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*Research article*

# Joint $\mathscr{A}$-operator seminorm bounds and refined joint $\mathscr{A}$-numerical radius inequalities

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**Abstract:** In this paper, we establish a new inequality relating the joint $\mathscr{A}$-operator seminorm of an $s$-tuple of operators $\mathbf{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s)$, where each $\mathcal{T}_k$ admits an $\mathscr{A}$-adjoint, to its joint $\mathscr{A}$-Euclidean operator seminorm. The obtained bound is expressed in terms of the $\mathscr{A}$-numerical radius of suitable operator products involving the components of the tuple and their $\mathscr{A}$-adjoints. Here, $\mathscr{A}$ is a nonzero positive operator on a complex Hilbert space $\mathscr{H}$ inducing a semi-Hilbertian structure, and $s$ is a positive integer. We further derive refined upper and lower bounds for the joint $\mathscr{A}$-numerical radius of $\mathbf{T}$, denoted by $\omega_{\mathscr{A}}(\mathbf{T})$, thereby improving its classical equivalence with the joint $\mathscr{A}$-operator seminorm. In particular, the estimate $\frac{1}{2\sqrt{s}}\|\mathbf{T}\|_{\mathscr{A}} \leq \omega_{\mathscr{A}}(\mathbf{T}) \leq \|\mathbf{T}\|_{\mathscr{A}}$ is sharpened by incorporating a term involving the $\mathscr{A}$-joint Crawford number of $\mathbf{T}^2$. Furthermore, we introduce a functional extension of the $\mathscr{A}$-joint numerical radius associated with a continuous nondecreasing function $\phi$ satisfying $\phi(0) = 0$, and establish several inequalities connecting this extension with the classical $\mathscr{A}$-joint numerical radius and the $\mathscr{A}$-operator seminorm. The main contribution of this work is the use of the $\mathscr{A}$-adjoint framework in the study of multivariable operators within semi-Hilbertian spaces. Our approach combines algebraic operator methods with nonlinear functional techniques, leading to strict improvements of the classical equivalence bounds through the incorporation of the $\mathscr{A}$-joint Crawford number.

**Keywords:** positive operator; semi-Hilbert space; joint $\mathscr{A}$-numerical radius; joint $\mathscr{A}$-operator seminorm; joint $\mathscr{A}$-Euclidean operator seminorm; Hilbert space

**Mathematics Subject Classification:** 47A10, 47A12, 47B20, 47B65, 46C05

## 1. Introduction and preliminaries

In recent years, the study of multivariable operators acting on Hilbert spaces has seen significant growth. Much of this progress stems from foundational concepts in single-variable operator theory, which continue to shape research in this area. Notably, early investigations into commuting tuples and joint normaloidity have provided essential tools for current research [1, 2]. Building on these foundations, recent contributions have focused on the spectral structure and algebraic properties of operator tuples [3–5]. In multivariable operator theory, unitary invariants play a pivotal role in classifying operator tuples [6]. Moreover, there has been a surge of interest in exploring these concepts within the framework of semi-Hilbert spaces and related unitary invariants [7–9].

This paper is dedicated to establishing new inequalities concerning the generalized Euclidean radius for tuples of operators on Hilbert spaces. The study of the Euclidean operator radius and its generalizations has a rich history [10–12], with notable extensions to various settings [13]. In particular, we investigate the impact of a semi-inner product structure generated by a positive operator within a complex Hilbert space.

First, we recall some essential notation and concepts. Let $\mathscr{H}$ denote a complex Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. The $C^*$-algebra of all bounded linear operators on $\mathscr{H}$ is denoted by $\mathbf{L}(\mathscr{H})$, with $I_{\mathscr{H}}$ representing the identity operator. For any operator $\mathcal{T}$, its range is denoted by $\operatorname{ran}(\mathcal{T})$, its null space by $\ker(\mathcal{T})$, and its adjoint by $\mathcal{T}^\star$. An operator is considered to be positive, written as $\mathcal{T} \geq \mathbf{0}$, if the inner product $\langle\mathcal{T}\xi,\xi\rangle$ is non-negative for every element $\xi$ in $\mathscr{H}$. When $\mathcal{T}$ is positive, its square root is denoted by $\sqrt{\mathcal{T}}$.

Throughout this paper, operators are considered as elements of $\mathbf{L}(\mathscr{H})$. We assume that $\mathcal{A}$ is a non-zero positive operator, which induces a semi-inner product defined by $\langle\xi,\xi_2\rangle_{\mathcal{A}} = \langle\mathcal{A}\xi,\xi_2\rangle$. This structure forms a semi-Hilbert space, denoted $(\mathscr{H},\|\cdot\|_{\mathcal{A}})$, where the seminorm $\|\cdot\|_{\mathcal{A}}$ is given by the square root of the semi-inner product. Specifically, the unit sphere corresponding to this semi-inner product is denoted by $\Sigma^1_{\mathcal{A}}$, which consists of all elements with a seminorm equal to one.

If $\mathcal{A} = I_{\mathscr{H}}$, then the unit sphere is denoted by $\Sigma^1$, corresponding to the standard Hilbert space structure. Note that $(\mathscr{H},\|\cdot\|_{\mathcal{A}})$ is generally neither a normed space nor a complete space. However, it becomes a Hilbert space if and only if $\mathcal{A}$ is injective and its range is closed. For further details, we refer the reader to [7].

Let $\mathbb{N}$ denote the set of positive integers. We define $\mathbf{L}(\mathscr{H})^s$ as the Cartesian product of $\mathbf{L}(\mathscr{H})$ taken $s$ times, where $s \in \mathbb{N}$. If $\mathbf{T} = (\mathcal{T}_1,\ldots,\mathcal{T}_s) \in \mathbf{L}(\mathscr{H})^s$, then $\mathbf{T}$ is called a commuting $s$-tuple if $[\mathcal{T}_i,\mathcal{T}_j] = \mathbf{0}$ for all $i, j \in \{1,\ldots,s\}$. Here, $[\cdot,\cdot]$ represents the commutator of two operators, defined as $[\mathcal{T}_1,\mathcal{T}_2] := \mathcal{T}_1\mathcal{T}_2 - \mathcal{T}_2\mathcal{T}_1$ for $\mathcal{T}_1,\mathcal{T}_2 \in \mathbf{L}(\mathscr{H})$.

Let $\mathbf{T} = (\mathcal{T}_1,\ldots,\mathcal{T}_s) \in \mathbf{L}(\mathscr{H})^s$ be an $s$-tuple of (not necessarily commuting) operators on the semi-Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle_{\mathcal{A}})$. The following two quantities were introduced in [14]:

$$\omega_{\mathcal{A}}(\mathbf{T}) := \sup_{\xi\in\Sigma^1_{\mathcal{A}}} \sqrt{\sum_{r=1}^{s} \left|\langle\mathcal{T}_r\xi,\xi\rangle_{\mathcal{A}}\right|^2} \quad \text{and} \quad \|\mathbf{T}\|_{\mathcal{A}} := \sup_{\xi\in\Sigma^1_{\mathcal{A}}} \sqrt{\sum_{r=1}^{s} \|\mathcal{T}_r\xi\|^2_{\mathcal{A}}},$$

where $\Sigma^1_{\mathcal{A}} = \{\xi \in \mathscr{H} : \|\xi\|_{\mathcal{A}} = 1\}$.

Both $\omega_{\mathcal{A}}(\mathbf{T})$ and $\|\mathbf{T}\|_{\mathcal{A}}$ may take infinite values, even when $s = 1$ (see [15]).

The special case $s = 1$ is of particular importance in the literature on operator inequalities in Hilbert and semi-Hilbert spaces. In this setting, for a single operator $\mathcal{T} \in \mathbf{L}(\mathscr{H})$, the general definitions reduce to the following well-known $\mathcal{A}$-numerical radius and $\mathcal{A}$-operator norm:

$$\omega_{\mathcal{A}}(\mathcal{T}) = \sup_{\xi\in\Sigma^1_{\mathcal{A}}}\left|\langle\mathcal{T}\xi, \xi\rangle_{\mathcal{A}}\right| \quad \text{and} \quad \|\mathcal{T}\|_{\mathcal{A}} = \sup_{\xi\in\Sigma^1_{\mathcal{A}}}\|\mathcal{T}\xi\|_{\mathcal{A}}.$$

Several classical and recent inequalities relating these two quantities have been established in this case. The foundational properties of the numerical range and classical inequalities can be found in [16, 17]. Significant advancements and sharper bounds for the numerical radius of Hilbert space operators have been extensively developed [18–20]. Further refinements, including convex treatments and estimates for the products of operators, are discussed in [21–23]. Moreover, extensions addressing weighted inequalities and specific operator matrices have been thoroughly investigated [24–26]. Additionally, classical inequalities of the Kato and Hermite-Hadamard type have been extended to operator sequences [27,28] as well as [29,30]. For further results and historical context in semi-Hilbert spaces, the reader is referred to [31–33] and the references therein.

To ensure that these quantities are finite, it is necessary to recall the concept of $\mathcal{A}$-adjoint operators, which play a key role in this work. This notion is drawn from [34]. For any $\mathcal{T} \in \mathbf{L}(\mathscr{H})$, an operator $\mathcal{R} \in \mathbf{L}(\mathscr{H})$ is called an $\mathcal{A}$-adjoint of $\mathcal{T}$ if, for all $\xi, \xi_2 \in \mathscr{H}$, the condition $\langle\mathcal{T}\xi, \xi_2\rangle_{\mathcal{A}} = \langle\xi, \mathcal{R}\xi_2\rangle_{\mathcal{A}}$ holds or, equivalently, if $\mathcal{A}\mathcal{R} = \mathcal{T}^\star\mathcal{A}$. Note that not every operator in $\mathbf{L}(\mathscr{H})$ possesses an $\mathcal{A}$-adjoint. Furthermore, even when an $\mathcal{A}$-adjoint operator exists, it may not be unique. This presents several challenges when analyzing operators in semi-Hilbert spaces.

Let $\mathbf{L}_{\mathcal{A}}(\mathscr{H})$ and $\mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})$ represent the sets of all operators that admit $\mathcal{A}$-adjoints and $\sqrt{\mathcal{A}}$-adjoints, respectively. Applying Douglas's theorem [35], we immediately observe the following characterizations:

$$\mathbf{L}_{\mathcal{A}}(\mathscr{H}) = \left\{\mathcal{T} \in \mathbf{L}(\mathscr{H})\,;\, \operatorname{ran}(\mathcal{T}^\star\mathcal{A}) \subseteq \operatorname{ran}(\mathcal{A})\right\}$$

and

$$\mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H}) = \left\{\mathcal{T} \in \mathbf{L}(\mathscr{H})\,;\, \exists\, c > 0 \text{ such that } \|\mathcal{T}\xi\|_{\mathcal{A}} \leq c\|\xi\|_{\mathcal{A}},\ \forall\xi \in \mathscr{H}\right\}.$$

The operators in $\mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})$ are referred to as $\mathcal{A}$-bounded operators. Furthermore, note that both $\mathbf{L}_{\mathcal{A}}(\mathscr{H})$ and $\mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})$ are subalgebras of $\mathbf{L}(\mathscr{H})$, but they are generally neither closed nor dense in the $C^*$-algebra $\mathbf{L}(\mathscr{H})$. Additionally, the following strict inclusions hold:

$$\mathbf{L}_{\mathcal{A}}(\mathscr{H}) \subseteq \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H}) \subseteq \mathbf{L}(\mathscr{H})$$

(see [15]). Now, let us consider $\mathbf{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$. It can be established that both $\|\mathbf{T}\|_{\mathcal{A}}$ and $\omega_{\mathcal{A}}(\mathbf{T})$ are finite in this context. Consequently, the functions $\|\cdot\|_{\mathcal{A}}$ and $\omega_{\mathcal{A}}(\cdot)$ define two seminorms on $\mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$, which are referred to as the joint operator $\mathcal{A}$-seminorm and the joint $\mathcal{A}$-numerical radius of the operators, respectively. Note that $\|\cdot\|_{\mathcal{A}}$ and $\omega_{\mathcal{A}}(\cdot)$ are equivalent seminorms. More specifically, it was demonstrated in [14] that the following inequalities hold for every $\mathbf{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$:

$$\frac{1}{2\sqrt{s}}\|\mathbf{T}\|_{\mathcal{A}} \leq \omega_{\mathcal{A}}(\mathbf{T}) \leq \|\mathbf{T}\|_{\mathcal{A}}. \tag{1.1}$$

Note that for every $\mathcal{T} \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})$, we have

$$\|\mathcal{T}\|_{\mathcal{A}} = \sup_{\xi,\eta\in\Sigma^1_{\mathcal{A}}} \left|\langle \mathcal{T}\xi, \eta\rangle_{\mathcal{A}}\right|, \tag{1.2}$$

as discussed in [14] and the references cited therein.

Applying Douglas's theorem, we can select a specific operator $\mathcal{T}^{\sharp_{\mathcal{A}}}$ from among the operators that admit $\mathcal{A}$-adjoints. If $\mathcal{A}^{\dagger}$ denotes the Moore-Penrose pseudoinverse of $\mathcal{A}$, then $\mathcal{T}^{\sharp_{\mathcal{A}}}$ is defined as $\mathcal{T}^{\sharp_{\mathcal{A}}} = \mathcal{A}^{\dagger}\mathcal{T}^{\star}\mathcal{A}$ (see [36]). Although $\mathcal{T}^{\sharp_{\mathcal{A}}}$ shares some properties with the standard adjoint $\mathcal{T}^{\star}$, they are not identical. For example, the equality $(\mathcal{T}^{\sharp_{\mathcal{A}}})^{\sharp_{\mathcal{A}}} = \mathcal{T}$ does not hold in general. However, this equivalence is true if and only if $\operatorname{ran}(\mathcal{T}) \subseteq \overline{\operatorname{ran}(\mathcal{A})}$. Utilizing the second equality in (1.2), it is straightforward to show that $\|\mathcal{T}\|_{\mathcal{A}} = \|\mathcal{T}^{\sharp_{\mathcal{A}}}\|_{\mathcal{A}}$ for all $\mathcal{T} \in \mathbf{L}_{\mathcal{A}}(\mathscr{H})$.

