Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 20899–20914.
DOI:10.3934/math.2026848
Received: 07 April 2026
Revised: 30 June 2026
Accepted: 08 July 2026
Published: 15 July 2026

Research article

Second order neutral delay differential equations: on oscillation via linearization

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Abstract: This paper investigates the oscillatory behavior of second-order neutral differential equations with a continuously distributed delay. New monotonic characteristics of non-oscillatory solutions of the considered neutral delay differential equation are established. We obtain new oscillation criteria by using these properties to linearize the problem and applying the comparison principle. The corresponding proofs are established under the canonical assumption. The results in this paper improve and generalize some known results in the existing works. Finally, numerical validation via illustrative examples are provided.

Keywords: second-order neutral delay differential equations; asymptotic properties; linearization; oscillation theory
Mathematics Subject Classification: 34K40, 34K25, 34K06, 34K11

1. Introduction

The purpose of this paper is to study the asymptotic and oscillatory of solutions to a class of second-order neutral delay differential equations (NDDEs) with continuously distributed delay:

$$\left(r\left(l\right)\left[\left(X\left(l\right)+\int_a^b p(l,\mu)X(\tau(l,\mu))d\mu\right)'\right]^{\alpha}\right)'+\sum_{j=1}^{n}q_j(l)F(X(\sigma_j(l)))=0,\quad l\geq l_0. \tag{1.1}$$

For notational convenience, we set $Y\left(l\right)=X\left(l\right)+\int_a^b p(l,\mu)X(\tau(l,\mu))d\mu$.

Throughout this study, we impose the following assumptions:

(H1) $r\left(l\right)\in C^1\left(\left[t_0,\infty\right),\left(0,\infty\right)\right)$, α is a quotient of two positive odd integers;

(H2) $p(l,\mu) \in C([l_0,\infty)\times[a,b],[0,\infty))$, $0 \le p(l) \equiv \int_a^b p(l,\mu)d\mu \le P < 1$, and $q_j(l) \in C([l_0,\infty),(0.\infty))$ for j=1, 2, ..., n;

(H3) $\tau(l,\mu) \in C([l_0,\infty)\times[a,b],\mathbb{R})$ and $\tau(l,\mu) \le l$;

(H4) $\sigma_j(l) \in C([l_0,\infty),\mathbb{R})$, $\sigma_j(l) \le l$, $\lim_{l\to\infty} \sigma_j(l) = \infty$ for j=1, 2, ..., n and $\min(\sigma_j(l)) = \sigma_*$;

(H5) $F \in C(\mathbb{R},(0,\infty))$ and $F(X)/X^\alpha \ge k$ for $X \ne 0$, where k is a positive constant.

Furthermore, it is assumed that

$$R(l) = \int_{l_0}^{l} \frac{ds}{r^{1/\alpha}(s)} \to \infty \quad \text{as} \quad l \to \infty, \tag{1.2}$$

which is known as the Canonical condition.

A solution of Eq (1.1) is defined as a function $X(l) \in C^1([L_0,\infty))$ with $L_0 \ge l_0$, which satisfies $r(l)(Y'(l))^\alpha \in C^1([L_0,\infty))$ and Eq (1.1) on $[L_0,\infty)$. We only consider those solutions $X(l)$ of Eq (1.1) that satisfy $\sup\{|X(l)| : l \ge L\} > 0$ for every $L \ge L_0$, and assume that such solutions exist. A solution of Eq (1.1) is said to be oscillatory if it has arbitrarily large zeros on $[L_0,\infty)$; otherwise, the solution is non-oscillatory. The equation itself is called oscillatory if all of its solutions are oscillatory.

Second-order delay differential equations(DDEs) have theoretical significance and practical applications, and have been extensively investigated. NDDEs constitute a class of DDEs in which the highest derivative of the solution appears both with and without delay. A qualitative analysis of neutral differential equations is a considerable practical applications alongside its theoretical importance. These equations arise in various situations such as including the vibration of masses attached to elastic bars, variational problems with time delays, and electrical networks that contain lossless transmission, such as in high speed computers where such lines interconnect switching circuits [1]. Among the prominent contributions to this field are the books by Agarwal, Grace, and O'Regan [2] and by Dosly and Rehak [3], which are devoted to the oscillation theory of linear and half-linear equations. Erbe, Kong, and Zhang [4] provided a comprehensive treatment of the oscillatory behavior of functional differential equations and established many fundamental results in this area. Kiguradze and Chanturia [5] investigated the asymptotic properties of solutions of nonautonomous ordinary differential equations. Furthermore, Ladde, Lakshmikantham, and Zhang [6] and Gyori and Ladas [7] developed the oscillation theory of differential equations with deviating arguments and delay differential equations, respectively.

Additionally, although the oscillation theory for NDDEs has been extensively developed, most of the existing results address either equations with constant coefficients or single-delay cases. In particular, Philos and Purnaras [8] studied the oscillatory behavior of higher order neutral differential equations, while Frasson and Tacuri [9] investigated the asymptotic behavior of linear NDDEs with periodic coefficients. In his seminal book, Farkas [10] provided a comprehensive treatment of periodic motions and systems with delays, primarily focusing on retarded-type equations and ordinary differential systems.

A well-known oscillation criterion for second-order DDEs, proposed by Koplatadze et al. [11], is based on the monotonic properties of positive solutions. In particular, it relies on the increasing behavior of $Y(t)$ and the decreasing behavior of $Y(t)/t$ for positive solutions of $Y^{(n)}(t)+c(t)Y(\tau(t)) = 0$. These properties have been widely used in the literature to analyze oscillatory behavior in various classes of DDEs.

