Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 20915–20929.
DOI: 10.3934/math.2026849
Received: 29 March 2026
Revised: 09 June 2026
Accepted: 18 June 2026
Published: 15 July 2026

Research article

New finite-time and global Mittag-Leffler stability criteria for fractional-order inertial Cohen–Grossberg neural networks with S-type distributed delays

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Abstract: This paper studied the finite-time stability (FTS) and global Mittag-Leffler stability (GMLS) of a class of fractional inertial Cohen-Grossberg neural networks (FICGNNs) with S-type distributed delays. A unified modeling framework based on Lebesgue-Stieltjes integrals was adopted to encompass both discrete and continuously distributed delays. By establishing transformation between the fractional-order integrals of the state function with and without S-type distributed delays, new criteria for the corresponding FTS and GMLS were rigorously derived under appropriate assumptions. The theoretical results were further verified through numerical simulations.

Keywords: fractional-order; S-type distributed delays; Cohen-Grossberg neural networks; finite-time stability; global Mittag-Leffler stability
Mathematics Subject Classification: 34K20, 92B20

1. Introduction

Fractional-order neural networks (FNNs) were first introduced by Arena in 1988. A significant milestone was reached in 2008 when Boroomand, in the context of circuit system applications, substituted the conventional integer-order capacitor in a Hopfield neural network (HNN) with a fractional-order counterpart, thereby deriving fractional-order Hopfield neural networks (FHNNs) [1]. This breakthrough sparked a surge of research interest in FNNs. Representative cutting-edge work includes [2], which investigates the Mittag-Leffler stability (MLS) of anti-periodic solutions (APS) for FNNs. Furthermore, recent advancements have significantly broadened the scope of fractional-order network applications and stability analyses, as demonstrated by studies on the Mittag-Leffler synchronization and finite-time synchronization control of complex-valued fractional-order systems [3, 4].

So far, research on FNNs has largely focused on systems governed by equations with a single fractional derivative. In contrast, fractional-order inertial neural networks (FINNs) are mathematically characterized by differential equations that involve two distinct fractional-order derivatives. With the progressive maturation of fractional calculus theory, the investigation into the dynamical properties of FINNs has gained substantial significance and value. Notable recent contributions include studies on the stability of APS for fractional-order inertial bidirectional associative memory neural networks (FIBAMNNs) with delays [5], global Mittag-Leffler synchronization stability for FIBAMNNs with delays [6], global Mittag-Leffler stability (GMLS) and global asymptotic ω-periodicity for FICGNNs with time-varying delays [7], global Mittag-Leffler synchronization and the boundedness for fractional inertial Cohen-Grossberg neural networks (FICGNNs) [8], dynamical analysis of FINNs [9], investigations into GMLS, and S-asymptotically ω-periodicity and the boundedness for delayed fuzzy FINNs [10].

The inertial term is consistently recognized as a primary factor responsible for generating complex dynamical behaviors, including bifurcation and chaos. Consequently, introducing inertial term into neural systems is of significant importance and has yielded a wealth of research outcomes. For instance, the dynamical behaviors of various models have been thoroughly investigated, including inertial delayed bidirectional associative memory neural networks (BAMNNs) [11–13], coupled inertial delayed NNs [14], inertial NNs with time-varying delays [15–17], and the stochastic asymptotic stability of stochastic inertial delayed Cohen–Grossberg neural networks (CGNNs) [18].

When describing real-world problems, neural network systems often require the incorporation of time-delays. Time-delays can be categorized into two mutually exclusive and independent types: Discrete delays and continuous distributed delays. However, both types can be unified under the framework of S-type distributed delays (SD) via Lebesgue-Stieltjes integrals. Consequently, NNs with SD encompass both discrete and continuous distributed delay models. Considerable research has been conducted on the dynamical behaviors of NNs with SD. For example, [19, 20] investigate the global exponential robustness and exponential stability of HNNs with SD. The existence, uniqueness, and stability of solutions for stochastic reaction-diffusion HNNs with SD are investigated in [21]. In [22], the existence of periodic solutions of stochastic reaction-diffusion NNs with SD is established. The stability of a class of BAMNNs with SD is examined in [23]. The global exponential stability for cellular NNs with SD is explored in [24]. In [25], the existence and asymptotic stability for stochastic competitive NNs with SD are discussed; employing Lyapunov functional methods and stochastic analysis techniques, the study investigates almost sure exponential stability and the global mean-square exponential robust stability of the NNs. Recently, the global asymptotic stability(GAS), finite-time stability (FTS) and GMLS of neutral-type NNs with SD are addressed in [26, 27]. In the stability analysis of solutions for NNs, GMLS focuses on the asymptotic behavior of the system over an infinite time horizon. It ensures that the FICGNNs will eventually converge to the equilibrium point at a Mittag-Leffler rate, regardless of the initial states. This provides the core theoretical guarantee for the long-term reliability and robustness of the neural network during continuous operation. In contrast, FTS specifically examines the transient performance of the system within a predefined finite time interval. In many practical engineering fields, such as secure communication, real-time control, or rapid synchronization, systems are often subject to hard time constraints. These applications require the system states to remain within specific safety thresholds during a given time limit, rather than merely seeking convergence as $t \to \infty$.

The aforementioned literature primarily focuses on FTS and GMLS of solutions for FICGNNs with either discrete delays or continuous distributed delays. Research on FTS and GMLS of solutions for FICGNNs within a unified framework that encompasses both discrete and continuously distributed delays remains sparse. Therefore, investigating both FTS and GMLS of solutions for FICGNNs with SD within a unified mathematical framework is of significant practical importance. It ensures the system's capability for rapid response under complex dynamic environments (characterized by FTS) while simultaneously guaranteeing long-term stability (ensured by GMLS). As such, the present study undertakes a comprehensive investigation with significant implications for both theory and practice. The ensuing sections provide a detailed elucidation of our core contributions:

- An FICGNNs with discrete and continuous distributed time-delays is formulated as unified S-type distributed delay FICGNNs within the framework of Lebesgue-Stieltjes integrals.
- An interval-partitioning-based method is derived to transform the fractional-order integral of the state variable with SD into a general fractional-order integral without such delays.
- Stability criteria for the FTS and GMLS of FICGNNs with SD are established under appropriate assumptions. These results complement each other and provide a new foundation for further extensions of neural network research and practical applications.

