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Research article

Apostol Bernoulli-Lucas polynomials, Apostol Euler-Lucas polynomials and their properties

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Abstract: Two new famlies of polynomials were defined, namely, Apostol Bernoulli-Lucas polynomials and Apostol Euler-Lucas polynomials. The properties of these two polynomials were investigated, including explicit relations, diffference equations, summation formulae, and differential recurrence relations.

Keywords: apostol Bernoulli-Lucas polynomials; apostol Euler-Lucas polynomials; Lucas sequence
Mathematics Subject Classification: 11B68, 11B83

1. Introduction

Bernoulli polynomials are important research objects in the fields of mathematical analysis, number theory, and combinatorics. With the deepening of research on Bernoulli polynomials, mathematicians have derived a series of generalized Bernoulli polynomials by extending their definition forms and scopes of application [1–3]. In [4–6], the Apostol-Bernoulli polynomials generalize the classical Bernoulli polynomials by introducing parameters.

The Fibonacci sequence is a classic linear recurrence sequence, with its mathematical definition as follows: Initial values $F_0 = 0$, $F_1 = 1$; for any positive integer $n \geq 2$, it satisfies

$$F_n = F_{n-1} + F_{n-2}.$$

The first 10 terms of the Fibonacci sequence are in turn: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... [7–9]. This sequence has been widely applied in various fields.

It is known that the golden ratio $\alpha = \frac{1+\sqrt{5}}{2}$ and its conjugate $\beta = \frac{1-\sqrt{5}}{2}$ (also sometimes referred to as the silver ratio) satisfy $\lim\limits_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$. The Fibonomial coefficients (or golden binomial coefficients) are defined as follows (see [10–12]): For $1 \leq k \leq n$, let

$$\binom{n}{k}_F = \frac{F_n!}{F_k!\,F_{n-k}!},$$

where $F_n! := F_nF_{n-1}\cdots F_2F_1$, with the convention $F_0! = 1$. Additionally, it is stipulated that $\binom{n}{0}_F = 1$ and $\binom{n}{k}_F = 0$ for $k > n$. Further characteristics of these coefficients can be found in reference [10].

Correspondingly, the F-analogue of $(x+y)_F^n$ is expressed as (see [13, 14])

$$(x+y)_F^n = \sum_{k=0}^{n} \binom{n}{k}_F (-1)^{\binom{k}{2}} x^k y^{n-k},$$

where $(x+y)_F^0 := 1$, and we denote $(x+y)_F := (x+y)_F^1$.

The F-analogues of the usual exponential functions are given as (see [9, 10, 15])

$$e_F^t = \sum_{n=0}^{\infty} \frac{t^n}{F_n!},$$

$$E_F^t = \sum_{n=0}^{\infty} (-1)^{\binom{n}{2}} \frac{t^n}{F_n!}.$$

It can be verified that $e_F^{xt}E_F^{yt} = e_F^{(x+y)_F t}$and $e_F^{xt}e_F^{yt} = e_F^{(x+_F y)t}$, where the operation$(x +_F y)^n$ is defined by (see [12, 16, 17])

$$(x +_F y)^n = \sum_{k=0}^{n} \binom{n}{k}_F x^{n-k} y^k.$$

Furthermore, the golden F-derivative operator is introduced (see [10, 15, 16]):

$$D_F^x[f(x)] = \frac{f(\alpha x) - f(\beta x)}{x(\alpha - \beta)},$$

which is a linear operator. From the definition, the following properties can be directly derived:

$$D_F^x[x^n] = F_n x^{n-1},$$

$$D_F^x[e_F^{xt}] = t e_F^{xt},$$

$$D_F^x[E_F^{xt}] = t E_F^{-xt}.$$

In recent years, a number of identities and matrix representations for special polynomials have been given [12, 17, 18]. For example, in [12] the identities and matrix representation of generalized Bernoulli–Fibonacci polynomials $R_n^{(m)}(x;F)$ of order m are explored, which are defined as

