f_i(x) - f_i(u)| < \epsilon\}$. It has been proved in [16] that all the integral spaces L^p $(1 < p < \infty)$ as well as the sequence spaces $l^p(1 < p < \infty)$, the Sobolev spaces $W^{p,n}(1 < p < \infty)$, and the Schatten trace ideals $\mathbf{C}^p(1 < p < \infty)$, are V-proximal trustworthy. We state from Theorem 4.1 in [13] the following characterization of V-p.l.n functions on V-proximal trustworthy Banach space.

Proposition 3.2. *Assume that X is q-uniformly convex and p-uniformly smooth Banach space which is V-proximal trustworthy, let $f : X \to \mathbb{R} \cup \{+\infty\}$ be any l.s.c. function, and let $\bar{x} \in domf$ and $\rho > 0$. Then, f is V-p.l.n at $\bar{x}$ if and only if there exist $\eta_0 > 0, c_0 > 0, t_0 > 0$ such that*

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq -t\langle J(x_1) - J(x_2), x_1 - x_2\rangle, \tag{3.4}$$

for all $t \geq t_0$, all $x_1, x_2 \in \bar{x} + \eta_0\mathbb{B}$, and all $x_i^* \in \partial^\pi f(x_i) \cap c_0 t\mathbb{B}_*$ $(i = 1, 2)$.

The following result extends Theorem 3.2 in [13], in which the authors prove the equality at only the point $\bar{x}$ where the set S is V-prox-regular. However, we prove it here for all points very close to $\bar{x}$.

Proposition 3.3. *Assume that X is V-proximal trustworthy Banach space. If S is V-prox-regular at $\bar{x}$, then the indicator function ψ_S is V-p.l.n. at $\bar{x}$, and we have $N^\pi(S, x) = N^{L\pi}(S, x)$ for all $x \in \bar{x} + \delta\mathbb{B}$, for some $\delta > 0$.*

Proof. Assume that S is V-prox-regular at $\bar{x} \in S$. By Definition 3.2, there exist $r > 0, \epsilon > 0$ such that $\forall x \in (\bar{x} + \epsilon\mathbb{B}) \cap S$, $\forall x^* \in N^\pi(S, x) \cap \epsilon\mathbb{B}_*$, and we have

$$V(J(x) + rx^*, x) \leq V(J(x) + rx^*, z), \quad \forall z \in (\bar{x} + \epsilon\mathbb{B}) \cap S. \tag{3.5}$$

Because V is convex differentiable on X with respect to the second variable with $\nabla^F V(z^*, \cdot)(z) = 2[J(z) - z^*]$, $\forall z \in X, \forall z^* \in X^*$, we may write

$$\langle 2[J(z) - z^*], y - z\rangle \leq V(z^*, y) - V(z^*, z), \quad \forall y, z \in X. \tag{3.6}$$

Applying this inequality with $y = x$ and $z^* = J(x) + rx^*$, we obtain

$$\langle 2[J(z) - J(x) - rx^*], x - z\rangle \leq V(J(x) + rx^*, x) - V(J(x) + rx^*, z), \quad \forall z, x \in X. \tag{3.7}$$

Combining this inequality with inequality (3.5), we obtain

$$\begin{array}{rcl} 2\langle J(z) - J(x), x - z\rangle - 2r\langle x^*, x - z\rangle & = & \langle 2[J(z) - J(x) - rx^*], x - z\rangle \\ & \leq & V(J(x) + rx^*, x) - V(J(x) + rx^*, z) \\ & \leq & 0, \quad \forall z \in (\bar{x} + \epsilon\mathbb{B}) \cap S. \end{array}$$

Therefore, we obtain

$$\langle J(z) - J(x), x - z\rangle \leq r\langle x^*, x - z\rangle, \quad \forall x \in (\bar{x} + \epsilon\mathbb{B}) \cap S,\ \forall x^* \in N^\pi(S, x) \cap \epsilon\mathbb{B}_*,\ \forall z \in X.$$

Interchanging the roles of x and z, we get

$$\langle J(x) - J(z), z - x\rangle \leq r\langle z^*, z - x\rangle, \quad \forall z \in (\bar{x} + \epsilon\mathbb{B}) \cap S,\ \forall z^* \in N^\pi(S, z) \cap \epsilon\mathbb{B}_*,\ \forall x \in X.$$