A class of FICGNNs with SD is investigated in this paper as follows

$$\mathcal{D}_t^{\alpha} x_p(t) = -\alpha_p(x_p(t))\Big[h_p(x_p(t)) - \sum_{q=1}^{n} a_{pq} f_q(x_q(t)) - \sum_{q=1}^{n} b_{pq} g_q\Big(\int_{-\tau}^{0} x_q(t+\theta)\, d\omega_q(\theta)\Big) - I_p\Big], \tag{1.1}$$

with initial values

$$x_p(t) = \varphi_p(t), \quad t \in [-\tau, 0], \quad p, q = 1, \cdots, n,$$

where $\mathcal{D}_t^{\alpha}$ denotes the α-order Caputo fractional derivative with $0 < \alpha < 1$; $x_p(t)$ represents the state of the p-th neuron at time t; $\alpha_p(x_p(t))$ stands for the amplification function of the p-th neuron; $h_p(x_p(t))$ denotes the behavior function of the p-th neuron; a_{pq} and b_{pq} are the connection weights between neurons; $f_q(\cdot)$ and $g_q(\cdot)$ represent the nonlinear activation functions of the q-th neuron; I_p indicates the output to the p-th neuron at time t; $\int_{-\tau}^{0} x_q(t+\theta)d\omega_q(\theta)$ is the Lebesgue-Stieltjes integral of $x_q(t+\theta)$ with respect to the nondecreasing bounded-variation function $\omega_q(\theta)$ over the interval $[-\tau, 0]$ $(t > 0,\ \tau > 0)$, which satisfies $\int_{-\tau}^{0} d\omega_q(\theta) = k_q > 0$; and $\varphi_p(t)$ is a bounded continuous function.

2. Preliminaries

Definition 2.1. *[28] The fractional integral is defined as*

$$\mathcal{I}_t^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau) d\tau,$$

for all $t \geq 0$, where $\Gamma(q) = \int_0^{+\infty} t^{q-1} e^{-t} dt$. When $f(x)$ is constant, then $\mathcal{I}_t^q C = \dfrac{Ct^q}{\Gamma(1+q)}$.

Definition 2.2. *[28] The Caputo fractional derivative is defined as*

$$^{C}\mathcal{D}_t^q f(t) = \mathcal{I}^{n-q}(\mathcal{D}^n f(t)) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q-n+1}} d\tau,$$

where $n-1 \le q < n,\ n \in \mathbb{N}^+$. *Specifically,* $^{C}\mathcal{D}_t^q C \equiv 0$, *where C is constant.*

Lemma 2.3. *([29] Bellman-Gronwall inequality) Assume that function* $y(t)$ *satisfies*

$$y(t) \le \int_0^t a(\tau)y(\tau)d\tau + b(t),$$

with $a(t)$ *and* $b(t)$ *being known real positive functions. If* $b(t)$ *is differentiable, then*

$$y(t) \le b(0)e^{\int_0^t a(\tau)d\tau} + \int_0^t \dot{b}(\tau)e^{\int_\tau^t a(r)dr} d\tau.$$

Lemma 2.4. *[30] If* $s(t) \in C[0, +\infty]$, *and there exist* $d_1 > 0, d_2 > 0$ *which satisfy* $s(t) \le -d_1 \mathcal{I}_t^q s(t) + d_2$, *then*

$$s(t) \le d_2 E_q(-d_1 t^q),$$

where $0 < q < 1$, *and* $E_q(\cdot)$ *denotes one-parameter Mittag-Leffler function.*

Definition 2.5. *The equilibrium point* $u^* = (u_1^*, \cdots, u_n^*)^T$ *of system* (1.1) *is FTS with respect to* $\{\delta, \varepsilon, t_0, J\}$ *(where* $0 < \delta < \varepsilon$, $J = [t_0, t_0 + T]$, $T > 0$, *and* t_0 *is the initial time), for every solution* $u(t, t_0, \varphi)$ *with initial condition* $u_p(s) = \varphi_p(s), -\tau \le s \le 0, p = 1, \cdots, n$, *if and only if,* $\|\varphi - u^*\| < \delta$ *implies* $\|u(t) - u^*\| < \varepsilon$ *for all* $t \in J$, *with the norms given by*

$$\|\varphi - u^*\| = \sup_{-\infty \le s \le 0} \sum_{p=1}^{n} |\varphi_p(s) - u_p^*|, \quad \|u(t) - u^*\| = \sum_{p=1}^{n} |u_p(t) - u_p^*|.$$

Definition 2.6. *The system* (1.1) *is GMLS provided that for any pair of solutions* $x(t) = (x_1(t), \cdots, x_n(t))^T$ *and* $\bar{x}(t) = (\bar{x}_1(t), \cdots, \bar{x}_n(t))^T$ *with initial values* $\varphi(t)$, $\bar{\varphi}(t)$ $(-\infty < t \le 0)$, *if there exists positive constant* ρ *such that*

$$\|x(t) - \bar{x}(t)\| \le M(\varphi - \bar{\varphi})E_\alpha(-\rho t^\alpha), \quad t \ge 0,$$

with $M(\cdot) \ge 0$, $M(0) = 0$ *and* $E_\alpha(\cdot)$ *being the Mittag-Leffler function of a single parameter.*

For the sake of convenience in the discussion of this paper, we make the following assumptions:

$\mathrm{H}_1:$ The amplification function $\alpha_p(x_p)$ is differentiable with respect to x_p and satisfies

$$0 < \underline{\alpha}_p \le \alpha_p(x_p) \le \bar{\alpha}_p, \quad |\alpha_p'(x_p)| \le {\alpha_p}^*, \quad \forall x_p \in R.$$

$\mathrm{H}_2:$ The behavior function $h_p(x_p)$ is monotonic and differentiable with respect to x_p, and satisfies

$$0 < \underline{h}_p^* \le h_p'(x_p) \le \bar{h}_p^*, \quad \forall x_p \in R.$$

H_3 : $u_p(x_p) = \alpha_p(x_p)h_p(x_p)$ is differentiable with respect to x_p and, there exist constants $\underline{u}_p^* > 0, \bar{u}_p^* > 0$ such that

$$0 < \underline{u}_p^* \leq u_p'(x_p) \leq \bar{u}_p^*, \quad \forall x_p \in R.$$

H_4 : The activation functions $f_q(\cdot)$ and $g_q(\cdot)$ are bounded and satisfy Lipschitz condition. For $\forall \xi_1, \xi_2 \in \mathbb{R}$, there exist positive constants $F_q, G_q, \bar{f}_q, \bar{g}_q$ such that

$$\begin{aligned} &\left|f_q(\xi_1) - f_q(\xi_2)\right| \leq F_q \left|\xi_1 - \xi_2\right|, \quad \left|f_q(\cdot)\right| \leq \bar{f}_q, \\ &\left|g_q(\xi_1) - g_q(\xi_2)\right| \leq G_q \left|\xi_1 - \xi_2\right|, \quad \left|g_q(\cdot)\right| \leq \bar{g}_q, \quad q = 1, \cdots, n. \end{aligned}$$

