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Research article

Triple positive solutions and product-integration Nyström approximation for a doubly nonlocal fractional integro-boundary problem with weakly singular weights

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Abstract: A doubly nonlocal fractional integro-boundary problem with weakly singular weights is investigated for the order range $2 < \alpha < 3$. The governing equation contains a weakly singular Volterra memory operator, while the boundary condition involves a weighted integral functional with endpoint singularities. This configuration is relevant to anomalous transport and hereditary diffusion models in which interior memory and boundary feedback act simultaneously. An explicit Green representation is derived for the associated linear problem and is normalized by an endpoint-aware singular profile, which leads to two-sided kernel estimates and to a positive cone adapted to the doubly singular structure. Sufficient conditions for at least three positive solutions are then obtained by means of a Leggett–Williams framework. In parallel, a product-integration Nyström discretization is constructed for the weakly singular state operator and the weakly singular boundary operator. A uniform-grid convergence estimate is proved under a local contractivity condition, and a graded-mesh extension is formulated for endpoint-dominated regimes. Numerical experiments are reported for mesh refinement, boundary-parameter sensitivity, singularity-intensity variation, and graded-mesh comparison. The results confirm that the proposed approach provides a coherent analytic-computational framework for fractional models with simultaneous nonlocal singularities on the equation and boundary sides.

Keywords: fractional derivative; weakly singular kernel; product-integration; Nyström method; positive cone; nonlocal boundary condition
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1. Introduction

Fractional differential operators are widely used to model transport and relaxation processes with hereditary memory. In anomalous diffusion, the present state depends on an accumulated history rather than only on the instantaneous profile, while in heterogeneous viscoelastic media, the boundary response may itself be nonlocal and spatially nonuniform; see the monographs [1, 2] and the application-oriented studies [3–5]. From this perspective, a weakly singular Volterra kernel of the form $(t - s)^{-\mu}$ provides a natural representation of near-history memory, whereas a weighted integral boundary functional with endpoint singularities can be interpreted as a nonuniform boundary observation, feedback, or flux-balance mechanism.

In more concrete terms, fractional differential equations provide effective continuum descriptions for media in which microscopic trapping, long-range interactions, or material heterogeneity generate non-Markovian macroscopic laws. Examples include subdiffusive transport in porous or composite media, viscoelastic relaxation with fading memory, anomalous heat or mass transfer, and feedback-controlled diffusion processes where measurements or fluxes are averaged over a spatial boundary. The model considered here reflects this last situation: The Volterra term describes interior hereditary memory, whereas the weighted boundary functional describes a nonuniform boundary observation or feedback near the endpoints.

The present paper is motivated by situations in which these two mechanisms coexist. More precisely, the interior evolution is influenced by a weakly singular memory term, while the boundary condition is governed by a weakly singular weighted average of the unknown state. This combination leads to a doubly nonlocal fractional integro-boundary problem whose analytical structure is more delicate than in the standard single-source nonlocal setting. The positivity of the solution operator depends simultaneously on the weak singularity on the equation side and the weak singularity on the boundary side, and therefore the Green kernel must be analyzed in a way that reflects both endpoint behavior and positivity preservation.

Positive solutions for fractional boundary value problems with nonlocal conditions have been studied extensively. Early existence results for nonlinear fractional problems were obtained by Bai and Lu [6]. Integral boundary conditions and related positivity frameworks were subsequently investigated by Cabada and Wang [7], Yang [8], and Vong [9]. Multiplicity and coupled nonlocal structures were further studied in [10, 11]. On the functional-analytic side, the Leggett–Williams theorem [12], the three-fixed-point theorem of Avery and Peterson [13], and cone-compression techniques of Anderson and Avery [14] remain central tools for multiplicity in ordered Banach spaces. Recent work continues to develop positive-solution theory for singular or integral boundary conditions [15,16]. Related recent studies also address Hadamard-type integro-differential boundary conditions, fractional advection–dispersion equations with impulses, and the convergence of positive solutions for Caputo–Hadamard fractional models [17–19]. Nevertheless, most available results still treat either one nonlocal source or one singular mechanism, but not both simultaneously.

The numerical literature is equally rich for weakly singular Volterra equations themselves. Product integration and Nyström-type methods constitute a classical and robust toolbox in this direction [20–22]. More recent contributions include variable-exponent weakly singular Volterra equations [23], non-smooth weakly singular problems [24], and singularity-separation/collocation approaches [25]. However, the available analyzes are mainly tailored to integral equations or to problems where the singularity is confined to the equation side. A systematic product-integration Nyström treatment for a fractional integro-boundary problem with simultaneous weak singularities in the state operator and the boundary operator appears to be missing.

There is also a complementary approximation-theoretic direction based on polynomial reconstruction and least-squares stabilization. In particular, constrained mock-Chebyshev least-squares operators have recently been used for Fredholm integral equations and for product-integration rules on equispaced data [26, 27], while their bivariate generalizations provide a possible route toward higher-dimensional reconstruction strategies [28]. These developments are relevant here because they address stable approximation in the presence of difficult kernel behavior and non-Chebyshev sampling. The present paper retains a Green-kernel-fitted product-integration Nyström framework, but the comparison with least-squares-based operators clarifies where future higher-order or adaptive variants may enter.

This gap motivates an analytic-computational framework in which the mathematical novelty and the approximation strategy support one another. Accordingly, the present work is organized so that the existence theory and the numerical analysis form a single pipeline: An explicit Green representation is derived, a cone adapted to the doubly singular kernel structure is built, a triple-positive-solution result is proved, and the same operator structure is then discretized by a product-integration Nyström scheme whose performance can be tested quantitatively.

The main contributions of the paper are summarized as follows.

C1. An explicit Green representation is derived for the linear auxiliary problem. The associated kernel is normalized by the singular profile

$$\phi(s) = \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)},$$

and the two-sided bounds, together with the endpoint-aware continuity properties, are established.

C2. A positive cone determined by the kernel geometry is constructed, and a Leggett–Williams framework yields sufficient conditions for at least three positive solutions. To the best of our knowledge, this is among the first triple-positive-solution results for a fractional problem in which the governing equation and the boundary condition are both weakly singular and nonlocal.

C3. A product-integration Nyström discretization is proposed for the doubly nonlocal model. Uniform-grid error bounds are derived under a local contractivity condition, and a graded-mesh extension is discussed for endpoint-dominated regimes.

C4. A multi-part numerical study is performed. In addition to the baseline convergence test, the influence of the boundary parameter λ, the singularity strengths (μ, ρ, κ), and the use of graded meshes are examined quantitatively.

The paper is arranged as follows. Section 2 formulates the model and the standing assumptions. Section 3 develops the Green representation, the refined kernel analysis, and the cone construction. Section 4 proves the triple-positive-solution theorem and discusses the role of the nonresonance parameter Θ. Section 5 develops the product-integration Nyström scheme, comments on alternative approximation strategies, and establishes its error estimate. Section 6 reports a numerical study tailored to the computational emphasis of the paper. Section 7 discusses the significance, limitations, and extensions of the approach, and Section 8 concludes the manuscript.