Combining these two inequalities gives

$$2\langle J(z) - J(x), x - z\rangle \leq r\langle z^* - x^*, z - x\rangle \tag{3.8}$$

for all $x, z \in (\bar{x}+\epsilon\mathbb{B})\cap S$, all $x^* \in N^\pi(S, x)\cap\epsilon\mathbb{B}_*$, and all $z^* \in N^\pi(S, z)\cap\epsilon\mathbb{B}_*$. Let $t_0 = 0$, $c_0 = \frac{r\epsilon}{2} > 0$, and $t \geq t_0$. Let $x_i \in (\bar{x} + \epsilon\mathbb{B}) \cap \operatorname{dom} \psi_S$, and let $x_i^* \in \partial^\pi\psi_S(x_i) \cap c_0 t\mathbb{B}_*$ $(i = 1, 2)$. Then, $x_i^* \in N^\pi(S, x_i) \cap c_0 t\mathbb{B}_*$, and $\|\frac{2x_i^*}{rt}\| \leq \frac{2c_0 t}{rt} = \epsilon$. Hence, $\frac{2x_i^*}{rt} \in N^\pi(S, x_i) \cap \epsilon\mathbb{B}_*$, and so, inequality (3.8) ensures

$$2\langle J(x_1) - J(x_2), x_2 - x_1\rangle \leq r\langle \frac{2x_1^*}{rt} - \frac{2x_2^*}{rt}, x_1 - x_2\rangle = \frac{2}{t}\langle x_1^* - x_2^*, x_1 - x_2\rangle.$$

Consequently, we obtain that for any $t \geq t_0$, any $x_i \in (\bar{x} + \epsilon\mathbb{B}) \cap \operatorname{dom} \psi_S$, and any $x_i^* \in \partial^\pi\psi_S(x_i) \cap c_0 t\mathbb{B}_*$ $(i = 1, 2)$, we have

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq -t\langle J(x_1) - J(x_2), x_1 - x_2\rangle.$$

This ensures by Proposition 3.2 that ψ_S is V-p.l.n. at $\bar{x}$. The second part of the proposition follows straightforwardly from Theorem 3.1, and hence, the proof is complete. □

Now, we proceed to prove the main result of the (f, λ)-generalized projection for V-p.l.n. functions. First, we start by recalling the following density theorem, which is a direct consequence of Theorem 2.1 in [10], by taking the set $S := \bar{x} + \rho\mathbb{B}$.

Theorem 3.2. *Assume that X is a reflexive Banach space with smooth dual norm, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be any l.s.c. function, and let $\bar{x} \in dom f$ and $\rho > 0$. Then, the set of points $x^* \in X^*$ admitting a unique generalized (f, λ)-projection on $\bar{x} + \rho\mathbb{B}$ is dense in X^*; that is, for any $x^* \in X^*$, there exists a sequence x_n^* converging to x^* with $\pi_{\bar{x},\rho}^{f,\lambda}(x_n^*) \neq \emptyset$.*

Using this density result and Proposition 3.1, we prove the Hölder continuity of $\pi_{\bar{x},\rho}^{f,\lambda}$ on its domain locally around $J(\bar{x})$.

Proposition 3.4. *Assume that X is a q-uniformly convex and p-uniformly smooth Banach space, and let $\rho > 0$. Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a l.s.c. function which is V-p.l.n. at $\bar{x} \in domf$ with positive real numbers $\eta_0 > 0, c_0 > 0, t_0 > 1$ as in Proposition 3.1. Assume that f is bounded below on $\bar{x} + \rho\mathbb{B}$ by $\beta \in \mathbb{R}$ and that $\rho \in (0, \min\{\eta_0, \frac{c_0(2c_{p'})^{p-1}}{8}\})$. Then, there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \frac{\lambda_0}{t_0}]$, and for any $x_1^*, x_2^* \in dom\, \pi_{\bar{x},\rho}^{f,\lambda} \cap \mathcal{N}(J(\bar{x}), \frac{C_{M_\rho} \cdot \rho^q}{4^{q+1}}, \frac{c_0 t_0 \lambda}{2}) \cap [J(\bar{x}) + \frac{c_0 t_0 \lambda}{2}\mathbb{B}_*]$, and for any $y_i \in \pi_{\bar{x},\rho}^{f,\lambda}(x_i^*)$, we have*

$$\|y_2 - y_1\| \leq \gamma_\lambda \|x_2^* - x_1^*\|^{\frac{1}{q-1}} \text{ for some } \gamma_\lambda > 0. \tag{3.9}$$