$$\left(\frac{z}{2}\right)^m \left(\frac{z}{e_F^z - 1}\right)^m = \left(\frac{z^2}{2e_F^z - 2}\right)^m e_F^{xz} = \sum_{n=0}^{\infty} R_n^{(m)}(x;F) \frac{z^n}{F_n!}, \quad |z| < \frac{2\pi}{\ln|e_F|},$$

where the parameters $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, and $R_n^{(m)}(F) := R_n^{(m)}(0;F)$ are generalized Bernoulli–Fibonacci numbers defined by

$$\left(\frac{z^2}{2e_F^z - 2}\right)^m = \sum_{n=0}^{\infty} R_n^{(m)}(F) \frac{z^n}{F_n!}, \quad |z| < \frac{2\pi}{\ln|e_F|}.$$

In [17], the Apostol-type Bernoulli–Fibonacci polynomials $\mathbb{B}_{w,F}^{(\alpha)}(\zeta;\lambda)$ of order α and the Apostol-type Euler–Fibonacci polynomials $\mathbb{E}_{w,F}^{(\alpha)}(\zeta;\lambda)$ of order α are investigated, which are defined by

$$\left(\frac{d}{\lambda e_F^d - 1}\right)^\alpha e_F^{\zeta d} = \sum_{w=0}^{\infty} \mathbb{B}_{w,F}^{(\alpha)}(\zeta;\lambda)\frac{d^w}{F_w!},$$

and

$$\left(\frac{2}{\lambda e_F^d + 1}\right)^\alpha e_F^{\zeta d} = \sum_{w=0}^{\infty} \mathbb{E}_{w,F}^{(\alpha)}(\zeta;\lambda)\frac{d^w}{F_w!},$$

respectively. [19] investigated the properties of three-dimensional q-Hermite-Appell polynomials, while [20] systematically established the matrix representations and related theories for U-Bernoulli, U-Euler, and U-Genocchi polynomials.

Earlier work looked at Apostol-type polynomials tied to the Fibonacci sequence. Since Fibonacci and Lucas sequences are closely related, we decided to push this idea further. We introduce two new families linked to the Lucas sequence: The Apostol Bernoulli-Lucas polynomials and the Apostol Euler-Lucas polynomials. Along with properties that are similar to those in [17] (like explicit formulas and difference equations), we also give some new results, including Theorem 2.5, Theorem 3.5, Theorem 4.1, and Theorem 4.2. So this work offers a natural way to extend Apostol-type polynomials to the Lucas setting. Its explicit representations and difference equations facilitate the derivation of unprecedented combinatorial identities, such as convolution sums, inversion formulas, matrix erpresentations, and Lucas-style combinatorial summations, and it will establish novel research subjects for the interdisciplinary investigation bridging special functions and combinatorial number theory.

The necessary preliminaries for the present work are given in the following.

The Lucas suquence $\{L_n\}_{n\geq 0}$ is defined by

$$L_{n+2} = L_{n+1} + L_n,$$

with $L_0 = 2$, $L_1 = 1$. Next we provide the Lucas coefficients. The L-factorial is given by

$$L_n! = L_n L_{n-1} \dots L_1, \quad L_0! = 2.$$

The Lucas coefficients $n \geq k \geq 1$ are defined as

$$\binom{n}{k}_L = \frac{L_n!}{L_k!\, L_{n-k}!}$$

with $\binom{n}{0}_L = \frac{1}{2}$ and $\binom{n}{k}_L = 0$ for $n < k$. The Lucas coefficients satisfy

$$\binom{n}{k}_L = \binom{n}{n-k}_L,$$

$$\binom{n}{k}_L \binom{k}{j}_L = \binom{n}{j}_L \binom{n-j}{k-j}_L,$$

$$\binom{n}{k}_L = \binom{n-1}{k-1}_L + \frac{L_n - L_k}{L_n + L_k}\binom{n-1}{k}_L.$$

It can be further obtained that

$$\binom{n}{k}_L = \sum_{i=1}^{n-k+1} \prod_{j=0}^{i-1} \frac{L_{n-j} - L_k}{L_{n-j} + L_k}\binom{n-i}{k-1}_L.$$

We now present a definition that lies at the core of this work; every subsequent reasoning rests on it.