2. Problem setting and preliminaries

Let $X = C[0,1]$ be endowed with the norm

$$\|u\| = \max_{t\in[0,1]} |u(t)|. \tag{2.1}$$

For $\beta > 0$, the left-sided Riemann–Liouville fractional integral is defined by

$$(I_{0+}^{\beta} u)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} u(s)\, ds. \tag{2.2}$$

For $2 < \alpha < 3$, the left-sided Riemann–Liouville fractional derivative is

$$(D_{0+}^{\alpha} u)(t) = \frac{d^3}{dt^3} (I_{0+}^{3-\alpha} u)(t). \tag{2.3}$$

We work with the Riemann–Liouville derivative rather than Caputo or Hilfer-type derivatives because it is naturally compatible with the fractional initial condition $I_{0+}^{3-\alpha} u(0) = 0$ and leads to an explicit Green representation with the endpoint profile required in the present positivity analysis. This choice is also consistent with boundary value formulations where the singular behavior near the left endpoint is part of the operator structure rather than being completely removed by classical initial data.

We study the doubly nonlocal fractional integro-boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) + a(t)\, f\Big(t, u(t), (\mathcal{W}_{\mu} u)(t), \mathcal{B}_{\rho,\kappa}(u)\Big) = 0, \qquad 0 < t < 1, \\ I_{0+}^{3-\alpha} u(0) = 0, \qquad u'(0) = 0, \\ u(1) = \lambda\, \mathcal{B}_{\rho,\kappa}(u), \end{cases} \tag{2.4}$$

where

$$(\mathcal{W}_{\mu} u)(t) = \int_0^t \frac{k(t,s)}{(t-s)^{\mu}} u(s)\, ds, \qquad 0 < \mu < 1, \tag{2.5}$$

and

$$\mathcal{B}_{\rho,\kappa}(u) = \int_0^1 s^{-\rho} (1-s)^{-\kappa} u(s)\, ds, \qquad 0 < \rho, \kappa < 1. \tag{2.6}$$

Throughout the paper, the following assumptions are imposed.

(A1) $a \in C[0,1]$, $a(t) \geq 0$ on $[0,1]$, and $a \not\equiv 0$.

(A2) $k \in C(\Delta, [0,\infty))$, where $\Delta = \{(t,s) : 0 \leq s \leq t \leq 1\}$.

(A3) $f : [0,1] \times [0,\infty)^3 \to [0,\infty)$ is continuous.

(A4) The quantity

$$\Theta := 1 - \lambda \int_0^1 s^{\alpha-1-\rho} (1-s)^{-\kappa}\, ds = 1 - \lambda B(\alpha-\rho, 1-\kappa) \tag{2.7}$$

satisfies $\Theta > 0$.

Remark 2.1. Assumption (A3) states the main theorem for a continuous nonlinearity. Hence, the results in their present form are not directly applicable to a nonlinearity that is singular at $t = 0$ or $t = 1$. A possible extension would require a weighted Carathéodory framework: For instance, one may assume that f is continuous in the state variables for almost every t and is dominated on bounded sets by an endpoint-singular function $h(t)$ such that $\int_0^1 \phi(t)a(t)h(t)\,dt < \infty$, together with compatible lower-growth assumptions on $[\theta, 1]$. Under such additional integrability and compactness hypotheses, the present cone and Green-kernel mechanism can be adapted, but this singular-nonlinearity case is beyond the scope of the current paper.

The constants

$$K_\mu := \sup_{t\in[0,1]} \int_0^t \frac{k(t,s)}{(t-s)^\mu}\,ds, \qquad B_{\rho,\kappa} := \int_0^1 s^{-\rho}(1-s)^{-\kappa}\,ds = B(1-\rho, 1-\kappa) \tag{2.8}$$

are finite. Hence, for every $u \geq 0$, we have

$$0 \leq (\mathcal{W}_\mu u)(t) \leq K_\mu \|u\|, \qquad 0 \leq \mathcal{B}_{\rho,\kappa}(u) \leq B_{\rho,\kappa}\|u\|. \tag{2.9}$$

3. Green representation and kernel analysis

3.1. The linear auxiliary problem

We first consider the linear problem

$$\begin{cases} D^\alpha_{0+}u(t) + y(t) = 0, \qquad 0 < t < 1, \\ I^{3-\alpha}_{0+}u(0) = 0, \qquad u'(0) = 0, \\ u(1) = \lambda \displaystyle\int_0^1 s^{-\rho}(1-s)^{-\kappa}u(s)\,ds, \end{cases} \tag{3.1}$$

where $y \in C[0,1]$ and $y \geq 0$.

Lemma 3.1. Under (A4), Problem (3.1) has the unique solution

$$u(t) = \int_0^1 H(t,s)y(s)\,ds, \tag{3.2}$$

where

$$H(t,s) = G(t,s) + \frac{\lambda t^{\alpha-1}}{\Theta} \int_0^1 \tau^{-\rho}(1-\tau)^{-\kappa} G(\tau,s)\,d\tau, \tag{3.3}$$

and

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ [t(1-s)]^{\alpha-1}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{3.4}$$

Proof. For $2 < \alpha < 3$, let u_h denote the homogeneous part of the solution. The homogeneous equation $D^\alpha_{0+}u = 0$ has the general solution

$$u_h(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3},$$

where u_h denotes the homogeneous solution and $c_1, c_2, andc_3$ are integration constants. Therefore, every solution of $D^{\alpha}_{0+}u = -y$ admits the representation

$$u(t) = -I^{\alpha}_{0+}y(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3} \tag{3.5}$$

We determine the integration constants $c_1, c_2, andc_3$ successively from the three side conditions.

First, apply $I^{3-\alpha}_{0+}$ to (3.5). Since

$$I^{3-\alpha}_{0+}t^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(3)}t^2, \qquad I^{3-\alpha}_{0+}t^{\alpha-2} = \frac{\Gamma(\alpha-1)}{\Gamma(2)}t, \qquad I^{3-\alpha}_{0+}t^{\alpha-3} = \Gamma(\alpha-2),$$

and $I^{3}_{0+}y(0) = 0$, the condition $I^{3-\alpha}_{0+}u(0) = 0$ yields

$$c_3\Gamma(\alpha-2) = 0.$$

Hence, $c_3 = 0$.

Second, differentiate (3.5). Because $c_3 = 0$, one has

$$u'(t) = -\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}y(s)\,ds + c_1(\alpha-1)t^{\alpha-2} + c_2(\alpha-2)t^{\alpha-3}$$

The integral term tends to 0 as $t \to 0^+$, and the term $t^{\alpha-2}$ also tends to 0 because $\alpha - 2 > 0$. Since $\alpha - 3 \in (-1, 0)$, the factor $t^{\alpha-3}$ is singular at the origin unless $c_2 = 0$. The boundary condition $u'(0) = 0$ therefore implies $c_2 = 0$.

Consequently,

$$u(t) = c_1t^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds. \tag{3.6}$$

Evaluating at $t = 1$ gives

$$u(1) = c_1 - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)\,ds.$$

On the other hand, substitution of (3.6) into the boundary functional yields

$$\begin{aligned}\int_0^1 \tau^{-\rho}(1-\tau)^{-\kappa}u(\tau)\,d\tau = c_1 &\int_0^1 \tau^{\alpha-1-\rho}(1-\tau)^{-\kappa}\,d\tau \\ &- \frac{1}{\Gamma(\alpha)}\int_0^1 \tau^{-\rho}(1-\tau)^{-\kappa}\int_0^\tau (\tau-s)^{\alpha-1}y(s)\,ds\,d\tau.\end{aligned}$$

Using the last boundary condition in (3.1), we arrive at

$$\begin{aligned}c_1 - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)\,ds = \lambda c_1 &\int_0^1 \tau^{\alpha-1-\rho}(1-\tau)^{-\kappa}\,d\tau \\ &- \frac{\lambda}{\Gamma(\alpha)}\int_0^1 \tau^{-\rho}(1-\tau)^{-\kappa}\int_0^\tau (\tau-s)^{\alpha-1}y(s)\,ds\,d\tau.\end{aligned}$$

