Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 20677–20710.
DOI: 10.3934/math.2026841
Received: 23 January 2026
Revised: 30 March 2026
Accepted: 07 April 2026
Published: 14 July 2026

Research article

Boundedness of truncated Havin–Maz'ya potentials in Herz spaces with applications to p(·)-Laplace equations

Zareen A. Khan[1], **Waqar Afzal**[2,3,4,*], **Mujahid Abbas**[5,6], **Daniel Breaz**[7] **and Luminiţa-Ioana Cotîrlă**[8]

[1] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia

[2] Abdus Salam School of Mathematical Sciences, Government College University, Lahore 54000, Pakistan

[3] Center for Theoretical Physics, Khazar University, 41 Mehseti Str., Baku, AZ1096, Azerbaijan

[4] International Center for Interdisciplinary Research in Sciences, The University of Lahore, Lahore 54792, Pakistan

[5] Department of Mechanical Engineering Science, Faculty of Engineering and the Built Environment, University of Johannesburg, Johannesburg 2092, South Africa

[6] Department of Medical Research, China Medical University, Taichung 40447, Taiwan

[7] Department of Mathematics, "1 Decembrie 1918" University of Alba Iulia, Alba Iulia, 510009, Romania

[8] Department of Mathematics, Technical University of Cluj-Napoca, Cluj-Napoca, 400114, Romania

* **Correspondence:** Email: waqar_afzal_22@sms.edu.pk.

Abstract: In this paper, we developed a variational framework based on Dirichlet-type energy integrals for the p(·)-Laplace equation, establishing coercivity and weak lower semicontinuity of the associated functional to guarantee existence and uniqueness of minimizers. The central contribution is a pointwise estimation of weak solutions via fractional truncated potentials, employing Havin–Maz'ya-type nonlinear operators in Herz spaces. Our approach subsumes several earlier results, with classical linear potential estimates recovered as limiting cases.

Keywords: analysis of differential equations; truncated Havin–Maz'ya–Wolff potentials; p(·)-Laplace equation; variational methods; variable Herz spaces; boundedness
Mathematics Subject Classification: 42B35, 26D15, 46B25, 47G10

1. Introduction

The p(·)-Laplace equation forms an important foundation of mathematical physics because it arises naturally from variational principles that apply to a broad range of physical processes. This equation emerges from minimizing energy functionals of Dirichlet type, finding applications in determining the stationary points of action functionals and thereby governing the equations of motion of physical systems. In classical mechanics, elasticity theory, fluid dynamics, and electromagnetism, the dynamics and equilibrium of physical models can be formulated rigorously by minimizing suitable energy or action functionals. The variational formulation not only provides a coherent and unified derivation of governing equations, but also reveals inherent structural characteristics including conservation laws, symmetries, and stability properties.

Mathematically, the p(·)-Laplace equation represents a natural extension of variational problems to nonlinear partial differential equations with variable growth. Variational methods provide a powerful approach for studying these equations based on the functional-analytic properties of coercivity, convexity, and weak lower semicontinuity of the energy functional. These properties play a fundamental role in determining existence and uniqueness of solutions as well as their qualitative behavior. Moreover, variational methods are particularly well-suited for investigating nonlinear equations with nonstandard growth, including those with variable exponents, fractional derivatives, and nonlocal interactions. Consequently, the p(·)-Laplace framework remains central to modern mathematical physics, providing a robust foundation for analytical representation of both classical and contemporary physical theories.

The p(·)-Laplace equation plays a key role in the study of variational problems of both classical and fractional type, representing the necessary conditions for extremals of Dirichlet-type energy functionals. Early work on fractional variational problems was formulated by Agrawal [1], who generalized classical variational frameworks to fractional derivatives. Gear et al. [2] discussed numerical integration and computational treatment of such equations, particularly with constraints, while Crampin [3] examined the differential geometric structure and inverse problems in Lagrangian dynamics. Subsequent studies investigated exact solutions of fractional equations [4] and formulations involving Caputo derivatives [5] with generalized transversality conditions [6]. In classical mechanics contexts, Ball [7] studied minimizers and variational aspects of nonlinear equations [8], and Rheinboldt and Rabier [9] developed numerical approaches for their solution. Al-Omari and Alqahtani [10] introduced a primal local soft closure operator in soft primal spaces and established its fundamental properties. Their framework provides useful tools for uncertainty modeling with applications to variational problems, optimization, and nonlinear analysis.

Research in fractional mechanics and numerical simulation continued with Klimek [11] and Groll et al. [12], comparing Euler–Euler and Euler–Lagrange methods. Adrees et al. [13] introduced the class of modified (p, h)-convex stochastic processes, establishing its fundamental algebraic properties and demonstrating that it unifies several existing notions of convexity in the stochastic setting and Sokolichin et al. [14] conducted dynamic simulations of gas-liquid flows. Herzallah and Baleanu [15] studied fractional-order Hamiltonian formulations and the Euler–Lagrange equations, and Khan et al. [16] established convex integral inequalities for two-dimensional convex interval-valued mappings under pseudo interval order relations by employing fractional integral operators with non-singular kernels. Li et al. [17] introduced nonlocal-in-time formulations of kinetic energy of non-conservative systems, and Lagrangian and Eulerian drag models that are consistent between Euler–Lagrange and Euler–Euler approaches were introduced by Balachandar [18]. Afzal [19] investigated the regularity of the Navier–Stokes equations in Herz spaces, demonstrating the applicability of Herz spaces to the analysis of fluid flow and nonlinear partial differential equations, and Kogan and Olver [20] had constructed invariant formulations of the Euler–Lagrange equations. More recently, Alghamdi et al. [21] introduced novel operators in the framework of primal topological spaces and investigated their fundamental properties. Such operators provide useful tools in variational problem analysis.

In recent years, the truncated Havin–Maz'ya–Wolff potential has emerged as a fundamental tool in nonlinear potential theory and modern harmonic analysis, playing a key role in investigating non-trivial regularity properties of solutions to partial differential equations (PDEs). Its formulation embodies a delicate interplay between nonlinearity and nonlocality, providing a natural framework for studying problems involving measure data, nonlinear elliptic operators, and quasilinear PDEs. Unlike classical Riesz potentials, which are primarily linear in nature, the truncated Havin–Maz'ya–Wolff potential is specifically designed to capture nonlinear phenomena, making it indispensable in the analysis of capacities, functional embeddings, and sharp regularity estimates.

Integral operators, including the Hardy–Littlewood maximal operator [22] and various fractional integral operators such as the Riesz [23], Bessel–Riesz [24], and Bochner–Riesz potentials [25], serve as essential tools in harmonic analysis and underpin numerous advances in the study of partial differential equations. Among these, the truncated Havin–Maz'ya–Wolff potential [26] stands out for its central role in nonlinear potential theory, effectively capturing the interplay between nonlocal effects and nonlinear growth phenomena. Almeida et al. [27] investigated the mapping properties of Riesz and Wolff potentials in variable exponent weak Lebesgue spaces and established optimal boundedness results for these nonlinear potential operators.

Let $\psi \in L^1_{loc}(\mathbf{R}^n)$ and let $R > 0$ be finite. For $0 < \alpha < \frac{n}{s}$ and $s > 1$, the truncated Havin–Maz'ya–Wolff potential $\mathcal{V}^R_{\alpha,s}$ is defined by

$$\mathcal{V}^R_{\alpha,s}(\psi)(\varepsilon) := \int_0^R \left(r^{\alpha s} \frac{1}{|\mathcal{B}(\varepsilon, r)|} \int_{\mathcal{B}(\varepsilon,r)} |\psi(\tau)|\, d\tau \right)^{\frac{1}{s-1}} \frac{dr}{r}.$$

Borowski et al. [28] established a sharp rearrangement estimate for generalized Wolff–Havin–Maz'ya potentials, which yields a reduction principle characterizing the boundedness of such operators via a Hardy-type inequality. A direct application of this principle yields local regularity results for very weak solutions to quasilinear elliptic PDEs with non-standard growth in rearrangement-invariant spaces.