Proof. Set $\lambda_0 := \min\left\{1, \frac{2\rho}{c_0}, \frac{C_{M_\rho} \cdot \rho^q}{2 \cdot 2^{q+1}[f(\bar{x}) - \beta]}\right\}$, and let $\lambda \in (0, \frac{\lambda_0}{t_0}]$. Let $x_1^*, x_2^* \in \operatorname{dom} \pi_{\bar{x},\rho}^{f,\lambda} \cap \mathcal{N}(J(\bar{x}), \frac{C_{M_\rho} \cdot \rho^q}{4^{q+1}}, \frac{c_0 t_0 \lambda}{2}) \cap [J(\bar{x}) + \frac{c_0 t_0 \lambda}{2}\mathbb{B}_*]$ and let $y_i \in \pi_{\bar{x},\rho}^{f,\lambda}(x_i^*)$ $(i = 1, 2)$. By Part (1) in Remark 2.2, we have $y_i \in \bar{x} + \frac{3\rho}{4}\mathbb{B}$. Set $y_i^* := \lambda^{-1}[x_i^* - J(y_i)]$. By (iii) in Proposition 2.2, we derive

$$\|J(y_i) - J(\bar{x})\| \leq \frac{4M_\rho^{2-p}}{(2c_{p'})^{p-1}}\|y_i - \bar{x}\|^{p-1} \leq \frac{4M_\rho^{2-p}}{(2c_{p'})^{p-1}}\left(\frac{3\rho}{4}\right)^{p-1} = 4M_\rho^{2-p}\left(\frac{3\rho}{8c_{p'}}\right)^{p-1}. \tag{3.10}$$

Set $t_\lambda := \frac{8M_\rho^{2-p}}{c_0\lambda}\left(\frac{3\rho}{8c_{p'}}\right)^{p-1}$ and $t_{0,\lambda} := \max\{t_0, t_\lambda\}$. Hence, we obtain

$$\begin{aligned}\|y_i^*\| &= \lambda^{-1}\|x_i^* - J(y_i)\| \leq \lambda^{-1}[\|x_i^* - J(\bar{x})\| + \|J(\bar{x}) - J(y_i)\|] \\ &\leq \lambda^{-1}\left(\frac{c_0 t_0 \lambda}{2} + 4M_\rho^{2-p}\left(\frac{3\rho}{8c_{p'}}\right)^{p-1}\right) = \frac{c_0 t_0}{2} + \frac{c_0 t_\lambda}{2} \leq c_0 t_{0,\lambda}.\end{aligned}$$

On the other hand, the inequalities $\lambda_0 \leq \frac{2\rho}{c_0}$ and $\lambda \leq \frac{\lambda_0}{t_0}$ give $t_0\lambda \leq \lambda_0 \leq \frac{2\rho}{c_0}$; that is, $\frac{c_0 t_0 \lambda}{2} \leq \rho$. This ensures that $\mathcal{N}(J(\bar{x}), \frac{C_{M_\rho} \cdot \rho^q}{4^{q+1}}, \frac{c_0 t_0 \lambda}{2}) \subset \mathcal{N}_{J(\bar{x})}(\rho)$. Thus, we apply Proposition 2.3 to get $y_i \in [J + \lambda\partial^\pi f]^{-1}(x_i^*)$, $(i = 1, 2)$; that is, $x_i^* \in J(y_i) + \lambda\partial^\pi f(y_i)$ $(i = 1, 2)$. First, we need to check if λ_0 is the same in Proposition 2.3. Therefore, we obtain $y_i^* \in \partial^\pi f(y_i) \cap c_0 t_{0,\lambda}\mathbb{B}$ with $y_i \in \bar{x} + \frac{3\rho}{4}\mathbb{B} \subset \bar{x} + \eta_0 B$ $(i = 1, 2)$. Thus, all the assumptions of Proposition 3.1 are satisfied, and so, we obtain for $t := t_{0,\lambda}$

$$\langle y_1^* - y_2^*, y_1 - y_2\rangle \geq -t_{0,\lambda}\langle J(y_1) - J(y_2), y_1 - y_2\rangle,$$

which ensures that

$$\langle [x_1^* - J(y_1)] - [x_2^* - J(y_2)], y_1 - y_2\rangle \geq -t_{0,\lambda}\lambda\langle J(y_1) - J(y_2), y_1 - y_2\rangle,$$

and so,

$$\langle x_1^* - x_2^*, y_1 - y_2\rangle \geq [1 - t_{0,\lambda}\lambda]\langle J(y_1) - J(y_2), y_1 - y_2\rangle.$$

By assumption, we have $\lambda \in \left(0, \frac{\lambda_0}{t_0}\right]$, and so, $\lambda t_0 < \lambda_0 < 1$. Also, by our assumption, we have

$$\rho < \frac{c_0(2c_{p'})^{p-1}}{8} \Leftrightarrow \rho^{p-1} = \rho \cdot \rho^{p-2} < \frac{c_0(2c_{p'})^{p-1}}{8}\rho^{p-2} < \frac{c_0(2c_{p'})^{p-1}}{8}M_\rho^{p-2} \Leftrightarrow \frac{8M_\rho^{2-p}\rho^{p-1}}{c_0(2c_{p'})^{p-1}} < 1.$$