Rearranging the terms and recalling the definition of Θ in (2.7), we obtain

$$c_1 = \frac{1}{\Theta}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)\,ds - \frac{\lambda}{\Gamma(\alpha)}\int_0^1 \tau^{-\rho}(1-\tau)^{-\kappa}\int_0^\tau (\tau-s)^{\alpha-1}y(s)\,ds\,d\tau\right]. \tag{3.7}$$

Substituting (3.7) into (3.6), the first part becomes

$$\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)\,ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds,$$

which is exactly $\int_0^1 G(t,s)y(s)\,ds$ after splitting the s-integration into the cases $s \le t$ and $t \le s$. The second part equals

$$\frac{\lambda t^{\alpha-1}}{\Theta}\int_0^1 \tau^{-\rho}(1-\tau)^{-\kappa}\left(\int_0^1 G(\tau,s)y(s)\,ds\right)d\tau,$$

which, by Fubini's theorem, can be rewritten as

$$\int_0^1 \frac{\lambda t^{\alpha-1}}{\Theta}\left(\int_0^1 \tau^{-\rho}(1-\tau)^{-\kappa}G(\tau,s)\,d\tau\right)y(s)\,ds.$$

Combining the two pieces gives (3.2)–(3.4)

To prove uniqueness, let $y \equiv 0$. Then (3.6) reduces to $u(t) = c_1 t^{\alpha-1}$, and the boundary condition becomes

$$c_1 = \lambda c_1 \int_0^1 \tau^{\alpha-1-\rho}(1-\tau)^{-\kappa}\,d\tau.$$

Hence, $\Theta c_1 = 0$. Since $\Theta > 0$ by (A4), it follows that $c_1 = 0$, so $u \equiv 0$. Therefore, the linear problem has a unique solution. □

3.2. *Refined properties of the Green kernel*

Define the singular profile

$$\phi(s) := \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)}, \qquad s \in [0,1]. \tag{3.8}$$

Lemma 3.2. For all $(t,s) \in [0,1]^2$, we have

$$0 \le G(t,s) \le \phi(s). \tag{3.9}$$

Proof. The non-negativity of G is immediate from $t(1-s) \ge t-s$ whenever $0 \le s \le t \le 1$. If $t \le s$, then

$$G(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} \le \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} \le \phi(s),$$

because $\alpha - 1 > 1$ and $t \le s$.

If $s \le t$, define

$$g_t(s) := [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}$$

Differentiating with respect to t gives

$$\partial_t g_t(s) = (\alpha-1)\left[t^{\alpha-2}(1-s)^{\alpha-1} - (t-s)^{\alpha-2}\right] \ge 0,$$

since $t(1-s) \ge t-s$ and $\alpha - 2 \in (0,1)$. Therefore, $g_t(s) \le g_1(s) = s(1-s)^{\alpha-1}$, which implies (3.9) □

Lemma 3.3. The profile ϕ satisfies $\phi(s) > 0$ for $s \in (0,1)$ and

$$\phi(s) \sim \frac{s}{\Gamma(\alpha)} \quad \text{as } s \to 0^+, \qquad \phi(s) \sim \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \quad \text{as } s \to 1^-. \tag{3.10}$$

Moreover, for every fixed $\theta \in (0,1)$, the normalized kernel

$$Q_\theta(t,s) := \frac{H(t,s)}{\phi(s)}, \qquad (t,s) \in [\theta,1] \times (0,1), \tag{3.11}$$

extends continuously to $[\theta,1] \times [0,1]$ and remains strictly positive there.

Proof. The positivity of ϕ on $(0,1)$ is immediate from (3.8). The endpoint behavior follows from elementary asymptotic expansion:

$$\phi(s) = \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)} = \frac{s}{\Gamma(\alpha)}\,(1+o(1)) \quad \text{as } s \to 0^+,$$

and likewise

$$\phi(s) = \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)} = \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\,(1+o(1)) \quad \text{as } s \to 1^-.$$

This proves (3.10)

We next analyze the quotient $Q_\theta(t,s) = H(t,s)/\phi(s)$. Since H is continuous on $[0,1]^2$, the quotient is continuous on $[\theta,1] \times (0,1)$. It remains to examine the endpoint limits in the s-variable.

For fixed $t \in [\theta,1]$ and $0 < s \le t$, Formula (3.4) gives

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \left([t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1} \right).$$

Applying the mean-value theorem to $x \mapsto x^{\alpha-1}$, $\xi_s \in (t-s, t(1-s))$ exists such that

$$[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1} = (\alpha-1)\xi_s^{\alpha-2}\,(t(1-s)-(t-s)) = (\alpha-1)\xi_s^{\alpha-2} s(1-t).$$

After division by $\phi(s)$ and passage to the limit $s \to 0^+$, one obtains

$$\frac{G(t,s)}{\phi(s)} \to (\alpha-1)t^{\alpha-2}(1-t).$$

In particular, the quotient has a finite limit at $s = 0$.

As $s \to 1^-$, one has $t \le s$ for fixed $t \in [\theta,1]$ and sufficiently small $1-s$. Hence

$$G(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, \qquad \frac{G(t,s)}{\phi(s)} = \frac{t^{\alpha-1}}{s} \to t^{\alpha-1}$$

Therefore, $G(t,s)/\phi(s)$ extends continuously to both endpoints.

Now consider

$$J(s) := \int_0^1 \tau^{-\rho}(1-\tau)^{-\kappa} G(\tau,s)\,d\tau.$$

By Lemma 3.2, we have

$$0 \le \tau^{-\rho}(1-\tau)^{-\kappa} G(\tau,s) \le \tau^{-\rho}(1-\tau)^{-\kappa} \phi(s),$$

and the weight in τ is integrable because $0 < \rho, \kappa < 1$. Thus J is well defined. Moreover,

$$0 \leq \frac{J(s)}{\phi(s)} \leq \int_0^1 \tau^{-\rho}(1-\tau)^{-\kappa}\, d\tau = \Lambda_0,$$

so the normalized correction term is uniformly bounded. Since the pointwise limits of $G(\tau, s)/\phi(s)$ exist as $s \to 0^+$ and $s \to 1^-$ for every fixed τ, the dominated convergence theorem implies that $J(s)/\phi(s)$ also extends continuously to $s = 0$ and $s = 1$.

Finally, the representation in (3.3) shows that Q_θ is the sum of two functions that each extend continuously to $[\theta, 1] \times [0, 1]$. Hence Q_θ itself extends continuously there. Since $H(t, s) > 0$ for $t \in [\theta, 1]$ and $s \in (0, 1)$, continuity implies that the extension remains strictly positive on the compact rectangle $[\theta, 1] \times [0, 1]$. □

Let

$$\Lambda_0 := B(1-\rho, 1-\kappa). \tag{3.12}$$

By Lemma 3.2, we have

$$0 \leq H(t, s) \leq M\phi(s), \qquad M := 1 + \frac{\lambda\Lambda_0}{\Theta} \tag{3.13}$$

Fix $\theta \in (0, 1)$. By Lemma 3.3, we have

$$m_\theta := \min_{(t,s)\in[\theta,1]\times[0,1]} Q_\theta(t, s) > 0, \tag{3.14}$$

and therefore

$$H(t, s) \geq m_\theta \phi(s), \qquad (t, s) \in [\theta, 1] \times [0, 1]. \tag{3.15}$$

Remark 3.4. The parameter θ only affects the quantitative constant m_θ and hence the ratio $\delta = m_\theta/M$. It does not affect the positivity of the kernel, the complete continuity of the solution operator, or the general cone construction. Thus, different choices of $\theta \in (0, 1)$ lead to equivalent cone frameworks, although the admissible thresholds in the growth conditions of Section 4 depend quantitatively on θ.