3. Main results

Theorem 3.1. *Suppose that Assumptions* H_1–H_4 *hold, and let the parameters* δ, ε, T *be given according to Definition 2.5. If the following two conditions are satisfied*

$$\begin{aligned} &(1) \quad \sum_{p=1}^{n} \frac{1}{\underline{h}_p^*} \max_{1\leq q\leq n} \{(|a_{pq}|\, F_q + |b_{pq}|\, G_q\, k_q)\} < 1, \\ &(2) \quad \delta e^{\frac{Vt^\alpha}{\Gamma(\alpha+1)}} \Big(1 + \sum_{p=1}^{n} \bar{\alpha}_p \max_{1\leq q\leq n}\{|b_{pq}|G_q k_q\} \frac{t^\alpha}{\Gamma(\alpha+1)}\Big) < \varepsilon, \end{aligned}$$

where

$$V = \max_{1\leq p\leq n} \{ -\underline{u}_p^* + \sum_{q=1}^{n} \bar{\alpha}_q (|a_{qp}|F_p + |b_{qp}|G_p k_p) + \bar{\alpha}_p^* (\sum_{q=1}^{n} |a_{pq}|\bar{f}_q + \sum_{q=1}^{n} |b_{pq}|\bar{g}_q + |I_p|)\} > 0,$$

then system (1.1) *possesses a unique equilibrium point, and this equilibrium is FTS.*

Proof. Since the Caputo fractional derivative of a constant is zero (i.e., $\mathcal{D}_t^\alpha x_p^* = 0$), the equilibrium point of system (1.1) satisfies

$$h_p(x_p^*) = \sum_{q=1}^{n} a_{pq} f_q(x_q^*) + \sum_{q=1}^{n} b_{pq} g_q \Big(\int_{-\tau}^{0} x_q^* \, d\omega_q(\theta) \Big) + I_p,$$

and function $h_p(x_p)$ is monotonic and differentiable with its derivative bounded by $0 < \underline{h}_p^* \leq h_p'(x_p) \leq \bar{h}_p^*$ for all $x_p \in \mathbb{R}$, and a differentiable inverse function h_p^{-1} exists. Consequently, for any $x, y \in \mathbb{R}$,

$$h_p^{-1}(x) - h_p^{-1}(y) = (h^{-1})'(\xi)(x - y) = \frac{x - y}{h_p'(\xi)} \leq \frac{|x - y|}{\underline{h}_p^*}, \quad \xi \in (x, y). \tag{3.1}$$

Let $x = (x_1, \cdots, x_n)^T$, $y = (y_1, \cdots, y_n)^T \in \mathbb{R}^n$ and define a mapping $P : \mathbb{R}^n \to \mathbb{R}^n$ by $P(x) = (P_{x_1}, \cdots, P_{x_n})^T$, where

$$P_{x_p} = h_p^{-1}\Big[\sum_{q=1}^{n} a_{pq} f_q(x_q) + \sum_{q=1}^{n} b_{pq} g_q \Big(\int_{-\tau}^{0} x_q \, d\omega_q(\theta) \Big) + I_p \Big], \quad p = 1, \cdots, n. \tag{3.2}$$

By (3.1) and (3.2), we deduce that

$$\begin{aligned}
\|P(x)-P(y)\| &= \sum_{p=1}^{n}|P_{x_p}-P_{y_p}| \\
&= \sum_{p=1}^{n}\Big\{h_p^{-1}\Big[\sum_{q=1}^{n}a_{pq}f_q(x_q)+\sum_{q=1}^{n}b_{pq}g_q\Big(\int_{-\tau}^{0}x_q\,d\omega_q(\theta)\Big)+I_p\Big] \\
&\quad -h_p^{-1}\Big[\sum_{q=1}^{n}a_{pq}f_q(y_q)+\sum_{q=1}^{n}b_{pq}g_q\Big(\int_{-\tau}^{0}y_q\,d\omega_q(\theta)\Big)+I_p\Big]\Big\} \\
&\le \sum_{p=1}^{n}\frac{1}{\underline{h}_p^*}\Big[\sum_{q=1}^{n}|a_{pq}||f_q(x_q)-f_q(y_q)|+\sum_{q=1}^{n}|b_{pq}|\Big|g_q\big(\int_{-\tau}^{0}x_q d\omega_q(\theta)\big)-g_q\big(\int_{-\tau}^{0}y_q d\omega_q(\theta)\big)\Big|\Big] \\
&\le \sum_{p=1}^{n}\frac{1}{\underline{h}_p^*}\Big[\sum_{q=1}^{n}|a_{pq}|\,F_q\,|x_q-y_q|+\sum_{q=1}^{n}|b_{pq}|\,G_q\,k_q\,|x_q-y_q|\Big] \\
&\le \sum_{p=1}^{n}\frac{1}{\underline{h}_p^*}\max_{1\le q\le n}\{(|a_{pq}|\,F_q+|b_{pq}|\,G_q\,k_q)\}\sum_{q=1}^{n}|x_q-y_q| \\
&\le \|x-y\|.
\end{aligned}$$

Therefore, the mapping P is contractive. By the contraction mapping principle, there exists a unique point $x^*=(x_1^*,\cdots,x_n^*)^T$ such that $P(x^*)=x^*$. Consequently, it satisfies

$$x_p^*=h_p^{-1}\Big[\sum_{q=1}^{n}a_{pq}f_q(x_q^*)+\sum_{q=1}^{n}b_{pq}g_q\Big(\int_{-\tau}^{0}x_q^*\,d\omega_q(\theta)\Big)+I_p\Big],$$

which implies

$$-h_p(x_p^*)+\sum_{q=1}^{n}a_{pq}f_q(x_q^*)+\sum_{q=1}^{n}b_{pq}g_q\Big(\int_{-\tau}^{0}x_q^*\,d\omega_q(\theta)\Big)+I_p=0.$$

Thus, x^* is the unique equilibrium point of system (1.1).

Next, we demonstrate the unique equilibrium point x^* is FTS. Let $e_p(t)=x_p(t)-x_p^*$. Consider a solution $x(t,t_0,\varphi)$ with initial condition $x_p(s)=\varphi_p(s)$ for $-\tau\le s\le 0$, such that $\|\varphi-x^*\|<\delta$ (where $0\le\delta<\varepsilon$).