In this framework, let $\psi : [0, \infty) \to [0, \infty)$ be a non-decreasing function, $\nu \geq 1$, and $\beta \in [0, \nu]$. Then there exists a constant $\mathfrak{c} = \mathfrak{c}(\nu, \beta) > 0$ such that

$$\left\| \mathcal{V}^R_{\beta,\psi} \varphi \right\|_{L^\infty(\mathbb{R}^\nu)} \leq \mathfrak{c} \int_0^{\omega_\nu R^\nu} t^{\frac{\beta}{\nu}-1}\, \psi\!\left(\omega_\nu^{1-\frac{\beta}{\nu}}\, t^{\frac{\beta}{\nu}}\, \varphi^{**}(t) \right) dt.$$

Here:

- ψ is the given non-decreasing function,
- φ denotes the function under the potential,
- ν is the spatial dimension,

- β is the order parameter of the potential,
- ω_ν is the measure of the unit ball in $\mathbb{R}^\nu$,
- $\mathfrak{c}$ is a positive constant depending only on ν and β,
- R is the truncation parameter,
- t is the integration variable,
- $\varphi^{**}(t)$ is the maximal function of the decreasing rearrangement of φ,
- $L^\infty(\mathbb{R}^\nu)$ denotes the space of essentially bounded measurable functions, and
- dt stands for the Lebesgue measure.

Nonlinear potential theory has been one of the key instruments of qualitative analysis of partial differential equations, especially in the analysis of elliptic and nonlocal equations with measure data and nonstandard growth conditions. One of the pillars of this theory is the Wolff potential that offers an effective basis to obtain pointwise and integral estimates of solutions of quasilinear elliptic equations. Maz'ya and Havin [29] have done early foundational work, including the development of a nonlinear potential-theoretic method of elliptic equations, and Cardona and Ruzhansky [30] investigated subelliptic Sobolev and Besov spaces on compact Lie groups associated with Hörmander vector fields and established embedding theorems between subelliptic and classical Besov spaces. This was followed by systematic research on Wolff potential estimates of elliptic equations with measures by Korte and Kuusi [31], Maly [32], and Lukkari, Maeda, and Marola [33], which indicated that the Wolff potential estimates were effective at fine properties of solutions. Over the past few years, equations of Orlicz growth and equations of (p, q)-growth have been the subject of substantial interest due to interest in materials science and nonlinear elasticity. Abbas et al. [34] established Jensen, Ostrowski, and Hermite–Hadamard-type inequalities for h-convex stochastic processes by employing the center–radius order relation for interval-valued stochastic processes. Baroni [35] generalized Riesz and Wolff potential estimates to a wide range of quasilinear equations, and Chlebicka, Youn, and Zatorska-Goldstein [36] studied problems in vectors where the growth and measure functions are of Orlicz type. Subsequent developments are local boundedness and regularity theorems of nonstandard growth of elliptic equations by Cupini, Marcellini, and Mascolo [37], and finer gradient estimates below the duality exponent by Mingione [38]. These advances are directly related to the theory of rearrangement-invariant and Orlicz spaces, developed by O'Neil [39], where Peetre [40] and Afzal et al. [41] offered the right functional analytic context on which potential estimates should be based. More recently, Kim, Lee, and Lee [42] obtained Wolff potential estimates and a Wiener-type condition in nonlocal equations with Orlicz growth and, in so doing, have shown that nonlinear potential techniques still apply in nonlocal equations. These findings support the strength of Wolff-type potentials as an analytical toolkit that unites the analysis of elliptic and nonlocal equations with a complicated growth behavior. Abd El-Latif, Alqahtani, and Gharib [43] developed a broader class of soft sets through the supra soft δ-closure operator and examined its essential properties.

While the literature on Wolff potentials remains relatively sparse, it is instructive to examine the boundedness of related operators that share direct connections with Wolff-type potentials. Investigations into Riesz potential operators and their boundedness properties have a long history. In a classical setting, pioneering results were obtained by Maas and van Neerven [44], who studied the mapping properties of Riesz transforms associated with elliptic operators on abstract Wiener spaces, opening new pathways in infinite-dimensional analysis. This line of inquiry has since evolved toward more generalized function spaces, including Orlicz–Morrey, Herz–Morrey, and variable exponent settings. For instance, Guliyev and Deringoz [45] analyzed the Riesz potential and its commutators in generalized Orlicz–Morrey spaces. Further contributions in variable exponent frameworks include the works of Wu [46], and Rafeiro and Samko [47], who established boundedness results in Herz–Morrey spaces, Herz spaces, and variable Hardy spaces, respectively. Ahmad et al. [48] introduced the concept of n-fractional polynomial s-like m-convexity associated with Raina's mapping and investigated its fundamental algebraic properties. Ameen and Alqahtani [49] introduced the concept of Baire category soft sets and investigated their symmetric local properties of operators.

Significant progress has also been made in the study of function spaces that generalize classical Morrey and Orlicz frameworks, particularly in the context of Musielak–Orlicz–Morrey spaces [50]. Sawano and Shimomura [51], and more recently Ohno and Shimomura [52], extended classical Sobolev embeddings and the theory of Riesz potential operators to more general settings, including non-doubling measures and unbounded metric spaces. In parallel, Aimar et al. [53] bridged potential theory with multilinear harmonic analysis, formulating a multilinear framework for Riesz potentials within general BMO_β spaces, thereby linking linear and multilinear analysis in a unified setting. Afzal et al. [54] investigated the boundedness of the multidimensional Katugampola operator in Campanato spaces.

Further generalizations have focused on Bessel- and B-type Riesz operators. Hasanov et al. [55] investigated commutators of B-Riesz potentials in B-Morrey spaces, while Burenkov [56] provided a comprehensive survey of developments in Morrey-type spaces. Al-Omari and Alqahtani [57] investigated primal structures with closure operators and established their basic characteristics and boundedness results in grand Herz–Morrey spaces [58]. Afzal et al. [59] resolved several open problems concerning the characterization of function spaces by introducing new geometric properties of Morrey spaces in the Besov framework with non-standard growth. Alqahtani and Abd El-Latif [60] introduced novel operators to characterize separation axioms in topological spaces and established their fundamental properties.

The interplay between such operators and PDE regularity has been highlighted by Afzal et al. [61], who introduced modified and enhanced Riesz potentials to obtain Sobolev-type estimates pertinent to elliptic regularity theory. For additional results on the boundedness and applications of these operators, the reader is referred to [62–65] and the references therein.

Motivation and novelty

The main contributions and novelty of this study are as follows:

1. The key motivation lies in the crucial role played by the boundedness of the truncated Havin–Maz'ya–Wolff potential operator in the analysis of various mathematical and physical evolution equations, particularly for verifying regularity, controllability, and related properties. For the first time, the weak formulation of Euler–Lagrange equations based on Dirichlet-type energy integrals is investigated through the boundedness of this nonlinear potential, establishing coercivity, weak lower semicontinuity, and the existence and uniqueness of minimizers. Previous studies have mainly focused on local frameworks, such as Lebesgue spaces [27] or rearrangement-invariant spaces [28]. In contrast, the present work studies the operator globally in Herz spaces with variable parameters, thereby generalizing earlier results and introducing new structural features.
2. Furthermore, the truncated Havin–Maz'ya potential is nonlinear and, for the first time, is

investigated in Herz spaces. Earlier works primarily utilized linear operators such as Riesz and Bessel potentials [55, 61], which can be viewed as special cases of the current study. This nonlinear operator is more capable of handling nonlinear differential models, whereas linear operators lack this flexibility.