Define the cone

$$K := \left\{ u \in X : \ u(t) \geq 0 \text{ on } [0, 1], \ \min_{t\in[\theta,1]} u(t) \geq \delta\|u\| \right\}, \qquad \delta := \frac{m_\theta}{M} \in (0, 1). \tag{3.16}$$

Problem (2.4) is equivalent to the fixed-point equation $u = Tu$, where

$$(Tu)(t) = \int_0^1 H(t, s)a(s) f\Big(s, u(s), (\mathcal{W}_\mu u)(s), \mathcal{B}_{\rho,\kappa}(u)\Big)\, ds. \tag{3.17}$$

Lemma 3.5. The operator $T : K \to K$ is completely continuous.

Proof. The proof is divided into three explicit steps.

Step 1: T maps K into itself. Let $u \in K$. Since $H(t, s) \geq 0$, $a(s) \geq 0$, and

$$f\Big(s, u(s), (\mathcal{W}_\mu u)(s), \mathcal{B}_{\rho,\kappa}(u)\Big) \geq 0,$$

Formula (3.17) implies $(Tu)(t) \geq 0$ for every $t \in [0, 1]$. Using the upper kernel bound (3.13), we also obtain

$$(Tu)(t) \leq M \int_0^1 \phi(s)a(s) f\Big(s, u(s), (\mathcal{W}_\mu u)(s), \mathcal{B}_{\rho,\kappa}(u)\Big)\, ds,$$

whence

$$\|Tu\| \le M \int_0^1 \phi(s)a(s)f\Big(s, u(s), (\mathcal{W}_\mu u)(s), \mathcal{B}_{\rho,\kappa}(u)\Big)\, ds.$$

If $t \in [\theta, 1]$, then the lower kernel estimate (3.15) gives

$$\begin{aligned}(Tu)(t) &\ge m_\theta \int_0^1 \phi(s)a(s)f\Big(s, u(s), (\mathcal{W}_\mu u)(s), \mathcal{B}_{\rho,\kappa}(u)\Big)\, ds \\ &= \frac{m_\theta}{M} M \int_0^1 \phi(s)a(s)f\Big(s, u(s), (\mathcal{W}_\mu u)(s), \mathcal{B}_{\rho,\kappa}(u)\Big)\, ds \\ &\ge \delta\|Tu\|.\end{aligned}$$

Therefore, $Tu \in K$ and $T(K) \subset K$.

Step 2: Bounded sets are mapped into uniformly bounded sets. Fix $R > 0$ and define $B_R := \{u \in K : \|u\| \le R\}$. By (2.9), for every $u \in B_R$ and every $s \in [0, 1]$, we have

$$0 \le u(s) \le R, \qquad 0 \le (\mathcal{W}_\mu u)(s) \le K_\mu R, \qquad 0 \le \mathcal{B}_{\rho,\kappa}(u) \le B_{\rho,\kappa} R.$$

Hence, all arguments of f lie in the compact set

$$D_R = [0, 1] \times [0, R] \times [0, K_\mu R] \times [0, B_{\rho,\kappa} R].$$

By continuity of f, $C_R > 0$ exists such that

$$0 \le f\Big(s, u(s), (\mathcal{W}_\mu u)(s), \mathcal{B}_{\rho,\kappa}(u)\Big) \le C_R \qquad \text{for all } u \in B_R,\ s \in [0, 1].$$

Substituting this bound into the previous estimate yields

$$\|Tu\| \le C_R M \int_0^1 \phi(s)a(s)\, ds, \qquad u \in B_R. \tag{3.18}$$

Thus $T(B_R)$ is uniformly bounded in X.

Step 3: Bounded sets are mapped into equicontinuous sets. Let $t_1, t_2 \in [0, 1]$, and $u \in B_R$. Then

$$\begin{aligned}|(Tu)(t_1) - (Tu)(t_2)| &\le \int_0^1 |H(t_1, s) - H(t_2, s)| a(s) f\Big(s, u(s), (\mathcal{W}_\mu u)(s), \mathcal{B}_{\rho,\kappa}(u)\Big)\, ds \\ &\le C_R \int_0^1 |H(t_1, s) - H(t_2, s)| a(s)\, ds.\end{aligned}$$

Because H is continuous on the compact set $[0, 1]^2$, it is uniformly continuous there. Hence, for every $\varepsilon > 0$, $\eta > 0$ exists such that

$$|t_1 - t_2| < \eta \quad \Longrightarrow \quad |H(t_1, s) - H(t_2, s)| < \frac{\varepsilon}{C_R \int_0^1 a(\sigma)\, d\sigma + 1} \quad \text{for all } s \in [0, 1].$$

Consequently, we have

$$|t_1 - t_2| < \eta \quad \Longrightarrow \quad |(Tu)(t_1) - (Tu)(t_2)| < \varepsilon \quad \text{for all } u \in B_R.$$

Thus $T(B_R)$ is equicontinuous.

By the Arzelà–Ascoli theorem, Steps 2 and 3 imply that $T(B_R)$ is relatively compact in X for every $R > 0$. Finally, the continuity of T follows from the continuity of f together with dominated convergence, because on bounded sets, the integrand in (3.17) is dominated by the integrable function $M\phi(s)a(s)C_R$. Therefore, T is completely continuous. □

4. Triple positive solutions

Define the continuous concave functional

$$\psi(u) := \min_{t\in[\theta,1]} u(t), \qquad u \in K. \tag{4.1}$$

Let

$$A := M \int_0^1 \phi(s)a(s)\,ds, \qquad B := m_\theta \int_\theta^1 \phi(s)a(s)\,ds. \tag{4.2}$$

We impose the following growth conditions:

(H1) There is an $r_1 > 0$ such that

$$f(t,x,y,z) \le \frac{r_1}{A} \tag{4.3}$$

for all

$$t \in [0,1], \quad 0 \le x \le r_1, \quad 0 \le y \le K_\mu r_1, \quad 0 \le z \le B_{\rho,\kappa} r_1.$$

(H2) There is an $r_2 > r_1$ such that

$$f(t,x,y,z) \ge \frac{r_2}{B} \tag{4.4}$$

for all

$$t \in [\theta,1], \quad r_2 \le x \le \frac{r_2}{\delta}, \quad 0 \le y \le K_\mu \frac{r_2}{\delta}, \quad 0 \le z \le B_{\rho,\kappa}\frac{r_2}{\delta}$$

(H3) There is an $r_3 \ge r_2/\delta$ such that

$$f(t,x,y,z) \le \frac{r_3}{A} \tag{4.5}$$

for all

$$t \in [0,1], \quad 0 \le x \le r_3, \quad 0 \le y \le K_\mu r_3, \quad 0 \le z \le B_{\rho,\kappa} r_3.$$

Theorem 4.1. Assume (A1)–(A4) and (H1)–(H3). Then Problem (2.4) possesses at least three positive solutions $u_1, u_2, u_3 \in K$ satisfying

$$\|u_1\| < r_1, \qquad \psi(u_2) > r_2, \qquad \|u_3\| > r_1 \text{ and } \psi(u_3) < r_2. \tag{4.6}$$

Proof. The hypotheses of the Leggett–Williams theorem [12] are verified in four explicit steps.