This ensures that $\lambda t_\lambda = \frac{8M_\rho^{2-p}}{c_0}\left(\frac{3\rho}{8c_{p'}}\right)^{p-1} < \frac{8M_\rho^{p-2}\rho^{p-1}}{c_0(2c_{p'})^{p-1}} < 1$. Consequently, we obtain $\lambda t_{t_0,\lambda} = \max\{\lambda t_0, \lambda t_\lambda\} < 1$, and hence, $1 - \lambda t_{t_0,\lambda} > 0$. Thus, by using inequality (2.10), we have

$$\langle J(y_1) - J(y_2), y_1 - y_2\rangle \geq C_{M_\rho}\|y_1 - y_2\|^q$$

with $C_{M_\rho} = \frac{2c_q}{4^{q-1}M_\rho^{q-2}} > 0$, and so

$$\begin{aligned} \|x_1^* - x_2^*\| \cdot \|y_1 - y_2\| &\geq \langle x_1^* - x_2^*, y_1 - y_2\rangle \\ &\geq [1 - \lambda t_{t_0,\lambda}]\langle J(y_1) - J(y_2), y_1 - y_2\rangle \\ &\geq [1 - \lambda t_{t_0,\lambda}]C_{M_\rho}\|y_1 - y_2\|^q, \end{aligned} \tag{3.11}$$

and hence,

$$\|y_1 - y_2\| \leq \gamma_\lambda\|x_1^* - x_2^*\|^{\frac{1}{q-1}} \text{ with } \gamma_\lambda := \left[\frac{1}{(1 - \lambda t_{t_0,\lambda})C_{M_\rho}}\right]^{\frac{1}{q-1}}.$$

This completes the proof of the proposition. □

Now, we are ready to prove the single-valuedness and the local Hölder continuity of the operator $\pi_{\bar{x},\epsilon}^{f,\lambda}$ around $J(\bar{x})$ for V-primal lower nice functions.

Theorem 3.3. *Assume that X is a q-uniformly convex and p-uniformly smooth Banach space, and let $\rho > 0$. Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a l.s.c. function which is V-p.l.n. at $\bar{x} \in dom f$ with positive real numbers $\eta_0 > 0, c_0 > 0, t_0 > 0$ as in Proposition 3.4. Assume that f is bounded below on $\bar{x} + \rho\mathbb{B}$ by $\beta \in \mathbb{R}$ and that $\rho \in \left(0, \min\left\{\eta_0, \frac{c_0(2c_{p'})^{p-1}}{8}\right\}\right)$. Then, there exists $\lambda_0 > 0$ such that for any $\lambda \in \left(0, \frac{\lambda_0}{t_0}\right)$, there exists $\delta_1 > 0$ such that for any $x^* \in J(\bar{x}) + \delta_1\mathbb{B}_*$, the (f, λ)-generalized projection $\pi_{\bar{x},\rho}^{f,\lambda}(x^*)$ exists and is unique, and for any $x_1^*, x_2^* \in J(\bar{x}) + \delta_1\mathbb{B}_*$, we have*

$$\|\pi_{\bar{x},\rho}^{f,\lambda}(x_2^*) - \pi_{\bar{x},\rho}^{f,\lambda}(x_1^*)\| \leq \gamma_\lambda\|x_2^* - x_1^*\|^{\frac{1}{q-1}} \tag{3.12}$$

for some $\gamma_\lambda > 0$.

Proof. We prove the conclusion of the theorem in two steps. Let λ_0 be as in Proposition 3.4, and let $\lambda \in \left(0, \frac{\lambda_0}{t_0}\right]$. Set $\mathcal{V}_{J(\bar{x})} := \mathcal{N}(J(\bar{x}), \frac{C_{M_\rho}\cdot\rho^q}{4^{q+1}}, \frac{c_0t_0\lambda}{2}) \cap [J(\bar{x}) + \frac{c_0t_0\lambda}{2}\text{int}(\mathbb{B}_*)]$. This set is an open neighborhood of $J(\bar{x})$, so there exists $\delta_1 > 0$ such that $J(\bar{x}) + 2\delta_1\mathbb{B}_* \subset \mathcal{V}_{J(\bar{x})}$.