From system (1.1), we have

$$\begin{aligned}
\mathcal{D}_t^{\alpha}e_p(t) &= \alpha_p(x_p^*)h_p(x_p^*)-\alpha_p(x_p(t))h_p(x_p(t))+\alpha_p(x_p(t))\Big[\sum_{q=1}^{n}a_{pq}\Big(f_q(x_q(t))-f_q(x_q^*)\Big) \\
&\quad +\sum_{q=1}^{n}b_{pq}\Big(g_q\big(\int_{-\tau}^{0}x_q(t+\theta)\,d\omega_q(\theta)\big)-g_q\big(\int_{-\tau}^{0}x_q^*\,d\omega_q(\theta)\big)\Big)\Big] \\
&\quad +\Big(\alpha_p(x_p(t))-\alpha_p(x_p^*)\Big)\Big[\sum_{q=1}^{n}a_{pq}\,f_q(x_q^*)+\sum_{q=1}^{n}b_{pq}\,g_q\big(\int_{-\tau}^{0}x_q^*\,d\omega_q(\theta)\big)+I_p\Big].
\end{aligned}\tag{3.3}$$

Let $u_p(x)=\alpha_p(x)h_p(x)$, and we obtain

$$\begin{aligned}
\alpha_p(x_p^*)h_p(x_p^*)-\alpha_p(x_p(t))h_p(x_p(t)) &= u_p(x_p^*)-u_p(x_p(t)) \\
&= u_p'(\xi_p)\big(x_p^*-x_p(t)\big)=-u_p'(\xi_p)e_p(t),\quad \xi_p\in(x_p,x_p^*).
\end{aligned}\tag{3.4}$$

Substituting (3.4) into (3.3), we get

$$\begin{aligned}
\mathcal{D}_t^{\alpha}|e_p(t)| &\leq sgn(e_p(t))\mathcal{D}_t^{\alpha}e_p(t)\\
&= sgn(e_p(t))\Big\{-u_p'(\xi_p)e_p(t)+\alpha_p(x_p(t))\Big[\sum_{q=1}^{n}a_{pq}(f_q(x_q(t))-f_q(x_q^*))\\
&\quad+\sum_{q=1}^{n}b_{pq}\Big(g_q(\int_{-\tau}^{0}x_q(t+\theta)\,d\omega_q(\theta))-g_q(\int_{-\tau}^{0}x_q^*\,d\omega_q(\theta))\Big)\Big]\\
&\quad+\alpha_p'(\eta_p)e_p(t)\Big[\sum_{q=1}^{n}a_{pq}\,f_q(x_q^*)+\sum_{q=1}^{n}b_{pq}\,g_q(\int_{-\tau}^{0}x_q^*\,d\omega_q(\theta))+I_p\Big]\Big\}\\
&\leq -\underline{u}_p^*|e_p(t)|+\bar{\alpha}_p\Big[\sum_{q=1}^{n}|a_{pq}|\,F_q\,|e_q(t)|+\sum_{q=1}^{n}|b_{pq}|\,G_q\int_{-\tau}^{0}|e_q(t+\theta)|d\omega_q(\theta)\Big]\\
&\quad+\bar{\alpha}_p^*|e_p(t)|(\sum_{q=1}^{n}|a_{pq}|\,\bar{f}_q+\sum_{q=1}^{n}|b_{pq}|\,\bar{g}_q+|I_p|)\\
&= \Big[-\underline{u}_p^*+\bar{\alpha}_p^*(\sum_{q=1}^{n}|a_{pq}|\,\bar{f}_q+\sum_{q=1}^{n}|b_{pq}|\,\bar{g}_q+|I_p|)\Big]|e_p(t)|+\bar{\alpha}_p\sum_{q=1}^{n}|a_{pq}|\,F_q|e_q(t)|\\
&\quad+\bar{\alpha}_p\sum_{q=1}^{n}|b_{pq}|\,G_q\int_{-\tau}^{0}|e_q(t+\theta)|d\omega_q(\theta),\quad \xi_p,\eta_p\in(x_p,x_p^*).
\end{aligned}\tag{3.5}$$

Taking the fractional integral $\mathcal{I}_t^{\alpha}$ on both sides of (3.5) and utilizing the property of Caputo derivative $\mathcal{I}_t^{\alpha}(\mathcal{D}_t^{\alpha}|e_p(t)|)=|e_p(t)|-|e_p(0)|$, we can conclude that

$$\begin{aligned}
\sum_{p=1}^{n}|e_p(t)| &\leq \sum_{p=1}^{n}|e_p(0)|+\sum_{p=1}^{n}\Big\{-\underline{u}_p^*+\bar{\alpha}_p^*(\sum_{q=1}^{n}|a_{pq}|\bar{f}_q+\sum_{q=1}^{n}|b_{pq}|\bar{g}_q+|I_p|)\Big\}\mathcal{I}_t^{\alpha}|e_p(t)|\\
&\quad+\sum_{p=1}^{n}\sum_{q=1}^{n}\bar{\alpha}_p|a_{pq}|F_q\mathcal{I}_t^{\alpha}|e_q(t)|+\sum_{p=1}^{n}\sum_{q=1}^{n}\bar{\alpha}_p\,|b_{pq}|G_q\mathcal{I}_t^{\alpha}\Big(\int_{-\tau}^{0}|e_q(t+\theta)|\,d\omega_q(\theta)\Big).
\end{aligned}\tag{3.6}$$

For $t+\theta\leq 0$,

$$\begin{aligned}
\mathcal{I}_t^{\alpha}\Big(\int_{-\tau}^{0}|e_q(t+\theta)|\,d\omega_q(\theta)\Big) &\leq \sup_{s\leq 0}|\varphi_q(s)-x_q^*|\,\mathcal{I}_t^{\alpha}(k_q)\\
&= \sup_{s\leq 0}|\varphi_q(s)-x_q^*|\,\frac{k_q\,t^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}\tag{3.7}$$

For $t+\theta\geq 0$,

$$\begin{aligned}
&\mathcal{I}_t^{\alpha}\Big(\int_{-\tau}^{0}|e_q(t+\theta)|\,d\omega_q(\theta)\Big)\\
=\;& \tfrac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}(\int_{-\tau}^{0}|e_q(s+\theta)|d\omega_q(\theta))ds\\
=\;& \tfrac{1}{\Gamma(\alpha)}\int_{-\tau}^{0}(\int_0^t(t-s)^{\alpha-1}|e_q(s+\theta)|ds)d\omega_q(\theta)\\
=\;& \tfrac{1}{\Gamma(\alpha)}\int_{-\tau}^{0}[\int_{\theta}^{t+\theta}(t+\theta-s)^{\alpha-1}|e_q(s)|ds]d\omega_q(\theta)\\
=\;& \tfrac{1}{\Gamma(\alpha)}\int_{-\tau}^{0}[\int_{\theta}^{0}(t+\theta-s)^{\alpha-1}|e_q(s)|ds+\int_0^{t+\theta}(t+\theta-s)^{\alpha-1}|e_q(s)|ds]d\omega_q(\theta)\\
\leq\;& \sup_{s\leq 0}|\varphi_q(s)-x_q^*|\frac{k_q\,t^{\alpha}}{\Gamma(\alpha+1)}+k_q\,\mathcal{I}_t^{\alpha}|e_q(t)|.
\end{aligned}\tag{3.8}$$

Combining (3.7) and (3.8), we arrive at

$$\mathcal{I}_t^{\alpha}\Big(\int_{-\tau}^{0}|e_q(t+\theta)|\,d\omega_q(\theta)\Big)\leq\sup_{s\leq 0}|\varphi_q(s)-x_q^*|\frac{k_q\,t^{\alpha}}{\Gamma(\alpha+1)}+k_q\,\mathcal{I}_t^{\alpha}|e_q(t)|.\tag{3.9}$$