$$(x \oplus y)^n = \sum_{k=0}^{n} \binom{n}{k}_L x^k y^{n-k}. \tag{1.1}$$

The L-exponential function e_L^t is defined as

$$e_L^t = \sum_{n=0}^{\infty} \frac{t^n}{L_n!}. \tag{1.2}$$

Based on (1.1) and (1.2) above, it follows that

$$e_L^x \cdot e_L^y = e_L^{x \oplus y}. \tag{1.3}$$

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. We denote by $\mathcal{A}$ the linear space of analytic functions defined on $\mathbb{C} \setminus \{0\}$, whose power series expansions

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad g(x) = \sum_{n=0}^{\infty} b_n x^n$$

are absolutely convergent at points x, αx, and βx. For any $h \in \mathcal{A}$, the operator $D_L^x : \mathcal{A} \to \mathcal{A}$ is defined as

$$D_L^x[h(x)] = \frac{h(\alpha x) + h(\beta x)}{x}.$$

Thus, we have

$$D_L^x[x^n] = \frac{(\alpha x)^n + (\beta x)^n}{x} = (\alpha^n + \beta^n)x^{n-1} := L_n\, x^{n-1},$$

$$D_L^x[e^x] = \sum_{n=0}^{\infty} L_{n+1} \frac{x^n}{(n+1)!},$$

$$D_L^x[e_L^{kx}] = k e_L^{kx}.$$

Also, the operator D_L^t is a linear operator.

Let $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$, and we define

$$\int_0^x h(t) d_L(t) = x \sum_{n=0}^{\infty} (-1)^n h\left(\frac{\beta^n}{\alpha^{n+1}} x\right) \frac{\beta^n}{\alpha^{n+1}}, \tag{1.4}$$

$$\int_a^b h(t)d_L(t) = \int_0^b h(t)d_L(t) - \int_0^a h(t)d_L(t). \tag{1.5}$$

These two definitions are of vital importance to the proofs of Theorems 2.5 and 3.5 presented in this work. We now furnish two valuable identities in the following.

$$\begin{aligned}\int_0^x t^m d_L(t) =& x\sum_{n=0}^{\infty}(-1)^n\left(\frac{\beta^n}{\alpha^{n+1}}x\right)^m\frac{\beta^n}{\alpha^{n+1}}\\ =& x^{m+1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\beta^n}{\alpha^{n+1}}\right)^m\frac{\beta^n}{\alpha^{n+1}}\\ =& \frac{x^{m+1}}{\alpha^{m+1}}\sum_{n=0}^{\infty}\left(-\frac{\beta^{m+1}}{\alpha^{m+1}}\right)^n\\ =& \frac{x^{m+1}}{\alpha^{m+1}}\cdot\frac{1}{1+\frac{\beta^{m+1}}{\alpha^{m+1}}}\\ =& \frac{x^{m+1}}{L_{m+1}},\end{aligned}$$

$$\begin{aligned}\int_0^x e_L^t d_L(t) =& x\sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{\infty}\frac{\left(\frac{\beta^n}{\alpha^{n+1}}x\right)^k}{L_k!}\frac{\beta^n}{\alpha^{n+1}}\\ =& \sum_{k=0}^{\infty}\frac{x^{k+1}}{L_k!}\frac{1}{\alpha^{k+1}}\sum_{n=0}^{\infty}\left(-\frac{\beta^{k+1}}{\alpha^{k+1}}\right)^n\\ =& \sum_{k=0}^{\infty}\frac{x^{k+1}}{L_k!}\frac{1}{\alpha^{k+1}}\frac{1}{1+\frac{\beta^{k+1}}{\alpha^{k+1}}}\\ =& \sum_{k=1}^{\infty}\frac{x^k}{L_k!}\\ =& e_L^x - \frac{1}{2}.\end{aligned}$$

2. The Apostol Bernoulli-Lucas polynomials

This chapter conducts an investigation into the properties of the Apostol Bernoulli-Lucas polynomials, drawing upon prior notions including Lucas coefficients and other associated definitions.