Next, we summarize some fundamental properties of the operator $\mathcal{T}^{\sharp_{\mathcal{A}}}$ based on [34, 36, 37]. For $\mathcal{T}_1, \mathcal{T}_2 \in \mathbf{L}_{\mathcal{A}}(\mathscr{H})$ and $\mu, \nu \in \mathbb{C}$, the linear combination $\mu\mathcal{T}_1 + \nu\mathcal{T}_2$ belongs to $\mathbf{L}_{\mathcal{A}}(\mathscr{H})$ and satisfies $(\mu\mathcal{T}_1 + \nu\mathcal{T}_2)^{\sharp_{\mathcal{A}}} = \overline{\mu}\mathcal{T}_1^{\sharp_{\mathcal{A}}} + \overline{\nu}\mathcal{T}_2^{\sharp_{\mathcal{A}}}$. Additionally, the product $\mathcal{T}_1\mathcal{T}_2$ is in $\mathbf{L}_{\mathcal{A}}(\mathscr{H})$ with $(\mathcal{T}_1\mathcal{T}_2)^{\sharp_{\mathcal{A}}} = \mathcal{T}_2^{\sharp_{\mathcal{A}}}\mathcal{T}_1^{\sharp_{\mathcal{A}}}$. Furthermore, for any $\mathcal{T} \in \mathbf{L}_{\mathcal{A}}(\mathscr{H})$, its powers $\mathcal{T}^p \in \mathbf{L}_{\mathcal{A}}(\mathscr{H})$ and $(\mathcal{T}^p)^{\sharp_{\mathcal{A}}} = (\mathcal{T}^{\sharp_{\mathcal{A}}})^p$ for all $p \in \mathbb{N}$.

We also note that $(\mathcal{T}^{\sharp_{\mathcal{A}}})^{\sharp_{\mathcal{A}}} = \mathbb{P}_{\overline{\operatorname{ran}(\mathcal{A})}}\mathcal{T}\mathbb{P}_{\overline{\operatorname{ran}(\mathcal{A})}}$ and $((\mathcal{T}^{\sharp_{\mathcal{A}}})^{\sharp_{\mathcal{A}}})^{\sharp_{\mathcal{A}}} = \mathcal{T}^{\sharp_{\mathcal{A}}}$, where $\mathbb{P}_{\overline{\operatorname{ran}(\mathcal{A})}}$ represents the orthogonal projection onto the closure of the range of $\mathcal{A}$, $\overline{\operatorname{ran}(\mathcal{A})}$. Finally, we have the identity

$$\|\mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}\|_{\mathcal{A}} = \|\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}}\|_{\mathcal{A}} = \|\mathcal{T}\|^2_{\mathcal{A}}, \quad \forall\, \mathcal{T} \in \mathbf{L}_{\mathcal{A}}(\mathscr{H}). \tag{1.3}$$

For additional details concerning the operator $\mathcal{T}^{\sharp_{\mathcal{A}}}$, the reader may consult [34, 36, 37].

Let $\mathbb{B}_s$ denote the open unit ball in $\mathbb{C}^s$ with respect to the Euclidean norm, defined as

$$\mathbb{B}_s := \left\{ \alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{C}^s \;:\; \|\alpha\|_2 := \left( \sum_{r=1}^{s} |\alpha_r|^2 \right)^{\frac{1}{2}} < 1 \right\}.$$

In [8], the authors introduced two specific seminorms on $\mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$. For a tuple $\mathbf{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$, they are defined as

$$\omega_{e,\mathcal{A}}(\mathbf{T}) = \sup_{\alpha\in\mathbb{B}_s} \omega_{\mathcal{A}}\Big(\sum_{r=1}^{s} \alpha_r \mathcal{T}_r\Big) \quad \text{and} \quad \|\mathbf{T}\|_{e,\mathcal{A}} = \sup_{\alpha\in\mathbb{B}_s} \Big\|\sum_{r=1}^{s} \alpha_r \mathcal{T}_r\Big\|_{\mathcal{A}}.$$

It is well established that $\omega_{\mathcal{A}}(\mathbf{T}) = \omega_{e,\mathcal{A}}(\mathbf{T})$ for every $\mathbf{T} \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$ [8]. However, $\|\mathbf{T}\|_{e,\mathcal{A}}$ is not generally equal to $\|\mathbf{T}\|_{\mathcal{A}}$. The seminorm $\|\cdot\|_{e,\mathcal{A}}$ on $\mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$ is equivalent to the joint operator $\mathcal{A}$-seminorm and satisfies the following bounds:

$$\frac{1}{\sqrt{s}}\|\mathbf{T}\|_{\mathcal{A}} \leq \|\mathbf{T}\|_{e,\mathcal{A}} \leq \|\mathbf{T}\|_{\mathcal{A}} \tag{1.4}$$

for all $\mathbf{T} \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$ [8, 14].

An $s$-tuple $\mathbf{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\mathcal{A}}(\mathscr{H})^s$ is considered $\mathcal{A}$-doubly commuting if $[\mathcal{T}_i, \mathcal{T}_j] = \mathbf{0}$ for all $i, j$ and $[\mathcal{T}_i^{\sharp_{\mathcal{A}}}, \mathcal{T}_j] = \mathbf{0}$ for all $i \neq j$. If, in addition, each operator $\mathcal{T}_i$ is $\mathcal{A}$-hyponormal, meaning that $[\mathcal{T}_i^{\sharp_{\mathcal{A}}}, \mathcal{T}_i] \geq_{\mathcal{A}} \mathbf{0}$ for all $i$, then it follows that $\|\mathbf{T}\|_{\mathcal{A}} = \|\mathbf{T}\|_{e,\mathcal{A}}$ (see [8]).

Next, we review some important classes of operators within semi-Hilbert spaces. For an operator $\mathcal{T} \in \mathbf{L}(\mathscr{H})$, we define the following properties: An operator $\mathcal{T}$ is called $\mathcal{A}$-self-adjoint if $\mathcal{A}\mathcal{T} = \mathcal{T}^{\star}\mathcal{A}$, which indicates that $\mathcal{AT}$ is a self-adjoint operator in the usual sense. We say that $\mathcal{T}$ is $\mathcal{A}$-positive, denoted $\mathcal{T} \geq_{\mathcal{A}} \mathbf{0}$, if $\mathcal{AT} \geq \mathbf{0}$. It was shown in [15] that if $\mathcal{T}$ is $\mathcal{A}$-self-adjoint, then $\mathcal{T} \in \mathbf{L}_{\mathcal{A}}(\mathscr{H})$ and the following equality holds:

$$\|\mathcal{T}\|_{\mathcal{A}} = \omega_{\mathcal{A}}(\mathcal{T}). \tag{1.5}$$

An operator $\mathcal{T} \in \mathbf{L}_{\mathcal{A}}(\mathscr{H})$ is classified as $\mathcal{A}$-normal if $[\mathcal{T}^{\sharp_{\mathcal{A}}}, \mathcal{T}] = \mathbf{0}$. Furthermore, $\mathcal{T}$ is termed $\mathcal{A}$-paranormal if the inequality $\|\mathcal{T}\xi\|_{\mathcal{A}}^2 \leq \|\mathcal{T}^2\xi\|_{\mathcal{A}}$ is satisfied for all $\xi \in \Sigma_{\mathcal{A}}^1$.

It is evident that $\mathcal{A}$-self-adjoint operators are a subset of $\mathbf{L}_{\mathcal{A}}(\mathscr{H})$. Because $\mathscr{H}$ is a complex Hilbert space, the condition $\mathcal{T} \geq_{\mathcal{A}} \mathbf{0}$ ensures that $\mathcal{T}$ is $\mathcal{A}$-self-adjoint. Generally, $\mathcal{A}$-self-adjoint operators are not always $\mathcal{A}$-normal (see [15] or [38, Example 5.1]). If $\mathcal{T}$ is $\mathcal{A}$-normal, it trivially follows that $\mathcal{T}$ is $\mathcal{A}$-hyponormal (since the condition $[\mathcal{T}^{\sharp_{\mathcal{A}}}, \mathcal{T}] = \mathbf{0}$ implies $[\mathcal{T}^{\sharp_{\mathcal{A}}}, \mathcal{T}] \geq_{\mathcal{A}} \mathbf{0}$). Conversely, $\mathcal{A}$-hyponormal operators do not necessarily have to be $\mathcal{A}$-normal, even in finite-dimensional spaces (see [15]).

Let $\mathcal{T}$ be an $\mathcal{A}$-self-adjoint operator. Note that the equality $\mathcal{T}^{\sharp_{\mathcal{A}}} = \mathcal{T}$ might not hold in general (see [34]). It is valid, however, if $\operatorname{ran}(\mathcal{T}) \subseteq \overline{\operatorname{ran}(\mathcal{A})}$ (see also [34]). Furthermore, if $\mathcal{T} \in \mathbf{L}(\mathscr{H})$ is an $\mathcal{A}$-self-adjoint operator, its adjoint $\mathcal{T}^{\sharp_{\mathcal{A}}}$ is also $\mathcal{A}$-self-adjoint, and it satisfies $(\mathcal{T}^{\sharp_{\mathcal{A}}})^{\sharp_{\mathcal{A}}} = \mathcal{T}^{\sharp_{\mathcal{A}}}$. For a discussion on the foundational properties of these operator classes, we direct the reader to [15] and the references provided therein.

For a tuple $\mathbf{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\mathcal{A}}(\mathscr{H})^s$, it is clear that $\sum_{r=1}^{s} \mathcal{T}_r^{\sharp_{\mathcal{A}}}\mathcal{T}_r \geq \mathbf{0}$, so according to (1.5), we obtain the expression

$$\|\mathbf{T}\|_{\mathcal{A}} = \sqrt{\left\| \sum_{r=1}^{s} \mathcal{T}_r^{\sharp_{\mathcal{A}}}\mathcal{T}_r \right\|_{\mathcal{A}}}. \tag{1.6}$$

For an in-depth analysis, please see [38].

The main objective of this work is to establish new inequalities for operator tuples acting on semi-Hilbert spaces. Using the framework of $\mathcal{A}$-adjoint operators, we derive estimates involving the $\mathcal{A}$-joint Crawford number and obtain refinements of known bounds relating the joint $\mathcal{A}$-numerical radius and the joint $\mathcal{A}$-operator seminorm. These results provide additional information on the relationship between operator seminorms and numerical radii in semi-Hilbert spaces.

The remainder of the paper is organized as follows.

In Section 2, motivated by the recent investigations in [39,40], we establish new inequalities relating the joint $\mathcal{A}$-operator seminorm $\|\mathbf{T}\|_{\mathcal{A}}$ of an $s$-tuple $\mathbf{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s)$ to its joint $\mathcal{A}$-Euclidean seminorm $\|\mathbf{T}\|_{e,\mathcal{A}}$. These estimates are expressed in terms of the $\mathcal{A}$-numerical radius of suitable operator products involving the components $\mathcal{T}_k$ and their $\mathcal{A}$-adjoints. We also derive refined bounds for the joint $\mathcal{A}$-numerical radius $\omega_{\mathcal{A}}(\mathbf{T})$, improving the classical inequality (1.1) through the incorporation of a term involving the $\mathcal{A}$-joint Crawford number of $\mathbf{T}^2$. Several of these results are new even in the classical Hilbert space case where $\mathcal{A} = I_{\mathscr{H}}$.

In Section 3, inspired by [10], we introduce a functional extension of the joint $\mathcal{A}$-numerical radius. More precisely, for a continuous nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(0) = 0$, we define a corresponding $\phi$-version of $\omega_{\mathcal{A}}(\mathbf{T})$. We then establish general inequalities linking this quantity with both the joint $\mathcal{A}$-numerical radius and the joint $\mathcal{A}$-operator seminorm.

Finally, Section 4 contains concluding remarks and several directions for future investigation.

## 2. Joint $\mathcal{A}$-seminorm bounds

The inequalities established in this section are obtained by combining the properties of the $\mathcal{A}$-adjoint with the semi-inner product structure. The analysis of the cross-terms arising in the operator expansions plays a central role in the proofs.

The following theorem establishes a general inequality linking the seminorm and numerical radius for the sum of $s$ operators, bounded in terms of standard vector norms.

**Theorem 2.1.** *Let $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s \in \mathbf{L}_{\mathcal{A}}(\mathcal{H})$ and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s) \in \mathbb{C}^s$. Then*

$$\begin{aligned}\left\|\sum_{r=1}^{s} \alpha_r \mathcal{T}_r\right\|_{\mathcal{A}}^2 &\leq \|\alpha\|_\infty^2 \left\|\sum_{r=1}^{s} \mathcal{T}_r^{\sharp_{\mathcal{A}}} \mathcal{T}_r\right\|_{\mathcal{A}} + (\|\alpha\|_1^2 - \|\alpha\|_2^2) \max_{1\leq j\neq r\leq s} \omega_{\mathcal{A}}\left(\mathcal{T}_j^{\sharp_{\mathcal{A}}} \mathcal{T}_r\right) \\ &\leq \|\alpha\|_2^2 \left(\left\|\sum_{r=1}^{s} \mathcal{T}_r^{\sharp_{\mathcal{A}}} \mathcal{T}_r\right\|_{\mathcal{A}} + (s-1) \max_{1\leq j\neq r\leq s} \omega_{\mathcal{A}}\left(\mathcal{T}_j^{\sharp_{\mathcal{A}}} \mathcal{T}_r\right)\right),\end{aligned}$$

*where $\|\alpha\|_1, \|\alpha\|_2$, and $\|\alpha\|_\infty$ represent the $\ell_1$, $\ell_2$, and $\ell_\infty$ norms of the scalar vector $\alpha$, respectively.*

*Proof.* Let $\mathcal{T} \in \mathbf{L}_{\mathcal{A}}(\mathcal{H})$. Applying (1.3) and the triangle inequality, we deduce

$$\begin{aligned}\left\|\sum_{r=1}^{s} \alpha_r \mathcal{T}_r\right\|_{\mathcal{A}}^2 &= \left\|\left(\sum_{r=1}^{s} \alpha_r \mathcal{T}_r\right)^{\sharp_{\mathcal{A}}} \left(\sum_{r=1}^{s} \alpha_r \mathcal{T}_r\right)\right\|_{\mathcal{A}} \\ &= \omega_{\mathcal{A}}\left(\left(\sum_{r=1}^{s} \alpha_r \mathcal{T}_r\right)^{\sharp_{\mathcal{A}}} \left(\sum_{r=1}^{s} \alpha_r \mathcal{T}_r\right)\right) \\ &= \omega_{\mathcal{A}}\left(\sum_{r=1}^{s} |\alpha_r|^2 \mathcal{T}_r^{\sharp_{\mathcal{A}}} \mathcal{T}_r + \sum_{1\leq j\neq r\leq s} \overline{\alpha_j} \alpha_r \mathcal{T}_j^{\sharp_{\mathcal{A}}} \mathcal{T}_r\right) \\ &\leq \omega_{\mathcal{A}}\left(\sum_{r=1}^{s} |\alpha_r|^2 \mathcal{T}_r^{\sharp_{\mathcal{A}}} \mathcal{T}_r\right) + \omega_{\mathcal{A}}\left(\sum_{1\leq j\neq r\leq s} \overline{\alpha_j} \alpha_r \mathcal{T}_j^{\sharp_{\mathcal{A}}} \mathcal{T}_r\right).\end{aligned}$$