Substituting (3.9) into (3.6) with $\sum\limits_{p=1}^{n}|e_p(0)| \leq \|\varphi - x^*\| \leq \delta$ yields

$$\begin{aligned}
& \sum_{p=1}^{n}|e_p(t)| \\
\leq\ & \sum_{p=1}^{n}\Big\{-\underline{u}_p^* + \sum_{q=1}^{n}\bar{\alpha}_q(|a_{qp}|F_p + |b_{qp}|G_pk_p) + \bar{\alpha}_p^*(\sum_{q=1}^{n}|a_{pq}|\bar{f}_q \\
& + \sum_{q=1}^{n}|b_{pq}|\bar{g}_q + |I_p|)\Big\}\mathcal{I}_t^{\alpha}|e_p(t)| + \sum_{p=1}^{n}\sum_{q=1}^{n}\bar{\alpha}_p\,|b_{pq}|G_q\sup_{s\leq 0}|\varphi_q(s) - x_q^*|\frac{k_q\, t^{\alpha}}{\Gamma(\alpha+1)} + \sum_{p=1}^{n}|e_p(0)| \\
\leq\ & \max_{1\leq p\leq n}\{-\underline{u}_p^* + \sum_{q=1}^{n}\bar{\alpha}_q(|a_{qp}|F_p + |b_{qp}|G_pk_p) + \bar{\alpha}_p^*(\sum_{q=1}^{n}|a_{pq}|\bar{f}_q + \sum_{q=1}^{n}|b_{pq}|\bar{g}_q \\
& + |I_p|)\}\mathcal{I}_t^{\alpha}(\sum_{p=1}^{n}|e_p(t)|) + \|\varphi - x^*\|\sum_{p=1}^{n}\bar{\alpha}_p\max_{1\leq q\leq n}\{|b_{pq}|G_qk_q\}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \sum_{p=1}^{n}|e_p(0)| \\
\leq\ & V\mathcal{I}_t^{\alpha}(\sum_{p=1}^{n}|e_p(t)|) + \delta\sum_{p=1}^{n}\bar{\alpha}_p\max_{1\leq q\leq n}\{|b_{pq}|G_qk_q\}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \delta \\
=\ & \frac{V}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\sum_{p=1}^{n}|e_p(s)|ds + \delta\Big(1 + \sum_{p=1}^{n}\bar{\alpha}_p\max_{1\leq q\leq n}\{|b_{pq}|G_qk_q\}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big),
\end{aligned} \tag{3.10}$$

where

$$V = \max_{1\leq p\leq n}\{-\underline{u}_p^* + \sum_{q=1}^{n}\bar{\alpha}_q(|a_{qp}|F_p + |b_{qp}|G_pk_p) + \bar{\alpha}_p^*(\sum_{q=1}^{n}|a_{pq}|\bar{f}_q + \sum_{q=1}^{n}|b_{pq}|\bar{g}_q + |I_p|)\}.$$

Applying Lemma 2.3 to (3.10) with $b(t) = \delta(1 + \sum\limits_{p=1}^{n}\bar{\alpha}_p\max_q\{|b_{pq}|G_qk_q\}\frac{t^{\alpha}}{\Gamma(\alpha+1)})$, it follows that

$$\begin{aligned}
\sum_{p=1}^{n}|e_p(t)| &\leq b(0)e^{\frac{V}{\Gamma(\alpha+1)}t^{\alpha}} + \int_0^t\dot{b}(\tau)e^{\frac{V}{\Gamma(\alpha)}\int_{\tau}^{t}(t-r)^{\alpha-1}dr}d\tau \\
&\leq \delta e^{\frac{V}{\Gamma(\alpha+1)}t^{\alpha}} + \delta\sum_{p=1}^{n}\bar{\alpha}_p\max_{1\leq q\leq n}\{|b_{pq}|G_qk_q\}\int_0^t\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}e^{\frac{V}{\Gamma(\alpha+1)}(t-\tau)^{\alpha}}d\tau \\
&\leq \delta e^{\frac{V}{\Gamma(\alpha+1)}t^{\alpha}}\left(1 + \sum_{p=1}^{n}\bar{\alpha}_p\max_{1\leq q\leq n}\{|b_{pq}|G_qk_q\}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right) \\
&< \varepsilon.
\end{aligned}$$

Therefore, according to Definition 2.5, the equilibrium point of system (1.1) is FTS, which completes the proof. □

Theorem 3.2. *Suppose that assumptions* $H_1 - H_4$ *hold, and if*

$$\eta = \min_{1\leq p\leq n}\Big\{\underline{u}_p^* - \alpha_p^*(\sum_{q=1}^{n}|a_{pq}|\,\bar{f}_q + \sum_{q=1}^{n}|b_{pq}|\,\bar{g}_q + |I_p|) - \sum_{q=1}^{n}\bar{\alpha}_q|a_{qp}|F_p\Big\} - \max_{1\leq p\leq n}(\sum_{q=1}^{n}\bar{\alpha}_q|b_{qp}|G_pk_p) > 0,$$

the system (1.1) *is GMLS for* $0 \leq t \leq T \leq +\infty$.

Proof. Let $x(t)$ and $\tilde{x}(t)$ be two solutions of system (1.1) satisfying initial conditions $x_p(s) =$

$\varphi_p(s),\ \tilde{x}_p(s)=\tilde{\varphi}_p(s)\ (-\tau\le s\le 0)$. Let $e_p(t)=x_p(t)-\tilde{x}_p(t)$; from (1.1), we obtain

$$\begin{aligned}\mathcal{D}_t^{\alpha}e_p(t) &= \alpha_p(\tilde{x}_p)h_p(\tilde{x}_p)-\alpha_p(x_p(t))h_p(x_p(t))+\alpha_p(x_p(t))\Big[\sum_{q=1}^{n}a_{pq}\Big(f_q(x_q(t))-f_q(\tilde{x}_q(t))\Big)\\ &\quad+\sum_{q=1}^{n}b_{pq}\Big(g_q(\int_{-\tau}^{0}x_q(t+\theta)\,d\omega_q(\theta))-g_q(\int_{-\tau}^{0}\tilde{x}_q(t+\theta)\,d\omega_q(\theta))\Big)\Big]\\ &\quad+\Big(\alpha_p(x_p(t))-\alpha_p(\tilde{x}_p(t))\Big)\Big[\sum_{q=1}^{n}a_{pq}\,f_q(\tilde{x}_q(t))\\ &\quad+\sum_{q=1}^{n}b_{pq}\,g_q(\int_{-\tau}^{0}\tilde{x}_q(t+\theta)\,d\omega_q(\theta))+I_p\Big].\end{aligned}\tag{3.11}$$