3. We also construct a nontrivial example in a tabulated framework demonstrating that, under specific choices of the exponent function, the Riesz potential may fail to be bounded, whereas the truncated Havin–Maz'ya potential remains well-behaved. Additionally, through a series of remarks, we show that our results generalize several earlier findings obtained under different parameter settings (see, e.g., [62–64]).
4. Finally, as observed in the literature [28, 33, 36, 42], the Wolff potential is directly connected with verifying well-posedness, controllability, and properties such as regularity and smoothing of solutions in classical function spaces. By extending this to the global Herz space framework, this work significantly contributes to the development of these properties in a more general and flexible setting compared to previous studies.

The remainder of this article is organized as follows. In Section 1, we provide a detailed introduction that highlights the main motivation of the study and situates our work within the broader context of boundedness problems for operators arising in harmonic analysis. Section 2 presents the necessary preliminary material, recalling essential definitions, lemmas, and auxiliary results that underpin our theoretical framework. In Section 3, we state and prove the principal results of the paper, emphasizing the mapping properties of truncated Havin–Maz'ya potentials within the considered functional setting. Section 4 is devoted to the investigation of existence, uniqueness, and regularity of weak solutions to the variable-exponent $\mathfrak{p}(\cdot)$-Laplace equation via the Euler–Lagrange framework, leveraging the boundedness of truncated Havin–Maz'ya potentials. Finally, Section 5 concludes the paper with a concise summary of the main findings and discusses potential directions for future research inspired by the present study.

2. Preliminaries

In this section, we lay the groundwork for the subsequent analysis by introducing essential definitions, outlining the core principles underlying operator boundedness and providing a detailed account of generalized Lebesgue and Herz spaces. For a more comprehensive treatment, we refer the reader to the standard references [59, 64].

We denote by $\mathbf{R}^{\mathfrak{n}}$ the $\mathfrak{n}$-dimensional Euclidean space and by $\mathbf{R}^{+}$ the set of positive real numbers. For $\varepsilon \in \mathbf{R}^{\mathfrak{n}}$ and $\mathfrak{r} > 0$, $\mathbf{B}(\varepsilon, \mathfrak{r})$ represents the open ball of radius $\mathfrak{r}$ centered at ε. For a measurable function φ, supp φ denotes its support, $|\omega|$ the Lebesgue measure of a set $\omega \subset \mathbf{R}^{\mathfrak{n}}$, and χ_{E} the characteristic function of a measurable set E. The abbreviation "a.e." stands for "almost everywhere".

Variable exponents are denoted by $\mathcal{P}_0^{\log}(\omega)$ for log-Hölder continuous functions and by $\mathcal{P}(\omega)$ for measurable functions $\mathfrak{q}(\cdot)$ satisfying $1 < \mathfrak{q}_- \le \mathfrak{q}(\varepsilon) \le \mathfrak{q}_+ < \infty$. The space $\mathrm{L}^{\mathfrak{q}(\cdot)}_{\mathrm{loc}}(\omega)$ consists of functions that are locally integrable with respect to $\mathfrak{q}(\cdot)$. We write $\psi \lesssim \phi$ if ψ is bounded by ϕ up to a constant and $\psi \approx \phi$ if both $\psi \lesssim \phi$ and $\phi \lesssim \psi$ hold. Continuous embedding of quasi-normed spaces is denoted by $\omega_1 \hookrightarrow \omega_2$.

The convolution $(\psi * \phi)(\varepsilon)$ is defined as

$$(\psi * \phi)(\varepsilon) := \int_{\mathbf{R}^n} \psi(\varepsilon - \nu)\, \phi(\nu)\, d\nu.$$

Finally, we employ the classical dyadic decomposition: for $\mathfrak{l} \in \mathbb{Z}$, set

$$\mathbf{B}_{\mathfrak{l}} := \mathbf{B}(0, 2^{\mathfrak{l}}), \quad \mathbf{R}_{\mathfrak{l}} := \mathbf{B}_{\mathfrak{l}} \setminus \mathbf{B}_{\mathfrak{l}-1}, \quad \chi_{\mathfrak{l}} := \chi_{\mathbf{R}_{\mathfrak{l}}},$$

which provides scale-wise localization in function spaces.

2.1. Spaces of variable integrability

We denote by $\mathcal{P}(\mathbf{R}^n)$ the set of all measurable functions $\mathfrak{p} : \mathbf{R}^n \to [1, \infty)$, called variable exponents. For a measurable set $\omega \subset \mathbf{R}^n$ and $\mathfrak{p} \in \mathcal{P}(\mathbf{R}^n)$, we define

$$\mathfrak{p}^+_\omega := \operatorname{ess\,sup}_{\tau\in\omega} \mathfrak{p}(\tau), \qquad \mathfrak{p}^-_\omega := \operatorname{ess\,inf}_{\tau\in\omega} \mathfrak{p}(\tau), \qquad \mathfrak{p}^+ := \mathfrak{p}^+_{\mathbf{R}^n}, \qquad \mathfrak{p}^- := \mathfrak{p}^-_{\mathbf{R}^n}.$$

Throughout, we consider bounded exponents. The variable exponent Lebesgue space $L^{\mathfrak{p}(\cdot)}(\mathbf{R}^n)$ [71] consists of measurable functions ψ such that the modular

$$\rho_{\mathfrak{p}(\cdot)}(\psi) := \int_{\mathbf{R}^n} |\psi(\tau)|^{\mathfrak{p}(\tau)}\, d\tau$$

is finite. Equipped with the norm

$$\|\psi\|_{\mathfrak{p}(\cdot)} := \inf\{\mu > 0 : \rho_{\mathfrak{p}(\cdot)}(\psi/\mu) \le 1\},$$

$L^{\mathfrak{p}(\cdot)}(\mathbf{R}^n)$ forms a Banach function space. For constant exponents $\mathfrak{p}(\tau) \equiv \mathfrak{p}$, this reduces to the classical space $L^{\mathfrak{p}}(\mathbf{R}^n)$. The modular and norm satisfy the unit ball property: $\rho_{\mathfrak{p}(\cdot)}(\psi) \le 1$ if and only if $\|\psi\|_{\mathfrak{p}(\cdot)} \le 1$.

Hölder's inequality generalizes to variable exponents as

$$\|\psi\phi\|_{\mathfrak{s}(\cdot)} \le 2\, \|\psi\|_{\mathfrak{p}(\cdot)}\, \|\phi\|_{\mathfrak{q}(\cdot)}, \qquad \text{with } \frac{1}{\mathfrak{s}(\tau)} = \frac{1}{\mathfrak{p}(\tau)} + \frac{1}{\mathfrak{q}(\tau)} \text{ a.e.},$$

recovering the classical case when $\mathfrak{s}(\cdot) \equiv 1$ and $\mathfrak{q}(\cdot) = \mathfrak{p}'(\cdot)$.

