Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 20986–21011.
DOI:10.3934/math.2026852
Received: 29 April 2026
Revised: 11 July 2026
Accepted: 13 July 2026
Published: 16 July 2026

Research article

Local total neighborhood antimagic labelings–a new variant of vertex colorings

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Abstract: Let $G = (V, E)$ be a simple graph of order p and size q. A local total neighborhood antimagic labeling of G is a bijection $f : V \cup E \to [1, p + q]$ such that, for every vertex $u \in V$, the induced vertex color $f^+_{tn}(u) = \sum_{ux\in E}[f(ux) + f(x)]$ assigns distinct integers (called colors) to adjacent vertices of G. We define $f^+_{tn}(u) = 0$ when u is an isolated vertex. A graph G is said to be local total neighborhood antimagic if it admits a local total neighborhood antimagic labeling. The local total neighborhood antimagic chromatic number of G, denoted by $\chi_{ltna}(G)$, is the minimum number of distinct induced colors taken over all such labelings of G. In this paper, we investigated properties of $\chi_{ltna}(G)$ for graphs without isolated vertices and provide results that characterize this chromatic parameter. We focused on several graph classes, including paths, cycles, complete graphs, complete bipartite graphs, wheel graphs, and various types of join graphs.

Keywords: local total neighborhood antimagic chromatic number; path; cycle; complete graph; complete bipartite graph; wheel graph; join product
Mathematics Subject Classification: 05C78, 05C15

1. Introduction

For integers $a < b$, let $[a, b] = \{a, a + 1, \ldots, b\}$. The disjoint union of graphs G and H is denoted by $G + H$, and nG is the disjoint union of $n \geq 2$ copies of G. The join of graphs G and H is denoted by $G \vee H$. For a given vertex x of G, let $d_G(x)$ (or $d(x)$ if no ambiguity) be the degree of x. If $uv \notin E(G)$ (respectively, $uv \in E(G)$), we let $G+uv$ (respectively, $G-uv$) be obtained from G by adding (respectively, deleting) the edge uv.

Let $G=(V,E)$ be a simple (p,q)-graph, i.e., G is of order p and size q. A bijective edge labeling $h: E \to [1,q]$ is a local antimagic labeling if every edge $uv \in E$ has $h^+(u) \neq h^+(v)$, where $h^+(u) = \sum_{e\sim u} h(e)$ (over all edges e incident to u) is called the induced vertex label of u. The smallest number of distinct induced vertex labels of G over all possible local antimagic labelings h of G is called the local antimagic chromatic number of G, denoted by $\chi_{la}(G)$ (see [1]). Interested readers may refer to [6–9] for related papers.

A bijective total labeling $f: V\cup E \to [1,p+q]$ is called a *total neighborhood antimagic labeling* if all vertices have distinct total weights where the total weight of a vertex u is $f^+_{tn}(u) = \sum_{ux\in E}[f(ux)+f(x)]$, over every neighbor x of u. If the total weights form an arithmetic sequence with first term a and common difference $d \geq 1$, we say f is an (a,d)-total neighborhood antimagic labeling. Note that if we allow $d=0$, then f is known as a total neighborhood magic labeling with magic constant a (see [10–12]). Motivated by this, we say f is a *local total neighborhood antimagic labeling* of G if f induces a vertex coloring $f^+_{tn}: V \to \mathbb{Z}$, where $f^+_{tn}(u) = \sum\limits_{ux\in E}[f(ux)+f(x)]$. If u is an isolated vertex, we let $f^+_{tn}(u)=0$. We say G is *local total neighborhood antimagic* if it admits a *local total neighborhood antimagic labeling f*.

We say $f^+_{tn}(u)$ is the induced color of u under f. For a local total neighborhood antimagic labeling f, the number of distinct induced colors is denoted by $c_{ltna}(f)$ so that f is also a local total neighborhood antimagic $c_{ltna}(f)$-coloring of G. The *local total neighborhood antimagic chromatic number* of G, denoted by $\chi_{ltna}(G)$, is

$$\min\{c_{ltna}(f) \mid f \text{ is a local total neighborhood antimagic labeling of } G\}.$$

Since we define the label of any isolated vertex by 0, we may define $\chi_{ltna}(O_n)=1$, where O_n is the empty graph of order n. Moreover, if G is a graph without isolated vertex and $H = G+O_n$, $n\geq 1$, then $\chi_{ltna}(H) = \chi_{ltna}(G)+1$. Thus, throughout this paper, we focus only on graph G, which is a simple graph of order $p \geq 2$ without isolated vertices and admits a local total neighborhood antimagic labeling f, unless stated otherwise. Moreover,

$$\chi_{ltna}(G) \geq \chi(G) \geq 2. \tag{1.1}$$

Similar to the local antimagic conjecture, we also have:

Conjecture 1.1. *Every graph admits a local total neighborhood antimagic labeling.*

Antimagic-type labelings have been widely studied as effective methods to distinguish vertices via induced weights, with substantial progress in total labelings and neighborhood-based weighting frameworks. The local total neighborhood antimagic labeling considered in this paper merges these ideas by assigning labels to vertices and edges and enforcing neighborhood-sum constraints that separate adjacent vertices. This yields a strengthened variant of local antimagic labeling. Motivated by the increasing interest in locally distinguishing colorings and their relevance to network identification, we also introduce and study the corresponding chromatic invariant, the local total neighborhood antimagic chromatic number. Examining this parameter for several fundamental graph classes not only extends the labeling theory but also provides structural insights that advance the study of neighborhood-based total labelings.

2. Preliminary results

We refer to [2] for concepts and notations not defined in this paper. We have some preliminary results on local total neighborhood antimagic chromatic number of G.

Lemma 2.1. *Suppose $k \geq 1$ and G has a vertex that is adjacent to k pendant vertices. If G admits a local total neighborhood antimagic labeling, then $\chi_{ltna}(G) \geq k+1$.*

Proof. Let v be adjacent to k pendant vertices $u_i, 1 \leq i \leq k$. Thus, under each local total neighborhood antimagic labeling f, we must have $f^+_{tn}(u_i) = f(u_iv) + f(v), 1 \leq i \leq k$. Therefore, $f^+_{tn}(u_i) \neq f^+_{tn}(u_j)$ for $i \neq j$. Moreover, $f^+_{tn}(v) \neq f^+_{tn}(u_i)$ for $1 \leq i \leq k$. Therefore, there are at least $k+1$ distinct induced color under f. The lemma holds. □

Lemma 2.2. *Suppose G is a (p,q)-graph and f is a local total neighborhood antimagic labeling of G. If, for any $u, v \in V(G)$,*

(i) $f^+_{tn}(u) = f^+_{tn}(v)$ implies that $d(u) = d(v)$, and

(ii) $f^+_{tn}(u) \neq f^+_{tn}(v)$ implies that $2(p+q+1)[d(u) - d(v)] \neq f^+_{tn}(u) - f^+_{tn}(v)$,

then $g = p+q+1-f$ is also a local total neighborhood antimagic labeling of G with $c_{ltna}(f) = c_{ltna}(g)$.

Proof. For any $u, v \in V(G)$, we have $g^+_{tn}(u) = 2d(u)(p+q+1) - f^+_{tn}(u)$ and $g^+_{tn}(v) = 2d(v)(p+q+1) - f^+_{tn}(v)$. Now, $g^+_{tn}(u) - g^+_{tn}(v) = 2(p+q+1)[d(u) - d(v)] - [f^+_{tn}(u) - f^+_{tn}(v)]$. If $f^+_{tn}(u) = f^+_{tn}(v)$, then condition (i) implies that $g^+_{tn}(u) = g^+_{tn}(v)$. If $f^+_{tn}(u) \neq f^+_{tn}(v)$, then condition (ii) implies that $g^+_{tn}(u) \neq g^+_{tn}(v)$. Thus, g is also a local total neighborhood antimagic labeling of G with $c_{ltna}(f) = c_{ltna}(g)$. □

Lemma 2.3. *Let G be a (p,q)-graph such that $\chi(G) = \chi_{ltna}(G) = t \geq 2$ and f be a local total neighborhood antimagic t-coloring of G with color classes $V_1, \ldots, V_t$. Suppose $uv \notin E(G)$, $\chi(G+uv) = t+1$, and $f^+_{tn}(u) + f(v) + p + q + 1$, $f^+_{tn}(v) + f(u) + p + q + 1$ and $f^+_{tn}(x)$ are mutually distinct for each $x \in (V(G) \setminus \{u,v\})$. Then*

(a) If there exists a color class $V_i = \{u,v\}$, then $\chi_{ltna}(G+uv) = t+1$.

(b) If there exist distinct singleton color classes $V_i = \{u\}$ and $V_j = \{v\}$, then $\chi_{ltna}(G+uv) = t$.

Proof. Define $g : V(G+uv) \cup E(G+uv) \to [1, p+q+1]$ such that $g(x) = f(x)$ for $x \in V(G) \cup E(G)$ and $g(uv) = p+q+1$. Now, $g^+_{tn}(x) = f^+_{tn}(x)$ if $x \notin \{u,v\}$, and that $g^+_{tn}(u) = f^+_{tn}(u) + f(v) + p + q + 1$, $g^+_{tn}(v) = f^+_{tn}(v) + f(u) + p + q + 1$. Thus, $g^+_{tn}(u)$, $g^+_{tn}(v)$, $g^+_{tn}(x)$ are all distinct for $x \in (V(G) \setminus \{u,v\})$. Under (a), g is a local total neighborhood antimagic $(t+1)$-coloring of $G+uv$ so that $t+1 = \chi(G+uv) \leq \chi_{ltna}(G+uv) \leq t+1$. Under (b), g is a local total neighborhood antimagic t-coloring of $G+uv$ so that $t \leq \chi(G) \leq \chi_{ltna}(G+uv) \leq t$. The lemma holds. □

Lemma 2.4. *Let G be a (p,q)-graph such that $\chi(G) = \chi_{ltna}(G) = t \geq 2$ and $uv \in E(G)$. If f is a local total neighborhood antimagic t-coloring of G such that (i) $f(uv) = p+q$, (ii) $f^+_{tn}(u) - f^+_{tn}(v) = f(v) - f(u)$, and (iii) u and v belong to distinct singleton color classes $V_i = \{u\}$ and $V_j = \{v\}$ and $f^+_{tn}(u) - f(v) - (p+q) \neq f^+_{tn}(x)$ for each $x \in (V(G) \setminus \{u,v\})$, then $\chi_{ltna}(G-uv) \leq t-1$ with equality holding if $\chi(G-uv) = t-1$.*

Proof. By definition, $f_{tn}^+(u) \neq f_{tn}^+(v)$. By condition (i), we can define a bijection $g : V(G-uv) \cup E(G-uv) \to [1, p+q-1]$ such that $g(x) = f(x)$ for $x \in V(G-uv) \cup E(G-uv)$. Now, $g_{tn}^+(x) = f_{tn}^+(x)$ for $x \notin \{u, v\}$, $g_{tn}^+(u) = f_{tn}^+(u) - f(v) - (p+q)$ and $g_{tn}^+(v) = f_{tn}^+(v) - f(u) - (p+q)$. By condition (ii), $g_{tn}^+(u) = g_{tn}^+(v)$. By condition (iii), g is a local total neighborhood antimagic $(t-1)$-coloring of $G-uv$. Thus, $\chi_{ltna}(G-uv) \le t-1$ with equality holding if $\chi(G-uv) = t-1$. □

Remark 1. *The simplest graph that satisfies Lemma 2.4 is K_2. Another example is the K_4 with vertices $u_i, 1 \le i \le 4$ and a local total neighborhood antimagic 4-coloring f given by $f(u_1) = 1$, $f(u_2) = 2$, $f(u_3) = 3$, $f(u_4) = 4$, $f(u_1u_2) = 10$, $f(u_1u_3) = 6$, $f(u_1u_4) = 9$, $f(u_2u_3) = 7$, $f(u_2u_4) = 8$, $f(u_3u_4) = 5$. Clearly, $f_{tn}^+(u_1) = 34$, $f_{tn}^+(u_2) = 33$, $f_{tn}^+(u_3) = 25$ and $f_{tn}^+(u_4) = 28$. It is easy to verify that $f_{tn}^+(u_1) - f_{tn}^+(u_2) = f(u_2) - f(u_1)$, and that K_4 and f also satisfy all other conditions of Lemma 2.4. Since $K_4 - u_1u_2 \cong K_{1,1,2}$, by Lemma 2.4, we have $\chi_{ltna}(K_{1,1,2}) = 3$. In Subsection 4.3, we prove that $\chi_{ltna}(K_{1,1,n}) = 3$ for $n \ge 1$ using Lemma 2.3(a).*

3. Copies of a graph

Theorem 3.1. *Let G be a (p, q)-graph with $\chi_{ltna}(G) = t \ge 2$. Suppose f is a local total neighborhood antimagic t-coloring of G. Let $n \ge 2$ be an integer. If, for any $u, v \in V(G)$,*

(a) $f_{tn}^+(u) = f_{tn}^+(v)$ implies that $d_G(u) = d_G(v)$, and

(b) $f_{tn}^+(u) \neq f_{tn}^+(v)$ implies that $n[f_{tn}^+(u) - f_{tn}^+(v)] \neq (d_G(u) - d_G(v))(n-1)$,

then $\chi_{ltna}(nG) \le t$ with equality holding if $\chi(G) = t$.