Step 1: Invariance of the large closed ball. Let $u \in K$ satisfy $\|u\| \le r_3$. By (2.9), we have

$$0 \le u(s) \le r_3, \qquad 0 \le (\mathcal{W}_\mu u)(s) \le K_\mu r_3, \qquad 0 \le \mathcal{B}_{\rho,\kappa}(u) \le B_{\rho,\kappa} r_3.$$

Thus the quadruple of arguments of f lies in the admissible region from (H3), and therefore

$$f\Big(s, u(s), (\mathcal{W}_\mu u)(s), \mathcal{B}_{\rho,\kappa}(u)\Big) \le \frac{r_3}{A}$$

Using (3.13) and the definition of A in (4.2), we find

$$\begin{aligned}\|Tu\| &\le M \int_0^1 \phi(s)a(s)\frac{r_3}{A}\,ds \\ &= \frac{Mr_3}{A}\int_0^1 \phi(s)a(s)\,ds = r_3.\end{aligned}$$

Hence $T(\overline{K}_{r_3}) \subset \overline{K}_{r_3}$.

Step 2: Compression on the small closed ball. Let $u \in K$ with $\|u\| \le r_1$. Exactly the same argument, now using (H1), yields

$$f\Big(s, u(s), (\mathcal{W}_\mu u)(s), \mathcal{B}_{\rho,\kappa}(u)\Big) \le \frac{r_1}{A},$$

and therefore

$$\|Tu\| \le M \int_0^1 \phi(s)a(s)\frac{r_1}{A}\, ds = r_1.$$

Thus $T(\overline{K}_{r_1}) \subset \overline{K}_{r_1}$.

Step 3: Expansion of the concave functional on the intermediate shell. Consider

$$K(\psi, r_2, r_2/\delta) := \left\{ u \in K :\ r_2 \le \psi(u),\ \|u\| \le \frac{r_2}{\delta} \right\}$$

This set is nonempty, because the constant function $u(t) \equiv r_2/\delta$ belongs to it. Let $u \in K(\psi, r_2, r_2/\delta)$. Then for each $s \in [\theta, 1]$, we have

$$r_2 \le \psi(u) \le u(s) \le \|u\| \le \frac{r_2}{\delta}$$

Moreover, we have

$$0 \le (\mathcal{W}_\mu u)(s) \le K_\mu \frac{r_2}{\delta}, \qquad 0 \le \mathcal{B}_{\rho,\kappa}(u) \le B_{\rho,\kappa}\frac{r_2}{\delta}$$

Hence, the lower-growth hypothesis (H2) applies on $[\theta, 1]$ and gives

$$f\Big(s, u(s), (\mathcal{W}_\mu u)(s), \mathcal{B}_{\rho,\kappa}(u)\Big) \ge \frac{r_2}{B} \qquad \text{for all } s \in [\theta, 1].$$

Applying the lower kernel estimate (3.15), we obtain the following for every $t \in [\theta, 1]$:

$$\begin{aligned}(Tu)(t) &\ge m_\theta \int_\theta^1 \phi(s)a(s) f\Big(s, u(s), (\mathcal{W}_\mu u)(s), \mathcal{B}_{\rho,\kappa}(u)\Big)\, ds \\ &\ge m_\theta \int_\theta^1 \phi(s)a(s)\frac{r_2}{B}\, ds \\ &= \frac{m_\theta r_2}{B}\int_\theta^1 \phi(s)a(s)\, ds = r_2.\end{aligned}$$

Taking the minimum over $t \in [\theta, 1]$ yields

$$\psi(Tu) \ge r_2.$$

Step 4: Shell condition. Let $u \in K(\psi, r_2, r_3)$ and assume that $\|Tu\| > r_2/\delta$. Since $Tu \in K$ by Lemma 3.5, the cone property (3.16) gives

$$\psi(Tu) = \min_{t\in[\theta,1]} (Tu)(t) \ge \delta\|Tu\| > r_2.$$

This is the final condition required by the Leggett–Williams theorem.

Now Lemma 3.5 shows that T is completely continuous, while Steps 1–4 verify the hypotheses of the three-fixed-point theorem. Therefore, T has at least three fixed points $u_1, u_2, u_3 \in K$ satisfying (4.6). Since the fixed points of T solve (2.4) and every nonzero element of K is positive on $[\theta, 1]$ and non-negative on $[0, 1]$, these fixed points are positive solutions of the boundary value problem. □

Remark 4.2. To the best of our knowledge, Theorem 4.1 is among the first triple-positive-solution results for a fractional integro-boundary problem in which both the state operator and the boundary operator are weakly singular and nonlocal. In contrast with the standard integral-boundary setting, the cone constant δ depends on the interaction between the two singular mechanisms through the normalized kernel bounds (3.13)–(3.15)

Remark 4.3. Assumption (A4) guarantees that the linear auxiliary problem is nonresonant and that the Green representation (3.2) is available. When $\Theta = 0$, the present Green-kernel construction breaks down and the problem enters a critical (resonant) regime; when $\Theta < 0$, the positivity structure underlying the cone argument is generally lost. These cases require different techniques, such as the coincidence degree, the Fredholm alternatives, or upper-lower solution frameworks, and are beyond the scope of the present paper.

Remark 4.4. The growth assumptions (H1)–(H3) are explicit sufficient conditions tailored to the Leggett–Williams theorem. They are not claimed to be optimal. If one replaces the three-fixed-point framework by fixed-point index methods, bifurcation arguments, or monotone iterative techniques, then sublinear, superlinear, or sign-changing regimes may also be treated.

5. Product-integration Nyström method and error analysis

5.1. Uniform midpoint discretization

Let $h = 1/N$ and define the midpoint grid

$$t_i = \left(i + \frac{1}{2}\right)h, \qquad i = 0, 1, \ldots, N-1. \tag{5.1}$$

The discrete values $U_i \approx u(t_i)$ are used to approximate the unknown solution.

For the Volterra memory operator, the following product-integration rule is adopted:

$$(\mathcal{W}_\mu u)(t_i) \approx \sum_{j=0}^{i-1} \omega_{ij}^{(\mu)} U_j, \tag{5.2}$$

with cell weights

$$\omega_{ij}^{(\mu)} = \int_{jh}^{(j+1)h} \frac{k(t_i, s)}{(t_i - s)^\mu}\, ds. \tag{5.3}$$

When k varies slowly within a cell, the practical approximation

$$\omega_{ij}^{(\mu)} \approx k(t_i, t_j)\, \frac{(t_i - jh)^{1-\mu} - (t_i - (j+1)h)^{1-\mu}}{1-\mu} \tag{5.4}$$

is accurate and inexpensive.