Step 1. In this step, we prove that $J(\bar{x}) + \delta_1\mathbb{B}_* \subset \text{dom}\,\pi_{\bar{x},\rho}^{f,\lambda}$. Let $x^* \in J(\bar{x}) + \delta_1\mathbb{B}_*$. Using the density theorem of $\pi_{\bar{x},\rho}^{f,\lambda}$ proved in Theorem 3.2, there exists a sequence x_n^* converging to x^* with $\pi_{\bar{x},\rho}^{f,\lambda}(x_n^*) \neq \emptyset$. Let $x_n \in \pi_{\bar{x},\rho}^{f,\lambda}(x_n^*)$. For n large enough, we have $x_n^* \in x^* + \delta_1\mathbb{B}_*$, and hence, $x_n^* \in J(\bar{x}) + \delta_1\mathbb{B}_* + \delta_1\mathbb{B}_* \subset J(\bar{x}) + 2\delta_1\mathbb{B}_* \subset \mathcal{V}_{J(\bar{x})}$. Thus, we found $x_n^* \in \text{dom}\,\pi_{\bar{x},\rho}^{f,\lambda} \cap \mathcal{N}(J(\bar{x}), \frac{C_{M_\rho}\cdot\rho^q}{4^{q+1}}, \frac{c_0t_0\lambda}{2}) \cap [J(\bar{x}) + \frac{c_0t_0\lambda}{2}\text{int}(\mathbb{B}_*)]$ and $x_n \in \pi_{\bar{x},\rho}^{f,\lambda}(x_n^*)$. Therefore, all the assumptions of Proposition 3.4 are satisfied, and so we have

$$\|x_n - x_m\| \leq \gamma\|x_n^* - x_m^*\|^{\frac{1}{q-1}}, \text{ for all } n \geq m \text{ very large}. \tag{3.13}$$

Because $x_n^* \to x^*$, we get the Cauchy property of the sequence $(x_n)_n$, and so it is convergent to some limit $\bar{y} \in \bar{x} + \frac{3\rho}{4}\mathbb{B}$. By construction, we have $x_n \in \pi_{\bar{x},\rho}^{f,\lambda}(x_n^*)$; that is,

$$f(x_n) + \frac{1}{2\lambda}V(x_n^*, x_n) \le f(z) + \frac{1}{2\lambda}V(x_n^*, z) \quad \forall z \in \bar{x} + \rho\mathbb{B}.$$

Taking the limit as $n \to \infty$ gives

$$f(\bar{y}) + \frac{1}{2\lambda}V(x^*, \bar{y}) \le f(z) + \frac{1}{2\lambda}V(x^*, z) \quad \forall z \in \bar{x} + \rho\mathbb{B};$$

that is, $\bar{y} \in \pi_{\bar{x},\rho}^{f,\lambda}(x^*)$, and hence, $x^* \in \operatorname{dom} \pi_{\bar{x},\rho}^{f,\lambda}$. Consequently, because x^* is taken as arbitrary in $J(\bar{x}) + \delta_1\mathbb{B}_*$, we obtain $J(\bar{x}) + \delta_1\mathbb{B}_* \subset \operatorname{dom} \pi_{\bar{x},\rho}^{f,\lambda}$.

Step 2. Fix now any $x^* \in J(\bar{x}) + \delta_1\mathbb{B}_*$. By Step 1, $\pi_{\bar{x},\rho}^{f,\lambda}(x^*) \neq \emptyset$. Let $y_1, y_2 \in \pi_{\bar{x},\rho}^{f,\lambda}(x^*)$. Using Step 1, we may write

$$\|y_2 - y_1\| \le \gamma\|x^* - x^*\|^{\frac{1}{q-1}} = 0.$$

This ensures that $y_1 = y_2$; that is, $\pi_{\bar{x},\rho}^{f,\lambda}(x^*)$ is a singleton on $J(\bar{x}) + \delta_1\mathbb{B}_*$.

Fix now any $x_1^*, x_2^* \in J(\bar{x}) + \delta_1\mathbb{B}_*$. By Step 1, we obtain $x_1^*, x_2^* \in \operatorname{dom} \pi_{\bar{x},\rho}^{f,\lambda} \cap \mathcal{V}_{J(\bar{x})}$ with $\pi_{\bar{x},\rho}^{f,\lambda}(x_i^*)$ $(i = 1, 2)$ as a singleton. Hence, by Proposition 3.4, we obtain

$$\|\pi_{\bar{x},\rho}^{f,\lambda}(x_2^*) - \pi_{\bar{x},\rho}^{f,\lambda}(x_1^*)\| \le \gamma\|x_2^* - x_1^*\|^{\frac{1}{q-1}}. \tag{3.14}$$

This completes the proof. □

We close this section with the following proposition proving the equality form in Proposition 2.3 for V-p.l.n. functions.