In 2019, Dzurina et al. [12], investigated the oscillatory behavior of second-order differential equations with several sub-linear neutral terms of the following form:

$$(a(t)(z'(t))' + q(t)x^{\beta}(\sigma(t)) = 0, \quad t \geq t_0 > 0,$$

where $m > 0$ is an integer, and

$$z(t) = x(t) + \sum_{i=1}^{m} p_i(l)x^{\alpha_i}(\tau_i(t)).$$

In the same year, Baculikova [13] studied the second-order nonlinear noncanonical differential equation

$$(a(t)(y'(t))' + p(t)f(y(\tau(t))) = 0, \quad t \geq t_0 > 0,$$

under the condition

$$\int^{\infty} \frac{1}{r(t)} dt < \infty.$$

Contrary to most existing results in the literature, the oscillation criteria for the studied equation were obtained using only one condition. Baculikova considered both delay and advanced differential equations, and a particular Euler-type equation was provided to illustrate the significance of the main results.

Baculikova and Dzurina [14] explored properties of a class of second-order DDEs by examining those of associated first-order delay equations. In [15, 16], a class of second-order nonlinear perturbed differential equations and their special cases were studied. Kusano and Naito [17] considered the functional differential equations with deviating arguments. Zhang et al. [18] were concerned with the oscillation of solutions to the following second-order neutral differential equation:

$$(r(t)\,|z'(t)|^{\alpha-1}\,|z'(t)|)' + q(t)\,|x(\sigma(t))|^{\alpha-1}\,x(\sigma(t)) = 0, \quad t \geq t_0 > 0,$$

where $m > 0$ is an integer, and

$$z(t) = x(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t)).$$

In 2019, Şenel et al. [19] investigated the existence of non-oscillatory solutions of second-order nonlinear neutral differential equations with distributed deviating arguments of the following form:

$$\left(r(t)\left(x(t) - \int_a^b p(t,\xi)x(t-\xi)\,d\xi\right)'\right)' + \int_{a_1}^{b_1} f_1(t, x(\sigma_1(t,\xi)))\,d\xi - \int_{a_2}^{b_2} f_2(t, x(\sigma_2(t,\xi)))\,d\xi = g(t).$$

The authors in [20] considered the oscillation properties of solutions to the following third-order neutral differential equations with several deviating arguments:

$$\left(r(\ell)(z''(\ell))^{\alpha}\right)' + \sum_{i=1}^{n} q_i(\ell)\,x^{\alpha}(\phi_i(\ell)) = 0, \quad \ell \geq \ell_0 > 0,$$

where

$$z(\ell) = x(\ell) + p(\ell)x(\varrho(\ell)),$$

and α is a quotient of odd positive integers.

Moreover, recent contributions to the oscillation theory can be found in the following works discussed. In [21], Temtek and Turarov studied the asymptotic and oscillatory behavior of solutions for second-order NDDEs:

$$(a(x)(v'(x))^m)' + C(x, \phi(x)) = 0,$$

where $v(x) = u(x) + b(x)u(\psi(x))$. In [22], Temtek considered the following differential equation:

$$\left(r(t)\left[\left(u(t) + \sum_{i=1}^{n} b_i(t)u(\psi_i(t))\right)'\right]^{\alpha}\right)' + Q(t, u(\tau(t))) = 0.$$

In [23], Moaaz and Al-Jaser established novel criteria to evaluate the oscillatory behavior of nonlinear second-order NDDEs with several delays. Specifically, they considered the following nonlinear NDDE:

$$\left(r(v)\psi(u(v))(Z'(v))^{\alpha}\right)' + \sum_{j=1}^{k} q_j(v)\, F(u(\sigma_j(v))) = 0, \quad v \geq v_0,$$

where $v \geq v_0$, and $Z = u + p(u \circ \tau)$.

Hua et al. [24] investigated the oscillatory behavior of solutions for a class of second-order neutral differential equations with multiple delays. By employing rigorous analytical methods, they established sufficient oscillation criteria that guarantee solutions exhibit oscillatory characteristics under prescribed conditions. The main results they obtained extended the existing theory by incorporating multiple delay terms and nonlinear effects, thus broadening the applicability of the oscillation theory to more complex neutral delay differential equations.

In 2024, El-Gaber et al. [25] established new oscillation criteria for the second-order damped neutral differential equation.

In 2025, Bazighifan et al. [26] investigated the oscillation criteria of nonlinear second-order neutral differential equations with multiple delays, thereby focusing on their noncanonical forms. By leveraging an innovative iterative technique, they established relationships that enhance the monotonic properties of positive solutions. In [27], the authors investigated the existence of non-oscillatory solutions for a class of higher-order nonlinear mixed neutral delay differential equations.

The aim of this paper is to develop new comparison theorems and oscillation conditions for second-order NDDEs by utilizing the monotonicity of non-oscillatory solutions. Baculikova and Dzurina [28] studied half-linear second-order DDEs of the following form:

$$(r(t)(Y'(t))^{\alpha})' + p(t)Y^{\alpha}(\tau(t)) = 0.$$

They established monotonic properties of non-oscillatory solutions and employed these properties to linearize the considered equation, thus leading to oscillation criteria. The equation studied here is more general than those considered in their paper. Motivated by earlier results [28], we establish new monotonic properties that enable the linearization of Eq (1.1). This linearization allows the oscillatory properties of the linearized equation to be transferred to the original nonlinear NDDEs.