Given that $\sum_{r=1}^{s} |\alpha_r|^2 \mathcal{T}_r^{\sharp_{\mathcal{A}}} \mathcal{T}_r \geq_{\mathcal{A}} \mathbf{0}$, we can use (1.5) and the subadditivity of $\omega_{\mathcal{A}}(\cdot)$ to obtain

$$\left\|\sum_{r=1}^{s} \alpha_r \mathcal{T}_r\right\|_{\mathcal{A}}^2 \leq \left\|\sum_{r=1}^{s} |\alpha_r|^2 \mathcal{T}_r^{\sharp_{\mathcal{A}}} \mathcal{T}_r\right\|_{\mathcal{A}} + \omega_{\mathcal{A}}\left(\sum_{1\leq j\neq r\leq s} \overline{\alpha_j} \alpha_r \mathcal{T}_j^{\sharp_{\mathcal{A}}} \mathcal{T}_r\right). \tag{2.1}$$

This establishes an intermediate upper bound. Now, let $\xi \in \Sigma_{\mathcal{A}}^1$. Because each operator $\mathcal{T}_r^{\sharp_{\mathcal{A}}} \mathcal{T}_r$ is $\mathcal{A}$-positive, it follows that $\langle \mathcal{T}_r^{\sharp_{\mathcal{A}}} \mathcal{T}_r \xi, \xi\rangle_{\mathcal{A}} = \|\mathcal{T}_r \xi\|_{\mathcal{A}}^2 \geq 0$. Utilizing the upper bound $|\alpha_r|^2 \leq \|\alpha\|_\infty^2$ for all $1 \leq r \leq s$, we find that

$$\left\langle \left(\sum_{r=1}^{s} |\alpha_r|^2 \mathcal{T}_r^{\sharp_{\mathcal{A}}} \mathcal{T}_r\right) \xi, \xi\right\rangle_{\mathcal{A}} = \sum_{r=1}^{s} |\alpha_r|^2 \langle \mathcal{T}_r^{\sharp_{\mathcal{A}}} \mathcal{T}_r \xi, \xi\rangle_{\mathcal{A}} \leq \|\alpha\|_\infty^2 \sum_{r=1}^{s} \langle \mathcal{T}_r^{\sharp_{\mathcal{A}}} \mathcal{T}_r \xi, \xi\rangle_{\mathcal{A}} = \|\alpha\|_\infty^2 \left\langle \left(\sum_{r=1}^{s} \mathcal{T}_r^{\sharp_{\mathcal{A}}} \mathcal{T}_r\right) \xi, \xi\right\rangle_{\mathcal{A}}.$$

Taking the supremum over all elements $\xi \in \Sigma^1_{\mathcal{A}}$ and using (1.5), we conclude that

$$\left\|\sum_{r=1}^{s} |\alpha_r|^2 \mathcal{T}_r^{\sharp_{\mathcal{A}}} \mathcal{T}_r\right\|_{\mathcal{A}} \leq \|\alpha\|_\infty^2 \left\|\sum_{r=1}^{s} \mathcal{T}_r^{\sharp_{\mathcal{A}}} \mathcal{T}_r\right\|_{\mathcal{A}}.$$

Applying the triangle inequality to the $\mathcal{A}$-numerical radius yields

$$\begin{aligned}\omega_{\mathcal{A}}\left(\sum_{1\leq j\neq r\leq s} \overline{\alpha_j}\alpha_r \mathcal{T}_j^{\sharp_{\mathcal{A}}}\mathcal{T}_r\right) &\leq \sum_{1\leq j\neq r\leq s} |\alpha_j||\alpha_r|\omega_{\mathcal{A}}\left(\mathcal{T}_j^{\sharp_{\mathcal{A}}}\mathcal{T}_r\right)\\ &\leq \left(\max_{1\leq j\neq r\leq s} \omega_{\mathcal{A}}\left(\mathcal{T}_j^{\sharp_{\mathcal{A}}}\mathcal{T}_r\right)\right) \sum_{1\leq j\neq r\leq s} |\alpha_j||\alpha_r|.\end{aligned}$$

Since the sum can be evaluated as

$$\sum_{1\leq j\neq r\leq s} |\alpha_j||\alpha_r| = \left(\sum_{r=1}^{s} |\alpha_r|\right)^2 - \sum_{r=1}^{s} |\alpha_r|^2 = \|\alpha\|_1^2 - \|\alpha\|_2^2,$$

we arrive at the first inequality.

For the second inequality, utilizing the relationship $\|\alpha\|_\infty \leq \|\alpha\|_2$ alongside the Cauchy-Schwarz vector inequality $\|\alpha\|_1 \leq \sqrt{s}\|\alpha\|_2$, we deduce that

$$\|\alpha\|_1^2 - \|\alpha\|_2^2 \leq (s-1)\|\alpha\|_2^2.$$

Substituting these relationships into the expression provides the desired result.

As a direct consequence of Theorem 2.1, we obtain the following bound relating $\|\mathbf{T}\|_{\mathcal{A}}$ to its joint $\mathcal{A}$-Euclidean operator seminorm $\|\mathbf{T}\|_{e,\mathcal{A}}$.

**Corollary 2.1.** *Let* $\mathbf{T} = (\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$. *Then*

$$\|\mathbf{T}\|_{e,\mathcal{A}} \leq \sqrt{\|\mathbf{T}\|_{\mathcal{A}}^2 + (s-1)\max_{1\leq j\neq r\leq s} \omega_{\mathcal{A}}\left(\mathcal{T}_j^{\sharp_{\mathcal{A}}}\mathcal{T}_r\right)}.$$

*Proof.* Taking the supremum over all vectors $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s) \in \mathbb{B}_s$ in Theorem 2.1, we deduce that

$$\|\mathbf{T}\|_{e,\mathcal{A}} \leq \sqrt{\left\|\sum_{r=1}^{s} \mathcal{T}_r^{\sharp_{\mathcal{A}}}\mathcal{T}_r\right\|_{\mathcal{A}} + (s-1)\max_{1\leq j\neq r\leq s} \omega_{\mathcal{A}}\left(\mathcal{T}_j^{\sharp_{\mathcal{A}}}\mathcal{T}_r\right)}.$$

Applying Eq (1.6), we obtain the final result.

By setting $\alpha_r = 1$, we derive the following unweighted inequality directly from the intermediate steps established in the preceding proof.

**Corollary 2.2.** *Let* $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s \in \mathbf{L}_{\mathcal{A}}(\mathscr{H})$. *Then*

$$\left\|\sum_{r=1}^{s} \mathcal{T}_r\right\|_{\mathcal{A}} \leq \sqrt{\left\|\sum_{r=1}^{s} \mathcal{T}_r^{\sharp_{\mathcal{A}}}\mathcal{T}_r\right\|_{\mathcal{A}} + \omega_{\mathcal{A}}\left(\sum_{j=1}^{s} \mathcal{T}_j^{\sharp_{\mathcal{A}}} \sum_{r=1}^{s} \mathcal{T}_r - \sum_{r=1}^{s} \mathcal{T}_r^{\sharp_{\mathcal{A}}}\mathcal{T}_r\right)}.$$

*Proof.* Setting $\alpha_r = 1$ for all $r = 1, \dots, s$ in Eq (2.1), we find that

$$\left\|\sum_{r=1}^{s} \mathcal{T}_r\right\|_{\mathcal{A}}^2 \leq \left\|\sum_{r=1}^{s} \mathcal{T}_r^{\sharp_{\mathcal{A}}} \mathcal{T}_r\right\|_{\mathcal{A}} + \omega_{\mathcal{A}}\left(\sum_{1\leq j\neq r\leq s} \mathcal{T}_j^{\sharp_{\mathcal{A}}} \mathcal{T}_r\right).$$

Since the cross-terms can be factored and rewritten as

$$\sum_{1\leq j\neq r\leq s} \mathcal{T}_j^{\sharp_{\mathcal{A}}} \mathcal{T}_r = \left(\sum_{j=1}^{s} \mathcal{T}_j^{\sharp_{\mathcal{A}}}\right)\left(\sum_{r=1}^{s} \mathcal{T}_r\right) - \sum_{r=1}^{s} \mathcal{T}_r^{\sharp_{\mathcal{A}}} \mathcal{T}_r,$$

we immediately obtain the desired inequality.

Let $\mathcal{T} \in \mathbf{L}_{\mathcal{A}}(\mathscr{H})$. We define the $\mathcal{A}$-real and $\mathcal{A}$-imaginary parts of $\mathcal{T}$ as

$$\mathfrak{R}_{\mathcal{A}}(\mathcal{T}) = \frac{\mathcal{T} + \mathcal{T}^{\sharp_{\mathcal{A}}}}{2} \quad \text{and} \quad \mathfrak{I}_{\mathcal{A}}(\mathcal{T}) = \frac{\mathcal{T} - \mathcal{T}^{\sharp_{\mathcal{A}}}}{2i}.$$

Note that $\mathcal{T} = \mathfrak{R}_{\mathcal{A}}(\mathcal{T}) + i\mathfrak{I}_{\mathcal{A}}(\mathcal{T})$ and that both components, $\mathfrak{R}_{\mathcal{A}}(\mathcal{T})$ and $\mathfrak{I}_{\mathcal{A}}(\mathcal{T})$, are $\mathcal{A}$-self-adjoint.

The next remark discusses a specific application of Corollary 2.2 to the case of two operators.

**Remark 2.1.** *Let $s = 2$ in Corollary 2.2. We then obtain the inequality*

$$\|\mathcal{T}_1 + \mathcal{T}_2\|_{\mathcal{A}}^2 \leq \left\|\mathcal{T}_1^{\sharp_{\mathcal{A}}}\mathcal{T}_1 + \mathcal{T}_2^{\sharp_{\mathcal{A}}}\mathcal{T}_2\right\|_{\mathcal{A}} + \omega_{\mathcal{A}}\left(\mathcal{T}_1^{\sharp_{\mathcal{A}}}\mathcal{T}_2 + \mathcal{T}_2^{\sharp_{\mathcal{A}}}\mathcal{T}_1\right). \tag{2.2}$$

*Applying Eq (2.2) to the adjoint operators $\mathcal{T}_1^{\sharp_{\mathcal{A}}}$ and $\mathcal{T}_2^{\sharp_{\mathcal{A}}}$ instead of $\mathcal{T}_1$ and $\mathcal{T}_2$, we find*

$$\left\|\mathcal{T}_1^{\sharp_{\mathcal{A}}} + \mathcal{T}_2^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}^2 \leq \left\|(\mathcal{T}_1^{\sharp_{\mathcal{A}}})^{\sharp_{\mathcal{A}}}\mathcal{T}_1^{\sharp_{\mathcal{A}}} + (\mathcal{T}_2^{\sharp_{\mathcal{A}}})^{\sharp_{\mathcal{A}}}\mathcal{T}_2^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}} + 2\left\|\mathfrak{R}_{\mathcal{A}}\left((\mathcal{T}_1^{\sharp_{\mathcal{A}}})^{\sharp_{\mathcal{A}}}\mathcal{T}_2^{\sharp_{\mathcal{A}}}\right)\right\|_{\mathcal{A}}.$$

*Thus, we obtain the expression*

$$\|\mathcal{T}_1 + \mathcal{T}_2\|_{\mathcal{A}}^2 \leq \left\|\mathcal{T}_1\mathcal{T}_1^{\sharp_{\mathcal{A}}} + \mathcal{T}_2\mathcal{T}_2^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}} + 2\left\|\mathfrak{R}_{\mathcal{A}}\left(\mathcal{T}_1\mathcal{T}_2^{\sharp_{\mathcal{A}}}\right)\right\|_{\mathcal{A}}. \tag{2.3}$$

*By setting $\mathcal{T}_1 = [\mathfrak{R}_{\mathcal{A}}(\mathcal{T})]^{\sharp_{\mathcal{A}}}$ and $\mathcal{T}_2 = -i[\mathfrak{I}_{\mathcal{A}}(\mathcal{T})]^{\sharp_{\mathcal{A}}}$ in Eq (2.3), and utilizing the facts that $\mathcal{T}_1^{\sharp_{\mathcal{A}}} = \mathcal{T}_1$ and $\mathcal{T}_2^{\sharp_{\mathcal{A}}} = -\mathcal{T}_2$ (since both $\mathfrak{R}_{\mathcal{A}}(\mathcal{T})$ and $\mathfrak{I}_{\mathcal{A}}(\mathcal{T})$ are $\mathcal{A}$-self-adjoint), we obtain*

$$\begin{aligned}
\left\|\mathcal{T}^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}^2 &= \left\|[\mathfrak{R}_{\mathcal{A}}(\mathcal{T})]^{\sharp_{\mathcal{A}}} - i[\mathfrak{I}_{\mathcal{A}}(\mathcal{T})]^{\sharp_{\mathcal{A}}}\right\|_{\mathcal{A}}^2 \\
&\leq \left\|\left([\mathfrak{R}_{\mathcal{A}}(\mathcal{T})]^{\sharp_{\mathcal{A}}}\right)^2 + \left([\mathfrak{I}_{\mathcal{A}}(\mathcal{T})]^{\sharp_{\mathcal{A}}}\right)^2\right\|_{\mathcal{A}} + 2\left\|\mathfrak{R}_{\mathcal{A}}\left([\mathfrak{R}_{\mathcal{A}}(\mathcal{T})]^{\sharp_{\mathcal{A}}}(i[\mathfrak{I}_{\mathcal{A}}(\mathcal{T})]^{\sharp_{\mathcal{A}}})\right)\right\|_{\mathcal{A}} \\
&= \left\|\left([\mathfrak{R}_{\mathcal{A}}(\mathcal{T})]^{\sharp_{\mathcal{A}}}\right)^2 + \left([\mathfrak{I}_{\mathcal{A}}(\mathcal{T})]^{\sharp_{\mathcal{A}}}\right)^2\right\|_{\mathcal{A}} + 2\left\|\mathfrak{I}_{\mathcal{A}}\left([\mathfrak{R}_{\mathcal{A}}(\mathcal{T})]^{\sharp_{\mathcal{A}}}[\mathfrak{I}_{\mathcal{A}}(\mathcal{T})]^{\sharp_{\mathcal{A}}}\right)\right\|_{\mathcal{A}}.
\end{aligned}$$

*Since it is known that $\|\mathcal{T}^{\sharp_{\mathcal{A}}}\|_{\mathcal{A}}^2 = \|\mathcal{T}\|_{\mathcal{A}}^2$ and*

$$\left\|\left([\mathfrak{R}_{\mathcal{A}}(\mathcal{T})]^{\sharp_{\mathcal{A}}}\right)^2 + \left([\mathfrak{I}_{\mathcal{A}}(\mathcal{T})]^{\sharp_{\mathcal{A}}}\right)^2\right\|_{\mathcal{A}} = \frac{1}{2}\left\|\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}} + \mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}\right\|_{\mathcal{A}},$$

*we determine that*

$$\|\mathcal{T}\|_{\mathcal{A}}^2 \leq \frac{1}{2}\left\|\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}} + \mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}\right\|_{\mathcal{A}} + 2\left\|\mathfrak{I}_{\mathcal{A}}\left([\mathfrak{R}_{\mathcal{A}}(\mathcal{T})]^{\sharp_{\mathcal{A}}}[\mathfrak{I}_{\mathcal{A}}(\mathcal{T})]^{\sharp_{\mathcal{A}}}\right)\right\|_{\mathcal{A}}.$$

*This simplifies to*

$$\|\mathcal{T}\|_{\mathcal{A}}^2 \leq \frac{1}{2}\left\|\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}} + \mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}\right\|_{\mathcal{A}} + 2\left\|\mathfrak{I}_{\mathcal{A}}\Big(\mathfrak{I}_{\mathcal{A}}(\mathcal{T})\mathfrak{R}_{\mathcal{A}}(\mathcal{T})\Big)\right\|_{\mathcal{A}}.$$

*This sequence of steps provides a correctly derived, reversed version of the bound*

$$\frac{1}{2}\left\|\mathcal{T}\mathcal{T}^{\sharp_{\mathcal{A}}} + \mathcal{T}^{\sharp_{\mathcal{A}}}\mathcal{T}\right\|_{\mathcal{A}} \leq \|\mathcal{T}\|_{\mathcal{A}}^2.$$

In the next result, we derive a weighted expansion by preserving the arbitrary scalars from the intermediate step of Theorem 2.1.