Proposition 3.5. *Under the assumptions of Theorem 3.3 with $t_0 > 1$, we have for all $\lambda \in \left(0, \min\left\{\frac{1}{t_0^2}, \frac{\lambda_0}{t_0}\right\}\right)$, there exists $\delta > 0$ such that*

$$\pi_{\bar{x},\rho}^{f,\lambda}(x^*) = \left[J + \lambda\partial^\pi f\right]^{-1}(x^*), \quad \forall x^* \in J(\bar{x}) + \delta\mathbb{B}_*. \tag{3.15}$$

Proof. The direct inclusion follows from Proposition 2.3. To prove (3.15), we need to prove that the right-hand side in the equality is at most single-valued. Let $\eta_0 > 0, c_0 > 0, t_0 > 1$ be given as in Definition 3.1. Let $\lambda \in \left(0, \min\left\{\frac{1}{t_0^2}, \frac{\lambda_0}{t_0}\right\}\right)$, and define the set-valued mapping $R_{\lambda,\eta_0} : X^* \to \bar{x} + \eta_0\mathbb{B}$ by $x^* \mapsto R_{\lambda,\eta_0}(x^*) := \left[J + \lambda\partial^\pi f \cap \frac{c_0}{t_0}\mathbb{B}_*\right]^{-1}(x^*)$. We proceed to show that R_{λ,η_0} is single-valued on its domain. Take $x_i^* \in \operatorname{dom} R_{\lambda,\eta_0}$, $i = 1, 2$. Choose $y_i \in R_{\lambda,\eta_0}(x_i^*)$; that is, $y_i \in \bar{x} + \eta_0\mathbb{B}$ and $x_i^* \in J(y_i) + \lambda\partial^\pi f(y_i) \cap \frac{c_0}{t_0}\mathbb{B}_*$, $i = 1, 2$. Set $y_i^* := \lambda^{-1}[x_i^* - J(y_i)], i = 1, 2$. Then, $y_i^* \in \partial^\pi f(y_i) \cap \frac{c_0}{\lambda t_0}\mathbb{B}_*$ with $y_i \in \bar{x} + \eta_0\mathbb{B}$, $i = 1, 2$. Because $\frac{1}{t_0\lambda} > t_0$, we apply Proposition 3.1 with $t = \frac{1}{t_0\lambda}$ to write

$$\langle \lambda^{-1}[x_1^* - J(y_1)] - \lambda^{-1}[x_2^* - J(y_2)], y_1 - y_2 \rangle \;=\; \langle y_1^* - y_2^*, y_1 - y_2 \rangle$$

$$\geq \quad -\frac{1}{t_0\lambda}\langle J(y_1) - J(y_2), y_1 - y_2\rangle, \qquad (3.16)$$

which ensures that

$$\langle x_1^* - x_2^*, y_1 - y_2\rangle \geq (1 - t_0^{-1})\langle J(y_1) - J(y_2), y_1 - y_2\rangle.$$

Let $M := \|\bar{x}\| + \eta_0$. Then, $\|y_i\| \leq \|\bar{x}\| + \eta_0 \leq M$ $(i = 1, 2)$. By (2.10), we obtain

$$\langle J(y_1) - J(y_2), y_1 - y_2\rangle \geq C_M\|y_1 - y_2\|^q$$

with $C_M = \dfrac{2c_q}{4^{q-1}M^{q-2}}$. Using once again the assumption $t_0 > 1$ to get $1 - t_0^{-1} > 0$, and so, we obtain

$$\langle x_1^* - x_2^*, y_1 - y_2\rangle \geq (1 - t_0^{-1})\langle J(y_1) - J(y_2), y_1 - y_2\rangle \geq (1 - t_0^{-1})C_M\|y_1 - y_2\|^q.$$

This ensures that

$$\|x_1^* - x_2^*\| \geq (1 - t_0^{-1})C_M\|y_1 - y_2\|^{q-1},$$

and so

$$\|y_1 - y_2\| \leq \left[\frac{4^{q-1}(\|\bar{x}\| + \eta_0)^{q-2}}{2c_q(1 - t_0^{-1})}\right]^{\frac{1}{q-1}} \|x_1^* - x_2^*\|^{\frac{1}{q-1}}.$$

Finally, we find constants $\eta_0 > 0, c_0 > 0, t_0 > 0$ such that for any $\lambda \in \left(0, \min\left\{t_0^{-2}, \dfrac{\lambda_0}{t_0}\right\}\right)$, any $x_i^* \in$ dom R_{λ,η_0}, $i = 1, 2$, and any $y_i \in R_{\lambda,\eta_0}(x_i^*)$, we have

$$\|y_1 - y_2\| \leq \left[\frac{4^{q-1}(\|\bar{x}\| + \eta_0)^{q-2}}{2c_q(1 - t_0^{-1})}\right]^{\frac{1}{q-1}} \|x_1^* - x_2^*\|^{\frac{1}{q-1}}. \qquad (3.17)$$