The first objective of this paper is to establish novel comparison theorems associated with Eq (1.1). In what follows, we establish new oscillation criteria based on the linearized forms of the equation.

Finally, we illustrate the applicability and effectiveness of the derived criteria by means of a concrete example.

The results obtained in this paper constitute a significant advancement of the existing oscillation theory and contribute to the development of a novel analytical framework for the investigation of second-order NDDEs.

2. Auxiliary results

We begin by presenting several useful lemmas concerning the monotonic properties of non-oscillatory solutions for the differential equations considered in this paper.

Lemma 1. *Let $X(l)$ be a positive solution of Eq (1.1). Then,*
(a) $r(l)(Y'(l))^{\alpha} > 0$;
(b) $\frac{Y(l)}{R(l)}$ is decreasing for $l \geq l_1 \geq l_0$. Moreover if

$$\int_{l_1}^{\infty} \sum_{j=1}^{n} q_j(s) R^{\alpha}(\sigma_i(s)) \left(1 - \int_a^b p(\sigma_j(s), \mu) d\mu\right)^{\alpha} ds = \infty \tag{2.1}$$

holds, then
(c) $\lim_{l\to\infty} \frac{Y(l)}{R(l)} = 0$.

Proof. Assume that $X(l)$ is a positive solution of Eq (1.1). Then

$$(r(l)(Y'(l))^{\alpha})' < 0,$$

and there exists $l_1 \geq l_0$ such that $r(l)(Y'(l))^{\alpha}$ has a constant sign for $l \geq l_1$.

(a) On the contrary, assume that $r(l)(Y'(l))^{\alpha} < 0$. Hence, there is a constant $L_1 > 0$ for which $r(l)(Y'(l))^{\alpha} \leq -L_1 < 0$. By integrating the preceding inequality from l_1 to l and using the canonical condition (1.2), it follows that

$$Y(l) \leq Y(l_1) - L_1^{1/\alpha} R(l) \to -\infty \quad as \quad l \to \infty.$$

Thus, assuming $r(l)(Y'(l))^{\alpha} < 0$ results in a contradiction, and it follows that $r(l)(Y'(l))^{\alpha} > 0$.

(b) Using the monotonicity of $r^{1/\alpha}(l) Y'(l)$, we obtain

$$Y(l) \geq \int_{l_1}^{l} \frac{r^{1/\alpha}(s) Y'(s)}{r^{1/\alpha}(s)} ds \geq r^{1/\alpha}(l) Y'(l) R(l), \tag{2.2}$$

which implies $\left(\frac{Y(l)}{R(l)}\right)' < 0$.

On the other hand, as $\frac{Y(l)}{R(l)}$ is a positive, decreasing function, there exists a constant δ such that

$$\lim_{l\to\infty} \frac{Y(l)}{R(l)} = \delta \geq 0.$$

(c) Assume, for the sake of contradiction, that $\delta > 0$. Then, $\frac{Y(l)}{R(l)} \geq \delta$ for $l \geq l_1$. From the definition of $Y(l)$ and (H3), we get

$$X(l) = Y(l) - \int_a^b p(l,\mu) X(\tau(l,\mu)) d\mu \geq Y(l) - \int_a^b p(l,\mu) Y(l) d\mu \geq Y(l)\left(1 - \int_a^b p(l,\mu) d\mu\right). \tag{2.3}$$

By integrating Eq (1.1) from l_1 to l, we obtain

$$r(l_1)(Y'(l_1))^{\alpha} \geq k\delta^{\alpha} \int_{l_1}^{l} \sum_{j=1}^{n} q_j(s) R^{\alpha}\left(\sigma_j(s)\right)\left(1 - \int_a^b p(\sigma_j(s), \mu) d\mu\right)^{\alpha} ds,$$

which for $l \to \infty$ contradicts with (2.1). Therefore, $\lim\limits_{l\to\infty} \frac{Y(l)}{R(l)} = 0$. The proof is complete. □

As $R(l)$ is increasing, there exists a constant $\lambda \geq 1$ that satisfies the following:

$$\frac{R(l)}{R(\sigma_*(l))} \geq \lambda. \tag{2.4}$$

Theorem 1. *Suppose (2.1) holds and there exists some $\beta > 1$ for which*

$$\frac{k}{\alpha} \sum_{j=1}^{n} q_j(l) R^{\alpha}\left(\sigma_j(l)\right) R(l) r^{1/\alpha}(l) \left(1 - \int_a^b p(\sigma_*(l), \mu) d\mu\right)^{\alpha} > \beta, \quad l \geq l_0. \tag{2.5}$$

If $X(l)$ is a positive solution of Eq (1.1), then

(1) $\frac{Y(l)}{R^{1-\beta}(l)}$ is decreasing for $l \geq l_1$,
(2) $\frac{Y(l)}{R^{\beta_0}(l)}$ is increasing for $l \geq l_1$, where $\beta_0 = \beta^{1/\alpha}\lambda^{\beta}$.