Similar to (3.6), we can deduce that

$$\begin{aligned}\mathcal{D}_t^{\alpha}|e_p(t)| &\le sgn(e_p(t))\mathcal{D}_t^{\alpha}e_p(t)\\ &= sgn(e_p(t))\Big\{-u_p'(\xi_p)e_p(t)+\alpha_p(x_p(t))\Big[\sum_{q=1}^{n}a_{pq}\Big(f_q(x_q(t))-f_q(\tilde{x}_q(t))\Big)\\ &\quad+\sum_{q=1}^{n}b_{pq}\Big(g_q(\int_{-\tau}^{0}x_q(t+\theta)\,d\omega_q(\theta))-g_q(\int_{-\tau}^{0}\tilde{x}_q(t+\theta)\,d\omega_q(\theta))\Big)\Big]\\ &\quad+\alpha_p'(\eta_p)e_p(t)\Big[\sum_{q=1}^{n}a_{pq}\,f_q(\tilde{x}_q(t))+\sum_{q=1}^{n}b_{pq}\,g_q(\int_{-\tau}^{0}\tilde{x}_q(t+\theta)\,d\omega_q(\theta))+I_p\Big]\Big\}\\ &\le -\underline{u}_p^*|e_p(t)|+\bar{\alpha}_p\Big[\sum_{q=1}^{n}|a_{pq}|F_q|e_q(t)|+\sum_{q=1}^{n}|b_{pq}|G_q\int_{-\tau}^{0}\sup_{-\tau\le s\le t}|x_q(s)-\tilde{x}_q(s)|d\omega_q(\theta)\Big]\\ &\quad+\alpha_p^*|e_p(t)|(\sum_{q=1}^{n}|a_{pq}|\bar{f}_q+\sum_{q=1}^{n}|b_{pq}|\bar{g}_q+|I_p|)\\ &= -\Big[\underline{u}_p^*-\alpha_p^*(\sum_{q=1}^{n}|a_{pq}|\,\bar{f}_q+\sum_{q=1}^{n}|b_{pq}|\,\bar{g}_q+|I_p|)-\sum_{q=1}^{n}\bar{\alpha}_q|a_{qp}|F_p\Big]|e_p(t)|\\ &\quad+\bar{\alpha}_p\sum_{q=1}^{n}|b_{pq}|G_qk_q\sup_{-\tau\le s\le t}|x_q(s)-\tilde{x}_q(s)|.\end{aligned}\tag{3.12}$$

From (3.12), it implies that

$$\begin{aligned}\sum_{p=1}^{n}|e_p(t)| &\le \sum_{p=1}^{n}\Big\{-\Big[\underline{u}_p^*-\alpha_p^*(\sum_{q=1}^{n}|a_{pq}|\,\bar{f}_q+\sum_{q=1}^{n}|b_{pq}|\,\bar{g}_q+|I_p|)-\sum_{q=1}^{n}\bar{\alpha}_q|a_{qp}|F_p\Big]\mathcal{I}_t^{\alpha}|e_p(t)|\\ &\quad+\mathcal{I}_t^{\alpha}\Big(\sum_{q=1}^{n}\bar{\alpha}_q|b_{qp}|G_pk_p[\sup_{-\tau\le s\le 0}|x_p(s)-\tilde{x}_p(s)|+\sup_{0\le s\le t}|x_p(s)-\tilde{x}_p(s)|]\Big)\Big\}.\end{aligned}\tag{3.13}$$

From (3.13), it follows that

$$\begin{aligned}&\sup_{0\le t\le T}\sum_{p=1}^{n}|e_p(t)|\\ \le\ & -\min_{1\le p\le n}\Big\{\underline{u}_p^*-\alpha_p^*(\sum_{q=1}^{n}|a_{pq}|\,\bar{f}_q+\sum_{q=1}^{n}|b_{pq}|\,\bar{g}_q+|I_p|)-\sum_{q=1}^{n}\bar{\alpha}_q|a_{qp}|F_p\Big\}\mathcal{I}_t^{\alpha}\Big(\sup_{0\le t\le T}\sum_{p=1}^{n}|e_p(t)|\Big)\\ &+\max_{1\le p\le n}(\sum_{q=1}^{n}\bar{\alpha}_q|b_{qp}|G_pk_p)\|\varphi-\tilde{\varphi}\|\tfrac{T^{\alpha}}{\Gamma(\alpha+1)}+\max_{1\le p\le n}(\sum_{q=1}^{n}\bar{\alpha}_q|b_{qp}|G_pk_p)\mathcal{I}_t^{\alpha}\Big(\sup_{0\le t\le T}\sum_{p=1}^{n}|e_p(t)|\Big)\\ \le\ & -\Big[\min_{1\le p\le n}\Big\{\underline{u}_p^*-\alpha_p^*(\sum_{q=1}^{n}|a_{pq}|\,\bar{f}_q+\sum_{q=1}^{n}|b_{pq}|\,\bar{g}_q+|I_p|)-\sum_{q=1}^{n}\bar{\alpha}_q|a_{qp}|F_p\Big\}\\ &-\max_{1\le p\le n}(\sum_{q=1}^{n}\bar{\alpha}_q|b_{qp}|G_pk_p)\Big]\cdot\mathcal{I}_t^{\alpha}\Big(\sup_{0\le t\le T}\sum_{p=1}^{n}|e_p(t)|\Big)+\max_{1\le p\le n}(\sum_{q=1}^{n}\bar{\alpha}_q|b_{qp}|G_pk_p)\|\varphi-\tilde{\varphi}\|\tfrac{T^{\alpha}}{\Gamma(\alpha+1)}\\ =\ & -\eta\mathcal{I}_t^{\alpha}\Big(\sup_{0\le t\le T}\sum_{p=1}^{n}|e_p(t)|\Big)+M(\varphi-\tilde{\varphi}),\end{aligned}\tag{3.14}$$

where

$$\eta = \min_{1\le p\le n}\Big\{\underline{u}_p^* - \alpha_p^*(\sum_{q=1}^{n}|a_{pq}|\,\bar{f}_q + \sum_{q=1}^{n}|b_{pq}|\,\bar{g}_q + |I_p|) - \sum_{q=1}^{n}\bar{\alpha}_q|a_{qp}|F_p\Big\} - \max_{1\le p\le n}(\sum_{q=1}^{n}\bar{\alpha}_q|b_{qp}|G_pk_p),$$

$$M(\varphi-\tilde{\varphi}) = \max_{1\le p\le n}(\sum_{q=1}^{n}\bar{\alpha}_q|b_{qp}|G_pk_p)\|\varphi-\tilde{\varphi}\|\frac{T^\alpha}{\Gamma(\alpha+1)}.$$