Finally, we take any $x^* \in$ dom R_{λ,η_0} and let $y_1, y_2 \in R_{\lambda,\eta_0}(x^*)$. The inequality (3.17) ensures that $y_1 = y_2$, and hence, the set-valued mapping R_{λ,η_0} is at most single-valued. This completes the proof. □

4. Conclusions

In summary, the Moreau envelope and its associated proximal mapping have proven to be indispensable tools in variational analysis, particularly in the study of constrained optimization problems in Hilbert spaces. The extension of these concepts to Banach spaces via the V-Moreau envelope and the (f, λ)-generalized projection has opened new avenues for analyzing nonconvex functions and sets in more general geometric settings. However, the existing theory has been primarily developed in a global framework, which limits its applicability to problems where only local information is available or relevant.

In this paper, we address this gap by introducing and developing a local theory for the V-Moreau envelope and its associated generalized projection. More precisely, for a given point $\bar{x}$ in the domain of f and a radius $\rho > 0$, we define the local V-Moreau envelope $e^V_{\lambda,\bar{x},\rho}f$ and the corresponding local (f, λ)-generalized projection $\pi^{f,\lambda}_{\bar{x},\rho}$. These objects allow us to study the variational properties of f and S in a neighborhood of $\bar{x}$ rather than globally on the whole space.

Our main contributions are as follows:

- We established the local Lipschitz continuity of $e^{V}_{\lambda,\bar{x},\rho}f$ in reflexive Banach spaces with smooth dual norms (Proposition 2.1).
- We proved that the existence of the local generalized projection is guaranteed whenever the local Moreau envelope is Fréchet subdifferentiable (Theorem 2.1).
- In the framework of uniformly convex and uniformly smooth Banach spaces, we investigated the relationship between $\pi^{f,\lambda}_{\bar{x},\rho}$ and the V-proximal subdifferential of f (Propositions 2.3 and 3.5).
- Our main result (Theorem 3.3) showed that for the class of V-primal lower nice functions, the local generalized projection was locally nonempty, single-valued, and Hölder continuous.
- We derived additional properties for this class of functions and for V-proximal sets, thereby extending several classical theorems from [12, 13, 26] to the Banach space setting.

These results not only generalize existing Hilbert space theories but also provide a flexible framework for tackling local constrained optimization problems in smooth Banach spaces. The local nature of our approach makes it particularly well-suited for applications involving variational inequalities, proximal point methods, and nonsmooth analysis in infinite-dimensional settings.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no conflicts of interest.

References

1. Y. Alber, I. Ryazantseva, *Nonlinear Ill-posed problems of monotone type*, Dordrecht: Springer, 2006. http://doi.org/10.1007/1-4020-4396-1
2. Y. I. Alber, A. I. Notik, On some estimates for projection operators in Banach spaces, *Communications on Applied Nonlinear Analysis*, **2** (1993), 47–55.

3. Y. I. Alber, Generalised projection operators in Banach spaces: properties and applications, *Functional Differential Equations*, **1** (1994), 1–21.

4. Y. I. Alber, A. N. Iusem, M. V. Solodov, Minimization of nonsmooth convex functionals in Banach spaces, *J. Convex Anal.*, **4** (1997), 235–255.

5. H. H. Bauschke, P. L. Combettes, *Convex analysis and monotone operator theory in Hilbert spaces*, 2 Eds., Cham: Springer, 2017. https://doi.org/10.1007/978-3-319-48311-5

6. F. Bernard, L. Thibault, N. Zlateva, Characterizations of prox-regular sets in uniformly convex Banach space, *J. Convex Anal.*, **13** (2006), 525–559.