Proof. Let f be a local total neighborhood antimagic t-coloring of G. Suppose $nG = G_1 + G_2 + \cdots + G_n$ such that $G_k \cong G, 1 \le k \le n$. For $1 \le i \le p$, let u_i be the i-th vertex of G and $u_{k,i}$ be the corresponding i-th vertex of G_k. Thus, nG has order np and size nq. Moreover, $d_G(u_i) = d_{nG}(u_{k,i}) = d_i$. For $1 \le i, j \le p$ and $1 \le k \le n$, define $g : V(nG) \cup E(nG) \to [1, n(p+q)]$ such that

(i) $g(u_{k,i}) = nf(u_i) + 1 - k$,

(ii) $g(u_{k,i}u_{k,j}) = n[f(u_iu_j) - 1] + k$ if $u_iu_j \in E(G)$.

Note that for every label x assigned to $V(G) \cup E(G)$, the corresponding labels assigned to $V(nG) \cup E(nG)$ are $(x-1)n+1$ to nx. Since $1 \le x \le p+q$, g is a bijective total labeling of nG. Without loss of generality, suppose u_i is adjacent to u_{i_a} in G, $1 \le a \le d_i$, so that the corresponding vertex $u_{k,i}$ is adjacent to u_{k,i_a} in G_k. We now have $f_{tn}^+(u_i) = \sum_{a=1}^{d_i} [f(u_{i_a}) + f(u_iu_{i_a})]$. Thus,

$$\begin{aligned} g_{tn}^+(u_{k,i}) &= \sum_{a=1}^{d_i} [g(u_{k,i_a}) + g(u_{k,i}u_{k,i_a})] \\ &= \sum_{a=1}^{d_i} [nf(u_{i_a}) + 1 - k] + \sum_{a=1}^{d_i} [n(f(u_iu_{i_a}) - 1) + k] \\ &= \sum_{a=1}^{d_i} n[f(u_{i_a}) + f(u_iu_{i_a})] - d_i(n-1) = nf_{tn}^+(u_i) - d_i(n-1). \end{aligned}$$

Let $1 \leq k_1, k_2 \leq n$. Now, $g^+_{tn}(u_{k_1,i}) - g^+_{tn}(u_{k_2,j}) = n(f^+_{tn}(u_i) - f^+_{tn}(u_j)) - (d_i - d_j)(n-1)$. Therefore, conditions (a) and (b) imply that $f^+_{tn}(u_i) = f^+_{tn}(u_j)$ if and only if $g^+_{tn}(u_{k_1,i}) = g^+_{tn}(u_{k_2,j})$ for all $1 \leq i, j \leq p$. Thus, g is a local total neighborhood antimagic t-coloring of nG so that $\chi_{ltna}(nG) \leq t$. Since $\chi(nG) = \chi(G)$, we conclude that $\chi_{ltna}(nG) = t$ if $\chi(G) = t$. This completes the proof. □

Corollary 3.2. *If G is a regular graph and $\chi_{ltna}(G) = t$, then $\chi_{ltna}(nG) \leq t$ with equality holding if $\chi(G) = t$.*

4. Some common graphs

4.1. Paths and cycles

For $m \geq 2$, let $P_m = u_1u_2\cdots u_m$ be the path of order m and let $e_i = u_iu_{i+1}$ for $1 \leq i \leq m-1$. For $m \geq 3$, let C_m be the cycle of order m obtained from P_m by adding an edge $e_m = u_mu_1$. In this section, we keep these notations.

Lemma 4.1. $\chi_{ltna}(P_n) = \begin{cases} 2 & \textit{for } n = 2, 5, \\ 3 & \textit{for } n = 3, 4, 6. \end{cases}$

Proof. We first show that there is a local total neighborhood antimagic labeling of P_n, where $2 \leq n \leq 6$. For P_2, label u_1u_2, u_1, u_2 by $1, 2, 3$ respectively. We show other labelings in the following figures.

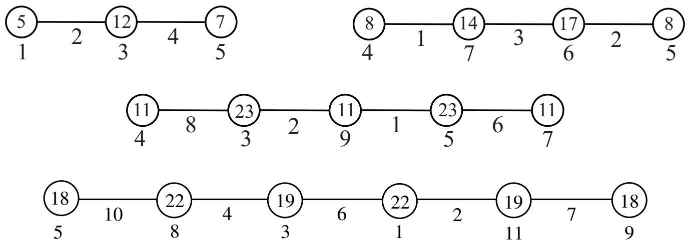


Figure 1. A local total neighborhood antimagic 2-coloring of P_5 and local total neighborhood antimagic 3-colorings of P_3, P_4 and P_6.

Thus, $\chi_{ltna}(P_n) \leq \begin{cases} 2 & \text{for } n = 2, 5, \\ 3 & \text{for } n = 3, 4, 6. \end{cases}$

Now, we show the lower bound of the $\chi_{ltna}(P_n)$ for $2 \leq n \leq 6$. Let f be a local total neighborhood antimagic labeling of P_n. By definition(1.1), $\chi_{ltna}(P_n) \geq \chi(P_n) = 2$ for $n = 2, 5$. Lemma 2.1 implies that $\chi_{ltna}(P_3) \geq 3$.

For P_4, $f^+_{tn}(u_2) + f^+_{tn}(u_3) = f(u_1) + f(e_1) + f(e_2) + f(u_3) + f(e_2) + f(u_2) + f(e_3) + f(u_4) = f^+_{tn}(u_1) + f^+_{tn}(u_4) + f(u_1) + f(u_4) + 2f(e_2) > f^+_{tn}(u_1) + f^+_{tn}(u_4)$. If $f^+_{tn}(u_2) = f^+_{tn}(u_4)$, then $f^+_{tn}(u_1), f^+_{tn}(u_2), f^+_{tn}(u_3)$ are distinct. If $f^+_{tn}(u_2) \neq f^+_{tn}(u_4)$, then $f^+_{tn}(u_2), f^+_{tn}(u_3), f^+_{tn}(u_4)$ are distinct. Thus, $\chi_{ltna}(P_4) \geq 3$.

Suppose that $\chi_{ltna}(P_6) = 2$. Let $a = f^+_{tn}(u_1)$ and $b = f^+_{tn}(u_2)$. Thus,

$$3(a+b) = \sum_{i=1}^{6} f^+_{tn}(u_i) = 2\sum_{j=1}^{11} j - f(u_1) - f(u_6) = 132 - (f(u_1) + f(u_6)) \leq 129,$$

$$2(a+b) = \sum_{i=2}^{5} f^+_{tn}(u_i) = \sum_{j=1}^{11} j + f(e_2) + f(u_3) + f(e_3) + f(u_4) + f(e_4)$$

$$= 66 + f(e_2) + f(u_3) + f(e_3) + f(u_4) + f(e_4) \geq 81.$$

Thus, $41 \leq a + b \leq 43$ and $f(u_1) + f(u_6) \equiv 0 \pmod 3$.

1) If $a + b = 41$, then $f(u_1) + f(u_6) = 9$. Then $f(e_2) + f(u_3) + f(e_3) + f(u_4) + f(e_4) = 16$. The only combination is $\{f(e_2), f(u_3), f(e_3), f(u_4), f(e_4)\} = \{1, 2, 3, 4, 6\}$. There is no solution for $f(u_1) + f(u_6) = 9$.

2) If $a + b = 42$, then $f(u_1) + f(u_6) = 6$. Then $f(e_2) + f(u_3) + f(e_3) + f(u_4) + f(e_4) = 18$. The combination for $\{f(e_2), f(u_3), f(e_3), f(u_4), f(e_4)\}$ is $\{1, 2, 3, 4, 8\}$, $\{1, 2, 3, 5, 7\}$ or $\{1, 2, 4, 5, 6\}$. There is no solution for $f(u_1) + f(u_6) = 6$.

3) If $a + b = 43$, then $f(u_1) + f(u_6) = 3$. Hence $\{f(u_1), f(u_6)\} = \{1, 2\}$. Now $f(e_2) + f(u_3) + f(e_3) + f(u_4) + f(e_4) = 20$. Let $m = \min\{f(e_2), f(u_3), f(e_3), f(u_4), f(e_4)\}$, then $20 = f(e_2) + f(u_3) + f(e_3) + f(u_4) + f(e_4) \geq m + (m + 1) + (m + 2) + (m + 3) + (m + 4) = 5m + 10$. Thus, $m \leq 2$. Since 1 and 2 are used, there is no solution.

   Hence, $\chi_{ltna}(P_6) \geq 3$. The lemma holds. □

From the proof of Lemma 4.1, we have the following corollaries:

Corollary 4.2. *A bipartite graph G with a component P_m $(m = 3, 4)$ has $\chi_{ltna}(G) \geq 3$.*

Corollary 4.3. *For $n \geq 2$, $\chi_{ltna}(nP_2) = 2$, and $\chi_{ltna}(nP_3) = \chi_{ltna}(nP_4) = 3$.*

Proof. By (1.1), $\chi_{ltna}(nP_2) \geq \chi(nP_2) = 2$. Corollary 4.2 implies that $\chi_{ltna}(nP_m) \geq 3$ for $m = 3, 4$. It is easy to check that the labeling f for $P_m, m = 2, 3, 4$ in the proof of Lemma 4.1 satisfies the conditions of Theorem 3.1. Thus, $\chi_{ltna}(nP_2) \leq 2$ and $\chi_{ltna}(nP_m) \leq 3$ for $m = 3, 4$. The corollary holds. □

Corollary 4.4. *For $n \geq 2$, $\chi_{ltna}(nP_6) \leq 3$.*

Proof. Since the labeling f for P_6 in the proof of Lemma 4.1 satisfies the conditions of Theorem 3.1, we have the corollary. □

Problem 4.1. *Determine $\chi_{ltna}(P_n)$ for $n \geq 7$, and $\chi_{ltna}(mP_n)$ for $m \geq 2, n \geq 5$.*

We now give some results on 2-regular graphs.

Theorem 4.5. *For $n, m \geq 1$, $\chi_{ltna}(nC_{2m+1}) = 3$.*

Proof. By (1.1), $\chi_{ltna}(nC_{2m+1}) \geq \chi(nC_{2m+1}) = 3$ for $n, m \geq 1$.

For $m \geq 1$, we define a total labeling $f : V(C_{2m+1}) \cup E(C_{2m+1}) \to [1, 4m + 2]$ such that

(a) $f(e_{2i-1}) = i$ for $i = 1, 2, \ldots, m + 1$ with the set of labels $[1, m + 1]$,

(b) $f(e_{2i}) = 3m + 2 + i$ for $i = 1, 2, \ldots, m$ with the set of labels $[3m + 3, 4m + 2]$,

(c) $f(u_{2i-1}) = 2m + 3 - i$ for $i = 1, 2, \ldots, m + 1$ with the set of labels $[m + 2, 2m + 2]$,

(d) $f(u_{2i}) = 3m + 3 - i$ for $i = 1, 2, \ldots, m$ with the set of labels $[2m + 3, 3m + 2]$.