By applying Lemma 2.4 to (3.14), we obtain

$$\|x-\bar{x}\| \le M(\varphi-\bar{\varphi})E_\alpha(-\eta t^\alpha).$$

Thus, by Definition 2.6, the system (1.1) is GMLS. □

Remark 3.1. *If we set $\alpha_p(x_p(t)) = 1, h_p(x_p(t)) = \beta_p x_p(t)$, then system* (1.1) *reduces to the following class of FINNs with SD:*

$$\mathcal{D}_t^\alpha x_p(t) = -\beta_p x_p(t) + \sum_{q=1}^{n} a_{pq}f_q(x_q(t)) + \sum_{q=1}^{n} b_{pq}g_q(\int_{-\tau}^{0} x_q(t+\theta)d\omega_q(\theta)) + I_p.$$

4. Numerical examples

We consider a class of FICGNNs with SD as follows:

$$\begin{cases} \mathcal{D}_t^\alpha x_1(t) = -\alpha_1(x_1(t))\Big[h_1(x_1(t)) - \sum_{q=1}^{2} a_{1q}f_q(x_q(t)) - \sum_{q=1}^{2} b_{1q}g_q(\int_{-\tau}^{0} x_q(t+\theta)d\omega_q(\theta)) - I_1\Big], \\ \mathcal{D}_t^\alpha x_2(t) = -\alpha_2(x_2(t))\Big[h_2(x_2(t)) - \sum_{q=1}^{2} a_{2q}f_q(x_q(t)) - \sum_{q=1}^{2} b_{2q}g_q(\int_{-\tau}^{0} x_q(t+\theta)d\omega_q(\theta)) - I_2\Big]. \end{cases} \tag{4.1}$$

Let

$$\alpha_1(x_1(t)) = 1 + \frac{0.5}{1+x_1^2},\ \alpha_2(x_2(t)) = 0.8 - \frac{0.5}{1+x_2^2},\ h_1(x_1(t)) = 1.6x_1,\ h_2(x_2(t)) = 2.5x_2,$$
$$f_1(x_1(t)) = 4sin(x_1(t)),\ f_1(x_2(t)) = 4sin(x_2(t)),\ g_1(x_1(t)) = 4x_1(t),\ g_2(x_2(t)) = 4x_2(t),$$
$$\omega_1(\theta) = \theta,\ \omega_2(\theta) = \theta,\ \alpha = 0.8,\ t = 0.5,\ \tau = 1,\ \varepsilon = 0.02,\ \delta = 0.005,\ k_1 = k_2 = 1,$$

then, it follows that $g_q(\int_{-\tau}^{0} x_q(t+\theta)d\omega_q(\theta)) = 4x_q(t-\tau),\ h_1'(x_1) = 1.6,\ h_2'(x_2) = 2.5,$

$$1 \le \alpha_1(x_1) \le 1.5,\ 0.3 \le \alpha_2(x_2) \le 0.8,\ |\alpha_1'(x_1)| = |\alpha_2'(x_2)| \le \frac{3\sqrt{3}}{16},$$
$$1.5 \le u_1'(x_1) = (\alpha_1(x_1)h_1(x_1))' \le 2.4,\ 0.75 \le u_2'(x_2) = (\alpha_2(x_2)h_2(x_2))' \le 2.15625,$$
$$\left|f_q(\xi_1) - f_q(\xi_2)\right| \le 4|\xi_1-\xi_2|,\ \left|f_q(\cdot)\right| \le 4,\ \left|g_q(\xi_1) - g_q(\xi_2)\right| \le 4|\xi_1-\xi_2|,\ \left|g_q(\cdot)\right| \le 4, q = 1,2.$$

Therefore, since $H_1 - H_4$ hold, we can get

$$\underline{\alpha}_1 = 1,\ \bar{\alpha}_1 = 1.5,\ \underline{\alpha}_2 = 0.3,\ \bar{\alpha}_2 = 0.8,\ \alpha_1{}^* = \alpha_2{}^* = \frac{3\sqrt{3}}{16},\ \underline{u}_1^* = 1.5,\ \underline{u}_2^* = 0.75,$$
$$F_1 = F_2 = 4,\ \bar{f}_1 = \bar{f}_2 = 4,\ G_1 = G_2 = 4,\ \bar{g}_1 = \bar{g}_2 = 4.$$

Example 4.1. *Let the parameters in system* (4.1) *be*

$$A = \begin{pmatrix} 0.03125 & 0.0625 \\ 0.0625 & 0.125 \end{pmatrix}, \quad B = \begin{pmatrix} 0.0625 & 0.125 \\ 0.03125 & 0.0625 \end{pmatrix}, \quad I = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}.$$

By calculation, we obtain

$$\begin{aligned}
&\sum_{p=1}^{2} \frac{1}{\underline{h}_p^*} \max_{1\le q\le 2}\{|a_{pq}|F_q + |b_{pq}|G_q k_q\} = 0.625\max\{0.375, 0.75\} + 0.4\max\{0.375, 0.75\} = 0.76875 < 1,\\
&\delta(1 + \sum_{p=1}^{2} \bar{\alpha}_p \max_{1\le q\le n}\{|b_{pq}|G_q k_q\}\frac{t^\alpha}{\Gamma(\alpha+1)})e^{\frac{Vt^\alpha}{\Gamma(\alpha+1)}} = \delta(1 + 0.575 \times \frac{0.5^{0.8}}{\Gamma(1.8)})e^{\frac{1.50273\times 0.5^{0.8}}{\Gamma(1.8)}} < 0.02,\\
&V = \max_{1\le p\le 2}\Big\{ -\underline{u}_p^* + \sum_{q=1}^{2} \bar{\alpha}_q(|a_{qp}|F_p + |b_{qp}|G_p\, k_p) + \alpha_p^*[\sum_{q=1}^{2}(|a_{pq}|\bar{f}_q + |b_{pq}|\bar{g}_q) + |I_p|]\Big\}\\
&\quad = \max\{-0.109766, 1.50273\} > 0,
\end{aligned}$$

which satisfies the condition of Theorem 3.1. Thus, system (4.1) possesses a unique equilibrium point that is FTS.

Figure 1 illustrates the dynamical behavior of Example 4.1. Specifically, subplots (a) and (b) demonstrate that the state trajectories rapidly converge to the unique equilibrium point within the theoretically predicted time bound, directly verifying the FTS criteria; while (c) and (d) display the corresponding phase-plane portrait and three-dimensional trajectories, respectively. The numerical simulations are in full agreement with the conclusions of Theorem 3.1, thereby confirming the validity of the theoretical analysis.