Let the convergence radius R be given by $R = \inf\left\{|z| > 0 : \lambda e_L^z = 1\right\}$, where we adopt the convention that $R = \infty$ if the set is empty. The Apostol Bernoulli-Lucas polynomials are given by

$$\frac{ze_L^{xz}}{\lambda e_L^z - 1} = \sum_{n=0}^{\infty} B_{n,L}(x;\lambda)\frac{z^n}{L_n!} \quad (|z| < R), \tag{2.1}$$

where $B_{n,L}(\lambda) = B_{n,L}(0;\lambda)$ are Apostol Bernoulli-Lucas numbers, which are generated by

$$\frac{z}{\lambda e_L^z - 1} = \sum_{n=0}^{\infty} B_{n,L}(\lambda)\frac{z^n}{L_n!} \quad (|z| < R). \tag{2.2}$$

The first four Apostol Bernoulli-Lucas numbers $B_{n,L}(\lambda)$ are

$$B_{0,L}(\lambda) = 0, \quad B_{1,L}(\lambda) = \frac{2}{\lambda - 2}, \quad B_{2,L}(\lambda) = -\frac{12\lambda}{(\lambda - 2)^2}, \quad B_{3L}(\lambda) = \frac{16\lambda(5\lambda + 2)}{(\lambda - 2)^3}.$$

The first four Apostol Bernoulli-Lucas polynomials $B_{n,L}(x;\lambda)$ are

$$B_{0,L}(x;\lambda) = 0, \quad B_{1,L}(x;\lambda) = \frac{1}{\lambda - 2}, \quad B_{2,L}(x;\lambda) = \frac{6x}{\lambda - 2} - \frac{12\lambda}{(\lambda - 2)^2},$$

$$B_{3,L}(x;\lambda) = \frac{8x^2(\lambda - 2)^2 - 48x\lambda(\lambda - 2) + 8\lambda(5\lambda + 2)}{(\lambda - 2)^3}.$$

Theorem 2.1. Let $\{B_{n,L}(x;\lambda)\}_{n\geq 0}$ be the Apostol Bernoulli-Lucas polynomials. Then,

$$B_{n,L}(x;\lambda) = \sum_{k=0}^{n} \binom{n}{k}_L B_{k,L}(\lambda)x^{n-k}.$$

Proof. From expansions (1.2), (2.1), and (2.2),

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,L}(x;\lambda)\frac{z^n}{L_n!} &= \frac{ze_L^{xz}}{\lambda e_L^z - 1} \\
&= \sum_{n=0}^{\infty} B_{n,L}(\lambda)\frac{z^n}{L_n!} \sum_{n=0}^{\infty} x^n \frac{z^n}{L_n!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}_L B_{k,L}(\lambda)x^{n-k} \frac{z^n}{L_n!}.
\end{aligned}$$

By comparing the coefficients of $\frac{z^n}{L_n!}$, the conclusion holds.

Theorem 2.2. Let $\{B_{n,L}(x;\lambda)\}_{n\geq 0}$ be the Apostol Bernoulli-Lucas polynomials. Thus,

$$\lambda B_{n,L}(x \oplus 1;\lambda) - B_{n,L}(x;\lambda) = L_n x^{n-1} \quad (n \geq 1).$$

Proof. Using expansions (1.3), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \lambda B_{n,L}(x \oplus 1;\lambda) - B_{n,L}(x;\lambda)\frac{z^n}{L_n!} &= \frac{\lambda z e_L^{(x\oplus 1)z} - ze_L^{xz}}{\lambda e_L^z - 1} \\
&= \frac{ze_L^{xz}(\lambda e_L^z - 1)}{\lambda e_L^z - 1} \\
&= \sum_{n=1}^{\infty} L_n x^{n-1} \frac{z^n}{L_n!}.
\end{aligned}$$

From this, it can be concluded that the above formula holds.