Now, we have $f_{tn}^{+}(u_1) = f(e_{2m+1}) + f(u_{2m+1}) + f(e_1) + f(u_2) = (m+1)+(m+2)+1+(3m+2) = 5m+6$. Similarly, we have $f_{tn}^{+}(u_{2i-1}) = 9m + 8$ for $2 \le i \le m + 1$, and $f_{tn}^{+}(u_{2i}) = 7m + 7$ for $1 \le i \le m$.

Thus, f is a local total neighborhood antimagic 3-coloring of C_{2m+1} and $\chi_{ltna}(C_{2m+1}) \le 3$. Thus, $\chi_{ltna}(C_{2m+1}) = 3$. By Corollary 3.2, the theorem holds. □

Example 4.1. *The local total neighborhood antimagic 3-coloring of* $3C_5$ *extended from the labeling in Theorem 4.5 is given in Figure 2.*

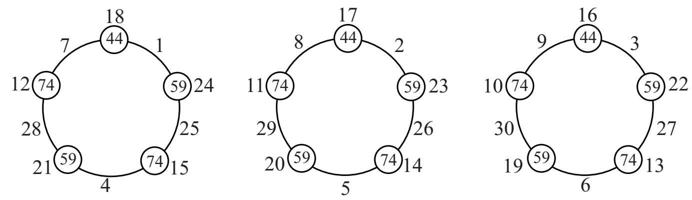


Figure 2. A local total neighborhood antimagic 3-coloring of $3C_5$.

Lemma 4.6. *If G is a bipartite graph with a component C_4, then $\chi_{ltna}(G) \ge 3$.*

Proof. Let f be a local total neighborhood antimagic of G and let $C_4 = u_1u_2u_3u_4u_1$. Let a, b, c, d be the edge labels of C_4 consecutively under f. Then we have $[f_{tn}^{+}(u_1) - f_{tn}^{+}(u_3)] + [f_{tn}^{+}(u_2) - f_{tn}^{+}(u_4)] = 2a - 2c \neq 0$. Thus, at least one difference is nonzero. Without loss of generality, we may assume $f_{tn}^{+}(u_1) \neq f_{tn}^{+}(u_3)$. Thus, $f_{tn}^{+}(u_1)$, $f_{tn}^{+}(u_2)$ and $f_{tn}^{+}(u_3)$ are distinct. Hence, $\chi_{ltna}(G) \ge 3$. □

Theorem 4.7. *For $n \ge 1$, $\chi_{ltna}(nC_4) = 3$.*

Proof. Figure 3 below gives a local total neighborhood antimagic 3-coloring of C_4. By Corollary 3.2, $\chi_{ltna}(nC_4) \le 3$.

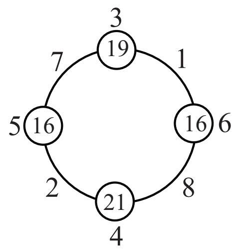


Figure 3. A local total neighborhood antimagic 3-coloring of C_4.

By Lemma 4.6, $\chi_{ltna}(nC_4) \ge 3$. Thus, the theorem holds. □

Example 4.2. *The local total neighborhood antimagic 3-coloring of* $3C_4$ *according to the labeling above is given in Figure 4.*

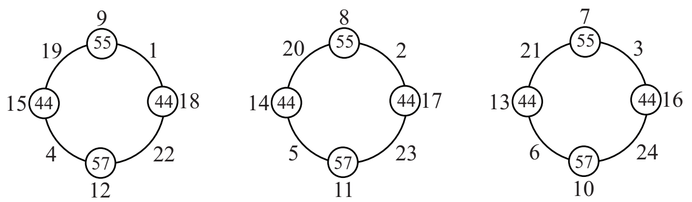


Figure 4. A local total neighborhood antimagic 3-coloring of $3C_4$.

Theorem 4.8. *For $n \geq 1$ and $m = 6, 8, 10$, $\chi_{ltna}(nC_m) = 2$.*

Proof. A local total neighborhood antimagic 2-coloring of C_m is given in Figure 5.

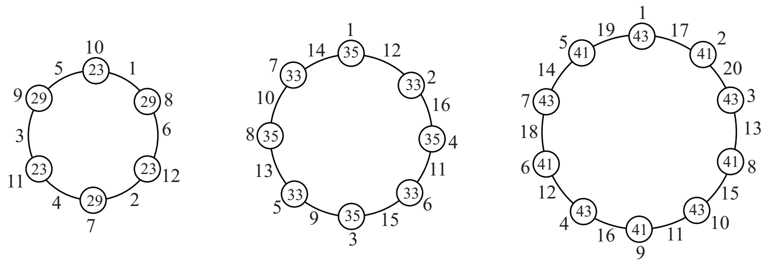


Figure 5. A local total neighborhood antimagic 2-coloring of $C_m, m = 6, 8, 10$.

Thus, $\chi_{ltna}(C_m) = 2$ for $m = 6, 8, 10$ since $\chi_{ltna}(C_m) \geq \chi(C_m) = 2$. By Corollary 3.2, the theorem holds. □

Conjecture 4.1. *For $n \geq 1$, $r \geq 6$, $\chi_{ltna}(nC_{2r}) = 2$.*

4.2. Complete graphs and complete bipartite graphs

Theorem 4.9. *For $n \geq 2$, $\chi_{ltna}(K_n) = \chi_{ltna}(mK_n) = n$.*

Proof. For $n = 2$, the result follows from Corollary 4.3. For $n \geq 3$, let $V(K_n) = \{v_i \mid 1 \leq i \leq n\}$. Without loss of generality, we may assume that K_n admits a local antimagic labeling $f : E(K_n) \to [1, \frac{n^2-n}{2}]$ such that $f^+(v_i) < f^+(v_j)$ for $1 \leq i < j \leq n$. For the existence of local antimagic labeling for connected graphs, please refer to [5]. Define a total labeling $g : V(K_n) \cup E(K_n) \to [1, \frac{n^2+n}{2}]$ such that $g(e) = f(e)$ for $e \in E(K_n)$, and $g(v_i) = \frac{n^2-n}{2} + n + 1 - i$ for $1 \leq i \leq n$. Thus, g is bijective, such that

$$\begin{aligned} g_{tn}^+(v_i) &= f^+(v_i) + \sum_{j=1}^{n} g(v_j) - g(v_i) = f^+(v_i) + \sum_{j=1}^{n} \left(\frac{n^2-n}{2} + j\right) - \left(\frac{n^2-n}{2} + n + 1 - i\right) \\ &= f^+(v_i) + \frac{n^3 - n^2 - 2}{2} + i. \end{aligned}$$

Therefore, $g_{tn}^+(v_i) < g_{tn}^+(v_j)$ for $1 \leq i < j \leq n$ so that g is a local total neighborhood antimagic n-coloring of K_n. Since $\chi(K_n) = n$, $\chi_{ltna}(K_n) = n$. By Corollary 3.2, we immediately have $\chi_{ltna}(mK_n) = n$. This completes the proof. □

Theorem 4.10. *For $n \geq 1$ and $m \geq 1$, $\chi_{ltna}(nK_{1,m}) = m + 1$.*

Proof. Since $nK_{1,1} \cong nP_2$ and $nK_{1,2} \cong nP_3$, the theorem holds for $m = 1, 2$ by Corollary 4.3. Consider $m \geq 3$. By Lemma 2.1, $\chi_{ltna}(K_{1,m}) \geq m+1$. Let $V(K_{1,m}) = \{u\} \cup \{v_i \mid 1 \leq i \leq m\}$ and $E(K_{1,m}) = \{uv_i \mid 1 \leq i \leq m\}$. We define a total labeling $f : V(K_{1,m}) \cup E(K_{1,m}) \to [1, 2m+1]$ such that $f(u) = 2m+1$, $f(v_i) = i$ and $f(uv_i) = 2m+1-i$, for $1 \leq i \leq m$. Thus, f is bijective. Moreover, for $1 \leq i \leq m$, $f_{tn}^+(v_i) = 4m+2-i$, $f_{tn}^+(u) = m(2m+1)$. Thus, f is a local total neighborhood antimagic $(m+1)$-coloring of $K_{1,m}$. Moreover, $\chi_{ltna}(K_{1,m}) \leq m + 1$. Consequently, $\chi_{ltna}(K_{1,m}) = m + 1$.

By Lemma 2.1, we also have $\chi_{ltna}(nK_{1,m}) \geq m + 1$. It is straightforward to check that f satisfies the conditions of Theorem 3.1 so that $\chi_{ltna}(nK_{1,m}) \leq m + 1$. Therefore, the theorem holds. □

Now, let us consider the complete bipartite graph $K_{m,n}$ for $m, n \geq 2$.

Theorem 4.11. *Suppose $m, n \geq 2$, $\chi_{ltna}(K_{m,n}) = 2$ except $\chi_{ltna}(K_{2,2}) = 3$.*

Proof. Since $K_{2,2} \cong C_4$, by Theorem 4.7, we get $\chi_{ltna}(K_{2,2}) = 3$.

Suppose $(m, n) \neq (2, 2)$. By (1.1), $\chi_{ltna}(K_{m,n}) \geq \chi(K_{m,n}) = 2$. Thus, define a bijective total labeling $f : V(K_{m,n}) \cup E(K_{m,n}) \to [1, t]$, where $t = mn + m + n$ such that f is a local total neighborhood antimagic 2-coloring. Let the vertex set and edge set of $K_{m,n}$ be $\{u_i, v_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ and $\{u_i v_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$, respectively.

Consider $m \equiv n \pmod 2$. Without loss of generality, we may assume that $n \geq m$. Let M be an $m \times n$ magic rectangle with entries $a_{i,j}$ in row i and column j, constant row sum $R = n(mn + 1)/2$ and constant column sum $C = m(mn + 1)/2$. For the existence of magic rectangle, please see [3, 4]. Define a total labeling $f : V(K_{m,n}) \cup E(K_{m,n}) \to [1, mn + m + n]$ such that

(1) $f(u_i v_j) = a_{i,j}$,

(2) $f(u_i) = mn + i, f(v_j) = mn + m + j$.

Thus, f is a bijection. Now,

$$f^+_{tn}(u_i) = R + \sum_{j=1}^{n} f(v_j) = n(mn + 1)/2 + n(mn + n) + n(n + 1)/2$$

and

$$f^+_{tn}(v_j) = C + \sum_{i=1}^{m} f(u_i) = m(mn + 1)/2 + m^2 n + m(m + 1)/2.$$

Thus, $f^+_{tn}(u_i) > f^+_{tn}(v_j)$. Therefore, f is a local total neighborhood antimagic 2-coloring.

Now, consider $m \not\equiv n \pmod 2$. We may assume $m = 2r$ and $n \geq 3$ is odd, where $r \geq 1$. We consider only $m, n \geq 2$. The case $n = 1$ is in Theorem 4.10.

Case A: Suppose $n = 4s + 3$, where $s \geq 0$. Now $t = 8rs + 8r + 4s + 3$. Let A_0 be the following $2r \times 3$ array

$1 + 4rs$	$4rs + 6r + 4s + 4$	$4rs + 6r + 2s + 1$
$\vdots$	$\vdots$	$\vdots$
$2i - 1 + 4rs$	$4rs + 6r + 4s + 2 + 2i$	$4rs + 6r + 2s + 5 - 4i$
$\vdots$	$\vdots$	$\vdots$
$2r - 1 + 4rs$	$4rs + 8r + 4s + 2$	$4rs + 2r + 2s + 5$
$4rs + 6r + 4s + 5$	$2 + 4rs$	$4rs + 6r + 2s - 1$
$\vdots$	$\vdots$	$\vdots$
$4rs + 6r + 4s + 3 + 2j$	$2j + 4rs$	$4rs + 6r + 2s + 3 - 4j$
$\vdots$	$\vdots$	$\vdots$
$4rs + 8r + 4s + 3$	$2r + 4rs$	$4rs + 2r + 2s + 3$

$1 \leq i, j \leq r$. Note that, if $r = 1$, then take the first and the last row of the array. The set of used integers is

$$[4rs + 1, 4rs + 2r] \cup [4rs + 6r + 4s + 4, 4rs + 8r + 4s + 3] \cup \{4rs + 2r + 2s + 1 + 2k \mid 1 \leq k \leq 2r\}.$$

Consider $s \geq 1$. Let us consider the additional $2r \times 4$ array A_k, where $1 \leq k \leq s$.