**Corollary 2.3.** *Let $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_s \in \mathbf{L}_{\mathcal{A}}(\mathscr{H})$ and let $\alpha_1, \alpha_2, \dots, \alpha_s \in \mathbb{C}$. Then*

$$\left\|\sum_{r=1}^{s}\alpha_r\mathcal{T}_r\right\|_{\mathcal{A}}^2 \leq \left\|\sum_{r=1}^{s}|\alpha_r|^2\mathcal{T}_r^{\sharp_{\mathcal{A}}}\mathcal{T}_r\right\|_{\mathcal{A}} + \omega_{\mathcal{A}}\left(\sum_{j=1}^{s}\overline{\alpha_j}\mathcal{T}_j^{\sharp_{\mathcal{A}}}\sum_{r=1}^{s}\alpha_r\mathcal{T}_r - \sum_{r=1}^{s}|\alpha_r|^2\mathcal{T}_r^{\sharp_{\mathcal{A}}}\mathcal{T}_r\right).$$

*Proof.* From the proof of Theorem 2.1, we previously established that

$$\left\|\sum_{r=1}^{s}\alpha_r\mathcal{T}_r\right\|_{\mathcal{A}}^2 \leq \left\|\sum_{r=1}^{s}|\alpha_r|^2\mathcal{T}_r^{\sharp_{\mathcal{A}}}\mathcal{T}_r\right\|_{\mathcal{A}} + \omega_{\mathcal{A}}\left(\sum_{1\leq j\neq r\leq s}\overline{\alpha_j}\alpha_r\mathcal{T}_j^{\sharp_{\mathcal{A}}}\mathcal{T}_r\right).$$

Because the sum of the cross-terms can be restructured as

$$\sum_{1\leq j\neq r\leq s}\overline{\alpha_j}\alpha_r\mathcal{T}_j^{\sharp_{\mathcal{A}}}\mathcal{T}_r = \left(\sum_{j=1}^{s}\overline{\alpha_j}\mathcal{T}_j^{\sharp_{\mathcal{A}}}\right)\left(\sum_{r=1}^{s}\alpha_r\mathcal{T}_r\right) - \sum_{r=1}^{s}|\alpha_r|^2\mathcal{T}_r^{\sharp_{\mathcal{A}}}\mathcal{T}_r,$$

we directly obtain the stated inequality.

To further refine our estimates, we introduce the concept of the $\mathcal{A}$-joint Crawford number in the following definition.

**Definition 2.1.** *Let $\mathbf{T} = (\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$. The $\mathcal{A}$-joint Crawford number of $\mathbf{T}$ is defined by*

$$m_{\mathcal{A}}(\mathbf{T}) = \inf\left\{\left(\sum_{r=1}^{s}|\langle\mathcal{T}_r\xi, \xi\rangle_{\mathcal{A}}|^2\right)^{\frac{1}{2}} : \xi \in \Sigma_{\mathcal{A}}^1\right\}.$$

The following proposition outlines the fundamental properties of the $\mathcal{A}$-joint Crawford number.

**Proposition 2.1.** *Let $\mathbf{T} = (\mathcal{T}_1, \dots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$. Then $m_{\mathcal{A}}(\mathbf{T}) = 0$ if and only if a sequence $(\xi_n)_{n\in\mathbb{N}} \subseteq \Sigma_{\mathcal{A}}^1$ exists such that*

$$\lim_{n\to\infty}\langle\mathcal{T}_r\xi_n, \xi_n\rangle_{\mathcal{A}} = 0 \quad \textit{for all } r = 1, \dots, s.$$

*Equivalently, $m_{\mathcal{A}}(\mathbf{T}) = 0$ if and only if $\mathbf{0}_s := (0, \dots, 0) \in \overline{W_{\mathcal{A}}(\mathbf{T})}$, where*

$$W_{\mathcal{A}}(\mathbf{T}) = \Big\{(\langle\mathcal{T}_1\xi, \xi\rangle_{\mathcal{A}}, \dots, \langle\mathcal{T}_s\xi, \xi\rangle_{\mathcal{A}}) : \xi \in \Sigma_{\mathcal{A}}^1\Big\} \subseteq \mathbb{C}^s$$

*is the joint $\mathcal{A}$-numerical range, and $\overline{W_{\mathcal{A}}(\mathbf{T})}$ denotes its closure.*

*Proof.* By the definition of the Crawford number, we have

$$m_{\mathcal{A}}(\mathbf{T}) = \inf_{\xi\in\Sigma^1_{\mathcal{A}}} \left( \sum_{r=1}^{s} \left|\langle \mathcal{T}_r\xi, \xi\rangle_{\mathcal{A}}\right|^2 \right)^{\frac{1}{2}}.$$

Suppose that $m_{\mathcal{A}}(\mathbf{T}) = 0$. According to the properties of the infimum, a sequence $(\xi_n)_{n\in\mathbb{N}} \subseteq \Sigma^1_{\mathcal{A}}$ must exist such that

$$\lim_{n\to\infty} \sum_{r=1}^{s} \left|\langle \mathcal{T}_r\xi_n, \xi_n\rangle_{\mathcal{A}}\right|^2 = 0.$$

Given that each term in the summation is non-negative, this condition requires that

$$\lim_{n\to\infty} \left|\langle \mathcal{T}_r\xi_n, \xi_n\rangle_{\mathcal{A}}\right|^2 = 0 \quad \text{for each } r = 1, \ldots, s.$$

Consequently, we see that

$$\lim_{n\to\infty} \langle \mathcal{T}_r\xi_n, \xi_n\rangle_{\mathcal{A}} = 0$$

for all $r = 1, \ldots, s$. The converse is straightforward and holds by simply reversing these steps.

For the second equivalence, we recall that a point $\lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbb{C}^s$ is an element of the closure of the joint $\mathcal{A}$-numerical range, $\overline{W_{\mathcal{A}}(\mathbf{T})}$, if and only if a sequence $(\xi_n)_{n\in\mathbb{N}} \subseteq \Sigma^1_{\mathcal{A}}$ exists such that

$$\lim_{n\to\infty} \langle \mathcal{T}_r\xi_n, \xi_n\rangle_{\mathcal{A}} = \lambda_r$$

for each $r = 1, \ldots, s$. If we let $\lambda = \mathbf{0}_s = (0, \ldots, 0)$, then the condition $\mathbf{0}_s \in \overline{W_{\mathcal{A}}(\mathbf{T})}$ is exactly equivalent to the existence of a sequence $(\xi_n)_{n\in\mathbb{N}} \subseteq \Sigma^1_{\mathcal{A}}$, where

$$\lim_{n\to\infty} \langle \mathcal{T}_r\xi_n, \xi_n\rangle_{\mathcal{A}} = 0$$

for all $r = 1, \ldots, s$. This completes the proof.

In the following theorem, we refine the classical estimates by incorporating the $\mathcal{A}$-joint Crawford number into the equivalence bounds.

**Theorem 2.2.** *Let* $\mathbf{T} = (\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$. *Then*

$$\|\mathbf{T}\|^2_{\mathcal{A}} + m_{\mathcal{A}}(\mathbf{T}^2) \leq 2\sqrt{s}\omega_{\mathcal{A}}(\mathbf{T})\|\mathbf{T}\|_{\mathcal{A}} \leq s\omega_{\mathcal{A}}(\mathbf{T})^2 + \|\mathbf{T}\|^2_{\mathcal{A}},$$

*where* $\mathbf{T}^2 = (\mathcal{T}_1^2, \mathcal{T}_2^2, \ldots, \mathcal{T}_s^2)$.

*Proof.* Let us define $\rho_r, \upsilon_r \in \mathbb{R}$ with $\rho_r \neq 0$ for each index $r = 1, 2, \ldots, s$. We observe that

$$\begin{aligned}
&\sum_{r=1}^{s} \|\mathcal{T}_r\xi\|^2_{\mathcal{A}} + \sum_{r=1}^{s} e^{2i\upsilon_r}\langle \mathcal{T}_r^2\xi, \xi\rangle_{\mathcal{A}} \\
&= \sum_{r=1}^{s} \Big\{ \frac{1}{2}\langle \rho_r e^{2i\upsilon_r}\mathcal{T}_r^2\xi + \rho_r^{-1}e^{i\upsilon_r}\mathcal{T}_r\xi, \rho_r e^{i\upsilon_r}\mathcal{T}_r\xi + \rho_r^{-1}\xi\rangle_{\mathcal{A}} \\
&\quad - \frac{1}{2}\langle \rho_r e^{2i\upsilon_r}\mathcal{T}_r^2\xi - \rho_r^{-1}e^{i\upsilon_r}\mathcal{T}_r\xi, \rho_r e^{i\upsilon_r}\mathcal{T}_r\xi - \rho_r^{-1}\xi\rangle_{\mathcal{A}} \Big\}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\Big|\sum_{r=1}^{s}\|\mathcal{T}_r\xi\|_{\mathcal{A}}^2+\sum_{r=1}^{s}e^{2i\upsilon_r}\langle\mathcal{T}_r^2\xi,\xi\rangle_{\mathcal{A}}\Big|\\
&\le\sum_{r=1}^{s}\frac{1}{2}\Big|\langle\rho_r e^{2i\upsilon_r}\mathcal{T}_r^2\xi+\rho_r^{-1}e^{i\upsilon_r}\mathcal{T}_r\xi,\rho_r e^{i\upsilon_r}\mathcal{T}_r\xi+\rho_r^{-1}\xi\rangle_{\mathcal{A}}\Big|\\
&\quad+\sum_{r=1}^{s}\frac{1}{2}\Big|\langle\rho_r e^{2i\upsilon_r}\mathcal{T}_r^2\xi-\rho_r^{-1}e^{i\upsilon_r}\mathcal{T}_r\xi,\rho_r e^{i\upsilon_r}\mathcal{T}_r\xi-\rho_r^{-1}\xi\rangle_{\mathcal{A}}\Big|\\
&\le\sum_{r=1}^{s}\frac{1}{2}\omega_{\mathcal{A}}(\mathcal{T}_r)\|\rho_r e^{i\upsilon_r}\mathcal{T}_r\xi+\rho_r^{-1}\xi\|_{\mathcal{A}}^2+\sum_{r=1}^{s}\frac{1}{2}\omega_{\mathcal{A}}(\mathcal{T}_r)\|\rho_r e^{i\upsilon_r}\mathcal{T}_r\xi-\rho_r^{-1}\xi\|_{\mathcal{A}}^2.
\end{aligned}$$

Since the algebraic inequality $|\langle\mathcal{T}_r\xi,\xi\rangle_{\mathcal{A}}|\le\left(\sum_{r=1}^{s}|\langle\mathcal{T}_r\xi,\xi\rangle_{\mathcal{A}}|^2\right)^{\frac{1}{2}}$ holds for any vector $\xi\in\mathscr{H}$, it logically follows that $\omega_{\mathcal{A}}(\mathcal{T}_r)\le\omega_{\mathcal{A}}(\mathbf{T})$. Therefore,

$$\begin{aligned}
&\Big|\sum_{r=1}^{s}\|\mathcal{T}_r\xi\|_{\mathcal{A}}^2+\sum_{r=1}^{s}e^{2i\upsilon_r}\langle\mathcal{T}_r^2\xi,\xi\rangle_{\mathcal{A}}\Big|\\
&\le\sum_{r=1}^{s}\frac{1}{2}\omega_{\mathcal{A}}(\mathbf{T})\|\rho_r e^{i\upsilon_r}\mathcal{T}_r\xi+\rho_r^{-1}\xi\|_{\mathcal{A}}^2+\sum_{r=1}^{s}\frac{1}{2}\omega_{\mathcal{A}}(\mathbf{T})\|\rho_r e^{i\upsilon_r}\mathcal{T}_r\xi-\rho_r^{-1}\xi\|_{\mathcal{A}}^2\\
&=\omega_{\mathcal{A}}(\mathbf{T})\sum_{r=1}^{s}\left\{\frac{1}{2}\|\rho_r e^{i\upsilon_r}\mathcal{T}_r\xi+\rho_r^{-1}\xi\|_{\mathcal{A}}^2+\frac{1}{2}\|\rho_r e^{i\upsilon_r}\mathcal{T}_r\xi-\rho_r^{-1}\xi\|_{\mathcal{A}}^2\right\}\\
&=\omega_{\mathcal{A}}(\mathbf{T})\sum_{r=1}^{s}\left\{\rho_r^2\|\mathcal{T}_r\xi\|_{\mathcal{A}}^2+\rho_r^{-2}\|\xi\|_{\mathcal{A}}^2\right\}.
\end{aligned}$$

Assume that $\mathcal{T}_r\xi\neq\mathbf{0}$ for all $r=1,2,\dots,s$. By carefully choosing $\upsilon_r$ such that $e^{2i\upsilon_r}\langle\mathcal{T}_r^2\xi,\xi\rangle_{\mathcal{A}}=|\langle\mathcal{T}_r^2\xi,\xi\rangle_{\mathcal{A}}|$ and setting $\rho_r=\sqrt{\frac{\|\xi\|_{\mathcal{A}}}{\|\mathcal{T}_r\xi\|_{\mathcal{A}}}}$ for all $r=1,2,\dots,s$, we derive the inequality

$$\sum_{r=1}^{s}\|\mathcal{T}_r\xi\|_{\mathcal{A}}^2+\sum_{r=1}^{s}|\langle\mathcal{T}_r^2\xi,\xi\rangle_{\mathcal{A}}|\quad\le\quad 2\omega_{\mathcal{A}}(\mathbf{T})\sum_{r=1}^{s}\|\mathcal{T}_r\xi\|_{\mathcal{A}}\|\xi\|_{\mathcal{A}}.$$