A function $\psi : \mathbf{R}^n \to \mathbf{R}$ is locally log-Hölder continuous if there exists $c_{\log} > 0$ such that

$$|\psi(\tau) - \psi(\varepsilon)| \le \frac{c_{\log}}{\log(e + 1/|\tau - \varepsilon|)} \quad \text{for all } \tau, \varepsilon \in \mathbf{R}^n.$$

We say ψ has log-Hölder continuity at the origin if

$$|\psi(\tau) - \psi(0)| \le \frac{c_{\log}}{\log(e + 1/|\tau|)}, \quad \tau \in \mathbf{R}^n,$$

and at infinity if there exist $\psi_\infty \in \mathbf{R}$ and $c_{\log} > 0$ such that

$$|\psi(\tau) - \psi_\infty| \le \frac{c_{\log}}{\log(e + |\tau|)}, \quad \tau \in \mathbf{R}^n.$$

We denote by $\mathcal{P}^{\log}_0(\mathbf{R}^n)$ the class of exponents with log-Hölder decay at the origin, by $\mathcal{P}^{\log}_\infty(\mathbf{R}^n)$ those with log-Hölder decay at infinity, and by $\mathcal{P}^{\log}(\mathbf{R}^n)$ the exponents which are locally log-Hölder continuous and have log decay at infinity. For $\mathfrak{p} \in \mathcal{P}^{\log}(\mathbf{R}^n)$, the conjugate exponent $\mathfrak{p}' \in \mathcal{P}^{\log}(\mathbf{R}^n)$ as well, with $(\mathfrak{p}')_\infty = (\mathfrak{p}_\infty)'$.

2.2. Some technical lemmas

Various important results have been established in the space $L^{p(\cdot)}(\mathbf{R}^n)$ under the assumption $p \in \mathcal{P}^{\log}(\mathbf{R}^n)$. Local log-Hölder continuity plays a key role in the analysis of classical operators in variable exponent spaces, as it ensures that

$$|\mathbf{B}|^{p_{\mathbf{B}}^-} \approx |\mathbf{B}|^{p_{\mathbf{B}}^+}$$

for balls $\mathbf{B}$ with a sufficiently small radius. The following auxiliary results illustrate that the log-Hölder decay conditions, either at the origin or at infinity, are sufficient to handle variable exponents defined on such domains.

Lemma 2.1 ([64]). *Let* $\beta \in L^\infty(\mathbf{R}^n)$ *and assume* $s_1 > 0$. *If* β *satisfies the log-Hölder continuity conditions near the origin and at infinity, then*

$$s_1^{\beta(\varepsilon)} \lesssim s_2^{\beta(\tau)} \begin{cases} \left(\dfrac{s_1}{s_2}\right)^{\beta^+}, & 0 < s_2 \leq \dfrac{s_1}{2}, \\ 1, & \dfrac{s_1}{2} < s_2 \leq 2s_1, \\ \left(\dfrac{s_1}{s_2}\right)^{\beta^-}, & s_2 > 2s_1, \end{cases}$$

for any $\varepsilon \in \mathbf{B}(0, s_1) \setminus \mathbf{B}(0, s_1/2)$ *and* $\tau \in \mathbf{B}(0, s_2) \setminus \mathbf{B}(0, s_2/2)$*, with implicit constants independent of* ε*,* τ*,* s_1*, and* s_2.

In many situations, one needs to estimate the norm of characteristic functions of balls or cubes when analyzing the behavior of operators in harmonic analysis. While in classical L^p spaces this is straightforward, the situation is more delicate for variable exponents. Nevertheless, for $p \in \mathcal{P}^{\log}(\mathbf{R}^n)$, the following estimates hold, with implied constants depending only on the log-Hölder continuity of p:

(i) For balls $\mathbf{B} \subset \mathbf{R}^n$ of small radius,

$$\|\chi_{\mathbf{B}}\|_{p(\cdot)} \approx |\mathbf{B}|^{1/p(\tau)}, \quad \tau \in \mathbf{B}.$$

(ii) For balls of large radius,

$$\|\chi_{\mathbf{B}}\|_{p(\cdot)} \approx |\mathbf{B}|^{1/p_\infty}.$$

Similar estimates hold for characteristic functions defined on (dyadic) annuli, even without full log-Hölder continuity at every point.

Lemma 2.2 ([64]). *Let* $p \in \mathcal{P}_\infty^{\log}(\mathbf{R}^n)$ *and let*

$$\mathbf{R} = \mathbf{B}(0, r) \setminus \mathbf{B}\Big(0, \frac{r}{2}\Big).$$

If $|\mathbf{R}| \geq 2^{-n}$*, then*

$$\|\chi_{\mathbf{R}}\|_{p(\cdot)} \approx |\mathbf{R}|^{1/p(\tau)} \approx |\mathbf{R}|^{1/p_\infty}, \quad \tau \in \mathbf{R},$$

with implicit constants independent of r *and* $\tau \in \mathbf{R}$. *Moreover, the left-hand side equivalence holds for all* $|\mathbf{R}| > 0$ *if, in addition,* $p \in \mathcal{P}_0^{\log}(\mathbf{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbf{R}^n)$.

Lemma 2.3 ([64]). *Let* $0 < \gamma < 1$ *and* $0 < q \le \infty$. *Consider a sequence of positive real numbers* $\{\varpi_j\}_{j\in\mathbb{Z}}$ *such that*

$$\left\|\{\varpi_j\}_{j\in\mathbb{Z}}\right\|_{\ell^q} = I < \infty.$$

Define two sequences:

$$\delta_l := \sum_{j\le l} \gamma^{l-j}\varpi_j, \qquad \upsilon_l := \sum_{j\ge l} \gamma^{j-l}\varpi_j, \quad l \in \mathbb{Z}.$$

Then $\{\delta_l\}_{l\in\mathbb{Z}}$ *and* $\{\upsilon_l\}_{l\in\mathbb{Z}}$ *also belong to* ℓ^q, *with the estimate*

$$\left\|\{\delta_l\}_{l\in\mathbb{Z}}\right\|_{\ell^q} + \left\|\{\upsilon_l\}_{l\in\mathbb{Z}}\right\|_{\ell^q} \le c\, I,$$

where $c > 0$ *depends only on* γ *and* q.

2.3. Variable exponent Herz spaces

Definition 2.1 ([66]). Let $p, q \in \mathcal{P}(\mathbf{R}^n)$ and $\alpha : \mathbf{R}^n \to \mathbf{R}$ with $\alpha \in L^\infty(\mathbf{R}^n)$.
The *inhomogeneous Herz space* $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ consists of all $\psi \in L^{p(\cdot)}_{\rm loc}(\mathbf{R}^n)$ such that

$$\|\psi\|_{K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \left\|\psi\, \chi_{\mathbf{B}_0}\right\|_{p(\cdot)} + \left(\sum_{l\geqslant 1} \left\|2^{l\alpha(\cdot)}\psi\, \chi_l\right\|^{q(\cdot)}_{p(\cdot)}\right)^{1/q(\cdot)} < \infty.$$

The *homogeneous Herz space* $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ is the set of all $\psi \in L^{p(\cdot)}_{\rm loc}(\mathbf{R}^n \setminus \{0\})$ such that

$$\|\psi\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \left(\sum_{l\in\mathbb{Z}} \left\|2^{l\alpha(\cdot)}\psi\, \chi_l\right\|^{q(\cdot)}_{p(\cdot)}\right)^{1/q(\cdot)} < \infty,$$

with the usual modification when $q(\cdot) = \infty$.
When α, p, and q are constants, these spaces reduce to the classical Herz spaces:

$$K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n) = K^{\alpha}_{p,q}(\mathbf{R}^n), \quad \dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n) = \dot{K}^{\alpha}_{p,q}(\mathbf{R}^n).$$

As in the classical case, these are quasinormed spaces, and they are norms if $p, q \in \mathcal{P}(\mathbf{R}^n)$ and $q \ge 1$.