Proof. Suppose that $X(l)$ is a positive solution of Eq (1.1). Condition (c) of Lemma 1 implies the following:

$$\lim_{l\to\infty} r^{1/\alpha}(l) Y'(l) = 0. \tag{2.6}$$

Therefore, an integration of Eq (1.1) yields the following:

$$r^{1/\alpha}(l) Y'(l) = \left(\int_l^{\infty} \sum_{j=1}^{n} q_j(s) F(X(\sigma_j(s))) ds\right)^{1/\alpha}. \tag{2.7}$$

It is easy to see that

$$\left[\left(r^{1/\alpha}(l) Y'(l)\right)^{\alpha}\right]' = \alpha \left(r^{1/\alpha}(l) Y'(l)\right)^{\alpha-1} \left(r^{1/\alpha}(l) Y'(l)\right)'. \tag{2.8}$$

Setting into Eq (1.1), we have either

$$\alpha \left(r^{1/\alpha}(l) Y'(l)\right)^{\alpha-1} \left(r^{1/\alpha}(l) Y'(l)\right)' + \sum_{j=1}^{n} q_j(l) F(X(\sigma_j(l))) = 0,$$

or

$$\left(r^{1/\alpha}(l) Y'(l)\right)' + \frac{1}{\alpha} \left(r^{1/\alpha}(l) Y'(l)\right)^{1-\alpha} \sum_{j=1}^{n} q_j(l) F(X(\sigma_j(l))) = 0. \tag{2.9}$$

Let $Z(l) = r^{1/\alpha}(l)\, Y'(l)$ in (2.9). Then, $Z(l) = r^{1/\alpha}(l)\, Y'(l)$ is positive decreasing and satisfies

$$Z'(l) + \frac{1}{\alpha} Z^{1-\alpha}(l) \sum_{j=1}^{n} q_j(l) F(X(\sigma_j(l))) = 0;$$

from (H5), we get

$$Z'(l) + \frac{k}{\alpha} Z^{1-\alpha}(l) \sum_{j=1}^{n} q_j(l) X^{\alpha}(\sigma_j(l)) \le 0. \tag{2.10}$$

On the other hand, (2.2) implies the following:

$$Y(l) \ge r^{1/\alpha}(l)\, Y'(l)\, R(l) \ge Z(l)\, R(l).$$

Then, by (2.3), it follows that

$$X(l) \ge Z(l)\, R(l) \left(1 - \int_a^b p(l,\mu)\, d\mu\right),$$

and take $\sigma_i(l)$ instead of l

$$X(\sigma_i(l)) \ge Z(\sigma_i(l)) R(\sigma_i(l)) \left(1 - \int_a^b p(\sigma_i(l),\mu) d\mu\right).$$

Since $\sigma_i(l) \le l$ and $Z(l)$ is positive decreasing,

$$X(\sigma_i(l)) \ge Z(l)\, R(\sigma_i(l)) \left(1 - \int_a^b p(\sigma_*(l),\mu) d\mu\right). \tag{2.11}$$

By substituting the last inequality into (2.10), we get

$$Z'(l) + \frac{k}{\alpha} \sum_{j=1}^{n} q_j(l) R^{\alpha}\left(\sigma_j(l)\right) \left(1 - \int_a^b p(\sigma_*(l),\mu) d\mu\right)^{\alpha} Z(l) \le 0,$$

and

$$Z'(l) + \frac{\beta}{R(l)\, r^{1/\alpha}(l)} Z(l) \le 0,$$

which implies

$$-Z'(l)\, R(l) \ge \frac{\beta}{r^{1/\alpha}(l)} Z(l) = \beta Y'(l).$$

The remainder of the proof is identical to that of Theorem 2.3 in [28] and is therefore omitted. □

In the previous results, no distinction was made between the cases $\alpha > 1$ and $\alpha < 1$. Nonetheless, each of these cases must be treated separately in order to derive the oscillation criteria for Eq (1.1).

3. Oscillation criteria for $\alpha > 1$

For brevity, we set the following:

$$\Gamma = \frac{k(1-\beta)^{1-\alpha}\lambda^{\beta(\alpha-1)}}{\alpha}.$$

Now, we present new comparison principles that substantially simplify the analysis of NDDEs.

Theorem 2. *Let $\alpha > 1$, and (2.1) and (2.5) hold. Then, Eq (1.1) is oscillatory if the associated equation*

$$\left(r^{1/\alpha}(l)\, Y'(l)\right)' + \Gamma R^{\alpha-1}(\sigma_*(l)) \sum_{j=1}^{n} q_j(l)\left(1 - \int_a^b p(\sigma_j(l),\mu)d\mu\right)^{\alpha} Y(\sigma_*(l)) = 0 \tag{3.1}$$

is oscillatory.

Proof. Assume that the desired result does not hold (i.e., $X(t)$ is a positive solution of Eq (1.1)). By Theorem 1(1), $\frac{Y(l)}{R^{1-\beta}(l)}$ is decreasing, we have the inequality

$$Y(l) \geq \frac{r^{1/\alpha}(l)\, Y'(l)}{(1-\beta)} R(l), \tag{3.2}$$

which for $\alpha > 1$ yields

$$Y^{1-\alpha}(l) \leq \frac{\left(r^{1/\alpha}(l)\, Y'(l)\right)^{1-\alpha}}{(1-\beta)^{1-\alpha}} R^{1-\alpha}(l).$$

Thus,

$$\left(r^{1/\alpha}(l)\, Y'(l)\right)^{1-\alpha} \geq (1-\beta)^{1-\alpha} \frac{Y^{1-\alpha}(l)}{R^{1-\alpha}(l)}. \tag{3.3}$$

According to Theorem 1(1), we obtain the following:

$$Y^{1-\alpha}(l) \geq \frac{Y^{1-\alpha}(\sigma_*(l))}{R^{(1-\alpha)(1-\beta)}(\sigma_*(l))} R^{(1-\alpha)(1-\beta)}(l). \tag{3.4}$$