Applying the Cauchy-Schwarz inequality, we find that

$$\sum_{r=1}^{s}\|\mathcal{T}_r\xi\|_{\mathcal{A}}^2+\sum_{r=1}^{s}|\langle\mathcal{T}_r^2\xi,\xi\rangle_{\mathcal{A}}|\quad\le\quad 2\sqrt{s}\omega_{\mathcal{A}}(\mathbf{T})\left(\sum_{r=1}^{s}\|\mathcal{T}_r\xi\|_{\mathcal{A}}^2\right)^{\frac{1}{2}}\|\xi\|_{\mathcal{A}}.\tag{2.4}$$

It is important to note that this inequality remains valid even if $\mathcal{T}_r\xi=0$ for some or all values of $r\in\{1,2,\dots,s\}$. Taking the supremum over all $\xi\in\Sigma_{\mathcal{A}}^1$ in Eq (2.4), we deduce that

$$\sum_{r=1}^{s}\|\mathcal{T}_r\xi\|_{\mathcal{A}}^2+\sum_{r=1}^{s}|\langle\mathcal{T}_r^2\xi,\xi\rangle_{\mathcal{A}}|\le 2\sqrt{s}\omega_{\mathcal{A}}(\mathbf{T})\left(\sum_{r=1}^{s}\|\mathcal{T}_r\xi\|_{\mathcal{A}}^2\right)^{\frac{1}{2}}\le 2\sqrt{s}\omega_{\mathcal{A}}(\mathbf{T})\|\mathbf{T}\|_{\mathcal{A}}.$$

Given that the $\ell_1$ norm provides an upper bound for the $\ell_2$ norm, we deduce that

$$\sum_{r=1}^{s} |\langle \mathcal{T}_r^2 \xi, \xi \rangle_{\mathcal{A}}| \geq \left( \sum_{r=1}^{s} |\langle \mathcal{T}_r^2 \xi, \xi \rangle_{\mathcal{A}}|^2 \right)^{\frac{1}{2}} \geq m_{\mathcal{A}}(\mathbf{T}^2).$$

Therefore, it follows that

$$\sum_{r=1}^{s} \|\mathcal{T}_r \xi\|_{\mathcal{A}}^2 + m_{\mathcal{A}}(\mathbf{T}^2) \quad \leq \quad 2\sqrt{s}\omega_{\mathcal{A}}(\mathbf{T}) \|\mathbf{T}\|_{\mathcal{A}}.$$

Taking the supremum over all vectors $\xi \in \Sigma_{\mathcal{A}}^1$, we arrive at the bound

$$\|\mathbf{T}\|_{\mathcal{A}}^2 + m_{\mathcal{A}}(\mathbf{T}^2) \leq 2\sqrt{s}\omega_{\mathcal{A}}(\mathbf{T})\|\mathbf{T}\|_{\mathcal{A}}.$$

The final inequality in the theorem statement follows directly from an application of the classical arithmetic-geometric mean inequality. This completes the proof.

The proof relies on the introduction of the auxiliary parameters $\rho_r$ and $\upsilon_r$, which allow us to separate the contributions of the operators and the vector $\xi$. This approach leads naturally to the appearance of the $\mathcal{A}$-joint Crawford number in the resulting estimate.

The quantity $m_{\mathcal{A}}(T^2)$ measures the minimal joint numerical contribution of the operators on the $\mathcal{A}$-unit sphere. Its incorporation into Theorem 2.2 yields a refinement of the classical equivalence bounds between the joint $\mathcal{A}$-operator seminorm and the joint $\mathcal{A}$-numerical radius.

The following remark explicitly details how Theorem 2.2 offers a distinct refinement of the classical inequality (1.1).

**Remark 2.2.** *Specifically, if we consider* $\mathbf{T} = (\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$ *under the condition that* $\|\mathbf{T}\|_{\mathcal{A}} \neq 0$*, we obtain the following sequence of bounds:*

$$\begin{aligned} \frac{1}{2\sqrt{s}}\|\mathbf{T}\|_{\mathcal{A}} \leq \frac{1}{2\sqrt{s}}\left(\|\mathbf{T}\|_{\mathcal{A}} + \frac{m_{\mathcal{A}}(\mathbf{T}^2)}{\|\mathbf{T}\|_{\mathcal{A}}}\right) \quad &\leq \quad \omega_{\mathcal{A}}(\mathbf{T}) = \sqrt{\frac{\omega_{\mathcal{A}}^2(\mathbf{T})}{\|\mathbf{T}\|_{\mathcal{A}}}\|\mathbf{T}\|_{\mathcal{A}}} \\ &\leq \quad \frac{1}{2}\left(\frac{\omega_{\mathcal{A}}^2(\mathbf{T})}{\|\mathbf{T}\|_{\mathcal{A}}} + \|\mathbf{T}\|_{\mathcal{A}}\right) \leq \|\mathbf{T}\|_{\mathcal{A}}. \end{aligned}$$

The next result follows directly as an application of Theorem 2.2.

**Corollary 2.4.** *Let* $\mathbf{T} = (\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$*. Then*

$$m_{\mathcal{A}}(\mathbf{T}^2) \leq s\omega_{\mathcal{A}}(\mathbf{T})^2.$$

The following corollary outlines the behavior of the Crawford number when a specific equality condition is met.

**Corollary 2.5.** *Let* $\mathbf{T} = (\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$ *with* $\|\mathbf{T}\|_{\mathcal{A}} \neq 0$*. If* $\frac{1}{2\sqrt{s}}\|\mathbf{T}\|_{\mathcal{A}} = \omega_{\mathcal{A}}(\mathbf{T})$*, then* $m_{\mathcal{A}}(\mathbf{T}^2) = 0$.

The next result outlines a power inequality concerning the joint $\mathcal{A}$-numerical radius for tuples of operators.

**Theorem 2.3.** *If* $\mathbf{T} = (\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$, *then*

$$\omega_{\mathcal{A}}(\mathbf{T}^p) \leq \sqrt{s}\omega^p_{\mathcal{A}}(\mathbf{T}).$$

*Proof.* Let us consider an arbitrary vector $\xi \in \Sigma^1_{\mathcal{A}}$. We observe that $|\langle \mathcal{T}_r\xi, \xi\rangle|_{\mathcal{A}} \leq \left(\sum_{k=1}^{s} |\langle \mathcal{T}_k\xi, \xi\rangle|^2_{\mathcal{A}}\right)^{\frac{1}{2}}$ and, as a direct consequence, $\omega_{\mathcal{A}}(\mathcal{T}_r) \leq \omega_{\mathcal{A}}(\mathbf{T})$ for every index $r = 1, \ldots, s$.

Initially, suppose that $\omega_{\mathcal{A}}(\mathbf{T}) \leq 1$. This assumption implies that $\omega_{\mathcal{A}}(\mathcal{T}_r) \leq 1$ for each $r = 1, \ldots, s$. Applying the well-established power inequality from [41, Proposition 3.10], it logically follows that $\omega_{\mathcal{A}}(\mathcal{T}^p_r) \leq 1$ for each $r = 1, \ldots, s$. Therefore, we calculate

$$\begin{aligned}
\omega_{\mathcal{A}}(\mathbf{T}^p) &= \sup_{\xi\in\Sigma^1_{\mathcal{A}}} \left(\sum_{r=1}^{s} |\langle \mathcal{T}^p_r\xi, \xi\rangle_{\mathcal{A}}|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{r=1}^{s} \sup_{\xi\in\Sigma^1_{\mathcal{A}}} |\langle \mathcal{T}^p_r\xi, \xi\rangle_{\mathcal{A}}|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{r=1}^{s} \omega^2_{\mathcal{A}}(\mathcal{T}^p_r)\right)^{\frac{1}{2}} \leq \sqrt{s}.
\end{aligned}$$

To address the general case where $\omega_{\mathcal{A}}(\mathbf{T}) > 1$, we define the scaled operators $\mathcal{T}'_r = \frac{\mathcal{T}_r}{\omega_{\mathcal{A}}(\mathbf{T})}$ for $r = 1, \ldots, s$, and let the tuple be $\mathbf{T}' = (\mathcal{T}'_1, \ldots, \mathcal{T}'_s)$. It follows that $\omega_{\mathcal{A}}(\mathbf{T}') = 1$, and applying the previously established estimate yields $\omega_{\mathcal{A}}((\mathbf{T}')^p) \leq \sqrt{s}$. Since $(\mathbf{T}')^p = \frac{\mathbf{T}^p}{\omega^p_{\mathcal{A}}(\mathbf{T})}$, we conclude that

$$\omega_{\mathcal{A}}(\mathbf{T}^p) \leq \sqrt{s}\,\omega^p_{\mathcal{A}}(\mathbf{T}).$$

As an immediate consequence of Theorem 2.3, we derive the following useful upper bound for the operator norm.

**Corollary 2.6.** *Let* $\mathbf{T} = (\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$. *If* $\omega_{\mathcal{A}}(\mathbf{T}) \leq 1$, *then*

$$\|\mathbf{T}^p\|_{\mathcal{A}} \leq 2s.$$

*Proof.* Assume that $\|\mathbf{T}^p\|_{\mathcal{A}} \neq 0$. Combining the results of Theorem 2.2 with Theorem 2.3, we obtain

$$\frac{\|\mathbf{T}^p\|_{\mathcal{A}}}{2\sqrt{s}} \leq \frac{1}{2\sqrt{s}}\left(\|\mathbf{T}^p\|_{\mathcal{A}} + \frac{m_{\mathcal{A}}(\mathbf{T}^{2p})}{\|\mathbf{T}^p\|_{\mathcal{A}}}\right) \leq \omega_{\mathcal{A}}(\mathbf{T}^p) \leq \sqrt{s}\omega^p_{\mathcal{A}}(\mathbf{T}) \leq \sqrt{s}.$$

From Theorem 2.2, we also obtain the following result, which evaluates specific equality conditions related to Corollary 2.6.

**Corollary 2.7.** *Let* $\mathbf{T} = (\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$. *If* $\omega_{\mathcal{A}}(\mathbf{T}) \leq 1$ *and* $\|\mathbf{T}^p\|_{\mathcal{A}} = 2s$, *then* $m_{\mathcal{A}}(\mathbf{T}^{2p}) = 0$, $\omega_{\mathcal{A}}(\mathbf{T}^p) = \sqrt{s}$ *and* $\omega^p_{\mathcal{A}}(\mathbf{T}) = 1$. *In particular, if* $\omega_{\mathcal{A}}(\mathbf{T}) < 1$, *we conclude that* $\|\mathbf{T}^p\|_{\mathcal{A}} < 2s$.

In the next theorem, we present a lower bound for the joint $\mathcal{A}$-numerical radius by explicitly utilizing the $\mathcal{A}$-joint Crawford number. For a tuple $\mathbf{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\mathcal{A}}(\mathscr{H})^s$, we naturally denote its $\mathcal{A}$-real and $\mathcal{A}$-imaginary parts by $\Re_{\mathcal{A}}(\mathbf{T}) = (\Re_{\mathcal{A}}(\mathcal{T}_1), \ldots, \Re_{\mathcal{A}}(\mathcal{T}_s))$ and $\Im_{\mathcal{A}}(\mathbf{T}) = (\Im_{\mathcal{A}}(\mathcal{T}_1), \ldots, \Im_{\mathcal{A}}(\mathcal{T}_s))$, respectively.

**Theorem 2.4.** *Let* $\mathbf{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\mathcal{A}}(\mathscr{H})^s$. *Then the following inequality holds:*

$$\omega_{\mathcal{A}}(\mathbf{T})^2 \geq \max\left\{\omega_{\mathcal{A}}(\Re_{\mathcal{A}}(\mathbf{T}))^2 + m_{\mathcal{A}}(\Im_{\mathcal{A}}(\mathbf{T}))^2,\ \omega_{\mathcal{A}}(\Im_{\mathcal{A}}(\mathbf{T}))^2 + m_{\mathcal{A}}(\Re_{\mathcal{A}}(\mathbf{T}))^2\right\}.$$

*Proof.* Select an arbitrary unit vector $\xi \in \Sigma^1_{\mathcal{A}}$. For each index $r \in \{1, \ldots, s\}$, the standard Cartesian decomposition $\mathcal{T}_r = \Re_{\mathcal{A}}(\mathcal{T}_r) + i\Im_{\mathcal{A}}(\mathcal{T}_r)$ implies that the inner product satisfies $\langle \mathcal{T}_r\xi, \xi\rangle_{\mathcal{A}} = \langle \Re_{\mathcal{A}}(\mathcal{T}_r)\xi, \xi\rangle_{\mathcal{A}} + i\langle \Im_{\mathcal{A}}(\mathcal{T}_r)\xi, \xi\rangle_{\mathcal{A}}$. Since both $\Re_{\mathcal{A}}(\mathcal{T}_r)$ and $\Im_{\mathcal{A}}(\mathcal{T}_r)$ are $\mathcal{A}$-self-adjoint operators, the individual terms $\langle \Re_{\mathcal{A}}(\mathcal{T}_r)\xi, \xi\rangle_{\mathcal{A}}$ and $\langle \Im_{\mathcal{A}}(\mathcal{T}_r)\xi, \xi\rangle_{\mathcal{A}}$ must evaluate to strictly real numbers. Computing the absolute square yields

$$|\langle \mathcal{T}_r\xi, \xi\rangle_{\mathcal{A}}|^2 = \langle \Re_{\mathcal{A}}(\mathcal{T}_r)\xi, \xi\rangle^2_{\mathcal{A}} + \langle \Im_{\mathcal{A}}(\mathcal{T}_r)\xi, \xi\rangle^2_{\mathcal{A}}.$$

Summing this relation over all indices $r = 1, \ldots, s$, we obtain the identity

$$\sum_{r=1}^{s} |\langle \mathcal{T}_r\xi, \xi\rangle_{\mathcal{A}}|^2 = \sum_{r=1}^{s}\langle \Re_{\mathcal{A}}(\mathcal{T}_r)\xi, \xi\rangle^2_{\mathcal{A}} + \sum_{r=1}^{s}\langle \Im_{\mathcal{A}}(\mathcal{T}_r)\xi, \xi\rangle^2_{\mathcal{A}}.$$

Taking the supremum over all $\xi \in \Sigma^1_{\mathcal{A}}$ on both sides of the equation and utilizing the elementary analytic property that $\sup(f + g) \geq \sup(f) + \inf(g)$, we deduce

$$\begin{aligned}
\omega_{\mathcal{A}}(\mathbf{T})^2 &= \sup_{\xi\in\Sigma^1_{\mathcal{A}}}\left(\sum_{r=1}^{s}\langle \Re_{\mathcal{A}}(\mathcal{T}_r)\xi, \xi\rangle^2_{\mathcal{A}} + \sum_{r=1}^{s}\langle \Im_{\mathcal{A}}(\mathcal{T}_r)\xi, \xi\rangle^2_{\mathcal{A}}\right) \\
&\geq \sup_{\xi\in\Sigma^1_{\mathcal{A}}}\left(\sum_{r=1}^{s}\langle \Re_{\mathcal{A}}(\mathcal{T}_r)\xi, \xi\rangle^2_{\mathcal{A}}\right) + \inf_{\xi\in\Sigma^1_{\mathcal{A}}}\left(\sum_{r=1}^{s}\langle \Im_{\mathcal{A}}(\mathcal{T}_r)\xi, \xi\rangle^2_{\mathcal{A}}\right) \\
&= \omega_{\mathcal{A}}(\Re_{\mathcal{A}}(\mathbf{T}))^2 + m_{\mathcal{A}}(\Im_{\mathcal{A}}(\mathbf{T}))^2.
\end{aligned}$$

By applying a symmetric argument, we obtain the complementary bound

$$\omega_{\mathcal{A}}(\mathbf{T})^2 \geq m_{\mathcal{A}}(\Re_{\mathcal{A}}(\mathbf{T}))^2 + \omega_{\mathcal{A}}(\Im_{\mathcal{A}}(\mathbf{T}))^2.$$

Combining these two inequalities provides the desired maximum bound stated in the theorem.