Remark 2.1. *Several special cases of Definition 2.1 recover known Herz-type spaces depending on the choice of exponents:*

- *If* $\beta(\cdot) = \beta$, $p(\cdot) = p$, *and* $q(\cdot) = q$, *we recover the classical Herz spaces.*
- *If* $\beta(\cdot) = \beta$, $p(\cdot)$ *is variable, and* $q(\cdot) = q$, *we obtain Herz-type spaces with variable* $p(\cdot)$.
- *If* $\beta(\cdot) = \beta$, $p(\cdot) = p$, *and* $q(\cdot) = q$, *the classical Herz spaces are recovered again.*

Proposition 2.1. *Let* $\beta \in L^\infty(\mathbf{R}^n)$ *and* $p, q \in \mathcal{P}(\mathbf{R}^n)$, *and consider the inhomogeneous Herz space* $K^{\beta(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$.

1. *If the exponent function* $\beta(\cdot)$ *is* log-Hölder continuous at infinity, *then the space is equivalent to the corresponding Herz space with a constant exponent at infinity:*

$$K^{\beta(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n) = K^{\beta_\infty}_{p(\cdot),q(\cdot)}(\mathbf{R}^n),$$

where $\beta_\infty := \lim_{|\varepsilon|\to\infty} \beta(\varepsilon)$.

2. *Moreover, if* $\beta(\cdot)$ *additionally satisfies the* logarithmic decay condition near the origin, *then for all* $\psi \in \dot{K}^{\beta(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$, *we have the quasi-norm equivalence*

$$\|\psi\|_{\dot{K}^{\beta(\cdot)}_{p(\cdot),q(\cdot)}} \approx \left\|\{2^{I\beta(0)}\,\psi\,\chi_I\}\right\|_{\ell^{q(\cdot)}_{<}(L^{p(\cdot)})} + \left\|\{2^{I\beta_\infty}\,\psi\,\chi_I\}\right\|_{\ell^{q(\cdot)}_{>}(L^{p(\cdot)})}.$$

Here, the constants implicit in the equivalence depend only on the log-Hölder constants of $\beta(\cdot)$ *and the variable exponents* $p(\cdot), q(\cdot)$.

3. The main results

This section is devoted to a detailed investigation of the boundedness properties of the truncated Havin–Maz'ya–Wolff potential $\mathcal{V}^R_{\alpha,s}$, which constitutes a nonlinear extension of several classical potential operators, recovering them as special or limiting cases. Such a framework plays a fundamental role in modern nonlinear potential theory and provides an effective analytical tool for describing fine regularity, integrability, and qualitative properties of solutions to nonlinear partial differential equations and variational problems.

Let $\psi \in L^1_{loc}(\mathbf{R}^n)$ and let $R > 0$ be finite. For $0 < \alpha < \frac{n}{s}$ and $s > 1$, the truncated Havin–Maz'ya–Wolff potential $\mathcal{V}^R_{\alpha,s}$ is defined by

$$\mathcal{V}^R_{\alpha,s}(\psi)(\varepsilon) := \int_0^R \left(r^{\alpha s}\frac{1}{|\mathcal{B}(\varepsilon,r)|}\int_{\mathcal{B}(\varepsilon,r)}|\psi(\tau)|\,d\tau\right)^{\frac{1}{s-1}}\frac{dr}{r}.$$

The structure of $\mathcal{V}^R_{\alpha,s}$ is intrinsically nonlinear. The localized averaging over balls $\mathcal{B}(\varepsilon, r)$, combined with the nonlinear exponent s, captures the interaction between nonlocality and nonlinearity within a finite spatial scale. The truncation parameter R is particularly significant in boundary value problems and variational models, where global information is unavailable or physically irrelevant.

3.1. Recoverability and comparison with existing operators

A fundamental feature of the Havin–Maz'ya–Wolff potential is its precise connection with classical linear potentials. In particular, it reduces to the Riesz potential in the limiting regime $s \to 1^+$, thereby establishing $\mathcal{V}^R_{\alpha,s}$ as a genuine nonlinear generalization of $\mathcal{I}_\alpha$. Reduction of the Havin–Maz'ya–Wolff potential occurs as $s \to 1^+$.

The truncated Havin–Maz'ya–Wolff potential is

$$\mathcal{V}_{\alpha,s}(\psi)(\varepsilon) = \int_0^\infty \left(r^{\alpha s}\frac{1}{|\mathcal{B}(\varepsilon,r)|}\int_{\mathcal{B}(\varepsilon,r)}|\psi(\tau)|\,d\tau\right)^{\frac{1}{s-1}}\frac{dr}{r}.$$

Since $|\mathcal{B}(\varepsilon, r)| \sim r^n$, the integrand can be written as

$$\left(r^{\alpha s-n}\int_{\mathcal{B}(\varepsilon,r)}|\psi(\tau)|\,d\tau\right)^{\frac{1}{s-1}}.$$

Introduce

$$\mathcal{A}(r) := r^{\alpha}\frac{1}{|\mathcal{B}(\varepsilon, r)|}\int_{\mathcal{B}(\varepsilon,r)} |\psi(\tau)|\, d\tau,$$

so that

$$\mathcal{V}_{\alpha,s}(\psi)(\varepsilon) = \int_0^{\infty} \exp\Big(\frac{\log \mathcal{A}(r)}{s-1}\Big)\frac{dr}{r}.$$

As $s \to 1^+$, we have

$$(\mathcal{A}(r))^{\frac{1}{s-1}} \longrightarrow \mathcal{A}(r),$$

and thus

$$\lim_{s\to 1^+} \mathcal{V}_{\alpha,s}(\psi)(\varepsilon) = \int_0^{\infty} \mathcal{A}(r)\frac{dr}{r} = \int_0^{\infty} r^{\alpha}\frac{1}{|\mathcal{B}(\varepsilon, r)|}\int_{\mathcal{B}(\varepsilon,r)} |\psi(\tau)|\, d\tau\frac{dr}{r}.$$

Applying Fubini's theorem: We can interchange the order of integration because the integrand is non-negative. That is,

$$\int_0^{\infty} \frac{r^{\alpha}}{|\mathcal{B}(\varepsilon, r)|}\int_{\mathcal{B}(\varepsilon,r)} |\psi(\tau)|\, d\tau\frac{dr}{r} = \int_{\mathbf{R}^n} |\psi(\tau)| \int_{r\geq|\varepsilon-\tau|} \frac{r^{\alpha}}{|\mathcal{B}(\varepsilon, r)|}\frac{dr}{r}\, d\tau.$$

Using the layer-cake representation: The inner integral over r is equivalent, up to a constant, to $|\varepsilon - \tau|^{-(n-\alpha)}$. Hence,

$$\int_{\mathbf{R}^n} |\psi(\tau)| \int_{r\geq|\varepsilon-\tau|} \frac{r^{\alpha}}{|\mathcal{B}(\varepsilon, r)|}\frac{dr}{r}\, d\tau \sim \int_{\mathbf{R}^n} \frac{|\psi(\tau)|}{|\varepsilon-\tau|^{n-\alpha}}\, d\tau.$$

Conclusion: We finally get

$$\boxed{\lim_{s\to 1^+} \mathcal{V}_{\alpha,s}(\psi)(\varepsilon) = \mathcal{I}_{\alpha}\psi(\varepsilon)},$$

showing that the Havin–Maz'ya–Wolff potential is a nonlinear generalization of the classical Riesz potential, retaining its scaling while extending to nonlinear and nonlocal settings. The structural progression, from the sublinear operator to the Riesz potential and its nonlinear generalization, the truncated Havin–Maz'ya–Wolff potential, is summarized below in Table 1 and illustrated in Figure 1.