By substituting (3.4) from (3.3), we have in view of (2.4) that

$$\begin{aligned}\left(r^{1/\alpha}(l)\, Y'(l)\right)^{1-\alpha} &\geq (1-\beta)^{1-\alpha} \frac{R^{(1-\alpha)(1-\beta)}(l)}{R^{(1-\alpha)(1-\beta)}(\sigma_*(l))} \frac{Y^{1-\alpha}(\sigma_*(l))}{R^{1-\alpha}(l)} \\ &\geq (1-\beta)^{1-\alpha} \lambda^{\beta(\alpha-1)} \frac{Y^{1-\alpha}(\sigma_*(l))}{R^{1-\alpha}(\sigma_*(l))}.\end{aligned} \tag{3.5}$$

By combining (2.10) and (3.5), it follows that $Y(l)$ satisfies the following linear differential inequality:

$$\left(r^{1/\alpha}(l)\, Y'(l)\right)' + \Gamma R^{\alpha-1}(\sigma_*(l)) \sum_{j=1}^{n} q_j(l)\left(1 - \int_a^b p(\sigma_j(l),\mu)d\mu\right)^{\alpha} Y(\sigma_*(l)) \leq 0. \tag{3.6}$$

In contrast, Corollary 1 in [17] guarantees that the corresponding Eq (3.1) admits a positive solution. This contradicts the earlier assumption and hence completes the proof. □

Using the results of the previous theorem, new oscillation criteria are established.

Theorem 3. *Let $\alpha > 1$, and (2.1) and (2.5) hold. If*

$$\limsup_{l\to\infty}\left\{R^{\beta-1}\left(\sigma_*\left(l\right)\right)\int_{l_1}^{\sigma_*(l)} R\left(s\right)R^{\alpha-\beta}\left(\sigma_*\left(s\right)\right)Q_0(s)ds + R^{\beta}\left(\sigma_*\left(l\right)\right)\int_{\sigma_*(l)}^{l} R^{\alpha-\beta}\left(\sigma_*\left(s\right)\right)Q_0(s)ds \right.$$
$$\left. + R^{1-\beta_0}\left(\sigma_*\left(l\right)\right)\int_{l}^{\infty} R^{\alpha+\beta_0-1}\left(\sigma_*\left(s\right)\right)Q_0(s)ds\right\} > \frac{1}{\Gamma},$$

then Eq (1.1) is oscillatory, where $Q_0(s) = \sum_{j=1}^{n} q_j(s)\left(1 - \int_a^b p(\sigma_j(s),\mu)d\mu\right)^{\alpha}$.

Proof. As the proof closely follows the approach adopted in Theorem 4.1 of [28], the complete proof is omitted. □

In many practical applications, there is often a need for criteria that are more readily verifiable. The following theorem establishes such conditions.

Theorem 4. *Let $\alpha > 1$, and (2.1) and (2.5) hold. If*

$$\liminf_{l\to\infty}\int_{\sigma_*(l)}^{l} R^{\alpha}\left(\sigma_*\left(s\right)\right)\sum_{j=1}^{n} q_j(s)\left(1 - \int_a^b p(\sigma_j(s),\mu)d\mu\right)^{\alpha} ds > \frac{1}{\Gamma e},$$

then Eq (1.1) is oscillatory.

Proof. Assume, contrary to the claim, that Eq (1.1) admits a positive solution $X(l)$. By Theorem 2, Eq (3.1) is non-oscillatory, and we can suppose that it admits an eventually positive solution $Y(l)$. Let us introduce the function $Z(l) = r^{1/\alpha}(l)Y'(l)$, which is positive and decreasing. Therefore, it follows that

$$Y\left(l\right) = Y\left(l_1\right) + \int_{l_1}^{l} Y'\left(s\right)ds \geq \int_{l_1}^{l} \frac{r^{1/\alpha}\left(s\right)Y'\left(s\right)}{r^{1/\alpha}\left(s\right)}ds \geq r^{1/\alpha}\left(l\right)Y'\left(l\right)\int_{l_1}^{l}\frac{ds}{r^{1/\alpha}\left(s\right)} = Z(l)R(l).$$

Substituting the expression for $Y(l)$ into Eq (3.1) yields the first order differential inequality

$$Z'\left(l\right) + \Gamma R^{\alpha}\left(\sigma_*\left(l\right)\right)\sum_{j=1}^{n} q_j(l)\left(1 - \int_a^b p(\sigma_j(l),\mu)d\mu\right)^{\alpha} Z\left(\sigma_*\left(l\right)\right) \leq 0, \tag{3.7}$$

which admits a positive solution $Z(l)$. In view of Theorem 2.1.1 in [6], (3.7) has no positive solution if

$$\liminf_{l\to\infty}\int_{\sigma_*(l)}^{l} \Gamma R^{\alpha}\left(\sigma_*\left(s\right)\right)\sum_{j=1}^{n} q_j(s)\left(1 - \int_a^b p(\sigma_j(s),\mu)d\mu\right)^{\alpha} ds > \frac{1}{e}.$$

This contradicts our assumption that Eq (1.1) admits a positive solution. This completes the proof. □

4. Oscillation criteria for $0 < \alpha < 1$

Now, we focus on the case $\alpha \in (0,1)$. This case requires different techniques owing to the distinct structure of the nonlinear term. The results presented below are parallel to those obtained in Section 3, while incorporating essential modifications to accommodate the altered parameter range. For simplicity, we introduce the following notation:

$$\Omega = \frac{\beta^{\frac{1-\alpha}{\alpha}}\lambda^{1-\alpha}}{\alpha\left(1-\beta_0\right)^{\frac{1-\alpha}{\alpha}}}.$$

Theorem 5. *Let* $0 < \alpha < 1$*, and (2.1), and (2.5) hold. Then, Eq (1.1) is oscillatory provided that*

$$\left(r^{1/\alpha}(l)\,Y'(l)\right)' + \Omega R^{\alpha-1}\left(\sigma_*(l)\right)\sum_{j=1}^{n} q_j(l)\left(1-\int_a^b p(\sigma_j(l),\mu)d\mu\right)^{\alpha} Y\left(\sigma_*(l)\right) = 0 \tag{4.1}$$

is oscillatory.