R_1	$1 + 4r(k-1)$	$t - r - 4r(k-1)$	$3r + 1 + 4r(k-1)$	$t - 2r - 4r(k-1)$
R_2	$2 + 4r(k-1)$	$t - r - 1 - 4r(k-1)$	$3r + 2 + 4r(k-1)$	$t - 2r - 1 - 4r(k-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
R_{r-1}	$r - 1 + 4r(k-1)$	$t - 2r + 2 - 4r(k-1)$	$4r - 1 + 4r(k-1)$	$t - 3r + 2 - 4r(k-1)$
R_r	$r + 4r(k-1)$	$t - 2r + 1 - 4r(k-1)$	$4r + 4r(k-1)$	$t - 3r + 1 - 4r(k-1)$
R_{r+1}	$t - 4r(k-1)$	$r + 1 + 4r(k-1)$	$t - 3r - 4r(k-1)$	$2r + 1 + 4r(k-1)$
R_{r+2}	$t - 1 - 4r(k-1)$	$r + 2 + 4r(k-1)$	$t - 3r - 1 - 4r(k-1)$	$2r + 2 + 4r(k-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
R_{2r-1}	$t - r + 2 - 4r(k-1)$	$2r - 1 + 4r(k-1)$	$t - 4r + 2 - 4r(k-1)$	$3r - 1 + 4r(k-1)$
R_{2r}	$t - r + 1 - 4r(k-1)$	$2r + 4r(k-1)$	$t - 4r + 1 - 4r(k-1)$	$3r + 4r(k-1)$

Similarly, if $r = 1$, then take the first and the last row of the array. It is easy to verify that the row sum is $R' = 2(t+1) = 8(m+1)(s+1)$ and the column sum is $C' = r(t+1) = 2m(m+1)(s+1)$. Now, the set of used integers is $[1, 4rs] \cup [t - 4rs + 1, t] = [1, 2ms] \cup [2ms + 4m + 4s + 4, 4ms + 4m + 4s + 3]$.

It is easy to verify that the row sum of A_0 is $R'' = 6(m+1)(s+1)$ and the column sum of A_0 is $C'' = 2m(m+1)(s+1)$.

Let $A = \begin{pmatrix} A_0 & A_1 & \cdots & A_s \end{pmatrix}$ be the $m \times (4s+3)$ array. Note that, when $s = 0$, $A = A_0$. Now, the row sum of A is $sR' + R'' = (8s+6)(m+1)(s+1)$ and the column sum is $2m(m+1)(s+1)$.

Note that the set of unused integers is $U_1 \cup U_2 \cup U_3$ where $U_1 = [2ms + m + 1, 2ms + m + 2s + 1]$, $U_2 = \{2ms + m + 2s + 2i \mid 1 \leq i \leq m\}$ and $U_3 = [2ms + 3m + 2s + 2, 2ms + 3m + 4s + 3]$ with $|U_1| = 2s + 1$ $|U_3| = 2s + 2$ and $|U_2| = m$.

We now define a bijective total labeling $f : V(K_{m,4s+3}) \cup E(K_{m,4s+3}) \to [1, t]$ such that

(a) $f(u_i v_j) = a_{i,j}$, where $a_{i,j}$ is the (i, j)-th entry of A.

(b) The vertices v_j, $1 \leq j \leq 4s+3$, are assigned the $4s+3$ integers in $U = U_1 \cup U_3$ randomly.

(c) The vertices u_i, $1 \leq i \leq m$, are assigned the m integers in U_2 randomly.

Now,

$$\begin{aligned}
f_{tn}^{+}(u_i) &= sR' + R'' + \sum_{x \in U} x \\
&= (8s+6)(m+1)(s+1) + \frac{2s+1}{2}(4ms + 2m + 2s + 2) + (s+1)(4ms + 6m + 6s + 5) \\
&= 16ms^2 + 28ms + 16s^2 + 13m + 28s + 12, \\
f_{tn}^{+}(v_j) &= C'' + \sum_{y \in U_2} y = 2m(m+1)(s+1) + \frac{m}{2}(4ms + 4m + 4s + 2) \\
&= 4m^2 s + 4m^2 + 4ms + 3m.
\end{aligned}$$

Now,

$$\begin{aligned}
f_{tn}^{+}(u_i) - f_{tn}^{+}(v_j) &= 16ms^2 - 4m^2 s + 16s^2 + 24ms - 4m^2 + 10m + 28s + 12 \\
&= 4m(4s - m + 2)(s+1) + 16s^2 + 2m + 28s + 12. \qquad (4.1)
\end{aligned}$$

When $4s \geq m-2$, (4.1) > 0.

When $4s \leq m-4$, i.e., $-m \leq -4s-4$, then

$$\begin{aligned}(4.1) &\leq (4m)(-2)(s+1) + 16s^2 + 2m + 28s + 12\\ &= 16s^2 - 8ms - 6m + 28s + 12\\ &\leq 16s^2 + 8(-4s-4)s + 6(-4s-4) + 28s + 12 = -16s^2 - 28s - 12 < 0.\end{aligned}$$

Thus, f is a local total neighborhood antimagic 2-coloring for $K_{2r,4s+3}$.

Case B: Suppose $n = 4s+5$, where $s \geq 0$. Now $t = 8rs + 12r + 4s + 5$.

Suppose $r = 1$. So that $t = 12s + 17$. Let

$$A = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 12s+16 & 3 & 12s+14 & \cdots & 4s+3 & 8s+14 & 8s+11\\ \hline 12s+17 & 2 & 12s+15 & 4 & \cdots & 8s+15 & 4s+4 & 4s+7\\ \hline\end{array}$$

be a $2\times(4s+5)$ matrix. Note that the set of entries of A is $[1, 4s+4] \cup [8s+14, 12s+17] \cup \{4s+7, 8s+11\}$. The row sum of A is $R = 24s^2 + 66s + 45$ and the column sum is $C = 12s + 18$. Define a total labeling $f : V(K_{2,4s+5}) \cup E(K_{2,4s+5}) \to [1, 12s+17]$ such that $f(u_i v_j) = a_{i,j}$, the (i, j)-th entry of A. Now, let $f(u_1) = 4s+5$, $f(u_2) = 4s+6$, and assign the remaining $4s+5$ elements in $[4s+8, 8s+10] \cup \{8s+12, 8s+13\}$ to v_j, $1 \leq j \leq 4s+5$, randomly. Thus, f is a bijection.

$$\begin{aligned} f_{tn}^{+}(u_i) &= R + \sum_{j=1}^{4s+5} f(v_j) \text{ for } 1 \leq i \leq 2\\ &= 24s^2 + 66s + 45 + (4s+3)(12s+18)/2 + (8s+12) + (8s+13)\\ &= 48s^2 + 136s + 97,\\ f_{tn}^{+}(v_j) &= C + f(u_1) + f(u_2) \text{ for } 1 \leq j \leq 4s+5\\ &= 12s + 18 + (4s+5) + (4s+6)\\ &= 20s + 29.\end{aligned}$$

Consider $r \geq 2$. Suppose $s = 0$, then $t = 12r + 5$. Let the following $2r \times 5$ array be A whose entries are $a_{i,j}$.

1	$t-2r+1$	$t-3r+1$	$4r$	$7r+2$
2	$t-2r+2$	$t-3r+2$	$4r-1$	$7r$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$t-2r+k$	$t-3r+k$	$4r+1-k$	$7r+4-2k$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
r	$t-r$	$t-2r$	$3r+1$	$5r+4$
$t-r+1$	$r+1$	$2r+1$	$t-4r+1$	$8r+1$
$t-r+2$	$r+2$	$2r+2$	$t-3r$	$7r-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t-r+l$	$r+l$	$2r+l$	$t-3r+2-l$	$7r+3-2l$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t-1$	$2r-1$	$3r-1$	$t-4r+3$	$5r+5$
t	$2r$	$3r$	$t-4r+2$	$5r+3$

where $2 \leq k, l \leq r$.

The column sum is $r(t+1)$ and the row sum is $2t+6r+5$. The labels used in the first 4 columns are $[1,4r] \cup [8r+6, 12r+5]$. The labels used in the last column are $[5r+3, 7r] \cup \{7r+2, 8r+1\}$. Now the unused integers are $[4r+1, 5r+2] \cup [7r+1, 8r+5] \setminus \{7r+2, 8r+1\}$. Thus, there are $2r+5$ unused labels.

Define a bijective total labeling $f : V(K_{2r,5}) \cup E(K_{2r,5}) \to [1, 12r+5]$ such that $f(u_iv_j) = a_{i,j}$, $1 \leq i \leq 2r$, $1 \leq j \leq 5$. We let $f(v_1) = 7r+1$, $f(v_j) = 8r+j$, $2 \leq j \leq 5$ and label u_i by the remaining unused labels randomly. Thus $\sum_{j=1}^{5} f(v_i) = 39r+15$ and $\sum_{i=1}^{2r} f(u_i) = \sum_{i=1}^{r+2}(4r+i) + \sum_{i=1}^{r-2}(7r+2+i) = 12r^2 - 3r$.

$$f_{tn}^{+}(u_i) = \text{row sum of } A + \sum_{j=1}^{5} f(v_j) = (30r+15) + (39r+15) = 69r+30,$$

$$f_{tn}^{+}(v_j) = \text{colum sum of } A + \sum_{i=1}^{2r} f(u_i) = r(12r+6) + (12r^2 - 3r) = 24r^2 + 3r.$$

Now, $f_{tn}^{+}(v_j) - f_{tn}^{+}(u_i) = 6(4r^2 - 11r - 5) \neq 0$, since the discriminant is 201 which is not a perfect square. Thus, f is a local total neighborhood antimagic 2-coloring for $K_{2r,5}$. Hence $\chi_{ltna}(K_{2r,5}) = 2$.

Suppose $s \geq 1$. Let A_k be as defined in Case A with the new value of $t = 8rs + 12r + 4s + 5$, where $1 \leq k \leq s$. Thus, the row sum is $R' = 2(t+1) = 16rs + 24r + 8s + 12$ and the column sum is $C' = r(t+1) = 12r^2 + 8r^2s + 4rs + 6r$. Now, the set of used integers is $[1, 4rs] \cup [t - 4rs + 1, t] = [1, 2ms] \cup [2ms + 6m + 4s + 6, 4ms + 6m + 4s + 5]$.

Let A_0 be the following $2r \times 5$ array

$1+4rs$	$10r+6+4rs+4s$	$9r+6+4rs+4s$	$4r+4rs$	$7r+2+4rs+2s$
$2+4rs$	$10r+7+4rs+4s$	$9r+7+4rs+4s$	$4r-1+4rs$	$7r+4rs+2s$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k+4rs$	$10r+5+k+4rs+4s$	$9r+5+k+4rs+4s$	$4r+1-k+4rs$	$7r+4-2k+4rs+2s$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r+4rs$	$11r+5+4rs+4s$	$10r+5+4rs+4s$	$3r+1+4rs$	$5r+4+4rs+2s$
$11r+6+4rs+4s$	$r+1+4rs$	$2r+1+4rs$	$8r+6+4rs+4s$	$8r+1+4rs+2s$
$11r+7+4rs+4s$	$r+2+4rs$	$2r+2+4rs$	$9r+5+4rs+4s$	$7r-1+4rs+2s$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$11r+5+l+4rs+4s$	$r+l+4rs$	$2r+l+4rs$	$9r+7-l+4rs+4s$	$7r+3-2l+4rs+2s$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$12r+4+4rs+4s$	$2r-1+4rs$	$3r-1+4rs$	$8r+8+4rs+4s$	$5r+5+4rs+2s$
$12r+5+4rs+4s$	$2r+4rs$	$3r+4rs$	$8r+7+4rs+4s$	$5r+3+4rs+2s$

where $2 \leq k, l \leq r$.