7. M. Bounkhel, *Regularity concepts in nonsmooth analysis: theory and applications*, New York: Springer, 2012. http://doi.org/10.1007/978-1-4614-1019-5

8. M. Bounkhel, Calculus rules for V-proximal subdifferentials in smooth Banach spaces, *J. Funct. Space.*, **2016** (2016), 1917387. https://doi.org/10.1155/2016/1917387

9. M. Bounkhel, Generalized Projections on closed nonconvex sets in uniformly convex and uniformly smooth Banach spaces, *J. Funct. Space.*, **2015** (2015), 478437. https://doi.org/10.1155/2015/478437

10. M. Bounkhel, Generalized (f, λ)-projection operator on closed nonconvex sets and its applications in reflexive smooth Banach spaces, *AIMS Mathematics*, **8** (2023), 29555–29568. http://doi.org/10.3934/math.20231513

11. M. Bounkhel, V-Moreau envelope of nonconvex functions on smooth Banach spaces, *AIMS Mathematics,* **9** (2024), 28589–28610. https://doi.org/10.3934/math.20241387

12. M. Bounkhel, M. Bachar, Generalized prox-regularity in reflexive Banach spaces with smooth dual norm, *J. Math. Anal. Appl.*, **475** (2019), 699–729. https://doi.org/10.1016/j.jmaa.2019.02.064

13. M. Bounkhel, M. Bachar, Primal lower nice functions in reflexive smooth Bnach spaces, *Mathematics*, **8** (2020), 2066. https://doi.org/10.3390/math8112066

14. M. Bounkhel, R. Al-Yusof, Proximal analysis in reflexive smooth Banach spaces, *Nonlinear Anal. Theor.*, **73** (2010), 1921–1939. https://doi.org/10.1016/j.na.2010.04.077

15. M. Bounkhel, M. Bachar, V-Prox-regular functions in smooth Banach spaces, *J. Funct. Space.*, **2020** (2020), 9465492. https://doi.org/10.1155/2020/9465492

16. M. Bounkhel, M. Bachar, V-Proximal trustworthy Banach spaces, *J. Funct. Space.*, **2020** (2020), 4274160. https://doi.org/10.1155/2020/4274160

17. F. H. Clarke, R. J. Stern, P. R. Wolenski, Proximal smoothness and the lower-C^2 property, *J. Convex Anal.*, **2** (1995), 117–144.

18. P. L. Combettes, V. R. Wajs, Signal recovery by proximal forward-backward splitting, *Multiscale Model. Sim.*, **4** (2005), 1168–1200. https://doi.org/10.1137/050626090

19. H. Federer, Curvature measures, *Trans. Amer. Math. Soc.*, **93** (1959), 418–491. https://doi.org/10.1090/S0002-9947-1959-0110078-1

20. W. L. Hare, R. A. Poliquin, Prox-regularity and stability of the proximal mapping, *J. Convex Anal.*, **14** (2007), 589–606.

21. W. L. Hare, C. Planiden, Parametrically prox-regular functions, *J. Convex Anal.*, **21** (2014), 901–923.

22. J. L. Li, The generalized projection operator on reflexive Banach spaces and its applications, *J. Math. Anal. Appl.*, **306** (2005), 55–71. https://doi.org/10.1016/j.jmaa.2004.11.007

23. J. L. Li, On the existence of solutions of variational inequalities in Banach spaces, *J. Math. Anal. Appl.*, **295** (2004), 115–126. https://doi.org/10.1016/j.jmaa.2004.03.010

24. J. J. Moreau, Fonctions convexes duales et points proximaux dans un espace Hilbertien, *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, **255** (1962), 2897–2899.

25. J. J. Moreau, Proximité et dualité dans un espace Hilbertien, *Bulletin de la Société Mathématique de France* **93** (1965), 273–299. https://doi.org/10.24033/bsmf.1625

26. R. A. Poliquin, R. T. Rockafellar, Prox-regular functions in variational analysis, *Trans. Amer. Math. Soc.*, **348** (1996), 1805–1838. https://doi.org/10.1090/s0002-9947-96-01544-9

27. R. A. Poliquin, R. T. Rockafellar, L. Thibault, Local differentiability of distance functions, *Trans. Amer. Math. Soc.*, **352** (2000), 5231–5249. https://doi.org/10.1090/s0002-9947-00-02550-2

28. R. T. Rockafellar, R. J. B. Wets, *Variational analysis*, 1 Eds., Heidelberg: Springer, 1998. https://doi.org/10.1007/978-3-642-02431-3

29. A. Shapiro, Existence and differentiability of metric projections in Hilbert spaces, *SIAM J. Optimiz.*, **4** (1994), 130–141. https://doi.org/10.1137/0804006

30. W. Takahashi, *Nonlinear functional analysis*, Yokohama: Yokohama Publishers, 2000.

31. K. Q. Wu, N. J. Huang, The generalized f-projection operator with an application, *B. Aust. Math. Soc.*, **73** (2006), 307–317. http://doi.org/10.1017/S0004972700038892

32. K. Q. Wu, N. J. Huang, Properties of the generalized f-projection operator and its application in Banach spaces, *Comput. Math. Appl.*, **54** (2007), 399–406. http://doi.org/10.1016/j.camwa.2007.01.029