The following theorem constitutes a vital step in examining how the quantity $\omega_{\mathcal{A}}$ behaves under the operation of operator multiplication.

**Theorem 2.5.** *Let* $\mathbf{T}_1 = (\mathcal{T}_{1,1}, \mathcal{T}_{1,2}, \ldots, \mathcal{T}_{1,s})$ *and* $\mathbf{T}_2 = (\mathcal{T}_{2,1}, \mathcal{T}_{2,2}, \ldots, \mathcal{T}_{2,s}) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$. *Then*

$$\omega_{\mathcal{A}}(\mathbf{T}_1\mathbf{T}_2) \leq 4s\omega_{\mathcal{A}}(\mathbf{T}_1)\,\omega_{\mathcal{A}}(\mathbf{T}_2).$$

*Proof.* First, we establish the submultiplicativity property for the $\mathcal{A}$-norm. Taking an arbitrary vector $\xi \in \Sigma^1_{\mathcal{A}}$, we observe that

$$\sum_{r=1}^{s} \|\mathcal{T}_{1,r}\mathcal{T}_{2,r}\xi\|^2_{\mathcal{A}} \leq \sum_{r=1}^{s} \|\mathcal{T}_{1,r}\|^2_{\mathcal{A}}\|\mathcal{T}_{2,r}\xi\|^2_{\mathcal{A}} \leq \|\mathbf{T}_1\|^2_{\mathcal{A}} \sum_{r=1}^{s} \|\mathcal{T}_{2,r}\xi\|^2_{\mathcal{A}},$$

where the final inequality is a consequence of the fact that $\|\mathcal{T}_{1,r}\|_{\mathcal{A}} \leq \|\mathbf{T}_1\|_{\mathcal{A}}$ for each index $r \in \{1, \ldots, s\}$.

Taking the square root of both sides of this expression, we obtain

$$\left(\sum_{r=1}^{s}\|\mathcal{T}_{1,r}\mathcal{T}_{2,r}\xi\|_{\mathcal{A}}^{2}\right)^{\frac{1}{2}} \leq \|\mathbf{T}_1\|_{\mathcal{A}}\left(\sum_{r=1}^{s}\|\mathcal{T}_{2,r}\xi\|_{\mathcal{A}}^{2}\right)^{\frac{1}{2}}.$$

Taking the supremum over all possible vectors $\xi \in \Sigma_{\mathcal{A}}^{1}$, we deduce that

$$\|\mathbf{T}_1\mathbf{T}_2\|_{\mathcal{A}} \leq \|\mathbf{T}_1\|_{\mathcal{A}}\|\mathbf{T}_2\|_{\mathcal{A}}. \tag{2.5}$$

Applying inequality (2.5) in conjunction with relation (1.1), we conclude that

$$\begin{aligned}\omega_{\mathcal{A}}(\mathbf{T}_1\mathbf{T}_2) &\leq \|\mathbf{T}_1\mathbf{T}_2\|_{\mathcal{A}} \leq \|\mathbf{T}_1\|_{\mathcal{A}}\|\mathbf{T}_2\|_{\mathcal{A}} \\ &\leq \left(2\sqrt{s}\omega_{\mathcal{A}}(\mathbf{T}_1)\right)\left(2\sqrt{s}\omega_{\mathcal{A}}(\mathbf{T}_2)\right) \\ &= 4s\omega_{\mathcal{A}}(\mathbf{T}_1)\,\omega_{\mathcal{A}}(\mathbf{T}_2).\end{aligned}$$

This completes the proof.

The next result extends our analysis to the class of $\mathcal{A}$-paranormal operators, providing bounds for their powers.

**Theorem 2.6.** *Let $\mathbf{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$. If each operator $\mathcal{T}_r$ for $(r = 1, 2, \ldots, s)$ is an $\mathcal{A}$-paranormal operator (which implies that $\|\mathcal{T}_r\xi\|_{\mathcal{A}}^2 \leq \|\mathcal{T}_r^2\xi\|_{\mathcal{A}}$ for all vectors $\xi \in \Sigma_{\mathcal{A}}^1$), then*

$$s^{\frac{1-p}{2}}\|\mathbf{T}\|_{\mathcal{A}}^{p} \leq \|\mathbf{T}^p\|_{\mathcal{A}} \leq \|\mathbf{T}\|_{\mathcal{A}}^{p}.$$

*Proof.* Let $\xi \in \Sigma_{\mathcal{A}}^1$. For all integers $p \in \mathbb{N}$, we will prove by mathematical induction that

$$\|\mathcal{T}_r{}^p\xi\|_{\mathcal{A}} \geq \|\mathcal{T}_r\xi\|_{\mathcal{A}}^{p}. \tag{2.6}$$

For the base case of $p = 1$, inequality (2.6) clearly holds in a trivial manner. Let $p \in \mathbb{N}$ be an arbitrary positive integer and assume the inductive hypothesis that $\|\mathcal{T}_r{}^p\xi\|_{\mathcal{A}} \geq \|\mathcal{T}_r\xi\|_{\mathcal{A}}^p$ holds for all vectors $\xi \in \Sigma_{\mathcal{A}}^1$. Assume that $\|\mathcal{T}_r\xi\|_{\mathcal{A}} \neq 0$. Under these conditions, we have

$$\begin{aligned}\left\|\mathcal{T}_r{}^{p+1}\xi\right\|_{\mathcal{A}} &= \left\|\mathcal{T}_r{}^p\left(\frac{\mathcal{T}_r\xi}{\|\mathcal{T}_r\xi\|_{\mathcal{A}}}\right)\right\|_{\mathcal{A}}\|\mathcal{T}_r\xi\|_{\mathcal{A}} \\ &\geq \left\|\mathcal{T}_r\left(\frac{\mathcal{T}_r\xi}{\|\mathcal{T}_r\xi\|_{\mathcal{A}}}\right)\right\|_{\mathcal{A}}^{p}\|\mathcal{T}_r\xi\|_{\mathcal{A}} \\ &= \left\|\mathcal{T}_r{}^2\xi\right\|_{\mathcal{A}}^{p}\|\mathcal{T}_r\xi\|_{\mathcal{A}}^{1-p} \\ &\geq \|\mathcal{T}_r\xi\|_{\mathcal{A}}^{p+1}.\end{aligned}$$

We observe that if $\|\mathcal{T}_r\xi\|_{\mathcal{A}} = 0$, then it is clearly true that $\left\|\mathcal{T}_r{}^{p+1}\xi\right\|_{\mathcal{A}} \geq \|\mathcal{T}_r\xi\|_{\mathcal{A}}^{p+1}$. Hence, the inequality $\|\mathcal{T}_r{}^p\xi\|_{\mathcal{A}} \geq \|\mathcal{T}_r\xi\|_{\mathcal{A}}^p$ has been rigorously established by mathematical induction. Moreover, applying the relation $\|\mathcal{T}_r{}^p\xi\|_{\mathcal{A}} \geq \|\mathcal{T}_r\xi\|_{\mathcal{A}}^p$, and taking into account that the polynomial function $f(t) = t^p$ is strictly convex on the interval $[0, \infty)$ for any $p \geq 1$, we can legitimately apply Jensen's inequality to the discrete sum to obtain

$$\begin{aligned} \|\mathbf{T}^p\|_{\mathcal{A}} &= \sup_{\xi \in \Sigma^1_{\mathcal{A}}} \sqrt{\sum_{r=1}^{s} \|\mathcal{T}_r{}^p \xi\|^2_{\mathcal{A}}} \\ &\geq \sup_{\xi \in \Sigma^1_{\mathcal{A}}} \sqrt{\sum_{r=1}^{s} \|\mathcal{T}_r \xi\|^{2p}_{\mathcal{A}}} \\ &\geq s^{\frac{1-p}{2}} \sup_{\xi \in \Sigma^1_{\mathcal{A}}} \sqrt{\left(\sum_{r=1}^{s} \|\mathcal{T}_r \xi\|^2_{\mathcal{A}}\right)^p} \\ &= s^{\frac{1-p}{2}} \|\mathbf{T}\|^p_{\mathcal{A}}. \end{aligned}$$

Additionally, applying the previously established result (2.5), we obtain the upper bound $\|\mathbf{T}^p\|_{\mathcal{A}} \leq \|\mathbf{T}\|^p_{\mathcal{A}}$. This concludes the proof.

The following remark confirms the sharpness of the inequalities presented in Theorem 2.6.

**Remark 2.3.** *Consider the specific case where $\mathcal{T}_r = \sqrt{s} I_{\mathscr{H}}$ for the indices $r = 1, 2, \ldots, s$. In this scenario, the first inequality stated in Theorem 2.6 simplifies to a strict equality. Similarly, if we let $\mathbf{T} = (I_{\mathscr{H}}, \mathbf{0}, \ldots, \mathbf{0})$, then the second inequality in Theorem 2.6 also resolves to an equality. Thus, the inequalities provided in Theorem 2.6 are indeed sharp.*

## 3. Generalization of the $\mathcal{A}$-joint numerical radius

In this section, we introduce a natural generalization of the $\mathcal{A}$-joint numerical radius. Our approach is motivated by the observation that the classical quantity

$$\omega_{\mathcal{A}}(\mathbf{T}) := \sup_{\xi \in \Sigma^1_{\mathcal{A}}} \sqrt{\sum_{r=1}^{s} |\langle \mathcal{T}_r \xi, \xi \rangle_{\mathcal{A}}|^2}$$

can be interpreted as the Euclidean norm of the vector

$$(|\langle \mathcal{T}_1 \xi, \xi \rangle_{\mathcal{A}}|, \ldots, |\langle \mathcal{T}_s \xi, \xi \rangle_{\mathcal{A}}|) \in \mathbb{R}^s,$$

maximized over all $\xi \in \Sigma^1_{\mathcal{A}}$. In other words, $\omega_{\mathcal{A}}(\mathbf{T})$ measures the joint numerical behavior of $\mathbf{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s)$ through the $\ell^2$-structure of $\mathbb{R}^s$.

The functional extension introduced in this section allows us to study the influence of the convexity and concavity properties of $\phi$ on the joint $\mathcal{A}$-numerical radius. The resulting inequalities provide new relations between the joint $\mathcal{A}$-numerical radius and the joint $\mathcal{A}$-operator seminorm.

This interpretation naturally suggests replacing the Euclidean norm by a more general aggregation procedure generated by a suitable function. Motivated by this idea, we introduce the following generalized notion.

We formally introduce this generalized concept in the following definition.

**Definition 3.1.** *Let* $\mathbf{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$ *and let* $\phi : [0, \infty) \to [0, \infty)$ *be a continuous, non-decreasing function satisfying* $\phi(0) = 0$*. The* $(\mathcal{A}, \phi)$*-joint numerical radius of* $\mathbf{T}$ *is defined by*

$$\omega_{\mathcal{A},\phi}(\mathbf{T}) = \sup_{\xi \in \Sigma^1_{\mathcal{A}}} \phi^{-1}\left(\sum_{r=1}^{s} \phi(|\langle \mathcal{T}_r \xi, \xi \rangle_{\mathcal{A}}|)\right).$$

In particular, if $\phi(t) = t^2$, then $\omega_{\mathcal{A},\phi}$ reduces to $\omega_{\mathcal{A}}$ , while for $\phi(t) = t^q$ with $q \geq 1$, we obtain the $(\mathcal{A}, q)$-joint numerical radius of $\mathbf{T}$, which can be explicitly stated as

$$\omega_{\mathcal{A},q}(\mathbf{T}) = \sup_{\xi \in \Sigma^1_{\mathcal{A}}} \sqrt[q]{\sum_{r=1}^{s} |\langle \mathcal{T}_r \xi, \xi \rangle_{\mathcal{A}}|^q}.$$

The next result provides a standard auxiliary lemma concerning the superadditivity of convex functions.

**Lemma 3.1.** *Let* $\phi : [0, \infty) \to [0, \infty)$ *be a convex function with* $\phi(0) = 0$*. Then* $\phi$ *is superadditive on* $[0, \infty)$*, that is,*

$$\phi\left(\sum_{r=1}^{s} a_r\right) \geq \sum_{r=1}^{s} \phi(a_r),$$

*for all* $a_r \geq 0$*, where* $r = 1, \ldots, s$*. If* $\phi$ *is concave, then the inequality is reversed.*

In the next result, we establish an upper bound relating $\omega_{\mathcal{A},\phi}$ to the standard $\mathcal{A}$-numerical radius.

**Theorem 3.1.** *Let* $\mathbf{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$*, and let* $\phi : [0, \infty) \to [0, \infty)$ *be a continuous, strictly increasing, and convex function satisfying* $\phi(0) = 0$*. Then*

$$\omega_{\mathcal{A},\phi}(\mathbf{T}) \leq \sum_{r=1}^{s} \omega_{\mathcal{A}}(\mathcal{T}_r). \tag{3.1}$$

*Proof.* Since $\phi$ is a continuous, strictly increasing, and convex function on $[0, \infty)$ with $\phi(0) = 0$, its inverse function $\phi^{-1}$ is increasing and concave on its domain. Applying the subadditivity principle for concave functions (as detailed in Lemma 3.1), we obtain

$$\phi^{-1}\left(\sum_{r=1}^{s} b_r\right) \leq \sum_{r=1}^{s} \phi^{-1}(b_r),$$

for any valid set of values where $b_r \geq 0$ and $r = 1, \ldots, s$.