Theorem 2.3. Let $\{B_{n,L}(x;\lambda)\}_{n\geq 0}$ be the Apostol Bernoulli-Lucas polynomials. This yields

$$B_{n,L}(x \oplus y;\lambda) = \sum_{k=0}^{n} \binom{n}{k}_L B_{k,L}(x;\lambda)y^{n-k}.$$

Proof. Using expansions (1.3) , we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,L}(x\oplus y;\lambda)\frac{z^n}{L_n!} &= \frac{ze_L^{(x\oplus y)z}}{\lambda e_L^z - 1}\\ &= \frac{ze_L^{xz}}{\lambda e_L^z - 1} e_L^{yz}\\ &= \sum_{n=0}^{\infty} B_{n,L}(x;\lambda)\frac{z^n}{L_n!}\sum_{n=0}^{\infty} y^n\frac{z^n}{L_n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} B_{k,L}(x;\lambda)\frac{z^k}{L_k!}y^{n-k}\frac{z^{n-k}}{L_{n-k}!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_L B_{k,L}(x;\lambda)y^{n-k}\frac{z^n}{L_n!}.\end{aligned}$$

Thus, it follows that the above equation holds.

Theorem 2.4. Let $\{B_{n,L}(x;\lambda)\}_{n\geq 0}$ be the Apostol Bernoulli-Lucas polynomials. Then,

$$D_L^x[B_{n,L}(x;\lambda)] = L_n B_{n-1,L}(x;\lambda) \quad (n \geq 1).$$

Proof. Using expansions (2.1), we have

$$\begin{aligned}D_L^x\left(\sum_{n=0}^{\infty} B_{n,L}(x;\lambda)\frac{z^n}{L_n!}\right) &= D_L^x\left(\frac{ze_L^{xz}}{\lambda e_L^z - 1}\right)\\ &= \frac{z}{\lambda e_L^z - 1} ze_L^{xz}\\ &= \sum_{n=0}^{\infty} B_{n,L}(x;\lambda)\frac{z^{n+1}}{L_n!}\\ &= \sum_{n=1}^{\infty} L_n B_{n-1,L}(x;\lambda)\frac{z^n}{L_n!}.\end{aligned}$$

Thus, we obtain the following

$$D_L^x[B_{n,L}(x;\lambda)] = L_n B_{n-1,L}(x;\lambda) \quad (n \geq 1).$$

Theorem 2.5. Let $\{B_{n,L}(x;\lambda)\}_{n\geq 0}$ be the Apostol Bernoulli-Lucas polynomials. Then,

$$\int_a^b B_{n,L}(t;\lambda)d_L(t) = \frac{B_{n+1,L}(b;\lambda) - B_{n+1,L}(a;\lambda)}{L_{n+1}}.$$

Proof. Using expansions (1.4) and (1.5), we have

$$\begin{aligned}&\int_a^b B_{n,L}(t;\lambda)d_L(t)\\ &= \int_0^b B_{n,L}(t;\lambda)d_L(t) - \int_0^a B_{n,L}(t;\lambda)d_L(t)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} \binom{n}{k}_L B_{k,L}(\lambda) \int_0^b t^{n-k} d_L(t) - \sum_{k=0}^{n} \binom{n}{k}_L B_{k,L}(\lambda) \int_0^a t^{n-k} d_L(t) \\
&= \sum_{k=0}^{n} \binom{n}{k}_L B_{k,L}(\lambda) \frac{b^{n-k+1}}{L_{n-k+1}} - \sum_{k=0}^{n} \binom{n}{k}_L B_{k,L}(\lambda) \frac{a^{n-k+1}}{L_{n-k+1}} \\
&= \frac{1}{L_{n+1}} \sum_{k=0}^{n+1} \binom{n+1}{k}_L B_{k,L}(\lambda) b^{n-k+1} - \frac{1}{L_{n+1}} \sum_{k=0}^{n+1} \binom{n+1}{k}_L B_{k,L}(\lambda) a^{n-k+1} \\
&= \frac{1}{L_{n+1}} \left[B_{n+1,L}(b;\lambda) - B_{n+1,L}(a;\lambda) \right].
\end{aligned}
$$

3. The Apostol Euler-Lucas polynomials

In this chapter, we investigate the Apostol Euler-Lucas polynomials and derive some of their interesting properties.