Proof. Let $X(l)$ be a positive solution of Eq (1.1). Taking the derivative of (2.7) with respect to l yields the following equation:

$$\left(r^{1/\alpha}(l)\,Y'(l)\right)' + \frac{1}{\alpha}\left(\int_l^\infty \sum_{j=1}^{n} q_j(s)F(X(\sigma_j(s)))ds\right)^{\frac{1-\alpha}{\alpha}} \sum_{j=1}^{n} q_j(l)F(X(\sigma_j(l))) = 0.$$

Employing the fact that $\frac{Y(l)}{R^{\beta_0}(l)}$ is an increasing function, the above inequality can be expressed as follows:

$$\left(r^{1/\alpha}(l)\,Y'(l)\right)' + \frac{k^{1/\alpha}}{\alpha}\frac{Y^{1-\alpha}\left(\sigma_*(l)\right)}{R^{(1-\alpha)\beta_0}\left(\sigma_*(l)\right)}\left(\int_l^\infty \sum_{j=1}^{n} q_j(s)R^{\alpha\beta_0}(\sigma_*(s))p_0^{\alpha}(s)ds\right)^{\frac{1-\alpha}{\alpha}} \sum_{j=1}^{n} q_j(l)\frac{Y^{\alpha}\left(\sigma_*(l)\right)}{p_0^{-\alpha}(l)} \leq 0,$$

where $p_0(l) = 1 - \int_a^b p(\sigma_*(l),\mu)d\mu$.

As a result, we deduce that $X(l)$ must satisfy the following linear differential inequality:

$$\left(r^{1/\alpha}(l)\,Y'(l)\right)' + \frac{k^{1/\alpha}}{\alpha}\left(\int_l^\infty \sum_{j=1}^{n} q_j(s)R^{\alpha\beta_0}(\sigma_*(s))p_0^{\alpha}(s)ds\right)^{\frac{1-\alpha}{\alpha}} \frac{\sum_{j=1}^{n} q_j(t)p_0^{\alpha}(l)}{R^{(1-\alpha)\beta_0}\left(\sigma_*(l)\right)}Y\left(\sigma_*(l)\right) \leq 0. \tag{4.2}$$

Additionally, by utilizing (2.4) and (2.5), we can derive the following:

$$\begin{aligned}\int_l^\infty \sum_{j=1}^{n} q_j(s)R^{\alpha\beta_0}(\sigma_*(s))p_0^{\alpha}(s)ds &\geq \frac{\alpha\beta}{k}\int_l^\infty \frac{R^{\alpha(\beta_0-1)}\left(\sigma_*(s)\right)}{r^{1/\alpha}(s)\,R(s)}ds \\ &\geq \frac{\alpha\beta}{k}\lambda^{\alpha\left(1-\beta_0\right)}\int_l^\infty \frac{R^{\alpha(\beta_0-1)-1}(s)}{r^{1/\alpha}(s)}ds \\ &= \frac{\beta\lambda^{\alpha\left(1-\beta_0\right)}}{k(1-\beta_0)}R^{\alpha(\beta_0-1)}(l)\,.\end{aligned}$$

By substituting into (4.2), we get

$$\left(r^{1/\alpha}(l)\,Y'(l)\right)' + \frac{k^{\frac{1}{\alpha}}\beta^{\frac{1-\alpha}{\alpha}}\lambda^{(1-\beta_0)(1-\alpha)}}{\alpha k^{\frac{1-\alpha}{\alpha}}(1-\beta_0)^{\frac{1-\alpha}{\alpha}}}\frac{R^{(1-\alpha)(\beta_0-1)}(l)}{R^{(1-\alpha)\beta_0}(\sigma_*(l))}\sum_{j=1}^{n} q_j(l)p_0^{\alpha}(l)\,Y(\sigma_*(l)) \le 0,$$

which, in view of (2.4), implies that $Y(l)$ is a positive solution of the differential inequality

$$\left(r^{1/\alpha}(l)\,Y'(l)\right)' + \frac{k\beta^{\frac{1-\alpha}{\alpha}}\lambda^{(1-\alpha)}}{\alpha(1-\beta_0)^{\frac{1-\alpha}{\alpha}}}R^{\alpha-1}(l)\sum_{j=1}^{n} q_j(l)p_0^{\alpha}(l)\,Y(\sigma_*(l)) \le 0.$$

Based on Corollary 1 in [17], the associated Eq (4.1) admits a positive solution, which contradicts the assumption that Eq (4.1) has no positive solutions. This completes the proof. □

Theorem 6. *Let* $0 < \alpha < 1$*, and (2.1) and (2.5) hold. If*

$$\limsup_{l\to\infty}\left\{R^{\beta-1}(\sigma_*(l))\int_{l_1}^{\sigma_*(l)} R^{\alpha}(s)\,R^{1-\beta}(\sigma_*(s))\,Q_0(s)ds + R^{\beta}(\sigma_*(l))\int_{\sigma_*(l)}^{l} R^{\alpha-1}(s)\,R^{1-\beta}(\sigma_*(s))\,Q_0(s)ds\right.$$
$$\left. + R^{1-\beta_0}(\sigma_*(l))\int_{l}^{\infty} R^{\alpha-1}(s)R^{\beta_0}(\sigma_*(s))\,Q_0(s)ds\right\} > \frac{1}{\Omega},$$

then Eq (1.1) is oscillatory, where $Q_0(s) = \sum_{j=1}^{n} q_j(s)\left(1 - \int_a^b p(\sigma_j(s),\mu)d\mu\right)^{\alpha}$.