Moreover, since $\phi$ is convex and satisfies $\phi(0) = 0$, it is superadditive. Thus, for any arbitrary sequence of non-negative numbers $a_1, \ldots, a_s$, we state that

$$\sum_{r=1}^{s} \phi(a_r) \leq \phi\left(\sum_{r=1}^{s} a_r\right).$$

Applying the increasing function $\phi^{-1}$ to both sides of the inequality, we obtain

$$\phi^{-1}\left(\sum_{r=1}^{s} \phi(a_r)\right) \leq \sum_{r=1}^{s} a_r.$$

Let $\xi \in \Sigma^1_{\mathcal{A}}$ and define the variables

$$a_r = |\langle \mathcal{T}_r \xi, \xi \rangle|_{\mathcal{A}}, \qquad r = 1, \ldots, s.$$

This specific substitution leads to the inequality

$$\phi^{-1}\left( \sum_{r=1}^{s} \phi(|\langle \mathcal{T}_r \xi, \xi \rangle|_{\mathcal{A}}) \right) \leq \sum_{r=1}^{s} |\langle \mathcal{T}_r \xi, \xi \rangle|_{\mathcal{A}}.$$

Taking the supremum over all unit vectors $\xi \in \Sigma^1_{\mathcal{A}}$, we deduce the final inequality

$$\omega_{\mathcal{A},\phi}(\mathbf{T}) \leq \sum_{r=1}^{s} \omega_{\mathcal{A}}(\mathcal{T}_r).$$

This completes the proof.

Building upon the foundation established by the previous bound, the next theorem derives a strictly stronger inequality by leveraging the monotonicity of the inverse function.

**Theorem 3.2.** *Let* $\mathbf{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$*, and let* $\phi : [0, \infty) \to [0, \infty)$ *be a continuous, strictly increasing and convex function satisfying* $\phi(0) = 0$*. Then*

$$\omega_{\mathcal{A},\phi}(\mathbf{T}) \leq \phi^{-1}\left( \sum_{r=1}^{s} \phi(\omega_{\mathcal{A}}(\mathcal{T}_r)) \right) \leq \sum_{r=1}^{s} \omega_{\mathcal{A}}(\mathcal{T}_r).$$

*Proof.* Assume that we have a vector $\xi \in \Sigma^1_{\mathcal{A}}$. Given the precise definition of the $\mathcal{A}$-numerical radius, we are guaranteed that the following relation holds:

$$\left| \langle \mathcal{T}_r \xi, \xi \rangle \right|_{\mathcal{A}} \leq \omega_{\mathcal{A}}(\mathcal{T}_r), \qquad r = 1, \ldots, s.$$

Since the function $\phi$ is strictly increasing across the interval $[0, \infty)$, it follows that the inequality is preserved when the function is applied to both sides

$$\phi\left( \left| \langle \mathcal{T}_r \xi, \xi \rangle \right|_{\mathcal{A}} \right) \leq \phi(\omega_{\mathcal{A}}(\mathcal{T}_r)), \qquad r = 1, \ldots, s.$$

Summing these terms over the index $r$, we obtain the following aggregate inequality:

$$\sum_{r=1}^{s} \phi\left( \left| \langle \mathcal{T}_r \xi, \xi \rangle \right|_{\mathcal{A}} \right) \leq \sum_{r=1}^{s} \phi(\omega_{\mathcal{A}}(\mathcal{T}_r)).$$

Subsequent application of the increasing inverse function $\phi^{-1}$ allows us to deduce that

$$\phi^{-1}\left( \sum_{r=1}^{s} \phi\left( \left| \langle \mathcal{T}_r \xi, \xi \rangle \right|_{\mathcal{A}} \right) \right) \leq \phi^{-1}\left( \sum_{r=1}^{s} \phi(\omega_{\mathcal{A}}(\mathcal{T}_r)) \right).$$

Calculating the supremum over all unit vectors $\xi \in \Sigma^1_{\mathcal{A}}$, we securely verify the first part of the inequality

$$\omega_{\mathcal{A},\phi}(\mathbf{T}) \leq \phi^{-1}\left( \sum_{r=1}^{s} \phi(\omega_{\mathcal{A}}(\mathcal{T}_r)) \right).$$

To establish the second half of the inequality, we recall that because $\phi$ is a convex and increasing function, its inverse $\phi^{-1}$ is concave and increasing, rendering it inherently subadditive (a property we thoroughly established in Theorem 3.1). Consequently, we find that

$$\phi^{-1}\left(\sum_{r=1}^{s}\phi(\omega_{\mathcal{A}}(\mathcal{T}_r))\right) \leq \sum_{r=1}^{s}\phi^{-1}(\phi(\omega_{\mathcal{A}}(\mathcal{T}_r))) = \sum_{r=1}^{s}\omega_{\mathcal{A}}(\mathcal{T}_r).$$

Combining the inequalities detailed above brings the proof to a close, effectively demonstrating that this new estimate provides a strict improvement over the previously stated boundary (3.1).

The following theorem demonstrates that the primary inequality found in (1.1) can be significantly improved for a carefully selected class of concave functions. It is important to note that this refinement is mathematically valid exclusively under the explicit assumption that the function $\phi$ is concave.

**Theorem 3.3.** *Let* $\mathbf{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$ *and let* $\phi : [0, \infty) \to [0, \infty)$ *be a continuous, strictly increasing, and concave function such that* $\phi(0) = 0$. *Then*

$$\frac{1}{2}\left\|\sum_{r=1}^{s}\mathcal{T}_r\right\|_{\mathcal{A}} \leq \omega_{\mathcal{A}}\left(\sum_{r=1}^{s}\mathcal{T}_r\right) \leq \omega_{\mathcal{A},\phi}(\mathbf{T}).$$

*Proof.* Let us define a vector $\xi \in \Sigma^1_{\mathcal{A}}$. Since the function $\phi$ is defined as concave and strictly increasing with $\phi(0) = 0$, its inverse function $\phi^{-1}$ will naturally be increasing and convex on its domain while preserving the property $\phi^{-1}(0) = 0$. Utilizing the results of Lemma 3.1, the convexity of $\phi^{-1}$ directly implies its superadditivity; therefore, we state that

$$\phi^{-1}\left(\sum_{r=1}^{s}\phi(|\langle\mathcal{T}_r\xi, \xi\rangle_{\mathcal{A}}|)\right) \geq \sum_{r=1}^{s}\phi^{-1}(\phi(|\langle\mathcal{T}_r\xi, \xi\rangle_{\mathcal{A}}|)).$$

Given that the composition $\phi^{-1} \circ \phi$ acts precisely as the identity function over the interval $[0, \infty)$, we deduce the relation

$$\phi^{-1}\left(\sum_{r=1}^{s}\phi(|\langle\mathcal{T}_r\xi, \xi\rangle_{\mathcal{A}}|)\right) \geq \sum_{r=1}^{s}|\langle\mathcal{T}_r\xi, \xi\rangle_{\mathcal{A}}|.$$

Applying the standard triangle inequality, we also deduce that

$$\sum_{r=1}^{s}|\langle\mathcal{T}_r\xi, \xi\rangle_{\mathcal{A}}| \geq \left|\sum_{r=1}^{s}\langle\mathcal{T}_r\xi, \xi\rangle_{\mathcal{A}}\right| = \left|\left\langle\left(\sum_{r=1}^{s}\mathcal{T}_r\right)\xi, \xi\right\rangle_{\mathcal{A}}\right|.$$

As a logical result of these preceding steps, we find that

$$\omega_{\mathcal{A},\phi}(\mathbf{T}) \geq \left|\left\langle\left(\sum_{r=1}^{s}\mathcal{T}_r\right)\xi, \xi\right\rangle_{\mathcal{A}}\right|.$$

Taking the supremum over all possible vectors $\xi \in \Sigma^1_{\mathcal{A}}$, we conclude that

$$\omega_{\mathcal{A},\phi}(\mathbf{T}) \geq \omega_{\mathcal{A}}\left(\sum_{r=1}^{s}\mathcal{T}_r\right).$$

The inequality presented on the left-hand side is derived directly from the classic operator estimate

$$\frac{1}{2}\|T\|_{\mathcal{A}} \leq \omega_{\mathcal{A}}(T), \qquad T \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H}),$$

which we apply to the sum operator $T = \sum_{r=1}^{s} \mathcal{T}_r$. This final step successfully brings the proof to its conclusion.

The next remark rigorously demonstrates that the second inequality detailed in Theorem 3.3 may entirely fail to hold if the function $\phi$ is assumed to be convex rather than concave.

**Remark 3.1.** *Let us define the space $\mathscr{H} = \mathbb{C}^2$ and establish the specific operator*

$$\mathcal{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \neq I_{\mathscr{H}}.$$

*The $\mathcal{A}$-inner product is defined by $\langle x, y\rangle_{\mathcal{A}} = \langle \mathcal{A}x, y\rangle$, and the associated unit sphere is*

$$\Sigma_{\mathcal{A}}^1 = \left\{\xi = (x_1, x_2) \in \mathbb{C}^2 : \ 2|x_1|^2 + |x_2|^2 = 1\right\}.$$

*Let us introduce the operators*

$$\mathcal{T}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{T}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

*Their sum is easily calculated as*

$$\mathcal{T}_1 + \mathcal{T}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

*For any vector $\xi = (x_1, x_2) \in \Sigma_{\mathcal{A}}^1$, we observe that the inner product evaluates to*

$$\langle (\mathcal{T}_1 + \mathcal{T}_2)\xi, \xi\rangle_{\mathcal{A}} = \langle \mathcal{A}(2x_1, 0), (x_1, x_2)\rangle = 4|x_1|^2.$$

*Given the constraint defining the sphere, $2|x_1|^2 + |x_2|^2 = 1$, it necessarily follows that the maximum possible value for $|x_1|^2$ is $|x_1|^2 \leq \frac{1}{2}$, and this precise maximum is reached exactly when $|x_1|^2 = \frac{1}{2}$. Therefore, the numerical radius is given by*

$$\omega_{\mathcal{A}}(\mathcal{T}_1 + \mathcal{T}_2) = 2.$$

*Let us now choose the convex function $\phi(t) = t^2$. With this choice, the extended formulation becomes*

$$\omega_{\mathcal{A}}(\mathbf{T}) = \omega_{\mathcal{A},\phi}(\mathbf{T}) = \sup_{\xi \in \Sigma_{\mathcal{A}}^1} \sqrt{|\langle \mathcal{T}_1\xi, \xi\rangle_{\mathcal{A}}|^2 + |\langle \mathcal{T}_2\xi, \xi\rangle_{\mathcal{A}}|^2}.$$

*We have the specific evaluations*

$$\langle \mathcal{T}_r\xi, \xi\rangle_{\mathcal{A}} = 2|x_1|^2, \qquad r = 1, 2,$$

*and on the basis of this, we compute the supremum as*

$$\omega_{\mathcal{A},\phi}(\mathbf{T}) = \sup_{\xi\in\Sigma^1_{\mathcal{A}}} 2\sqrt{2}\,|x_1|^2 = \sqrt{2}.$$

*Consequently, we have shown that*

$$\omega_{\mathcal{A}}(\mathcal{T}_1 + \mathcal{T}_2) = 2 \qquad \textit{and} \qquad \omega_{\mathcal{A},\phi}(\mathbf{T}) = \sqrt{2},$$

*which demonstrates the strict inequality*

$$\omega_{\mathcal{A}}(\mathcal{T}_1 + \mathcal{T}_2) > \omega_{\mathcal{A},\phi}(\mathbf{T}).$$

*This counterexample proves that the generalized inequality*

$$\omega_{\mathcal{A}}\left(\sum_{r=1}^{s} \mathcal{T}_r\right) \le \omega_{\mathcal{A},\phi}(\mathbf{T})$$

*does not hold in the general case when the generating function $\phi$ exhibits convexity.*

Despite the additional flexibility provided by these functional extensions, it is important to recognize an essential structural limitation of the approach. As shown by the previous counterexample, the effectiveness of the obtained bounds may fail when the generating function $\phi$ is convex. This observation indicates that the stability and subadditivity properties of the generalized numerical radius depend strongly on the convexity behavior of the function $\phi$.

The subsequent proposition provides highly useful lower estimates for the functional $\omega_{\mathcal{A},\phi}(\cdot)$.

**Proposition 3.1.** *Let $\mathbf{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$ and let $\phi : [0,\infty) \to [0,\infty)$ be an increasing and convex function. Then*

$$\omega_{\mathcal{A},\phi}(\mathbf{T}) \ge \sup_{|\lambda_r|\le 1} \omega_{\mathcal{A}}\left(\sum_{r=1}^{s} \frac{\lambda_r}{s}\mathcal{T}_r\right) \ge \frac{1}{2} \sup_{|\lambda_r|\le 1} \left\|\sum_{r=1}^{s} \frac{\lambda_r}{s}\mathcal{T}_r\right\|_{\mathcal{A}}. \tag{3.2}$$

*Proof.* Define arbitrary complex scalars $\lambda_1, \ldots, \lambda_s \in \mathbb{C}$ subject to the constraint $|\lambda_r| \le 1$, and select a vector $\xi \in \Sigma^1_{\mathcal{A}}$. Since the chosen function $\phi$ is both convex and increasing, while fulfilling the requirement that $\phi(0) = 0$, a direct application of Jensen's inequality provides the initial bound

$$\phi^{-1}\left(\sum_{r=1}^{s} \phi(|\langle \mathcal{T}_r\xi, \xi\rangle_{\mathcal{A}}|)\right) \ge \frac{1}{s}\sum_{r=1}^{s} |\langle \mathcal{T}_r\xi, \xi\rangle_{\mathcal{A}}|.$$

Applying the triangle inequality, we logically deduce that

$$\frac{1}{s}\sum_{r=1}^{s} |\langle \mathcal{T}_r\xi, \xi\rangle_{\mathcal{A}}| \ge \left|\sum_{r=1}^{s} \langle \frac{\lambda_r}{s}\mathcal{T}_r\xi, \xi\rangle_{\mathcal{A}}\right| = \left|\langle \sum_{r=1}^{s} \frac{\lambda_r}{s}\mathcal{T}_r\xi, \xi\rangle_{\mathcal{A}}\right|.$$

Taking the supremum over all available unit vectors $\xi \in \Sigma^1_{\mathcal{A}}$, we systematically obtain the following relation:

$$\omega_{\mathcal{A},\phi}(\mathbf{T}) \ge \omega_{\mathcal{A}}\left(\sum_{r=1}^{s} \frac{\lambda_r}{s}\mathcal{T}_r\right).$$

Given that this inequality holds true for every possible valid combination of the parameters $(\lambda_1, \ldots, \lambda_s)$ where $|\lambda_r| \leq 1$, we conclude that

$$\omega_{\mathcal{A},\phi}(\mathbf{T}) \geq \sup_{|\lambda_r|\leq 1} \omega_{\mathcal{A}}\left(\sum_{r=1}^{s} \frac{\lambda_r}{s}\mathcal{T}_r\right).$$

The final and remaining estimate presented in the theorem follows directly from a well-known application of the classical inequality which asserts that $\omega_{\mathcal{A}}(S) \geq \frac{1}{2}\|S\|_{\mathcal{A}}$.