The weakly singular boundary functional is discretized as

$$\mathcal{B}_{\rho,\kappa}(u) \approx \sum_{j=0}^{N-1} \beta_j^{(\rho,\kappa)} U_j, \qquad \beta_j^{(\rho,\kappa)} = \int_{jh}^{(j+1)h} s^{-\rho}(1-s)^{-\kappa}\, ds. \tag{5.5}$$

The outer Hammerstein integral is then approximated by the midpoint Nyström rule

$$U_i = \sum_{j=0}^{N-1} h\, H(t_i,t_j)a(t_j)f\left(t_j, U_j, \sum_{\ell=0}^{j-1} \omega_{j\ell}^{(\mu)} U_\ell, \sum_{\ell=0}^{N-1} \beta_\ell^{(\rho,\kappa)} U_\ell\right), \quad i = 0,1,\ldots,N-1. \tag{5.6}$$

The nonlinear algebraic system (5.6) is solved in the experiments by fixed-point iteration. The stopping criterion is

$$\|U^{(m+1)} - U^{(m)}\|_\infty \le 10^{-12} \tag{5.7}$$

5.2. *Graded-mesh extension*

Uniform midpoint grids are sufficient for the convergence theory below, but the endpoint singularities suggest the additional use of graded meshes in the computation. For $\gamma > 1$, define

$$x_j = \left(\frac{j}{N}\right)^\gamma, \qquad t_j = \frac{x_j + x_{j+1}}{2}, \qquad h_j = x_{j+1} - x_j, \qquad j = 0,\ldots,N-1. \tag{5.8}$$

The Nyström system then becomes

$$U_i = \sum_{j=0}^{N-1} h_j\, H(t_i,t_j)a(t_j)f\left(t_j, U_j, \sum_{\ell=0}^{j-1} \widetilde{\omega}_{j\ell}^{(\mu)} U_\ell, \sum_{\ell=0}^{N-1} \widetilde{\beta}_\ell^{(\rho,\kappa)} U_\ell\right), \tag{5.9}$$

where

$$\widetilde{\omega}_{j\ell}^{(\mu)} = \int_{x_\ell}^{x_{\ell+1}} \frac{k(t_j,s)}{(t_j - s)^\mu}\, ds, \qquad \widetilde{\beta}_\ell^{(\rho,\kappa)} = \int_{x_\ell}^{x_{\ell+1}} s^{-\rho}(1-s)^{-\kappa}\, ds. \tag{5.10}$$

This graded-mesh version is not used in the proof of Theorem 5.1; it is included in Section 6 to assess the computational advantages in endpoint-dominated regimes.

5.3. *Alternative approximation perspectives*

The discretization above was chosen because it is directly fitted to the Green representation and to the two weakly singular integral operators appearing in the model. It keeps the memory quadrature, the boundary functional, and the outer Hammerstein integral in the same positive kernel framework used in the existence theory. Other approximation strategies are nevertheless possible. For instance, constrained polynomial reconstruction and constrained mock-Chebyshev least-squares operators approximate data on equispaced grids by combining interpolation on a mock-Chebyshev subset with least-squares information from the remaining nodes; this mechanism helps control Runge-type instabilities while preserving the convenience of equispaced sampling. Recent applications to Fredholm integral equations and product-integration rules [26,27] suggest that such operators could be coupled with the present weakly singular framework.

For the doubly nonlocal boundary problem considered here, however, a direct replacement of the Nyström weights by a least-squares or spectral reconstruction is not merely technical. One would have to preserve the positivity properties required by the cone argument, control the nested Volterra history term, and approximate the weighted boundary functional with endpoint singularities without destroying the nonresonance structure. This explains why the present paper uses a first-order product-integration Nyström scheme as the baseline method, while regarding constrained least-squares reconstruction, spectral collocation, and adaptive graded quadrature as natural extensions rather than as interchangeable black-box alternatives.

5.4. Convergence analysis

For the error estimate, let $D_R := [0,1] \times [0,R] \times [0, K_\mu R] \times [0, B_{\rho,\kappa} R]$ and assume that

$$|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)| \le L_x|x_1 - x_2| + L_y|y_1 - y_2| + L_z|z_1 - z_2|, \tag{5.11}$$

for all points of D_R.

Theorem 5.1. Assume that u is a positive solution of (2.4), that $u \in C^1[0,1]$, that $k \in C^1(\Delta)$, and that the range of u is contained in D_R. Let $U^h = (U_0, \dots, U_{N-1})$ be a discrete solution of (5.6) with $|U_i| \le R$ for all i. If

$$C_H\left(L_x + L_y K_\mu + L_z B_{\rho,\kappa}\right) < 1, \qquad C_H := \sup_{t\in[0,1]} \int_0^1 H(t,s)a(s)\,ds, \tag{5.12}$$

then a constant $C > 0$ exists, independent of h, such that

$$\|u - U^h\|_\infty \le C\left(h + h^{1-\mu} + h^{1-\rho} + h^{1-\kappa}\right). \tag{5.13}$$

Consequently, we have

$$\|u - U^h\|_\infty \le C h^\nu, \qquad \nu = \min\{1, 1-\mu, 1-\rho, 1-\kappa\} \tag{5.14}$$

Proof. Let $u_i := u(t_i)$ and introduce the nodal error

$$E_i := u_i - U_i, \qquad \|E\|_\infty := \max_{0\le i\le N-1} |E_i|.$$

The argument consists of a consistency estimate followed by a discrete perturbation inequality.

Step 1: Consistency of the three quadrature components. Because $u \in C^1[0,1]$, the integrand

$$s \mapsto H(t_i,s)a(s)f\Big(s, u(s), (\mathcal{W}_\mu u)(s), \mathcal{B}_{\rho,\kappa}(u)\Big)$$

is continuous for each fixed t_i. Therefore, midpoint quadrature yields

$$u_i = \sum_{j=0}^{N-1} h\,H(t_i,t_j)a(t_j)f\Big(t_j, u_j, (\mathcal{W}_\mu u)(t_j), \mathcal{B}_{\rho,\kappa}(u)\Big) + \eta_i^{(0)}, \tag{5.15}$$

with

$$|\eta_i^{(0)}| \le C_0 h. \tag{5.16}$$

For the weakly singular Volterra term, standard product-integration analysis gives

$$(\mathcal{W}_\mu u)(t_j) = \sum_{\ell=0}^{j-1} \omega_{j\ell}^{(\mu)} u_\ell + \eta_j^{(1)}, \qquad |\eta_j^{(1)}| \le C_1 h^{1-\mu} \tag{5.17}$$

Likewise, the weakly singular boundary quadrature satisfies

$$\mathcal{B}_{\rho,\kappa}(u) = \sum_{\ell=0}^{N-1} \beta_\ell^{(\rho,\kappa)} u_\ell + \eta^{(2)}, \qquad |\eta^{(2)}| \le C_2\left(h^{1-\rho} + h^{1-\kappa}\right). \tag{5.18}$$

Step 2: Subtraction of the discrete scheme. Subtract (5.6) from (5.15). Then

$$E_i = \eta_i^{(0)} + \sum_{j=0}^{N-1} h\, H(t_i,t_j)a(t_j)\Bigg[f\Big(t_j, u_j, (\mathcal{W}_\mu u)(t_j), \mathcal{B}_{\rho,\kappa}(u)\Big) - f\left(t_j, U_j, \sum_{\ell=0}^{j-1} \omega_{j\ell}^{(\mu)} U_\ell, \sum_{\ell=0}^{N-1} \beta_\ell^{(\rho,\kappa)} U_\ell\right)\Bigg].$$

Taking the absolute values and applying the Lipschitz estimate (5.11), we obtain

$$|E_i| \le |\eta_i^{(0)}| + \sum_{j=0}^{N-1} h\, H(t_i,t_j)a(t_j)\Bigg(L_x|E_j| + L_y\left|(\mathcal{W}_\mu u)(t_j) - \sum_{\ell=0}^{j-1} \omega_{j\ell}^{(\mu)} U_\ell\right| + L_z\left|\mathcal{B}_{\rho,\kappa}(u) - \sum_{\ell=0}^{N-1} \beta_\ell^{(\rho,\kappa)} U_\ell\right|\Bigg).$$

Now substitute (5.17) and (5.18) into the last expression. This gives

$$|E_i| \le |\eta_i^{(0)}| + \sum_{j=0}^{N-1} h\, H(t_i,t_j)a(t_j)\Bigg(L_x|E_j| + L_y\left|\sum_{\ell=0}^{j-1} \omega_{j\ell}^{(\mu)} (u_\ell - U_\ell) + \eta_j^{(1)}\right| + L_z\left|\sum_{\ell=0}^{N-1} \beta_\ell^{(\rho,\kappa)} (u_\ell - U_\ell) + \eta^{(2)}\right|\Bigg).$$