It is easy to verify that the row sum of A_0 is $R'' = 20rs + 30r + 10s + 15$ and the column sum of A_0 is $C'' = 12r^2 + 8r^2 + 4rs + 6r$. Let $A = \begin{pmatrix} A_0 & A_1 & \cdots & A_s \end{pmatrix}$ be the $m \times (4s+5)$ array. Now, the row sum of A is $sR' + R'' = (4s+5)(m+1)(2s+3)$ and the column sum is $m(m+1)(2s+3)$.

Now, the sets of unused integers are U_1 and U_2, where $U_1 = [4rs + 4r + 1, 4rs + 5r + 2s + 2]$ and $U_2 = [4rs+7r+2s+1, 4rs+8r+4s+5] \setminus \{4rs+7r+2s+2, 4rs+8r+2s+1\}$. Note that $|U_1| = r+2s+2$ and $|U_2| = r + 2s + 3$.

We now define a bijective total labeling $f : V(K_{m,4s+5}) \cup E(K_{m,4s+5}) \to [1, t]$ such that

(a) $f(u_i v_j) = a_{i,j}$, where $a_{i,j}$ is the (i, j)-th entry of A.

(b) $f(u_i) = 4rs + 4r + i$ for $1 \le i \le r + 2$ and $f(u_{r+2+l}) = 4rs + 7r + 2s + 2 + l$ for $1 \le l \le r - 2$ (if $r = 2$, then the last part is omitted).

(c) The vertices v_j, $1 \le j \le 4s + 5$, are assigned the remaining $4s + 5$ unused integers randomly.

Now,

$$\sum_{i=1}^{2r} f(u_i) = \sum_{i=1}^{r+2}(4rs + 4r + i) + \sum_{l=1}^{r-2}(4rs + 7r + 2s + 2 + l) = 8r^2 s + 12r^2 + 2rs - 3r - 4s,$$

$$\begin{aligned}\sum_{j=1}^{4s+3} f(v_j) &= \sum_{j=1}^{2s}(4rs + 5r + 2 + j) + \sum_{j=1}^{2s+4}(4rs + 8r + 2s + 1 + j) + (4rs + 7r + 2s + 1)\\ &= 16rs^2 + 46rs + 8s^2 + 39r + 26s + 15.\end{aligned}$$

Thus,

$$\begin{aligned}f^+_{tn}(u_i) &= sR' + R'' + 16rs^2 + 46rs + 8s^2 + 39r + 26s + 15\\ &= (4s + 5)(2r + 1)(2s + 3) + 16rs^2 + 46rs + 8s^2 + 39r + 26s + 15\\ &= 32rs^2 + 90rs + 16s^2 + 69r + 48s + 30,\\ f^+_{tn}(v_j) &= C'' + 8r^2 s + 12r^2 + 2rs - 3r - 4s\\ &= 2r(2r + 1)(2s + 3) + 8r^2 s + 12r^2 + 2rs - 3r - 4s\\ &= 16r^2 s + 24r^2 + 6rs + 3r - 4s.\end{aligned}$$

Note that, these two formula also agree with $s = 0$. We have

$$\begin{aligned}f^+_{tn}(u_i) - f^+_{tn}(v_j) &= -16r^2 s + 32rs^2 - 24r^2 + 16s^2 + 84rs + 66r + 52s + 30\\ &= 16rs(2s - r) + 4(2s - r)(2s + r) + 20r(2s - r) + 44rs + 66r + 52s + 30. \quad (4.2)\end{aligned}$$

Suppose $n \ge m = 2r$. Since n is odd and m is even, it is equivalent to $2s \ge r - 2$.

$$(4.2) \ge -32rs - 8(2r - 2) - 40r + 44rs + 66r + 52s + 30 = 12rs + 10r + 52s + 46 > 0.$$

Suppose $2s = r - 3$, then $(4.2) = -2r^2 + 14r - 12 = -2(r - 1)(r - 6) \ne 0$, since s is a positive integer. Suppose $2s - r \le -4$, i.e., $r \ge 2s + 4$.

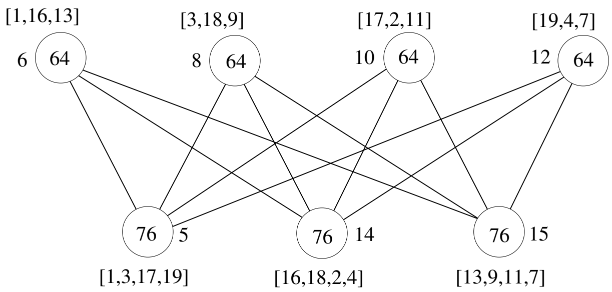
$$\begin{aligned}(4.2) &\le 16rs(-4) + 4(-4)(2s + r) + 20r(-4) + 44rs + 66r + 52s + 30\\ &= -20rs - 30r + 20s + 30 = -(20s + 30)(r - 1) < 0,\end{aligned}$$

Thus, f is a local total neighborhood antimagic 2-coloring for $K_{2r,4s+5}$. Hence, $\chi_{ltna}(K_{2r,4s+5}) = 2$.
This completes the proof. □

Example 4.3. *Consider $K_{4,3}$. Now $s = 0$ and $r = 2$. From the proof of Theorem 4.11, we have the array*

Column Sum	40	40	40	*Row Sum*
	1	16	13	30
	3	18	9	30
	17	2	11	30
	19	4	7	30

The labeling for $K_{4,3}$ is

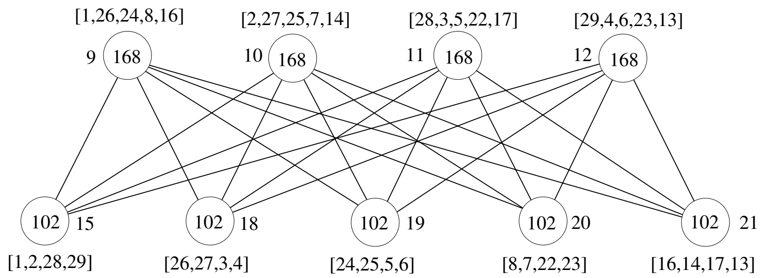


The numbers in the brackets near the vertex are labels for the edges incident to that vertex.

Consider $K_{4,5}$. Now $s = 0$ and $r = 2$ and $t = 29$. From the proof of Theorem 4.11, we have the array

Column Sum	60	60	60	60	60	*Row Sum*
	1	26	24	8	16	75
	2	27	25	7	14	75
	28	3	5	22	17	75
	29	4	6	23	13	75

The labeling for $K_{4,5}$ is



□

Corollary 4.12. *For $\alpha, m, n \geq 2$ and $(m, n) \neq (2, 2)$, $\chi_{ltna}(\alpha K_{m,n}) = 2$.*

Proof. Keep all the notations defined in the proof of Theorem 4.11. Now, $d(u_i) = n$ and $d(v_j) = m$.

When $m \equiv n \pmod 2$, $(m, n) \neq (2, 2)$ and $n \geq m$.

$$\begin{aligned}
& f_{tn}^{+}(u_i) - f_{tn}^{+}(v_j) \\
&= [n(mn+1)/2 + n(mn+n) + n(n+1)/2] - [m(mn+1)/2 + m^2 n + m(m+1)/2] \\
&= \frac{1}{2}[(n-m)(3mn+m+n+2) + 2n^2] > n - m.
\end{aligned}$$

Now we assume m is even and n is odd.

Case A: Let $n = 4s + 3$, $m = 2r$. Suppose $n \geq m$. It is equivalent to $4s \geq m - 2$. From (4.1), we have

$$f^+_{tn}(u_i) - f^+_{tn}(v_j) \geq 16s^2 + 2m + 28s + 12 = n^2 + 2m + n > n - m > 0.$$

Suppose $n \leq m - 1$. From (4.1), we have

$$\begin{aligned} f^+_{tn}(v_j) - f^+_{tn}(u_i) &\geq 8ms + 6m - 16s^2 - 28s - 12 = n(2m - n - 1) \\ &> n(m - n) > m - n > 0. \end{aligned}$$

Case B: Let $n = 4s + 5$, $m = 2r$. Suppose $r = 1$. Clearly

$$f^+_{tn}(u_i) - f^+_{tn}(v_j) = 48s^2 + 136s + 97 - 20s - 29 = 48s^2 + 116s + 68 > 4s + 3 = n - m > 0$$

Suppose $r \geq 2$. Suppose $n \geq m$. From the proof of Theorem 4.11

$$f^+_{tn}(u_i) - f^+_{tn}(v_j) > n - m > 0.$$

Suppose $m = n + 1$. From the proof of Theorem 4.11, $|f^+_{tn}(u_i) - f^+_{tn}(v_j)| > 1$.
Suppose $n \leq m - 3$, i.e., $r \geq 2s + 4$. From the proof of Theorem 4.11, we have

$$\begin{aligned} f^+_{tn}(v_j) - f^+_{tn}(u_i) &\geq (2r - 2)(10s + 15) = m(10s + 15) - 2(10s + 15) \\ &= 10sm + 15m - 20s - 30 > 15m - 20s - 50 = 15m - 5n - 25 > m - n > 0. \end{aligned}$$

Thus, for each case, we have $|f^+_{tn}(u_i) - f^+_{tn}(v_j)| > |m - n|$. Hence, $\alpha|f^+_{tn}(u_i) - f^+_{tn}(v_j)| > \alpha|m - n| \geq |m - n|(\alpha - 1)$. Therefore, condition (b) in the hypothesis of Theorem 3.1 holds. Thus, condition (a) also holds. By Theorem 3.1, we have the corollary. □

4.3. Stars join with a vertex

Suppose $m, s \geq 1$. Let $V(mK_{1,s} \vee K_1) = \{w_i, v_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq s\} \cup \{z\}$ and $E(mK_{1,s} \vee K_1) = \{w_i z, v_{i,j} z, w_i v_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq s\}$. Note that the order and the size of $mK_{1,s} \vee K_1$ are $m(s + 1) + 1$ and $m(2s + 1)$, respectively. Note that $K_{1,s} \vee K_1 \cong K_{1,1,s}$.

Theorem 4.13. *For $m, s \geq 1$, $\chi_{ltna}(mK_{1,s} \vee K_1) = 3$.*

Proof. Let $G = mK_{1,s} \vee K_1$, $m, s \geq 1$. Note that $K_{1,1} \vee K_1 \cong C_3$. The result follows from Theorem 4.5. Consider $(m, s) \neq (1, 1)$. By (1.1), $\chi_{ltna}(G) \geq \chi(G) = 3$. For $1 \leq i \leq m$, $1 \leq j \leq s$, define a total labeling $f : V(G) \cup E(G) \to [1, m(3s + 2) + 1]$ such that

(1) $f(v_{i,j}) = (j - 1)m + i$, $j \neq s$, with the set of labels $[1, m(s - 1)]$.

(2) $f(v_{i,s}) = m(s - 1) + 2i - 1$ and $f(w_i) = m(s - 1) + 2i$ with the set of labels $[m(s - 1) + 1, m(s + 1)]$.

(3) $f(w_i v_{i,j}) = m(2s + 2 - j) + 1 - i$ with the set of labels $[m(s + 1) + 1, m(2s + 1)]$.

(4) $f(v_{i,j} z) = m(2s + 1 + j) + 1 - i$ with the set of labels $[m(2s + 1) + 1, m(3s + 1)]$.

(5) $f(w_i z) = m(3s + 2) + 1 - i$ with the set of labels $[m(3s + 1) + 1, m(3s + 2)]$.

(6) $f(z) = m(3s + 2) + 1$.