Let the convergence radius R be given by $R = \inf\left\{|z| > 0 : \lambda e_L^z = -1\right\}$, where we adopt the convention that $R = \infty$ if the set is empty. The Apostol Euler-Lucas polynomials are given by

$$
\frac{2e_L^{xz}}{\lambda e_L^z + 1} = \sum_{n=0}^{\infty} E_{n,L}(x;\lambda) \frac{z^n}{L_n!} \quad (|z| < R), \tag{3.1}
$$

where $E_{n,L}(\lambda) = E_{n,L}(0;\lambda)$ are Apostol Euler-Lucas numbers, which are generated by

$$
\frac{2}{\lambda e_L^z + 1} = \sum_{n=0}^{\infty} E_{n,L}(\lambda) \frac{z^n}{L_n!} \quad (|z| < R). \tag{3.2}
$$

The first four Apostol Euler-Lucas numbers $E_{n,L}(\lambda)$ are

$$
E_{0,L}(\lambda) = \frac{8}{\lambda+2}, \quad E_{1,L}(\lambda) = -\frac{8}{(\lambda+2)^2}, \quad E_{2,L}(\lambda) = \frac{8\lambda(5\lambda-2)}{(\lambda+2)^3},
$$

$$
E_{3,L}(\lambda) = \frac{8\lambda(-33\lambda^2+28\lambda-4)}{(\lambda+2)^4}.
$$

The first four Apostol Euler-Lucas polynomials $E_{n,L}(x;\lambda)$ are

$$
E_{0,L}(x;\lambda) = \frac{8}{\lambda+2}, \quad E_{1,L}(x;\lambda) = \frac{4x}{\lambda+2} - \frac{4}{(\lambda+2)^2},
$$

$$
E_{2,L}(x;\lambda) = \frac{4x^2}{\lambda+2} - \frac{24x}{(\lambda+2)^2} + \frac{4\lambda(5\lambda-2)}{(\lambda+2)^3},
$$

$$
E_{3,L}(x;\lambda) = \frac{4x^3}{\lambda+2} - \frac{32x^2}{(\lambda+2)^2} + \frac{32\lambda(5\lambda-2)x}{(\lambda+2)^3} + \frac{4\lambda(-33\lambda^2+28\lambda-4)}{(\lambda+2)^4}.
$$

Theorem 3.1. Let $\{E_{n,L}(x;\lambda)\}_{n\geq 0}$ be the sequence of the Apostol Euler-Lucas polynomials. Then, the following is satisfied:

$$
E_{n,L}(x;\lambda) = \sum_{k=0}^{n} \binom{n}{k}_L E_{k,L}(\lambda) x^{n-k}.
$$

Proof. Using expansions (3.1) and (3.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,L}(x;\lambda)\frac{z^n}{L_n!} &= \frac{2e_L^{xz}}{\lambda e_L^z + 1}\\ &= \sum_{n=0}^{\infty} E_{n,L}(\lambda)\frac{z^n}{L_n!}\sum_{n=0}^{\infty} x^n\frac{z^n}{L_n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_L E_{k,L}(\lambda)x^{n-k}\frac{z^n}{L_n!}.\end{aligned}$$

Equating coefficients of $\frac{z^n}{L_n!}$, the conclusion holds.

Theorem 3.2. Let $\{E_{n,L}(x;\lambda)\}_{n\geq 0}$ be the sequence of the Apostol Euler-Lucas polynomials. Then,

$$\lambda E_{n,L}(x\oplus 1;\lambda) + E_{n,L}(x;\lambda) = 2x^n.$$

Proof. Using expansions (1.3), we have

$$\sum_{n=0}^{\infty}\left[\lambda E_{n,L}(x\oplus 1;\lambda) + E_{n,L}(x;\lambda)\right]\frac{z^n}{L_n!} = \frac{2\lambda e_L^{(x\oplus 1)z} + 2e_L^{xz}}{\lambda e_L^z + 1} = 2e_L^{xz} = \sum_{n=0}^{\infty} 2x^n\frac{z^n}{L_n!}.$$

Thus, it follows that the above equation holds.