Table 1. Recoverability of classical operators from the truncated Havin–Maz'ya–Wolff potential.

Parameter Settings	Recovered / Refined Operator	Reference
$s \to 1^+$, $0 < \alpha < n$, $R \to \infty$	Classical Riesz potential $\mathcal{I}_{\alpha}$	[72]
$\alpha = 0$, $s \to 1^+$, $R \to \infty$	Classical sublinear operator	[67]

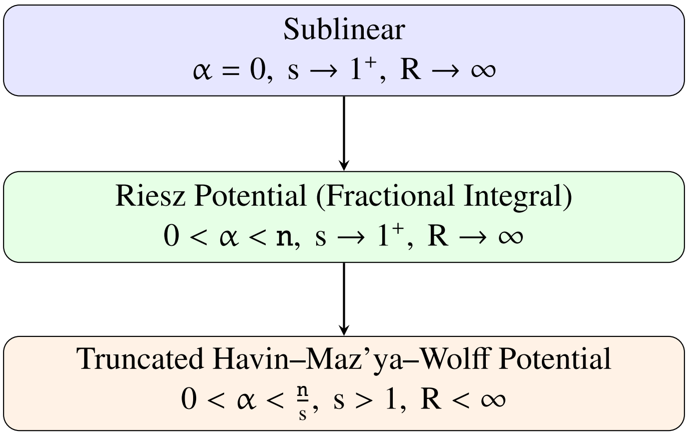

Figure 1. Structural progression from the sublinear operator to the Riesz potential, and its nonlinear generalization, the truncated Havin–Maz'ya–Wolff potential.

Distinctive characteristics of the truncated Havin–Maz'ya–Wolff potential: The classical Riesz potential, being linear, does not fully capture nonlinear interactions and can fail to control certain borderline behaviors in function spaces, such as self-improvement of integrability. In contrast, the truncated Havin–Maz'ya–Wolff potential $\mathcal{V}^R_{\alpha,s}$ introduces a nonlinear averaging mechanism through the exponent $\frac{1}{s-1}$, combined with localized averaging over balls of radius r up to a truncation scale R. This structure enhances both local and global behavior of the potential and provides stronger regularizing effects.

A comparative perspective highlighting the improved properties of the truncated Havin–Maz'ya–Wolff potential $\mathcal{V}^R_{\alpha,s}$ relative to the classical Riesz potential is provided in Table 2.

Table 2. Comparison of the classical Riesz potential and the truncated Havin–Maz'ya–Wolff potential in variable Herz spaces.

Setting	**Classical Riesz Potential** $\mathcal{I}_\alpha$	**Truncated HMW Potential** $\mathcal{V}^R_{\alpha,s}$
Ambient Space	Variable Herz space $\mathbf{K}^{\beta(\cdot)}_{r(\cdot),s(\cdot)}(\mathbb{R}^n)$	Same as left
Exponents	$r(\mu) = 2 + \frac{1}{1+\lvert\mu\rvert} \in (2,3),\ s(\mu) \equiv 4,\ \beta(\mu) \equiv \frac{1}{2}$	Same as left
Potential Parameters	Riesz order $\alpha = 1$	Radius $R > 0$, $\alpha = 1$, $s = 3$ $(1/(s-1) = 1/2)$
Kernel	$\mathcal{I}_1\psi(\varepsilon) = \int_{\mathbb{R}^n} \frac{\psi(\tau)}{\lvert\varepsilon - \tau\rvert^{n-1}} d\tau$	$\mathcal{V}^R_{1,3}(\psi)(\varepsilon) = \int_0^R \left(r^3 \frac{1}{\lvert\mathcal{B}(\varepsilon,r)\rvert} \int_{\mathcal{B}(\varepsilon,r)} \lvert\psi\rvert \right)^{1/2} \frac{dr}{r}$
Result	Insufficient decay; fails to control dyadic tails	Tail self-improvement; stronger integrability

3.2. Truncated Havin–Maz'ya–Wolff potential and its boundedness in variable Herz spaces

We now state the main results.

Theorem 3.1. *Let* $n \in \mathbb{N}$, *with* $1 < p^- \leq p^+ < \infty$ *and* $0 < q \leq \infty$. *Suppose that* $p(\cdot) \in \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ *is a variable exponent function satisfying*

$$1 < p^- \leq p^+ < \infty.$$

Further, let $a(\cdot) \in L^\infty(\mathbb{R}^n)$ *be log-Hölder continuous at infinity, and denote its limiting value at infinity* *by* a_∞*, which is assumed to satisfy*

$$-\frac{n}{p_\infty} < a_\infty < \frac{n}{p'_\infty}.$$

For $\varepsilon \in \mathbb{R}^n$ *and* $r > 0$*, let*

$$\mathcal{B}(\varepsilon, r) := \{\tau \in \mathbb{R}^n : |\tau - \varepsilon| < r\}$$

denote the open ball of radius r *centered at* ε*.*

Let $s > 1$ *and* $\sigma > 0$*, and let* $\psi : \omega \subset \mathbb{R}^n \to \mathbb{R}^m$*, with* $m \geq 1$*, belong to* $L^1_{loc}(\omega)$*. For a truncation radius* $R > 0$ *such that* $\mathcal{B}_R(\varepsilon) \subset \omega$*, define the truncated Havin–Maz'ya–Wulff potential by*

$$\mathcal{V}^R_{\alpha,s}\psi(\varepsilon) := \int_0^R \left(r^{\alpha s} \int_{\mathcal{B}_r(\varepsilon)} |\psi(\tau)| \, d\tau \right)^{\frac{1}{s-1}} \frac{dr}{r}, \quad \varepsilon \in \omega.$$

Then, the operator $\mathcal{V}^R_{\alpha,s}$ *extends boundedly to the space* $\mathbf{K}^{a(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$*; that is, there exists a constant* $\mathcal{M} > 0$ *such that*

$$\left\| \mathcal{V}^R_{\alpha,s}(\psi) \right\|_{\mathbf{K}^{a(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)} \leq \mathcal{M} \, \|\psi\|_{\mathbf{K}^{a(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)},$$

for all $\psi \in \mathbf{K}^{a(\cdot)}_{p(\cdot),q}(\mathbb{R}^n) \cap L^{p(\cdot)}(\mathbb{R}^n)$*.*

Proof. Let $\mathcal{V}^R_{\alpha,s}$ be the truncated Havin–Maz'ya potential defined by

$$\mathcal{V}^R_{\alpha,s}(\psi)(\varepsilon) := \int_0^R \left(r^{\alpha s} \frac{1}{|\mathcal{B}(\varepsilon, r)|} \int_{\mathcal{B}(\varepsilon,r)} |\psi(\tau)| \, d\tau \right)^{\frac{1}{s-1}} \frac{dr}{r}, \quad 0 < \alpha < \frac{n}{s}.$$

The operator $\mathcal{V}^R_{\alpha,s}$ is bounded on the space $L^{p(\cdot)}(\mathbb{R}^n)$. Let $a(\cdot) \in L^\infty(\mathbb{R}^n)$ be a function exhibiting log-Hölder continuity at infinity. With this property, the operator $\mathcal{V}^R_{\alpha,s}$ can be continuously extended and is bounded on the homogeneous Herz space $\mathbf{K}^{a(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$.