As a direct and logical consequence of the results established in Proposition 3.1, we present the following comprehensive collection of lower bounds for the functional $\omega_{\mathcal{A},\phi}(\mathbf{T})$.

**Corollary 3.1.** *Let* $\mathbf{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_s) \in \mathbf{L}_{\sqrt{\mathcal{A}}}(\mathscr{H})^s$ *and let* $\phi : [0, \infty) \to [0, \infty)$ *be an increasing convex function. Then*

$$\omega_{\mathcal{A},\phi}(\mathbf{T}) \geq \frac{1}{s} \max_{1\leq r\leq s} \omega_{\mathcal{A}}(\mathcal{T}_r) \geq \frac{1}{2s} \max_{1\leq r\leq s} \|\mathcal{T}_r\|_{\mathcal{A}}.$$

*Moreover,*

$$\omega_{\mathcal{A},\phi}(\mathbf{T}) \geq \frac{1}{s} \max_{\varepsilon_r=\pm 1} \omega_{\mathcal{A}}\left(\sum_{r=1}^{s} \varepsilon_r\mathcal{T}_r\right) \geq \frac{1}{2s} \max_{\varepsilon_r=\pm 1} \left\|\sum_{r=1}^{s} \varepsilon_r\mathcal{T}_r\right\|_{\mathcal{A}}.$$

*Proof.* From the proof structures utilized in Proposition 3.1, we inherently know that

$$\omega_{\mathcal{A},\phi}(\mathbf{T}) \geq \sup_{|\lambda_r|\leq 1} \omega_{\mathcal{A}}\left(\sum_{r=1}^{s} \frac{\lambda_r}{s}\mathcal{T}_r\right).$$

If we purposefully select the values $\lambda_i = 1$ and subsequently set $\lambda_r = 0$ for all indices $r \neq i$, we are left with the simplified relation

$$\omega_{\mathcal{A},\phi}(\mathbf{T}) \geq \frac{1}{s}\, \omega_{\mathcal{A}}(\mathcal{T}_i), \qquad 1 \leq i \leq s.$$

Taking the maximum over all $1 \leq i \leq s$, we readily construct the first required estimate in the sequence. The corresponding norm bound naturally follows from the application of the classical inequality that states $\omega_{\mathcal{A}}(S) \geq \frac{1}{2}\|S\|_{\mathcal{A}}$.

On the other hand, setting $\lambda_r = \varepsilon_r$ with the explicit condition that $\varepsilon_r \in \{\pm 1\}$, we determine that

$$\omega_{\mathcal{A},\phi}(\mathbf{T}) \geq \frac{1}{s}\omega_{\mathcal{A}}\left(\sum_{r=1}^{s} \varepsilon_r\mathcal{T}_r\right).$$

Calculating the maximum over every conceivable configuration of signs $\varepsilon_r \in \{\pm 1\}$ yields the second collection of proposed inequalities. The final corresponding norm estimate is generated once again through the reliance on the foundational relationship between $\omega_{\mathcal{A}}(\cdot)$ and $\|\cdot\|_{\mathcal{A}}$. This fully completes the logical progression required for the proof.

## 4. Conclusions

In this work, we established new estimates for the joint $\mathcal{A}$-numerical radius and related seminorms of the $s$-tuples of operators acting on semi-Hilbertian spaces. Our results provide sharper bounds and highlight deeper relationships among the $\mathcal{A}$-numerical radius, the $\mathcal{A}$-operator seminorm, and the $\mathcal{A}$-Euclidean seminorm. In several situations, the obtained inequalities also improve previously known results in the classical Hilbert space setting.

We further introduced a functional perspective through the notion of the $(\mathcal{A}, \phi)$-joint numerical radius. This framework shows that suitable nonlinear transformations may lead to stronger and more flexible inequalities. In particular, the concavity of $\phi$ plays a central role in deriving refined estimates, while the presented counterexamples illustrate the limitations that arise when $\phi$ is convex.

Beyond the derivation of new inequalities, our results also provide several structural insights into multivariable operators on semi-inner product spaces. First, they show that the gap between the $\mathcal{A}$-Euclidean seminorm and the joint $\mathcal{A}$-numerical radius is precisely controlled by the $\mathcal{A}$-joint Crawford number, which naturally appears as a correction term measuring the joint numerical dispersion of the system. Second, the functional extension introduced in this work demonstrates that the concavity of the generating function is essential for preserving the subadditivity properties underlying numerical radius inequalities. Finally, although the $\mathcal{A}$-adjoint framework offers a powerful algebraic tool for handling the degenerate geometry of semi-Hilbertian spaces, the analysis also reveals that particular care is required when extending these results through nonlinear functional constructions.

Therefore, the sequence of results established in this paper provides a rigorous theoretical framework connecting the algebraic structure of multivariable operators with the analytic and geometric properties of semi-inner product spaces.

The ideas developed here open several directions for future research. One important limitation of the present approach is the requirement that the operators involved admit an $\mathcal{A}$-adjoint. This condition restricts the applicability of the obtained bounds to the algebra $\mathbf{L}_{\mathcal{A}}(\mathscr{H})$, rather than the full operator space $\mathbf{L}(\mathscr{H})$.

A natural continuation of this work is to investigate the structural and geometric properties of the $(\mathcal{A}, \phi)$-joint numerical radius for wider classes of functions $\phi$ or to develop alternative approaches that avoid the $\mathcal{A}$-adjoint assumption altogether. It would also be interesting to determine optimal choices of $\phi$ leading to the sharpest possible estimates, as well as to explore applications to other operator inequalities and related concepts in semi-Hilbertian spaces. We hope that this functional framework will encourage further developments in the study of numerical radii and operator theory.

## Author contributions

Salma Aljawi, Ahad Hamoud Alotaibi, Cristian Conde and Kais Feki: Investigation, Writing–original draft, Writing–review & editing. The authors declare that they have contributed equally to this paper. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that they have no competing interests.

## References


1. M. Chō, M. Takaguchi, Some classes of commuting n-tuples of operators, *Stud. Math.*, **80** (1984), 245–259. https://doi.org/10.4064/sm-80-3-245-259
2. S. M. Patel, Joint normaloidity of operators, *Glas. Mat.*, **15** (1980), 373–376.
3. C. Benhida, R. E. Curto, S. H. Lee, J. Yoon, The spectral picture and joint spectral radius of the generalized spherical Aluthge transform, *Adv. Math.*, **408** (2022), 108602. https://doi.org/10.1016/j.aim.2022.108602
4. P. Grover, S. Singla, A distance formula for tuples of operators, *Linear Algebra Appl.*, **650** (2022), 267–285. https://doi.org/10.1016/j.laa.2022.06.002
5. T. Le, Decomposing algebraic $m$-isometric tuples, *J. Funct. Anal.*, **278** (2020), 108424. https://doi.org/10.1016/j.jfa.2019.108424
6. G. Popescu, *Unitary invariants in multivariable operator theory*, Providence, RI: American Mathematical Society, 2009.
7. N. Altwaijry, S. S. Dragomir, K. Feki, On $A$-normaloid $d$-tuples of operators and related questions, *Quaest. Math.*, **47** (2024), 1305–1326. https://doi.org/10.2989/16073606.2024.2353387
8. N. Altwaijry, K. Feki, N. Minculete, A new seminorm for $d$-tuples of $A$-bounded operators and their applications, *Mathematics*, **11** (2023), 685. https://doi.org/10.3390/math11030685
9. S. Chavan, K. Feki, Spherical symmetry of some unitary invariants for commuting tuples, *Oper. Matrices*, **15** (2021), 1131–1139. https://doi.org/10.7153/oam-2021-15-70
10. M. W. Alomari, M. Sababheh, C. Conde, H. R. Moradi, Generalized Euclidean operator radius, *Georgian Math. J.*, **31** (2024), 369–380. https://doi.org/10.1515/gmj-2023-2079
11. S. S. Dragomir, Some inequalities for the Euclidean operator radius of two operators in Hilbert spaces, *Linear Algebra Appl.*, **419** (2006), 256–264. https://doi.org/10.1016/j.laa.2006.04.017
12. S. S. Dragomir, The hypo-Euclidean norm of an n-tuple of vectors in inner product spaces and applications, *J. Inequal. Pure Appl. Math.*, **8** (2007), 52.
13. M. S. Moslehian, M. Sattari, K. Shebrawi, Extensions of Euclidean operator radius inequalities, *Math. Scand.*, **120** (2017), 129–144. https://doi.org/10.7146/math.scand.a-25509

14. H. Baklouti, K. Feki, O. A. M. S. Ahmed, Joint numerical ranges of operators in semi-Hilbertian spaces, *Linear Algebra Appl.*, **555** (2018), 266–284. https://doi.org/10.1016/j.laa.2018.06.021

15. K. Feki, Spectral radius of semi-Hilbertian space operators and its applications, *Ann. Funct. Anal.*, **11** (2020), 929–946. https://doi.org/10.1007/s43034-020-00064-y

16. K. E. Gustafson, D. K. M. Rao, *Numerical range*, New York: Springer, 1997. https://doi.org/10.1007/978-1-4613-8498-4

17. T. Kato, Notes on some inequalities for linear operators, *Math. Ann.*, **125** (1952), 208–212. https://doi.org/10.1007/BF01343117

18. F. Kittaneh, A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, *Stud. Math.*, **158** (2003), 11–17. https://doi.org/10.4064/sm158-1-2

19. F. Kittaneh, Numerical radius inequalities for Hilbert space operators, *Stud. Math.*, **168** (2005), 73–80. https://doi.org/10.4064/sm168-1-5

20. F. Kittaneh, H. R. Moradi, M. Sababheh, Sharper bounds for the numerical radius, *Linear Multilinear Algebra*, **73** (2025), 2309–2319. https://doi.org/10.1080/03081087.2023.2177248

21. Z. Heydarbeygi, M. Sababheh, H. R. Moradi, A convex treatment of numerical radius inequalities, *Czech. Math. J.*, **72** (2022), 601–614. https://doi.org/10.21136/CMJ.2022.0068-21

22. F. Kittaneh, H. R. Moradi, M. Sababheh, On the numerical radius of the product of Hilbert space operators, *Hokkaido Math. J.*, **53** (2024), 395–420. https://doi.org/10.14492/hokmj/2023-701

23. H. R. Moradi, M. Sababheh, New estimates for the numerical radius, *Filomat*, **35** (2021), 4957–4962. https://doi.org/10.2298/FIL2114957M

24. O. Hirzallah, F. Kittaneh, K. Shebrawi, Numerical radius inequalities for certain $2 \times 2$ operator matrices, *Integr. Equ. Oper. Theory*, **71** (2011), 129–147. https://doi.org/10.1007/s00020-011-1893-0

25. M. Sababheh, H. R. Moradi, More accurate numerical radius inequalities (I), *Linear Multilinear Algebra*, **69** (2021), 1964–1973. https://doi.org/10.1080/03081087.2019.1651815

26. S. Sheybani, M. Sababheh, H. R. Moradi, Weighted inequalities for the numerical radius, *Vietnam J. Math.*, **51** (2023), 363–377. https://doi.org/10.1007/s10013-021-00533-4

27. S. S. Dragomir, Some inequalities of Kato type for sequences of operators in Hilbert spaces, *Publ. Res. Inst. Math. Sci.*, **48** (2012), 937–955. https://doi.org/10.2977/prims/92

28. A. El Farissi, Simple proof and refinement of Hermite-Hadamard inequality, *J. Math. Inequal.*, **4** (2010), 365–369.

29. M. I. Abbas, M. A. Ragusa, Solvability of Langevin equations with two Hadamard fractional derivatives via Mittag-Leffler functions, *Appl. Anal.*, **101** (2022), 3231–3245. https://doi.org/10.1080/00036811.2020.1839645

30. C. Dellacherie, S. Martínez, J. San Martín, Hadamard functions of inverse $M$-matrices, In: *Inverse M-matrices and ultrametric matrices*, Cham: Springer, 2014, 165–213. https://doi.org/10.1007/978-3-319-10298-6_6

31. C. Pearcy, An elementary proof of the power inequality for the numerical radius, *Michigan Math. J.*, **13** (1966), 289–291.

32. P. Bhunia, S. S. Dragomir, M. S. Moslehian, K. Paul, *Lectures on numerical radius inequalities*, Cham: Springer, 2022. https://doi.org/10.1007/978-3-031-13670-2

33. A. Saddi, $A$-normal operators in semi Hilbertian spaces, *Aust. J. Math. Anal. Appl.*, **9** (2012), 1–12.

34. M. L. Arias, G. Corach, M. C. Gonzalez, Partial isometries in semi-Hilbertian spaces, *Linear Algebra Appl.*, **428** (2008), 1460–1475. https://doi.org/10.1016/j.laa.2007.09.031

35. R. G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert space, *Proc. Amer. Math. Soc.*, **17** (1966), 413–415. https://doi.org/10.1090/S0002-9939-1966-0203464-1

36. M. L. Arias, G. Corach, M. C. Gonzalez, Metric properties of projections in semi-Hilbertian spaces, *Integral Equ. Oper. Theory*, **62** (2008), 11–28.

37. M. L. Arias, G. Corach, M. C. Gonzalez, Lifting properties in operator ranges, *Acta Sci. Math. (Szeged)*, **75** (2009), 635–653.

38. H. Baklouti, K. Feki, S. A. O. A. Mahmoud, Joint normality of operators in semi-Hilbertian spaces, *Linear Multilinear Algebra*, **68** (2020), 845–866. https://doi.org/10.1080/03081087.2019.1593925

39. S. Jana, P. Bhunia, K. Paul, Euclidean operator radius inequalities of $d$-tuple operators and operator matrices, *Math. Slovaca*, **74** (2024), 947–962. https://doi.org/10.1515/ms-2024-0070

40. Z. I. Ismailov, P. I. Al, H. R. Moradi, M. Sababheh, A new collection of $n$-tuple operator inequalities, *Georgian Math. J.*, 2026. https://doi.org/10.1515/gmj-2026-3011

41. M. S. Moslehian, Q. Xu, A. Zamani, Seminorm and numerical radius inequalities of operators in semi-Hilbertian spaces, *Linear Algebra Appl.*, **591** (2020), 299–321. https://doi.org/10.1016/j.laa.2020.01.015