Theorem 3.3. Let $\{E_{n,L}(x;\lambda)\}_{n\geq 0}$ be the sequence of the Apostol Euler-Lucas polynomials. Then, the following holds true

$$E_{n,L}(x\oplus y;\lambda) = \sum_{k=0}^{n}\binom{n}{k}_L E_{k,L}(x;\lambda)y^{n-k}.$$

Proof. Using expansions (1.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,L}(x\oplus y;\lambda)\frac{z^n}{L_n!} &= \frac{2e_L^{(x\oplus y)z}}{\lambda e_L^z + 1}\\ &= \sum_{n=0}^{\infty} E_{n,L}(x;\lambda)\frac{z^n}{L_n!}\sum_{n=0}^{\infty} y^n\frac{z^n}{L_n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} E_{k,L}(x;\lambda)\frac{z^k}{L_k!}y^{n-k}\frac{z^{n-k}}{L_{n-k}!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_L E_{k,L}(x;\lambda)y^{n-k}\frac{z^n}{L_n!}.\end{aligned}$$

Equating coefficients of $\frac{z^n}{L_n!}$, the conclusion holds.

Theorem 3.4. Let $\{E_{n,L}(x;\lambda)\}_{n\geq 0}$ be the sequence of the Apostol Euler-Lucas polynomials. Then, we have

$$D_L^x[E_{n,L}(x;\lambda)] = L_n E_{n-1,L}(x;\lambda) \quad (n\geq 1).$$

Proof. Using expansions (3.1),

$$D_L^x\left(\sum_{n=0}^{\infty} E_{n,L}(x;\lambda)\frac{z^n}{L_n!}\right) = D_L^x\left(\frac{2e_L^{xz}}{\lambda e_L^z + 1}\right)$$

$$
\begin{aligned}
&= z\left(\frac{2e_L^{xz}}{\lambda e_L^z + 1}\right)\\
&= \sum_{n=0}^{\infty} E_{n,L}(x;\lambda)\frac{z^{n+1}}{L_n!}\\
&= \sum_{n=1}^{\infty} L_n E_{n-1,L}(x;\lambda)\frac{z^n}{L_n!}.
\end{aligned}
$$

Thus, it follows that the above equation holds.

Theorem 3.5. Let $\{E_{n,L}(x;\lambda)\}_{n\geq 0}$ be the sequence of the Apostol Euler-Lucas polynomials. Then, we have

$$\int_a^b E_{n,L}(t;\lambda)d_L(t) = \frac{E_{n+1,L}(b;\lambda) - E_{n+1,L}(a;\lambda)}{L_{n+1}}.$$

Proof. Using expansions (1.4) and (1.5), we have

$$
\begin{aligned}
\int_a^b E_{n,L}(t;\lambda)d_L(t) &= \int_0^b E_{n,L}(t;\lambda)d_L(t) - \int_0^a E_{n,L}(t;\lambda)d_L(t)\\
&= \int_0^b \sum_{k=0}^{n} \binom{n}{k}_L E_{k,L}(\lambda)t^{n-k}d_L(t) - \int_0^a \sum_{k=0}^{n} \binom{n}{k}_L E_{k,L}(\lambda)t^{n-k}d_L(t)\\
&= \sum_{k=0}^{n} \binom{n}{k}_L E_{k,L}(\lambda)\frac{b^{n-k+1} - a^{n-k+1}}{L_{n-k+1}}\\
&= \frac{1}{L_{n+1}} \sum_{k=0}^{n+1} \binom{n+1}{k}_L E_{k,L}(\lambda)b^{n-k+1} - \frac{1}{L_{n+1}} \sum_{k=0}^{n+1} \binom{n+1}{k}_L E_{k,L}(\lambda)a^{n-k+1}\\
&= \frac{1}{L_{n+1}}\left[E_{n+1,L}(b;\lambda) - E_{n+1,L}(a;\lambda)\right].
\end{aligned}
$$

4. The relationship between the Apostol Bernoulli-Lucas polynomials and the Apostol Euler-Lucas polynomials

In the following, we will give the relationship between the Apostol Bernoulli-Lucas polynomials $B_{n,L}(x;\lambda)$ and the Apostol Euler-Lucas polynomials $E_{n,L}(x;\lambda)$.