Note that for $s = 1$, we refer only to (2) to (6). Thus, f is bijective. We now have:

(a) $f_{tn}^{+}(v_{i,j}) = f(w_i) + f(z) + f(w_i v_{i,j}) + f(v_{i,j} z) = [m(s-1) + 2i] + [m(3s+2) + 1] + [m(2s+2-j) + 1 - i] + [m(2s+1+j) + 1 - i] = 8ms + 4m + 3.$

(b) For $s = 1$, $f_{tn}^{+}(w_i) = f(v_{i,1}) + f(w_i v_{i,1}) + f(w_i z) + f(z) = (2i-1) + (3m+1-i) + (5m+1-i) + (5m+1) = 13m + 2.$

(c) For $s \geq 2$,

$$\begin{aligned}
f_{tn}^{+}(w_i) &= \Big[\sum_{j=1}^{s-1} f(v_{i,j})\Big] + f(v_{i,s}) + \Big[\sum_{j=1}^{s} f(w_i v_{i,j})\Big] + f(w_i z) + f(z) \\
&= \sum_{j=1}^{s-1} \Big[(j-1)m + i\Big] + [m(s-1) + 2i - 1] + \sum_{j=1}^{s} \Big[m(2s+2-j) + 1 - i\Big] \\
&\quad + [m(3s+2) + 1 - i] + m(3s+2) + 1 \\
&= \left(\sum_{j=1}^{s-1} (2ms + m + 1)\right) + (ms + 2m + 1 - i) + (7ms + 3m + i + 1) \\
&= (s-1)(2ms + m + 1) + 8ms + 5m + 2 \\
&= 2ms^2 + 7ms + 4m + s + 1,
\end{aligned}$$

which agrees when $s = 1$.

(d)

$$\begin{aligned}
f_{tn}^{+}(z) &= \sum_{k=1}^{m(3s+2)} k - \sum_{k=m(s+1)+1}^{m(2s+1)} k \\
&= \frac{m(3s+2)(m(3s+2)+1)}{2} - \frac{(m(2s+1) + m(s+1) + 1)(ms)}{2} \\
&= m(3ms^2 + 5ms + 2m + s + 1).
\end{aligned}$$

It is routine to check that $f_{tn}^{+}(v_{i,j})$, $f_{tn}^{+}(w_i)$, and $f_{tn}^{+}(z)$ are mutually distinct when $(m, s) \neq (1, 1)$. Thus, f is a local total neighborhood antimagic 3-coloring of G so that $\chi_{ltna}(G) \leq 3$. This completes the proof. □

Corollary 4.14. *For $n, m, s \geq 1$, $\chi_{ltna}(n(mK_{1,s} \vee K_1)) = 3$.*

Proof. We keep all notations as defined in the proof of Theorem 4.13. By Theorem 4.5, we need only to deal with $(m, s) \neq (1, 1)$. It suffices to show that the labeling f satisfies the conditions (a) and (b) of Theorem 3.1.

Thus, f satisfies (a). In the following, we prove that $|f_{tn}^{+}(u) - f_{tn}^{+}(v)| \geq |d(u) - d(v)|$ for all distinct vertices u, v, which implies condition (b) of Theorem 3.1.

Now,

$$f_{tn}^{+}(w_i) - f_{tn}^{+}(v_{i,j}) = 2ms^2 + 7ms + 4m + s + 1 - (8ms + 4m + 3)$$

$$
\begin{aligned}
&= 2ms^2 - ms + s - 2 = (s-1)(ms+s+2) + (m-1)s^2 \\
&\geq s - 1 = d(w_i) - d(v_{i,j}) > 0.
\end{aligned}
$$

$$
\begin{aligned}
f^+_{tn}(z) - f^+_{tn}(v_{i,j}) &= m(3ms^2 + 5ms + 2m + s + 1) - (8ms + 4m + 3) \\
&= 3m^2s^2 + 5m^2s + 2m^2 - 7ms - 3m - 3 \\
&\geq 3ms^2 + 5ms + 2m - 7ms - 3m - 3 \\
&= m(3s+1)(s-1) - 3. \qquad (4.3)
\end{aligned}
$$

Suppose $s = 1$, then $m \geq 2$. Thus $f^+_{tn}(z) - f^+_{tn}(v_{i,j}) = 10m^2 - 10m - 3 > 2m - 2 = d(z) - d(v_{i,j}) > 0$.
Suppose $s \geq 2$, (4.3) $\geq m(3s+1) - 3 = m(s+1) + 2ms - 3 > m(s+1) - 2 = d(z) - d(v_{i,j}) > 0$

$$
\begin{aligned}
f^+_{tn}(z) - f^+_{tn}(w_i) &= m(3ms^2 + 5ms + 2m + s + 1) - (2ms^2 + 7ms + 4m + s + 1) \\
&= ms[(m-1)(2s+5) + (ms-2)] + m(2m-4) + (m-1)(s+1) \qquad (4.4)
\end{aligned}
$$

When $m = 1$, from (4.4), we have $|f^+_{tn}(z) - f^+_{tn}(w_i)| = |s(s-2) - 2| \geq 1$.
When $m \geq 2$, (4.4) $\geq (m-1)(s+1) = d(z) - d(w_i) > 0$.

Thus $|f^+_{tn}(u) - f^+_{tn}(v)| \geq |d(u) - d(v)|$ for all distinct vertices u, v. This completes the proof. □

Note that $mK_{1,1} \vee K_1 \cong mP_2 \vee K_1 \cong f_m$ which is also known as the friendship graph, and $mK_{1,2} \vee K_1 \cong mP_3 \vee K_1$. So we have

Corollary 4.15. *For $m \geq 1$, $\chi_{ltna}(f_m) = \chi_{ltna}(mP_3 \vee K_1) = 3$.*

Consider the total labeling f in the proof of Theorem 4.13. For $m \geq 2$, the friendship graph f_m has induced vertex weights $12m + 3$, $13m + 2$ and $2m(5m + 1)$ with corresponding vertex degrees 2, 2, $2m$, whereas the $mP_3 \vee K_1$ has induced vertex weights $20m+3$, $26m+3$ and $3m(8m+1)$ with corresponding vertex degrees 2, 3, $3m$. It is easy to show that the labeling f defined in the proof of Theorem 4.13 satisfies the conditions of Theorem 3.1 when $s = 1$, $m \geq 2$ and when $s = 2$, $m \geq 1$.

Corollary 4.16. *Suppose $n \geq 2$. If $G \cong f_m$, $m \geq 2$ or $G \cong mP_3 \vee K_1$, $m \geq 1$, then $\chi_{ltna}(nG) = 3$.*

Conjecture 4.2. *$\chi_{ltna}(mP_n \vee K_1) = 3$ for $m \geq 1$, $n \geq 4$.*

4.4. Wheel graphs

Consider the wheel $W_m = C_m \vee K_1$, where $m \geq 3$, $C_m = u_1u_2 \cdots u_mu_1$ and $V(K_1) = \{x\}$.

Theorem 4.17. *For $n \geq 1$, $\chi_{ltna}(W_{2n+1}) = 4$.*

Proof. By (1.1), we have $\chi_{ltna}(W_{2n+1}) \geq \chi(W_{2n+1}) = 4$. Since $W_3 \cong K_4$, we have $\chi_{ltna}(W_3) = 4$. Suppose $n \geq 2$. Define a total labeling $f : V(W_{2n+1}) \cup E(W_{2n+1}) \to [1, 6n+4]$ such that

(1) $f(u_{2i-1}) = i$ for $1 \leq i \leq n+1$ with the set of labels $[1, n+1]$.

(2) $f(u_{2i}) = n + 1 + i$ for $1 \leq i \leq n$ with the set of labels $[n+2, 2n+1]$.

(3) $f(u_iu_{i+1}) = 4n + 2 - i$ for $0 \leq i \leq 2n$ with the set of labels $[2n+2, 4n+2]$, here $u_0 = u_{2n+1}$.

(4) $f(xu_{2i-1}) = 4n + 3 + 2i$, $f(xu_{2i}) = 4n + 2 + 2i$ for $1 \le i \le n$, and $f(xu_{2n+1}) = 6n + 4$ with the set of labels $[4n + 4, 6n + 4]$.

(5) $f(x) = 4n + 3$.

Thus, f is bijective. Moreover,

$$f^+_{tn}(x) = \sum_{k=1}^{2n+1} k + \sum_{k=4n+4}^{6n+4} k = (2n + 1)(6n + 5) = 12n^2 + 16n + 5,$$

and

$$\begin{aligned} f^+_{tn}(u_{2n+1}) &= f(u_1) + f(u_{2n}) + f(u_{2n+1}u_1) + f(u_{2n}u_{2n+1}) + f(xu_{2n+1}) + f(x) \\ &= 1 + (2n + 1) + (4n + 2) + (2n + 2) + (6n + 4) + (4n + 3) = 18n + 13. \end{aligned}$$

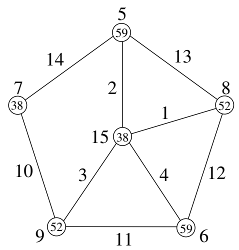
Similarly, we have $f^+_{tn}(u_1) = 18n + 14$, $f^+_{tn}(u_{2i-1}) = 18n + 14$ for $2 \le i \le n$, and $f^+_{tn}(u_{2i}) = 16n + 11$ for $1 \le i \le n$.

It is routine to check that $f^+_{tn}(x)$, $f^+_{tn}(u_{2i-1})$, $f^+_{tn}(u_{2i})$ $(1 \le i \le n)$, and $f^+_{tn}(u_{2n+1})$ are mutually distinct. Thus, f is a local total neighborhood antimagic 4-coloring of W_{2n+1} so that $\chi_{ltna}(W_{2n+1}) \le 4$. This completes the proof. □

Theorem 4.18. *For $n \ge 1$, if G is W_{2n+1} with a spoke deleted, then*

$$\chi_{ltna}(G) = \begin{cases} 3 & \text{for } n = 1, 2, \\ 4 & \text{for } n \ge 3. \end{cases}$$

Proof. Let $G = W_{2n+1} - xu_{2n+1}$. Now, G is of order $2n + 2$ and size $4n + 1$. For $n = 1$, the graph is $K_{1,1,2}$, which has $\chi_{ltna}(G) = 3$. For $n = 2$, the graph below and the corresponding labeling shows that $\chi_{ltna}(G) = 3$.



Suppose $n \ge 3$. Since u_1 to u_{2n} must have distinct weight with x, we need to consider only $f^+_{tn}(x)$ and $f^+_{tn}(x_{2n+1})$. If $\chi_{ltna}(G) = 3$, then $f^+_{tn}(u_{2n+1}) = f^+_{tn}(x)$. Since u_{2n+1} and x have common neighbors u_1, u_{2n}, we must have $f(u_{2n+1}u_1) + f(u_{2n}u_{2n+1}) = f(xu_1) + f(xu_{2n}) + \sum_{i=2}^{2n-1} [f(u_i) + f(xu_i)]$. Now, the left hand side is $L \le (6n+3) + (6n+2) = 12n+5$ and right hand side is $R \ge \sum_{i=1}^{4n-2} i = (2n-1)(4n-1) = 8n^2 - 6n + 1 > 24n - 6n + 1 = 18n + 1 > L$. Thus, $\chi_{ltna}(G) \ge 4$.

Using the labeling f of W_{2n+1} in the proof of Theorem 4.17, we delete xu_{2n+1} and the label $6n + 4$. Now, all u_i weights for $i = 1, \ldots, 2n$ remain the same so that $f^+_{tn}(u_{2i-1}) = 18n + 14$, $f^+_{tn}(u_{2i}) = 16n + 11$ for $1 \le i \le n$, whereas $f^+_{tn}(x) = 12n^2 + 9n$ and $f^+_{tn}(u_{2n+1}) = 8n + 6$. Moreover, every two adjacent vertex weights are distinct. Thus, G admits a local total neighborhood antimagic 4-coloring and $\chi_{ltna}(G) \le 4$. Consequently, the theorem holds. □

Theorem 4.19. *For even $m \geq 4$, $\chi_{ltna}(W_m) = 3$.*

Proof. By (1.1), we have $\chi_{ltna}(W_m) \geq \chi(W_m) = 3$. Therefore, it suffices to construct a local total neighborhood antimagic 3-coloring for W_m.

Suppose $m = 4n + 2 \geq 6$. Define a total labeling $f : V(W_{4n+2}) \cup E(W_{4n+2}) \to [1, 12n + 7]$ such that

(1) $f(u_{4i-3}) = i$ for $1 \leq i \leq n + 1$ with the set of labels $[1, n + 1]$.

(2) $f(u_{4i-1}) = n + 1 + i$ for $1 \leq i \leq n$ with the set of labels $[n + 2, 2n + 1]$.

(3) $f(u_{4i-2}) = 3n + 3 - i$ for $1 \leq i \leq n + 1$ with the set of labels $[2n + 2, 3n + 2]$.