Proof. The proof of this theorem can be made in a similar way to Theorem 4.2 in [28]. Therefore, the proof has been omitted. □

5. Examples

Example 1. *Consider the following second-order NDDE:*

$$\left(l^m\left[\left(X(l) + \int_a^b p_0 X(\tau_0 l)d\mu\right)'\right]^{\alpha}\right)' + \sum_{j=1}^{n}\frac{c_j}{l^{\alpha-m+1}}X^{\alpha}(b_j l) = 0, \quad l \ge l_0, \tag{5.1}$$

where $r(l) = l^m$*,* $p(l,\mu) = p_0$*,* $0 \le p_0 < 1$*,* $\tau(l,\mu) = \tau_0 l$*,* $\tau_0 \in (0,1)$*,* $q_j(l) = c_j/l^{\alpha-m+1}, c_j > 0$*,* $\sigma_j(l) = b_j l$*,* $b_j \in (0,1)$*,* $min(b_j l) = \sigma_0 l$*,* α *is the ratio of two positive odd integers and* $F(X) = X^{\alpha}$ *for* $k = 1$*.*

Let us consider the following special case of Eq (5.1) with $\alpha = 5 > 1$:

$$\left(l^3\left[\left(X(l) + \int_0^1 0.2X(0.4l)d\mu\right)'\right]^5\right)' + \sum_{j=1}^{2}\frac{c_j}{l^3}X^5(b_j l) = 0, \quad l \ge l_0, \tag{5.2}$$

where $m = 3$, $n = 2$, $a = 0$, $b = 1$, $p_0 = 0.2$, $\tau_0 = 0.4$, $c_1 = 2.0$, $c_2 = 1.0$, $b_1 = 0.6$, $b_2 = 0.9$, and $\sigma_0 = \min_{j=1,2}(b_1, b_2) = 0.6$. Then,

$$R(l) = \int_{l_0}^{l}\frac{ds}{s^{3/5}} \to \infty \text{ as } l \to \infty, \quad R\left(b_j l\right) = \int_{l_0}^{l}\frac{ds}{(b_j s)^{3/5}} \approx \frac{5}{2}(b_j l)^{\frac{2}{5}}.$$

Conditions (2.1) and (2.5) also hold, i.e.,

$$\int_{l_1}^{\infty} \sum_{j=1}^{2} q_j(s)\, R^{\alpha}(\sigma_j(s)) \left(1 - \int_a^b p(\sigma_j(s), \mu)\, d\mu\right)^{\alpha} ds$$

$$= \int_{l_1}^{\infty} \sum_{j=1}^{2} \frac{c_j}{s^3} \left(\frac{5}{2}\right)^5 b_j^2 s^2 (1 - p_0)^5\, ds = 32 \int_{l_1}^{\infty} \sum_{j=1}^{2} c_j b_j^2 \frac{ds}{s} = \infty,$$

and

$$\frac{k}{\alpha} \sum_{j=1}^{2} q_j(l)\, R^{\alpha}(\sigma_j(l)) R(l)\, r^{1/\alpha}(l) \left(1 - \int_a^b p(\sigma_*(l), \mu)\, d\mu\right)^{\alpha}$$

$$= \frac{1}{5} \sum_{j=1}^{2} \frac{c_j}{l^3} \left(\frac{5}{2}\right)^5 b_j^2 l^2 \left(\frac{5}{2}\right) l^{2/5} l^{3/5} (1 - p_0)^5 = \frac{1}{5} \sum_{j=1}^{2} \left(\frac{5}{2}\right)^6 c_j b_j^2 (1 - p_0)^5 \approx 24.48 \approx \beta > 1,\ \ l \geq l_0, respectively.$$

Moreover,

$$R(l) \approx \frac{5}{2} l^{2/5}, \quad R(\sigma_0 l) \approx \frac{5}{2} (0.6 l)^{2/5},$$

$$\lambda = \frac{R(l)}{R(\sigma_0 l)} = \sigma_0^{\left(\frac{3}{5} - 1\right)}, \quad \lambda = (0.6)^{-2/5} \approx 1.2266 > 1.$$

Since $\beta \approx 24.48$ and $\Gamma \approx 236$, the condition of Theorem 4 is satisfied, i.e.,

$$\liminf_{l \to \infty} \int_{0.6l}^{l} \left(\frac{5}{2} (0.6 s)^{2/5}\right)^5 \sum_{j=1}^{2} \frac{c_j}{s^3} (1 - p_0)^5\, ds > \frac{1}{\Gamma e} \implies 17.66 > 0.00156.$$

Therefore, Eq (5.2) is oscillatory by Theorem 4.