To establish this result, we investigate the Herz quasi-norm of $\mathcal{V}^R_{\alpha,s}(\psi)$ by employing a scale-wise decomposition of the domain. Specifically, for a given $l \in \mathcal{Z}$, we decompose the function ψ into its components corresponding to each scale,

$$\psi = \psi \cdot \zeta_{\mathcal{B}_{l-1}} + \psi \cdot \zeta_{\mathcal{B}_{l+1} \setminus \mathcal{B}_{l-1}} + \psi \cdot \zeta_{\mathcal{B}_R \setminus \mathcal{B}_{l+1}},$$

where ζ is the characteristic function and $\mathcal{B}_l := \mathcal{B}(0, 2^l)$.

This leads to the decomposition:

$$\left| \mathcal{V}^R_{\alpha,s}\psi(\varepsilon) \right| \leq \left| \mathcal{V}^R_{\alpha,s}\left(\psi \zeta_{\mathcal{B}_{l-1}}\right)(\varepsilon) \right| + \left| \mathcal{V}^R_{\alpha,s}\left(\psi \zeta_{\mathcal{B}_{l+1} \setminus \mathcal{B}_{l-1}}\right)(\varepsilon) \right| + \left| \mathcal{V}^R_{\alpha,s}\left(\psi \zeta_{\mathcal{B}_R \setminus \mathcal{B}_{l+1}}\right)(\varepsilon) \right|.$$

Next, we estimate the contribution of the first term, $\mathcal{V}^R_{\alpha,s}(\psi \, \zeta_{\mathcal{B}_{l-1}})$, for points $\varepsilon \in \mathcal{A}_l := \mathcal{B}_l \setminus \mathcal{B}_{l-1}$.

Note that for any $\tau \in \mathcal{B}_{l-1}$, we have $|\tau| < 2^{l-1}$, while for $\varepsilon \in \mathcal{A}_l$, $|\varepsilon| \geq 2^{l-1}$, we obtain the following estimate:

$$|\varepsilon - \tau| \gtrsim |\varepsilon| \approx 2^l.$$

Hence, for $0 < r \leq 2^{l-1}$ (where the inner integral is nontrivial), the kernel can be estimated by

$$r^{\alpha s} \frac{1}{|\mathcal{B}(\varepsilon, r)|} \int_{\mathcal{B}(\varepsilon,r)} |\psi(\tau)| \, d\tau \lesssim 2^{l\alpha s} \frac{1}{|\mathcal{B}_{l-1}|} \int_{\mathcal{B}_{l-1}} |\psi(\tau)| \, d\tau.$$

Therefore, the first term satisfies the pointwise estimate

$$\left|\mathcal{V}^{R}_{\alpha,s}\left(\psi\zeta_{\mathcal{B}_{l-1}}\right)(\varepsilon)\right| \lesssim \left(\int_0^{2^{l-1}}\left[2^{l\alpha s}\frac{1}{|\mathcal{B}_{l-1}|}\int_{\mathcal{B}_{l-1}}|\psi(\tau)|\,d\tau\right]^{\frac{1}{s-1}}\frac{dr}{r}\right) \lesssim 2^{\frac{l\alpha s}{s-1}}\frac{1}{|\mathcal{B}_{l-1}|^{\frac{1}{s-1}}}\int_{\mathcal{B}_{l-1}}|\psi(\tau)|\,d\tau.$$

By applying Hölder's inequality adapted to spaces with nonstandard growth on the exponent, we obtain

$$\int_{\mathcal{B}_{l-1}}|\psi(\tau)|\,d\tau \leq \left\|\psi\zeta_{\mathcal{B}_{l-1}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\cdot\left\|\zeta_{\mathcal{B}_{l-1}}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

Observing that $\mathcal{B}_{l-1}$ is a ball, the norm $\left\|\zeta_{\mathcal{B}_{l-1}}\right\|_{L^{p'(\cdot)}}$ can be effectively estimated by $|\mathcal{B}_{l-1}|^{\frac{1}{p_*}}$, where p_* denotes an appropriate averaged value of p' over the ball. This leads to the following bound:

$$\left|\mathcal{V}^{R}_{\alpha,s}(\psi\zeta_{\mathcal{B}_{l-1}})(\varepsilon)\right| \lesssim 2^{\frac{l\alpha s}{s-1}}\left\|\psi\zeta_{\mathcal{B}_{l-1}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Analogous reasoning can be applied to estimate the contributions of the remaining terms. We now turn our attention to the truncated Havin–Maz'ya potential acting on the "inner portion" of the function:

$$\mathcal{V}^{R}_{\alpha,s}(\psi\,\zeta_{\mathcal{B}_{l-2}\setminus\mathcal{B}_{-2}})(\varepsilon).$$

Consider a point $\varepsilon \in \mathcal{A}_l := \mathcal{B}_l \setminus \mathcal{B}_{l-1}$ and a point $\tau \in \mathcal{B}_{l-2} \setminus \mathcal{B}_{-2}$. For any dyadic annulus $\mathcal{A}_j$ with $j \leq l-2$, it follows that

$$|\varepsilon - \tau| \gtrsim |\varepsilon| \approx 2^l.$$

This observation allows us to control the contribution of the inner part in terms of the scale of the outer annulus. Thus, for $\varepsilon \in \mathcal{A}_l$, the truncated Havin–Maz'ya kernel satisfies

$$\left(r^{\alpha s}\frac{1}{|\mathcal{B}(\varepsilon,r)|}\int_{\mathcal{B}(\varepsilon,r)}|\psi(\tau)|\,d\tau\right)^{\frac{1}{s-1}} \lesssim 2^{\frac{l\alpha s}{s-1}}\int_{\mathcal{A}_j}|\psi(\tau)|\,d\tau.$$

Hence, we obtain the pointwise estimate:

$$\left|\mathcal{V}^{R}_{\alpha,s}(\psi\,\zeta_{\mathcal{B}_{l-2}\setminus\mathcal{B}_{-2}})(\varepsilon)\right| \lesssim 2^{\frac{l\alpha s}{s-1}}\sum_{j=-1}^{l-2}\int_{\mathcal{A}_j}|\psi(\tau)|\,d\tau.$$

Next, we apply Hölder's inequality on each annular layer and combine this with the standard estimate for the associated characteristic functions:

$$\int_{\mathcal{A}_j}|\psi(\tau)|\,d\tau \leq \left\|\psi\zeta_{\mathcal{A}_j}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\cdot\left\|\zeta_{\mathcal{A}_j}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

so that

$$\left|\mathcal{V}^{R}_{\alpha,s}(\psi\,\zeta_{\mathcal{B}_{l-2}\setminus\mathcal{B}_{-2}})(\varepsilon)\right| \lesssim \sum_{j=-1}^{l-2}2^{\frac{l\alpha s}{s-1}}\left\|\psi\zeta_{\mathcal{A}_j}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\left\|\zeta_{\mathcal{A}_j}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

By the boundedness of $\mathcal{V}^{R}_{\alpha,s}$ on $L^{p(\cdot)}$, we have

$$\left\|\zeta_{\mathcal{A}_l}\mathcal{V}^{R}_{\alpha,s}(\psi\,\zeta_{\mathcal{B}_{l-2}\setminus\mathcal{B}_{-2}})\right\|_{L^{p(\cdot)}} \lesssim \left\|\psi\zeta_{\mathcal{B}_{l-2}\setminus\mathcal{B}_{-2}}\right\|_{L^{p(\cdot)}}.$$