(4) $f(u_{4i}) = 4n + 3 - i$ for $1 \leq i \leq n$ with the set of labels $[3n + 3, 4n + 2]$.

(5) $f(u_{2i+1}u_{2i+2}) = 4n + 2 + i$ for $1 \leq i \leq 2n$ with the set of labels $[4n + 3, 6n + 2]$.

(6) $f(u_1u_2) = 6n + 3$, $f(u_2u_3) = 6n + 4$.

(7) $f(u_{2i+2}u_{2i+3}) = 8n + 5 - i$ for $1 \leq i \leq 2n$ where $u_{4n+3} = u_1$ with the set of labels $[6n + 5, 8n + 4]$.

(8) $f(xu_{2i}) = 10n + 5 - i$ for $1 \leq i \leq 2n$ with the set of labels $[8n + 5, 10n + 4]$.

(9) $f(xu_{2i+3}) = 10n + 5 + i$ for $1 \leq i \leq 2n - 1$ with the set of labels $[10n + 6, 12n + 4]$.

(10) $f(xu_1) = 12n + 5$, $f(xu_3) = 12n + 6$, $f(xu_{4n+2}) = 10n + 5$, $f(x) = 12n + 7$.

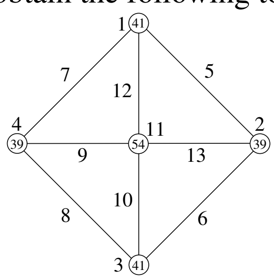
Thus, f is bijective. We now have the following vertex weights:

$$\begin{aligned}
f^+_{tn}(x) &= \sum_{k=1}^{4n+2} k + \sum_{k=8n+5}^{12n+6} k = (4n + 2)(12n + 7) = 48n^2 + 52n + 14,\\
f^+_{tn}(u_1) &= f(u_{4n+2}u_1) + f(u_1u_2) + f(xu_1) + f(u_{4n+2}) + f(u_2) + f(x)\\
&= (6n + 5) + (6n + 3) + (12n + 5) + (2n + 2) + (3n + 2) + (12n + 7) = 41n + 24.
\end{aligned}$$

Similarly, we have $f^+_{tn}(u_2) = 35n+21$, $f^+_{tn}(u_3) = 41n+24$, $f^+_{tn}(u_{4n+2}) = 35n+21$, $f^+_{tn}(u_{4i-3}) = 41n+24$ for $i \in [2, n+1]$, $f^+_{tn}(u_{4i-2}) = 35n+21$ for $i \in [2, n]$, $f^+_{tn}(u_{4i-1}) = 41n+24$ for $i \in [2, n]$, and $f^+_{tn}(u_{4i}) = 35n+21$ for $i \in [1, n]$.

As such, every two adjacent vertex weights are distinct. Thus, f is a local total neighborhood antimagic 3-coloring of W_{4n+2}, and so $\chi_{ltna}(W_{4n+2}) = 3$.

Suppose $m = 4n \geq 4$. For $n = 1$, we obtain the following total labeling of W_4.



Now assume $n \geq 2$. For convenience, we let $u_0 = u_{4n}$. Define a total labeling $f : V(W_{4n}) \cup E(W_{4n}) \to [1, 12n + 1]$ such that

(1) $f(u_{2i-1}) = i$ for $1 \leq i \leq 2n$ with the set of labels $[1, 2n]$.

(2) $f(u_{2i}) = 2n + 1 + i$ for $0 \leq i \leq 2n - 1$ with the set of labels $[2n + 1, 4n]$.

(3) $f(x) = 7n + 1$.

(4) $f(u_{2i-1}u_{2i}) = 6n + 1 - i$ for $1 \leq i \leq 2n$ with the set of labels $[4n + 1, 6n]$.

(5) $f(u_{2i}u_{2i+1}) = \begin{cases} 6n + 2 + i, & 0 \leq i \leq n - 2; \\ 6n + 3 + i, & n - 1 \leq i \leq 2n - 2; \\ 6n + 1, & i = 2n - 1; \end{cases}$

with the set of labels $[6n + 1, 7n] \cup [7n + 2, 8n + 1]$.

(6) $f(xu_i) = \begin{cases} 12n - i, & 0 \leq i \leq 2n - 3; \\ 12n - 1 - i, & 2n - 2 \leq i \leq 4n - 3, \end{cases}$

with the set of labels $[8n + 2, 10n + 1] \cup [10n + 3, 12n]$.

(7) $f(xu_{4n-2}) = 10n + 2$, $f(xu_{4n-1}) = 12n + 1$.

Thus, f is bijective. We now have the following vertex weights.

$$\begin{aligned} f_{tn}^{+}(u_{2i}) &= f(x) + f(u_{2i-1}) + f(u_{2i+1}) + f(u_{2i-1}u_{2i}) + f(u_{2i}u_{2i+1}) + f(xu_{2i}) \\ &= (7n + 1) + i + (i + 1) + (6n + 1 - i) + (6n + 2 + i) + (12n - 2i) \\ &= 31n + 5, \quad 1 \leq i < n - 2, \\ f_{tn}^{+}(u_{2n-2}) &= f(x) + f(u_{2n-3}) + f(u_{2n-1}) + f(u_{2n-2}u_{2n-1}) + f(u_{2n-3}u_{2n-2}) + f(xu_{2n-2}) \\ &= (7n + 1) + (n - 1) + n + (7n + 2) + (5n + 2) + (10n + 1) = 31n + 5. \end{aligned}$$

Similarly, we get $f_{tn}^{+}(u_{2i}) = 31n + 5$ for $n \leq i \leq 2n$.

$$\begin{aligned} f_{tn}^{+}(u_1) &= f(x) + f(u_2) + f(u_{4n}) + f(u_1u_2) + f(u_{4n}u_1) + f(xu_1) \\ &= (7n + 1) + (2n + 2) + (2n + 1) + 6n + (6n + 2) + (12n - 1) = 35n + 5. \end{aligned}$$

Similarly, we obtain $f_{tn}^{+}(u_{2i-1}) = 35n + 5$ for $2 \leq i \leq 2n$. Finally, we have $f_{tn}^{+}(x) = \sum_{k=1}^{4n} k + \sum_{k=8n+2}^{12n+1} k = 8n(6n + 1)$.

Thus, f is a local total neighborhood antimagic 3-coloring of W_{4n} so that $\chi_{ltna}(W_{4n}) = 3$.

This completes the proof. □

It is easy to check that the labeling f in the proof of Theorems 4.17, 4.18, or 4.19 satisfies the conditions of Theorem 3.1. We give the following corollary with the proof omitted.

Corollary 4.20. *For $\alpha \geq 2$ and $m \geq 3$,*

$$\chi_{ltna}(\alpha W_m) = \begin{cases} 4, & m \text{ is odd}; \\ 3, & m \text{ is even}, \end{cases}$$

and $\chi_{ltna}(\alpha(W_m - e)) = 3$ for $m = 3, 5$, where e is a spoke of W_m.

5. Self join graphs

Theorem 5.1. *Suppose G is a (p, q)-graph with $p \geq 3$ and $\chi_{ltna}(G) = t \geq 2$. For all $u, v \in V(G)$, suppose f is a local total neighborhood antimagic t-coloring of G satisfying the following conditions*

(a) $f^+_{tn}(u) = f^+_{tn}(v)$ implies that $d_G(u) = d_G(v)$.

(b) $f^+_{tn}(u) \neq f^+_{tn}(v)$ implies that $f^+_{tn}(u) - f^+_{tn}(v) \neq 2(p + q)[d_G(v) - d_G(u)]$.

(c) $f^+_{tn}(u) - f^+_{tn}(v) \neq (p + q)(2d_G(v) - p)$.

Then, $\chi_{ltna}(G \vee G) \leq 2t$. Moreover, the equality holds if $\chi(G) = t$.

Proof. Let the two copies of G be G_1 and G_2. Let $u_i \in V(G_1)$ and $v_i \in V(G_2)$ be the vertices corresponding to the i-th vertex x_i of G, $1 \leq i \leq p$, respectively. Let $d_i = d_G(x_i)$ so that

$$d_{G_1}(u_i) = d_{G_2}(v_i) = d_i$$

for all i. Define a total labeling $g : V(G_1 \vee G_2) \cup E(G_1 \vee G_2) \to [1, 2p + 2q + p^2]$ such that

(i) $g(x) = f(x)$ for $x \in V(G_1) \cup E(G_1)$ with the set of labels $[1, p + q]$.

(ii) $g(x) = f(x) + p + q$ for $x \in V(G_2) \cup E(G_2)$ with the set of labels $[p + q + 1, 2p + 2q]$.

(iii) $g(u_iv_j) = a_{i,j} + 2p + 2q$ where $a_{i,j}$ is the (i, j)-entry of a $p \times p$ magic square, with the set of labels $[2p + 2q + 1, 2p + 2q + p^2]$.

Now, for $1 \leq i \leq p$, we have

(1) $g^+_{tn}(u_i) = f^+_{tn}(x_i) + p(p^2 + 1)/2 + p(2p + 2q) + p(p + q) + \sum_{k=1}^{p} f(x_k)$.

(2) $g^+_{tn}(v_i) = f^+_{tn}(x_i) + 2d_i(p + q) + p(p^2 + 1)/2 + p(2p + 2q) + \sum_{k=1}^{p} f(x_k)$.

From (1), we see that $g^+_{tn}(u_i) = f^+_{tn}(x_i) + k_1$, where k_1 is a constant. Thus, $g^+_{tn}(u_i) = g^+_{tn}(u_j)$ if and only if $f^+_{tn}(x_i) = f^+_{tn}(x_j)$ for $1 \leq i \neq j \leq p$. From (2), we see that $g^+_{tn}(v_i) = f^+_{tn}(x_i) + 2d_i(p + q) + k_2$, where k_2 is a constant. Thus, conditions (a) and (b) imply that $g^+_{tn}(v_i) = g^+_{tn}(v_j)$ if and only if $f^+_{tn}(x_i) = f^+_{tn}(x_j)$. Now,

$$g^+_{tn}(u_i) - g^+_{tn}(v_j) = f^+_{tn}(x_i) - f^+_{tn}(x_j) + (p + q)(p - 2d_j).$$

Observe that $g^+_{tn}(u_i) - g^+_{tn}(v_j) \neq 0$ if and only if $f^+_{tn}(x_i) - f^+_{tn}(x_j) + (p + q)(p - 2d_j) \neq 0$ if and only if $f^+_{tn}(x_i) - f^+_{tn}(x_j) \neq (p + q)(2d_j - p)$. Thus, condition (c) implies that $g^+_{tn}(u_i) \neq g^+_{tn}(v_j)$ for $1 \leq i, j \leq p$. Therefore, g is a local total neighborhood antimagic $2t$-coloring of $G \vee G$. So that $\chi_{ltna}(G \vee G) \leq 2t$. Clearly, if $\chi(G) = t$, then $\chi_{ltna}(G \vee G) \geq \chi(G \vee G) = 2t$ so that $\chi_{ltna}(G \vee G) = 2t$. This completes the proof. □

We can also obtain a new set of conditions so that the conclusion of Theorem 5.1 holds.

Theorem 5.2. *Suppose G is a (p, q)-graph with $p \geq 3$ and $\chi_{ltna}(G) = t \geq 2$. Suppose f is a local total neighborhood antimagic t-coloring of G satisfying the following conditions:*

(a) $f^+_{tn}(u) = f^+_{tn}(v)$ *implies that* $d_G(u) = d_G(v)$*.*

(b) $f^+_{tn}(u) \neq f^+_{tn}(v)$ *implies that* $f^+_{tn}(u) - f^+_{tn}(v) \neq (2p + 3q + p^2)[d_G(v) - d_G(u)]$*.*

(c) $f^+_{tn}(u) - f^+_{tn}(v) \neq d_G(v)(2p + 3q + p^2) - p(p + 2q + p^2)$ *for all* $u, v \in V(G)$*.*

Then, $\chi_{ltna}(G \vee G) \leq 2t$*. Moreover, the equality holds if* $\chi(G) = t$*.*

Proof. Let the two copies of G be G_1 and G_2. Let $u_i \in V(G_1)$ and $v_i \in V(G_2)$ be the vertices corresponding to the i-th vertex of G, $1 \leq i \leq p$, respectively. Define a total labeling $g : V(G_1 \vee G_2) \cup E(G_1 \vee G_2) \to [1, 2p + 2q + p^2]$ by

(i) $g(x) = f(x)$ for $x \in V(G_1) \cup E(G_1)$ with the set of labels $[1, p + q]$.