Theorem 4.1. Suppose that n denotes a nonnegative integer. The equality below holds

$$B_{n,L}(x;\lambda) = \frac{L_n}{2}E_{n-1,L}(x;\lambda) + \sum_{k=0}^{n} \binom{n}{k}_L E_{k,L}(x;\lambda)B_{n-k,L}(\lambda).$$

Proof. Using expansions (2.1) and (3.1), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} B_{n,L}(x;\lambda)\frac{z^n}{L_n!} &= \frac{ze_L^{xz}}{\lambda e_L^z - 1}\\
&= \frac{z}{2}\frac{2e_L^{xz}}{\lambda e_L^z + 1}\left(1 + \frac{2}{\lambda e_L^z - 1}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} E_{n,L}(x;\lambda)\frac{z^n}{L_n!}\left(\frac{z}{2}+\sum_{n=0}^{\infty} B_{n,L}(\lambda)\frac{z^n}{L_n!}\right)\\
&= \sum_{n=0}^{\infty} \frac{L_n}{2}E_{n-1,L}(x,\lambda)\frac{z^n}{L_n!}+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_L E_{k,L}(x;\lambda)B_{n-k,L}(\lambda)\frac{z^n}{L_n!}\\
&= \sum_{n=0}^{\infty}\left[\frac{L_n}{2}E_{n-1,L}(x;\lambda)+\sum_{k=0}^{n}\binom{n}{k}_L E_{k,L}(x;\lambda)B_{n-k,L}(\lambda)\right]\frac{z^n}{L_n!}.
\end{aligned}
$$

Theorem 4.2. Let $B_{n,L}(\lambda)$, $E_{n,L}(\lambda)$ be Apostol Bernoulli-Lucas numbers and Apostol Euler-Lucas numbers, respectively, then we have

$$
\sum_{k=0}^{n}\binom{n}{k}_L E_{k,L}(x;\lambda)B_{n-k,L}(\lambda)=\sum_{k=0}^{n}\binom{n}{k}_L E_{k,L}(\lambda)B_{n-k,L}(x;\lambda).
$$

Proof. On the one hand,

$$
\begin{aligned}
\sum_{n=0}^{\infty} E_{n,L}(x;\lambda)\frac{z^n}{L_n!}\frac{z}{\lambda e_L^z-1} &= \sum_{n=0}^{\infty} E_{n,L}(x;\lambda)\frac{z^n}{L_n!}\sum_{n=0}^{\infty} B_{n,L}(\lambda)\frac{z^n}{L_n!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_L E_{k,L}(x;\lambda)B_{n-k,L}(\lambda)\frac{z^n}{L_n!}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\sum_{n=0}^{\infty} E_{n,L}(x;\lambda)\frac{z^n}{L_n!}\frac{z}{\lambda e_L^z-1} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_L E_{k,L}(\lambda)x^{n-k}\frac{z^n}{L_n!}\sum_{n=0}^{\infty} B_{n,L}(\lambda)\frac{z^n}{L_n!}\\
&= \sum_{n=0}^{\infty} E_{k,L}(\lambda)\frac{z^n}{L_n!}\sum_{n=0}^{\infty} x^n\frac{z^n}{L_n!}\sum_{n=0}^{\infty} B_{n,L}(\lambda)\frac{z^n}{L_n!}\\
&= \sum_{n=0}^{\infty} E_{n,L}(\lambda)\frac{z^n}{L_n!}\sum_{n=0}^{\infty} B_{n,L}(x;\lambda)\frac{z^n}{L_n!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_L E_{k,L}(\lambda)B_{n-k,L}(x;\lambda)\frac{z^n}{L_n!}.
\end{aligned}
$$

5. Conclusions

This paper constructs two novel polynomial families related to Lucas sequences, namely, Apostol Bernoulli-Lucas polynomials and Apostol Euler-Lucas polynomials. We systematically derive their basic properties as well as several original theorems extending existing Fibonacci-related Apostol-type polynomial results. Future research can explore more generalized extensions of the proposed polynomials.

Author contributions

C. Li: Conceptualization, methodology, formal analysis, writing – original draft, writing – review and editing; Z. Zhang: Conceptualization, validation, writing – review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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