Now, we consider $0 < \alpha < 1$ in Eq (5.1). Let us assume that $\alpha = \frac{1}{3}$;

$$\left(l^{0.2} \left[\left(X(l) + \int_0^1 0.25 X(0.5 l) d\mu\right)'\right]^{\frac{1}{3}}\right)' + \sum_{j=1}^{2} \frac{c_j}{l^{1.13333}} X^{\frac{1}{3}}(b_j l) = 0, \quad l \geq l_0, \tag{5.3}$$

where $m = 0.2$, $n = 2$, $a = 0$, $b = 1$, $p_0 = 0.25$, $\tau_0 = 0.5$, $c_1 = 0.1$, $c_2 = 0.061$, $b_1 = 0.4$, $b_2 = 0.6$, and $\sigma_0 = \min_{j=1,2}(b_1, b_2) = 0.4$. Then

$$R(l) = \int_{l_0}^{l} \frac{ds}{s^{0.6}} \to \infty \quad \text{as } l \to \infty, \quad R\left(b_j l\right) = \int_{l_0}^{l} \frac{ds}{(b_j s)^{0.6}} \approx \frac{1}{0.4} (b_j l)^{0.4},$$

and conditions (2.1) and (2.5) also hold, i.e.,

$$\int_{l_1}^{\infty} \sum_{j=1}^{2} q_j(s)\, R^{\alpha}(\sigma_j(s)) \left(1 - \int_a^b p(\sigma_j(s), \mu)\, d\mu\right)^{\alpha} ds$$

$$= \int_{l_1}^{\infty} \sum_{j=1}^{2} \frac{c_j}{s^{17/15}} \left(\frac{1}{0.4}\right)^{1/3} (b_j s)^{2/15} (1 - p_0)^{1/3}\, ds = \left(\frac{1}{0.4}\right)^{1/3} (1 - 0.25)^{1/3} \int_{l_1}^{\infty} \sum_{j=1}^{2} c_j b_j^{2/15} \frac{ds}{s} = \infty,$$

and

$$\begin{aligned}
&\frac{k}{\alpha}\sum_{j=1}^{2} q_j(l)\, R^{\alpha}(\sigma_j(l))R(l)\, r^{1/\alpha}(l)\left(1-\int_a^b p(\sigma_*(l),\mu)\, d\mu\right)^{\alpha}\\
=&3\sum_{j=1}^{2}\frac{c_j}{l^{17/15}}\left(\frac{1}{0.4}\right)^{1/3}(b_j l)^{2/15}\frac{l^{0.4}}{0.4}(l^{0.2})^3(1-p_0)^{1/3} = 3\sum_{j=1}^{2} c_j b_j^{2/15}\frac{(1-0.25)^{1/3}}{(0.4)^{4/3}}\\
\approx&1.3446 \approx \beta > 1, \quad l \geq l_0, respectively.
\end{aligned}$$

Furthermore,

$$R(l) \approx \frac{1}{0.4} l^{0.4}, \quad R(\sigma_0 l) \approx \frac{1}{0.4}(0.4l)^{0.4},$$

$$\lambda = \frac{R(l)}{R(\sigma_0 l)}, \quad \lambda \approx (0.4)^{-0.4} \approx 1.443 > 1.$$

Since $\beta \approx 1.345$ and $\Omega \approx 0.777$, the condition of Theorem 6 holds (i.e., since the first side of the condition in Theorem 6 goes to infinity and $1/\Omega \approx 1.287$, the condition of Theorem 6 is satisfied). Therefore, Eq (5.3) is oscillatory.

In the paper of Baculikova and Dzurina [28], an Euler-type equation was employed in the examples section to demonstrate that the assumptions of the established oscillation theorems are satisfied and, consequently, that the considered differential equation is oscillatory. However, examining the differential equations presented in our examples reveals a more general structure. Indeed, when $p_0 = 0$, the neutral term vanishes and the derivative part of Eq (1.1) reduces to a form closely related to the equation studied by Baculikova and Dzurina. Furthermore, the term expressed through the summation operator in our equation represents a broader class of delayed nonlinearities than the non-derivative term considered in their study. Moreover, the differential Eqs (5.2) and (5.3), which are introduced as illustrative examples for Theorems 4 and 6, respectively, possess a more general structure than the equation investigated by Baculikova and Dzurina. Nevertheless, the methodology used to establish oscillation in these examples is conceptually similar, as both approaches verify the validity of the theorem assumptions and then apply the corresponding oscillation criteria. Therefore, the primary purpose of presenting our examples is to demonstrate that Eq (1.1) is oscillatory whenever the conditions of the developed theorems are fulfilled. Therefore, our examples should not be viewed merely as repetitions of the Euler-type test equation considered in the literature; rather, they illustrate the applicability of the obtained oscillation criteria to a substantially more general class of neutral differential equations while preserving the same verification philosophy adopted in previous studies.

6. Conclusions

In this paper, we established new oscillation criteria for second-order NDDEs by examining the monotonic properties of non-oscillatory solutions. Our approach extends previous results by considering a neutral term and a more general nonlinearity. The theorems presented herein offer effective tools to analyze the oscillatory behavior of such equations. Observe that when $p(l,\mu) = 0$, that is, when the neutral term is eliminated, and $\sum_{j=1}^{n} q_j(l)F(X(\sigma_j(l))) = p(l)y^{\alpha}(\tau(l))$, the results presented here coincide with those reported in [28]. Thus, the consistency of our approach is confirmed, and its applicability to more complex equations is demonstrated.

Author contributions

Pakize Temtek wrote the main manuscript; Nagehan Kılınç Geçer carried out the language revision and editing. All authors reviewed the final manuscript.

Use of Generative-AI tools declaration

The authors used ChatGPT during the preparation of this manuscript to assist with language refinement and readability. All content was subsequently reviewed and edited by the authors, who take full responsibility for the published article.

Conflict of interest

The authors declare that they have no competing interests.

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