Step 3: Derivation of the perturbation inequality. Because all quadrature weights are nonnegative,

$$\sum_{\ell=0}^{j-1} \omega_{j\ell}^{(\mu)} |E_\ell| \le \left(\sum_{\ell=0}^{j-1} \omega_{j\ell}^{(\mu)}\right) \|E\|_\infty \le K_\mu \|E\|_\infty,$$

and

$$\sum_{\ell=0}^{N-1} \beta_\ell^{(\rho,\kappa)} |E_\ell| \le B_{\rho,\kappa}\|E\|_\infty.$$

Therefore, we have

$$|E_i| \le C_0 h + \sum_{j=0}^{N-1} h\, H(t_i,t_j)a(t_j)[\, L_x\|E\|_\infty + L_y\left(K_\mu\|E\|_\infty + C_1 h^{1-\mu}\right) + L_z\left(B_{\rho,\kappa}\|E\|_\infty + C_2(h^{1-\rho} + h^{1-\kappa})\right)].$$

Using the definition of C_H in (5.12), we arrive at

$$|E_i| \le C_0 h + C_H\left(L_x + L_y K_\mu + L_z B_{\rho,\kappa}\right)\|E\|_\infty + C_H L_y C_1 h^{1-\mu} + C_H L_z C_2\left(h^{1-\rho} + h^{1-\kappa}\right).$$

Hence, there exists a constant $C_3 > 0$, independent of h, such that

$$|E_i| \le C_3\left(h + h^{1-\mu} + h^{1-\rho} + h^{1-\kappa}\right) + q\|E\|_\infty,$$

where

$$q := C_H \left(L_x + L_y K_\mu + L_z B_{\rho,\kappa} \right).$$

Taking the maximum over i yields the discrete perturbation inequality

$$\|E\|_\infty \le C_3 \left(h + h^{1-\mu} + h^{1-\rho} + h^{1-\kappa} \right) + q\|E\|_\infty.$$

Step 4: Absorption of the contractive term. By Assumption (5.12), one has $q < 1$. Therefore, the last term on the right-hand side can be absorbed into the left-hand side, and we obtain

$$(1-q)\|E\|_\infty \le C_3 \left(h + h^{1-\mu} + h^{1-\rho} + h^{1-\kappa} \right).$$

Consequently,

$$\|E\|_\infty \le \frac{C_3}{1-q} \left(h + h^{1-\mu} + h^{1-\rho} + h^{1-\kappa} \right),$$

which proves (5.13). The estimate (5.14) follows immediately from the definition of

$$\nu = \min\{1, 1-\mu, 1-\rho, 1-\kappa\}$$

The proof is complete. □

Remark 5.2. To the best of our knowledge, Theorem 5.1 is among the first convergence results for a Nyström-type discretization of a doubly nonlocal fractional integro-boundary problem with simultaneous weak singularities in the state and boundary operators. The estimate is stated for a uniform grid, but the graded-mesh formulation (5.8)–(5.10) is included because it is computationally natural when endpoint singularities dominate the error.

Remark 5.3. The local contractivity condition (5.12) is used only in the final absorption step of the perturbation argument. It is therefore a sufficient stability condition for the convergence proof, not a structural requirement of the analytic existence theory. In more general nonlinear regimes, one may replace this contraction argument by a combination of consistency, a priori bounds, and a fixed-point index or topological degree for the discrete operator, or by a local invertibility estimate for the linearized discrete problem along an isolated stable solution branch. Such approaches could cover non-globally Lipschitz or mixed-growth nonlinearities, but they require additional monotonicity, spectral, or branch-isolation assumptions and are left for future work.

6. Numerical experiments

This section is organized to match the computational emphasis of the manuscript. First, the assumptions of the existence theorem are verified for a concrete model. Second, the discrete solver is tested on a sequence of uniform meshes. Third, the influence of the boundary parameter λ and the singularity parameters (μ, ρ, κ) is examined. Finally, uniform and graded meshes are compared in a more singular regime.

6.1. Benchmark problem and theorem verification

Consider

$$\begin{cases} D_{0+}^{5/2}u(t) + (1+t)\,f(t, u(t), (\mathcal{W}_{0.3}u)(t), \mathcal{B}_{0.2,0.2}(u)) = 0, \qquad 0 < t < 1, \\ I_{0+}^{1/2}u(0) = 0, \qquad u'(0) = 0, \\ u(1) = 0.18\,\mathcal{B}_{0.2,0.2}(u), \end{cases} \tag{6.1}$$

where

$$(\mathcal{W}_{0.3}u)(t) = \int_0^t \frac{1+t-s}{(t-s)^{0.3}}u(s)\,ds, \qquad \mathcal{B}_{0.2,0.2}(u) = \int_0^1 s^{-0.2}(1-s)^{-0.2}u(s)\,ds, \tag{6.2}$$

and

$$f(t, x, y, z) = g(x) + 0.01y + 0.01z, \tag{6.3}$$

with

$$g(x) = \begin{cases} 5x, & 0 \le x \le 0.01, \\ 0.05 + \dfrac{3.3 - 0.05}{0.02}(x - 0.01), & 0.01 < x < 0.03, \\ 3.3 - 0.1(x - 0.03), & x \ge 0.03. \end{cases} \tag{6.4}$$

With $\theta = 0.3$, one computes

$$\Theta = 1 - 0.18\,B(2.3, 0.8) \approx 0.8887 > 0, \qquad K_{0.3} \approx 2.0168, \qquad B_{0.2,0.2} = B(0.8, 0.8) \approx 1.5170.$$

Numerical evaluation of the kernel bounds gives

$$M \approx 1.3072, \qquad m_\theta \approx 0.1121, \qquad \delta = \frac{m_\theta}{M} \approx 0.0857.$$

Moreover, we have

$$A = M \int_0^1 \phi(s)(1+s)\,ds \approx 0.1623, \qquad B = m_\theta \int_{0.3}^1 \phi(s)(1+s)\,ds \approx 0.0107.$$

Choosing

$$r_1 = 0.01, \qquad r_2 = 0.03, \qquad r_3 = 0.6,$$

one obtains

$$\frac{r_1}{A} \approx 0.0616, \qquad \frac{r_2}{B} \approx 2.8096, \qquad \frac{r_3}{A} \approx 3.6960,$$

and the inequalities in (H1)–(H3) are verified directly. Hence, Theorem 4.1 yields at least three positive solutions for (6.1)

6.2. Baseline convergence on uniform meshes

System (5.6) was solved by fixed-point iteration on midpoint grids with $N = 40, 80, 160$. The resulting residuals, peak amplitudes, and inter-grid differences are listed in Table 1. Here

$$\|U^{(N)} - \mathcal{I}_{2N}U^{(2N)}\|_\infty \tag{6.5}$$

denotes the difference between the N-point solution and the linearly interpolated $2N$-point solution evaluated at the coarse nodes.

Table 1. Uniform-grid results for the benchmark problem.

N	Iterations	$\|R_N\|_\infty$	$\max_i U_i$	$\|U^{(N)} - \mathcal{I}_{2N}U^{(2N)}\|_\infty$
40	12	5.21×10^{-14}	0.288603	2.51×10^{-4}
80	12	8.99×10^{-15}	0.288375	5.48×10^{-5}
160	11	5.88×10^{-14}	0.288428	—

The inter-grid ratio gives the empirical order

$$\log_2\left(\frac{2.51 \times 10^{-4}}{5.48 \times 10^{-5}}\right) \approx 2.19.$$

For the finest grid, the interpolated values

$$u(0.25) \approx 0.12850, \qquad u(0.50) \approx 0.26279, \qquad u(0.75) \approx 0.26845$$

show a smooth positive profile whose peak is approximately 0.28843.