Example 4.2. *To provide a clearer demonstration of the GMLS of the system* (4.1)*, setting the parameters in system* (4.1) *gives*

$$A = \begin{pmatrix} 0.0078125 & 0.015625 \\ 0.015625 & 0.03125 \end{pmatrix}, \quad B = \begin{pmatrix} 0.015625 & 0.03125 \\ 0.0078125 & 0.015625 \end{pmatrix}, \quad I = \begin{pmatrix} 0.125 \\ 0.125 \end{pmatrix}.$$

By calculation, we obtain

$$\begin{aligned}
\eta &= \min_{1\le p\le 2}\Big\{\underline{u}_p^* - \alpha_p^*(\sum_{q=1}^{2}(|a_{pq}|\,\bar{f}_q + |b_{pq}|\,\bar{g}_q) + |I_p|) - \sum_{q=1}^{2}\bar{\alpha}_q|a_{qp}|F_p\Big\} - \max_{1\le p\le n}(\sum_{q=1}^{n}\bar{\alpha}_q|b_{qp}|G_p k_p)\\
&= \min\{1.2713, 0.47435\} - \max\{0.11875, 0.2375\} = 0.47435 - 0.2375 = 0.23685 > 0,
\end{aligned}$$

which satisfies the condition of Theorem 3.2. Thus, the system (4.1) is GMLS.

The simulation results for Example 4.2 are presented in Figure 2. Specifically, subplot (a) depicts the trajectory of $x_1(t)$, subplot (b) shows $x_2(t)$, subplot (c) displays the phase-plane behavior, and subplot (d) provides a three-dimensional view of the trajectories. These results collectively demonstrate consistency with Theorem 3.2, thus verifying the theoretical derivation.

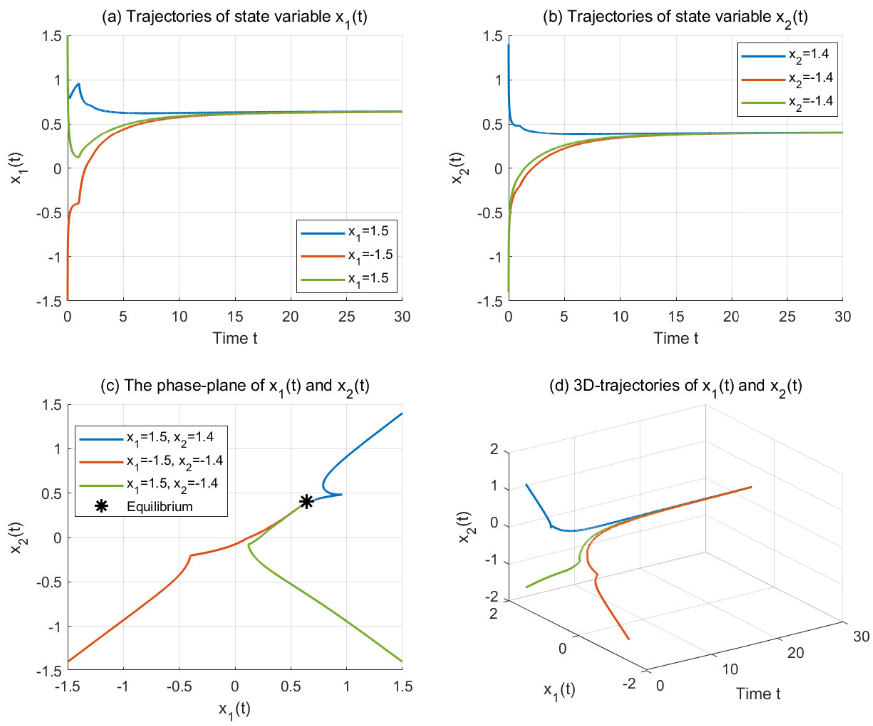

Figure 1. The relations of $x_1(t)$ and $x_2(t)$.

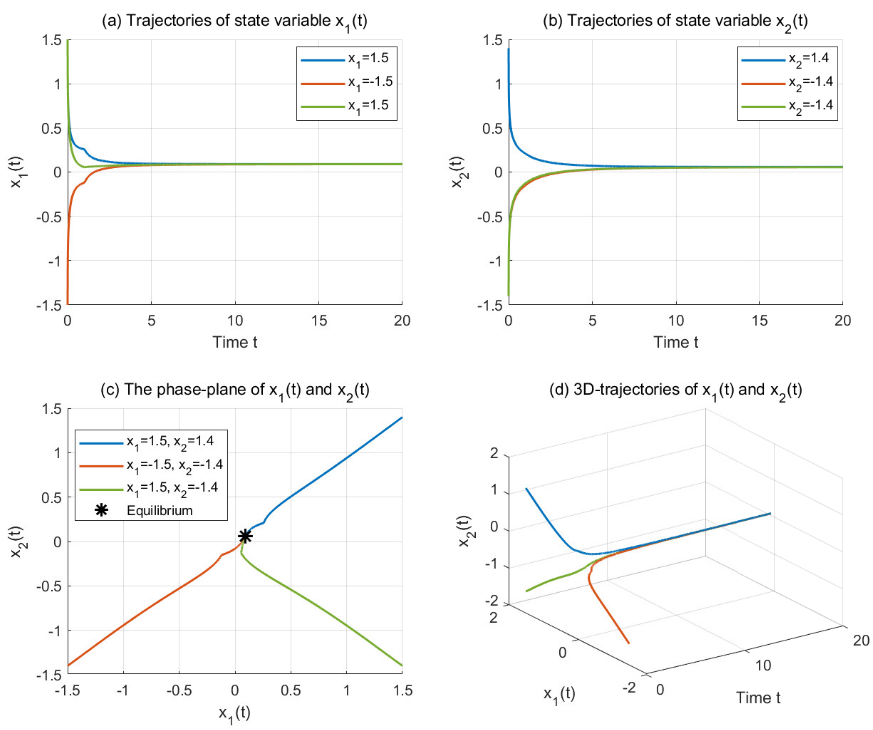

Figure 2. The relations of $x_1(t)$ and $x_2(t)$.

5. Conclusions

This paper mainly investigates the stability of a class of FICGNNs with SD. By transforming the fractional-order integral of the state variable with SD into a general fractional-order integral without such delays based on the integral interval partitioning method, stability criteria are derived for the FTS and GMLS of the NNs under certain assumptions, namely Theorems 3.1 and 3.2.

Currently, research on FINNs mainly focuses on two categories: the first involves networks whose two distinct fractional derivatives share a commensurate (proportional) relationship. For example, studies such as [5–7, 30] have examined the GAS, FTS, MLS, and APS stability of such NNs. The second category deals with NNs whose two fractional derivatives are incommensurate (non-proportional), as seen in [31, 32].

Following the analytical approach and methodology developed in this work, future research may extend to other network types, such as investigating the FTS, the MLS of solutions for incommensurate FIHNNs with SD, and the stability of periodic solutions for such NNs.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions

Wei Liu: Conceptualization, methodology, writing – original draft, writing – review & editing; Danning Xu: Methodology, formal analysis; Qiuyang Zhang: Software; Huawen Liu: Visualization; Tao Jiang and Hui Pan: Supervision, project administration. All authors have read and approved the final version of the manuscript for publication.

Acknowledgments

The authors acknowledge The Science Project of Zhejiang Educational Department (Y202454310).

Conflict of interest

No potential conflict of interest is reported by the authors.

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