(ii) $g(x) = f(x) + p + q$ for $x \in E(G_2)$ with the set of labels $[p + q + 1, p + 2q]$.

(iii) $g(u_iv_j) = a_{i,j} + p + 2q$, where $a_{i,j}$ is the (i, j)-entry of a $p \times p$ magic square, with the set of labels $[p + 2q + 1, p + 2q + p^2]$.

(iv) $g(x) = f(x) + p + 2q + p^2$ for $x \in V(G_2)$ with the set of labels $[p + 2q + p^2 + 1, 2p + 2q + p^2]$.

Now, for $1 \leq i \leq p$, we have

(1) $$g^+_{tn}(u_i) = f^+_{tn}(x_i) + p(p^2 + 1)/2 + p(p + 2q) + p(p + 2q + p^2) + \sum_{k=1}^{p} f(x_k)$$
$$= f^+_{tn}(x_i) + k_1.$$

(2) $$\begin{aligned} g^+_{tn}(v_i) = {} & f^+_{tn}(x_i) + d_{G_2}(v_i)(p + 2q + p^2) + d_{G_2}(v_i)(p + q) \\ & + p(p^2 + 1)/2 + p(p + 2q) + \sum_{k=1}^{p} f(x_k) \\ = {} & f^+_{tn}(x_i) + d_{G_2}(v_i)(2p + 3q + p^2) + k_2. \end{aligned}$$

Here, k_1, k_2 are constants. Similar to the proof of Theorem 5.1, the theorem holds. □

Remark 2.

(1) One can check that conditions (a) to (c) of Theorem 5.1 are equivalent to the following two conditions:

(1a) $f^+_{tn}(u) = f^+_{tn}(v)$ *implies that* $d_G(u) = d_G(v) \neq p/2$*.*

(1b) $f^+_{tn}(u) \neq f^+_{tn}(v)$ *implies that* $f^+_{tn}(u) - f^+_{tn}(v) \notin \{2(p + q)[d_G(v) - d_G(u)], (p + q)(p - 2d_G(v))\}$*.*
Particularly, if $d_G(u) = d_G(v) = d$*, then we only need to ensure that* $f^+_{tn}(u) \neq f^+_{tn}(v)$ *implies* $f^+_{tn}(u) - f^+_{tn}(v) \neq (p + q)(p - 2d)$*.*

(2) Similarly, the conditions (a) to (c) of Theorem 5.2 are equivalent to the following two conditions:

(2a) $f^+_{tn}(u) = f^+_{tn}(v)$ *implies that* $d_G(u) = d_G(v) \neq \frac{p(p+2q+p^2)}{(2p+3q+p^2)}$*.*

(2b) $f^+_{tn}(u) \neq f^+_{tn}(v)$ *implies that* $f^+_{tn}(u) - f^+_{tn}(v) \notin \{(2p + 3q + p^2)[d_G(v) - d_G(u)], d_G(v)(2p + 3q + p^2) - p(p + 2q + p^2)\}$*.*
Particularly, if $d_G(u) = d_G(v) = d$*, then we only need to ensure that* $f^+_{tn}(u) \neq f^+_{tn}(v)$ *implies* $f^+_{tn}(u) - f^+_{tn}(v) \neq d(2p + 3q + p^2) - p(p + 2q + p^2)$*.*

(3) Suppose $f_{tn}^+(u) > f_{tn}^+(v)$ always implies $d(u) \geq d(v)$. Then condition (b) for both theorems always holds.

Corollary 5.3. *Suppose G is an r-regular graph of order $p \geq 3$ and $\chi_{ltna}(G) = t$. If there is a local total neighborhood antimagic t-coloring f of G such that for all vertices u and v (i) $|f_{tn}^+(u) - f_{tn}^+(v)| \neq |\frac{p(2r-p)(2+r)}{2}|$ or (ii) $|f_{tn}^+(u) - f_{tn}^+(v)| \neq |(2pr + \frac{3pr^2}{2}) - (p^2 + p^3)|$, then $\chi_{ltna}(G \vee G) \leq 2t$ with equality holding if $\chi(G) = t$.*

Proof. Since $2q = pr$, and by Theorems 5.1 and 5.2, we have the corollary. □

We now consider the common graphs in Section 4.

Corollary 5.4. *For $m, n \geq 1$, $\chi_{ltna}(nC_{2m+1} \vee nC_{2m+1}) = 6$.*

Proof. The conditions of Corollary 5.3 for nC_{2m+1} becomes
(i) $|f_{tn}^+(u) - f_{tn}^+(v)| \neq |2n(2m + 1)(n(2m + 1) - 4)|$.
(ii) $|f_{tn}^+(u) - f_{tn}^+(v)| \neq n(2m + 1)[n^2(2m + 1)^2 + n(2m + 1) - 10]$.

We use the labeling f for C_{2m+1} defined in Theorem 4.5 and its extension g for nC_{2m+1} as defined in Theorem 3.1. It is known that $g_{tn}^+(u_{k_1,i}) - g_{tn}^+(u_{k_2,j}) = n(f_{tn}^+(u_i) - f_{tn}^+(u_j))$ for all vertices u_i, u_j and $1 \leq k_1, k_2 \leq n$.

From the proof of Theorem 4.5 we have $|f_{tn}^+(u_1) - f_{tn}^+(u_{2i-1})| = 4m + 2$, $|f_{tn}^+(u_1) - f_{tn}^+(u_{2j})| = 2m + 1$ and $|f_{tn}^+(u_{2i-1}) - f_{tn}^+(u_{2j})| = 2m + 1$ for all i, j when $m \geq 2$; and $|f_{tn}^+(u) - f_{tn}^+(v)| \leq 2$ for distinct vertices u and v when $m = 1$. Thus, the labeling g satisfies condition (i) of Corollary 5.3 when $m \neq 2$ and $n > 1$.

When $(m, n) = (2, 1)$, the labeling g satisfies condition (ii) of Corollary 5.3.

Thus, by Corollary 5.3, we have the corollary. □

Corollary 5.5. *For $n \geq 1$ and $m = 6, 8, 10$, $\chi_{ltna}(nC_m \vee nC_m) = 4$.*

Proof. The conditions of Corollary 5.3 becomes
(i) $|f_{tn}^+(u) - f_{tn}^+(v)| \neq 2mn(mn - 4)$.
(ii) $|f_{tn}^+(u) - f_{tn}^+(v)| \neq |mn(m^2n^2 + mn - 10)|$.
It is easy the check that the labelings defined in Theorem 4.8 and its extended labelings satisfy the conditions of Corollary 5.3. Thus, the corollary holds. □

Corollary 5.6. *For $s \geq 2$, $\chi_{ltna}(K_{1,1,s} \vee K_{1,1,s}) = 6$.*

Proof. Note that $K_{1,1,s} \cong K_{1,s} \vee K_1$ and $\chi_{ltna}(K_{1,1,s}) = 3$. Keep the notation defined in the proof of Theorem 4.13. We let $m = 1$ and let $u_1 = z$, $u_2 = w_1$ and $v_j = v_{1,j}$, $1 \leq j \leq s$. Then $d(u_1) = d(u_2) = s+1$, $d(v_j) = 2$, $f_{tn}^+(u_1) = 3s^2 + 6s + 3$, $f_{tn}^+(u_2) = 2s^2 + 8s + 5$ and $f_{tn}^+(v_j) = 8s + 7$. In the following, we check the conditions of Theorem 5.2.

Thus, f satisfies condition (a). Since $f_{tn}^+(u) > f_{tn}^+(v)$ implies $d(u) \geq d(v)$ for all $u, v \in V(K_{1,1,s})$. By Remark 2, condition (b) holds.

Condition (c) becomes

$$
\begin{aligned}
f_{tn}^+(u) - f_{tn}^+(v) &\neq d(v)(s^2 + 12s + 11) - (s^3 + 11s^2 + 26s + 16) \\
&= \begin{cases} 2s^2 - 3s - 5 & \text{if } d(v) = s + 1; \\ -s^3 - 9s^2 - 2s + 6 & \text{if } d(v) = 2. \end{cases}
\end{aligned}
$$

Now,

$$\begin{aligned}
-(f^+_{tn}(u_2) - f^+_{tn}(u_1)) = f^+_{tn}(u_1) - f^+_{tn}(u_2) &= s^2 - 2s - 2 \neq 2s^2 - 3s - 5,\\
f^+_{tn}(u_2) - f^+_{tn}(v_j) &= 2s^2 - 2 > 0 > -s^3 - 9s^2 - 2s + 6,\\
f^+_{tn}(u_1) - f^+_{tn}(v_j) &= 3s^2 - 2s - 4 > 0 > -s^3 - 9s^2 - 2s + 6,\\
f^+_{tn}(v_{j'}) - f^+_{tn}(v_j) &= 0 \neq -s^3 - 9s^2 - 2s + 6, \text{ where } 1 \leq j \neq j' \leq s,\\
f^+_{tn}(v_j) - f^+_{tn}(u_1) &= -3s^2 + 2s + 4 \leq -4 < 2s^2 - 3s - 5,\\
f^+_{tn}(v_j) - f^+_{tn}(u_2) &= -2s^2 + 2 < 2s^2 - 3s - 5.
\end{aligned}$$

Thus, f satisfies condition (c).

By Theorem 5.2, we have the corollary. □

Corollary 5.7. *For $n \geq 2$, $\chi_{ltna}(W_{2n-1} \vee W_{2n-1}) = 8$ and $\chi_{ltna}(W_{2n} \vee W_{2n}) = 6$. Moreover, $\chi_{ltna}((W_3 - e) \vee (W_3 - e)) = \chi_{ltna}((W_5 - e) \vee (W_5 - e)) = 6$, where e is a spoke of W_{2n-1} for $n = 2, 3$.*

Proof. It is routine to check that the labeling f in the proof of Theorems 4.17, 4.18 or 4.19, satisfies either Theorem 5.1 or Theorem 5.2. □

6. Conclusions and future work

In this paper, we introduced and explored the concept of the local total neighborhood antimagic labeling of simple graphs G and the associated chromatic parameter $\chi_{ltna}(G)$. We investigated the existence of such labelings and determined the values of $\chi_{ltna}(G)$ for various classes of graphs without isolated vertices, including paths, cycles, complete graphs, and various forms of join graphs. Our results highlight the graph operations and vertex connectivity in influencing the behavior of $\chi_{ltna}(G)$. This study contributes to the broader study of graph labelings and opens avenues for further investigations on labeling techniques and coloring-related graph invariants. In a subsequent paper, we will show that (i) for $k \geq 2$, $\chi_{ltna}(C_{2k} \times P_2) = 2$, $\chi_{ltna}((C_{2k} \times P_2) \vee (C_{2k} \times P_2)) = 4$, (ii) for $s, n \geq 1$, $\chi_{ltna}(C_{4n+2} \vee O_{2s+1}) = 3$, and (iii) for $r, s \geq 1$, $\chi_{ltna}(rK_2 \vee O_{2s+1}) = 3$.

We now question and suggest some open problems that may serve as potential directions for further investigation.

- Can general lower and upper bounds be determined for $\chi_{ltna}(G)$ in terms of specific graph parameters such as order, size, and degree?
- One could extend the concept of local total neighborhood antimagic labeling to random graphs, and determine whether random graphs typically admit local total neighborhood antimagic labelings.
- Can this concept be extended to directed graphs or even multigraphs [13, 14]? If so, how should the induced coloring be appropriately defined to reflect the directional properties and structural complexities of such graphs?
- Given the complexity of the problem, can efficient algorithms be developed in determining whether a given graph admits a local total neighborhood labeling for specific classes of graphs?

Author contributions

Zhen Bin Gao, Gee-Choon Lau, Wai Chee Shiu and Kai Siong Yow: Conceptualization, Writing-original draft, Writing-review and editing. All authors of this article have been contributed equally. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that he has no conflict of interest.

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