6.3. Sensitivity with respect to the boundary parameter

To illustrate the role of the doubly nonlocal boundary feedback, the parameter λ is varied while the remaining coefficients of (6.1) are kept fixed. Table 2 shows that larger values of λ decrease the nonresonance margin Θ and lead to a noticeable increase in both the solution's amplitude and the boundary functional.

Table 2. Sensitivity with respect to the boundary parameter λ on a uniform mesh with $N = 160$.

λ	Θ	$\max_i U_i$	$\mathcal{B}_{\rho,\kappa}(U)$	$U(0.5)$	Iterations
0.10	0.938191	0.277518	0.239775	0.255123	11
0.18	0.888744	0.288428	0.253157	0.262795	11
0.26	0.839297	0.300901	0.268122	0.271373	11

This trend is consistent with the analytic role of Θ in the Green representation: Stronger boundary feedback amplifies the nonlocal contribution and shifts the positive branch upward.

6.4. Influence of singularity strength and graded meshes

The singularity parameters are next varied simultaneously, still with a uniform mesh and the same nonlinear function. The results in Table 3 indicate that stronger singularity on both the equation side and the boundary side produces a larger positive profile and a smaller nonresonance margin.

Table 3. Influence of the singularity parameters on a uniform mesh with $N = 160$ and $\lambda = 0.16$.

μ	ρ	κ	Θ	$\max_i U_i$	$U(0.5)$	Iterations
0.20	0.10	0.10	0.920853	0.282016	0.258292	11
0.30	0.20	0.20	0.901106	0.285569	0.260798	11
0.45	0.35	0.35	0.858191	0.292976	0.265968	11

For the stronger singularity regime $(\mu, \rho, \kappa) = (0.45, 0.35, 0.35)$, uniform and graded meshes were compared against a reference solution computed on a graded mesh with $N = 1600$ and $\gamma = 3$. The results are reported in Table 4. On the coarsest grid, the graded mesh with $\gamma = 2$ reduces the maximum error from 1.79×10^{-4} to 1.19×10^{-4}, which is consistent with the expectation that endpoint clustering helps in the presence of stronger singular behavior.

Table 4. Uniform versus graded meshes for the stronger singularity regime $(\mu, \rho, \kappa) = (0.45, 0.35, 0.35)$ with $\lambda = 0.16$.

Mesh	N	$\max_i U_i$	$\|U^{(N)} - U^{\mathrm{ref}}\|_\infty$	Iterations
Uniform	40	0.293112	1.79×10^{-4}	10
Uniform	80	0.292957	2.49×10^{-5}	12
Uniform	160	0.292976	5.26×10^{-6}	11
Graded ($\gamma = 2$)	40	0.292515	1.19×10^{-4}	12
Graded ($\gamma = 2$)	80	0.292894	2.44×10^{-5}	12
Graded ($\gamma = 2$)	160	0.292968	5.91×10^{-6}	11
Graded ($\gamma = 3$)	40	0.293083	3.12×10^{-4}	12
Graded ($\gamma = 3$)	80	0.292722	6.17×10^{-5}	11
Graded ($\gamma = 3$)	160	0.292911	1.16×10^{-5}	11

Taken together, Tables 1–4 show that the proposed solver is robust, that the boundary parameter has a predictable amplifying effect, and that moderate grading can be beneficial in endpoint-dominated configurations.

7. Discussion

The paper was written so that the theoretical and computational components reinforce one another. On the theoretical side, the decisive point is that the Green kernel can be normalized by an endpoint-aware singular profile, which, in turn, permits the construction of a positive cone whose geometry records the interaction between the memory on the equation side and the weighted singularity on the boundary side. On the computational side, the same structure determines the quadrature weights appearing in the Nyström system. This dual role of the kernel is the main reason why the existence theory and the numerical analysis fit together naturally.

Several observations follow from the numerical study. First, the discrete solver is stable over the tested parameter range and converges in a small number of fixed-point iterations. Second, the parameter λ acts as a genuine nonlocal feedback intensity: As λ increases and Θ decreases, the solution branch moves upward. Third, stronger singularity on either side of the model has a visible quantitative effect on the solution's amplitude. Finally, the graded-mesh experiments suggest that moderate endpoint clustering is worthwhile when the singularity parameters are large, although the best grading parameter is problem-dependent.

The present analysis also has clear limitations. Only positive solutions are addressed, so sign-changing or oscillatory branches are not covered. The parameter λ is examined numerically but not yet via bifurcation theory. The convergence proof is based on a local contractivity assumption, which is convenient and transparent but not optimal. Moreover, the graded-mesh discussion is computational rather than fully theoretical: The experiments show the expected trend, but a sharp graded-mesh convergence theorem remains open.

The local contractivity assumption should thus be interpreted as a transparent way to express discrete stability. It may be relaxed if one can prove uniform a priori bounds and isolate a stable discrete branch by degree theory, fixed-point index, monotone iteration, or a suitable one-sided Lipschitz estimate. This would be especially useful for nonlinearities with stronger growth in the Volterra or boundary variables, where a global contraction is too restrictive but local stability around a positive branch may still hold.

The present method should also be viewed within a broader family of approximation paradigms. Least-squares and constrained mock-Chebyshev reconstruction could improve the polynomial accuracy while retaining equispaced data; spectral or collocation methods may be efficient after subtracting the dominant endpoint singular profile; and adaptive discretizations could choose the mesh grading from residual indicators or from the exponents (μ, ρ, κ). The main analytical challenge in all these alternatives is to retain the positivity and stability properties that the Green-kernel-fitted Nyström formulation provides naturally.

These limitations point to several concrete research directions. A first extension is to replace the Leggett–Williams framework by fixed-point index or Krasnosel skii-type arguments in order to treat sign-changing nonlinearities. A second is to use λ as a bifurcation parameter and study the transition from the nonresonant regime $\Theta > 0$ to the critical regime $\Theta = 0$. A third is to combine graded meshes with higher-order local reconstruction so as to improve accuracy for endpoint-dominated solutions. In particular, constrained least-squares product-integration and spectral reconstruction provide two plausible routes for such higher-order extensions. Finally, systems with asymmetric singularities on the equation and boundary sides may reveal a richer kernel geometry than the scalar problem considered here.

8. Conclusions

A doubly nonlocal fractional integro-boundary problem with weakly singular weights has been studied from both the analytic and computational viewpoints. An explicit Green representation and refined kernel bounds were derived, and these were used to construct a positive cone and to obtain a triple-positive-solution theorem. A product-integration Nyström discretization was then analyzed, and the resulting solver was tested on a multi-part numerical study that included mesh refinement, parameter sensitivity, and graded-mesh comparisons. The results show that the proposed framework is suitable not only for existence theory but also for reliable computation of positive solution branches in doubly nonlocal weakly singular fractional models.

Author contributions

Zhao-sheng Wang: Conceptualization, methodology, formal analysis, writing—original draft. Guan-fa Li: Validation, theoretical verification, supervision, writing—review and editing. Shiyu Li: Numerical experiments, investigation, data curation, writing—review and editing.

Use of Generative-AI tools declaration

The authors declare that no generative AI tools were used in the writing or research of this paper.

Conflict of interest

The authors declare that they have no conflict of interest.

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