Using the quasi-triangle inequality for $L^{p(\cdot)}$ norms:

$$\left\|\psi\, \zeta_{\mathcal{B}_{l-2}\setminus\mathcal{B}_{-2}}\right\|_{L^{p(\cdot)}} \lesssim \sum_{j=-1}^{l-2} \left\|\psi\, \zeta_{\mathcal{A}_j}\right\|_{L^{p(\cdot)}}.$$

Hence,

$$\left\|\zeta_{\mathcal{A}_l}\, \mathcal{V}^{R}_{\alpha,s}\left(\psi\, \zeta_{\mathcal{B}_{l-2}\setminus\mathcal{B}_{-2}}\right)\right\|_{L^{p(\cdot)}} \lesssim \sum_{j=-1}^{l-2} \left\|\psi\, \zeta_{\mathcal{A}_j}\right\|_{L^{p(\cdot)}}.$$

Taking the ℓ^{q}-quasinorm over $l \in \mathcal{Z}$ and applying a generalized Minkowski inequality, we get

$$\begin{aligned}
&\left\{\sum_{l\in\mathcal{Z}} 2^{l\alpha_\infty q} \left\|\zeta_{\mathcal{A}_l}\, \mathcal{V}^{R}_{\alpha,s}\left(\psi\, \zeta_{\mathcal{B}_{l-2}\setminus\mathcal{B}_{-2}}\right)\right\|^{q}_{L^{p(\cdot)}}\right\}^{\frac{1}{q}} \\
&\lesssim \left\{\sum_{l\in\mathcal{Z}} 2^{l\alpha_\infty q} \left(\sum_{j=-1}^{l-2} \left\|\psi\, \zeta_{\mathcal{A}_j}\right\|_{L^{p(\cdot)}}\right)^{q}\right\}^{\frac{1}{q}} \\
&\lesssim \left\{\sum_{j\in\mathcal{Z}} \left(\sum_{l=j+2}^{\infty} 2^{l\alpha_\infty q}\right)^{\frac{1}{q}} \left\|\psi\, \zeta_{\mathcal{A}_j}\right\|_{L^{p(\cdot)}}\right\}^{\frac{1}{q}}.
\end{aligned}$$

The inner sum converges for $\alpha_\infty q < 0$, and assuming the correct scaling for the Herz space together with the log-Hölder continuity, the final result is bounded by the Herz norm of ψ.

Estimation of $\mathcal{V}^{R}_{\alpha,s}\left(\psi\, \zeta_{\mathbb{R}^n\setminus\mathbf{B}_{l+2}}\right)(\varepsilon)$:

We now consider the contribution of the term where the support of the function is located far from the point of evaluation. Let $\varepsilon \in \mathcal{A}_l$ and $\tau \in \mathbb{R}^n \setminus \mathbf{B}_{l+2}$, so that $\tau \in \mathcal{A}_j$ with $j \geq l + 2$.

By the definition of the truncated Havin–Maz'ya potential,

$$\mathcal{V}^{R}_{\alpha,s}(\psi\, \zeta_{\mathbb{R}^n\setminus\mathbf{B}_{l+2}})(\varepsilon) = \int_0^R \left(r^{\alpha s} \frac{1}{|\mathbf{B}(\varepsilon, r)|} \int_{\mathbf{B}(\varepsilon,r)\cap(\mathbb{R}^n\setminus\mathbf{B}_{l+2})} |\psi(\tau)|\, d\tau\right)^{\frac{1}{s-1}} \frac{dr}{r}.$$

For $\tau \in \mathcal{A}_j$ with $j \geq l + 2$, we have

$$|\varepsilon - \tau| \geq |\tau| - |\varepsilon| \gtrsim 2^{j},$$

so that nontrivial contributions occur only for $r \gtrsim 2^{j}$. In this case, the inner integral can be bounded as

$$\int_{\mathbf{B}(\varepsilon,r)\cap\mathcal{A}_j} |\psi(\tau)|\, d\tau \lesssim \int_{\mathcal{A}_j} |\psi(\tau)|\, d\tau.$$

Hence, the far-field term satisfies the pointwise estimate

$$\left|\mathcal{V}^{R}_{\alpha,s}(\psi\, \zeta_{\mathbb{R}^n\setminus\mathbf{B}_{l+2}})(\varepsilon)\right| \lesssim \sum_{j\geq l+2} \left(2^{j\alpha s} \int_{\mathcal{A}_j} |\psi(\tau)|\, d\tau\right)^{\frac{1}{s-1}}.$$

Applying Hölder's inequality and the standard characteristic function estimate:

$$\int_{\mathcal{A}_j} |\psi(\tau)|\, d\tau \le \left\|\psi\, \zeta_{\mathcal{A}_j}\right\|_{L^{p(\cdot)}} \cdot \left\|\zeta_{\mathcal{A}_j}\right\|_{L^{p'(\cdot)}} \lesssim \left\|\psi\, \zeta_{\mathcal{A}_j}\right\|_{L^{p(\cdot)}} \cdot 2^{\frac{jn}{p'_\infty}}.$$

Thus, we arrive at the estimate

$$2^{l\alpha_\infty} \left\|\zeta_{\mathcal{A}_l} \mathcal{V}^R_{\alpha,s}(\psi\, \zeta_{\mathbb{R}^n \setminus \mathbf{B}_{l+2}})\right\|_{L^{p(\cdot)}} \lesssim 2^{l\alpha_\infty} \sum_{j\ge l+2} \left(2^{j\alpha s} \cdot 2^{\frac{jn}{p'_\infty}} \left\|\psi\, \zeta_{\mathcal{A}_j}\right\|_{L^{p(\cdot)}}\right)^{\frac{1}{s-1}} \cdot 2^{\frac{ln}{p_\infty}}.$$

This provides a truncated Havin–Maz'ya version of the far-field estimate, suitable for subsequent Herz quasi-norm summation.

Estimate for $\mathcal{V}^R_{\alpha,s}(\psi)$ on $\mathbf{B}_0$:

Decompose ψ as

$$\psi = \psi\, \zeta_{\mathbf{B}_0} + \psi\, \zeta_{\mathcal{R}_1} + \psi\, \zeta_{\mathbb{R}^n \setminus \mathbf{B}_1}.$$

Then, by the triangle inequality,

$$\left|\mathcal{V}^R_{\alpha,s}(\psi)(\varepsilon)\right| \le \left|\mathcal{V}^R_{\alpha,s}(\psi\, \zeta_{\mathbf{B}_0})(\varepsilon)\right| + \left|\mathcal{V}^R_{\alpha,s}(\psi\, \zeta_{\mathcal{R}_1})(\varepsilon)\right| + \left|\mathcal{V}^R_{\alpha,s}(\psi\, \zeta_{\mathbb{R}^n \setminus \mathbf{B}_1})(\varepsilon)\right|.$$

Local Part: By boundedness on variable Lebesgue spaces,

$$\left\|\mathcal{V}^R_{\alpha,s}(\psi\, \zeta_{\mathbf{B}_0})\, \zeta_{\mathbf{B}_0}\right\|_{L^{q(\cdot)}} \lesssim \left\|\psi\, \zeta_{\mathbf{B}_0}\right\|_{L^{p(\cdot)}} \le \|\psi\|_{\dot{\mathbf{K}}^{\alpha(\cdot)}_{p(\cdot),q}}.$$

Near Shell: Similarly, for $\mathcal{R}_1$,

$$\left\|\mathcal{V}^R_{\alpha,s}(\psi\, \zeta_{\mathcal{R}_1})\, \zeta_{\mathbf{B}_0}\right\|_{L^{q(\cdot)}} \lesssim \left\|\psi\, \zeta_{\mathcal{R}_1}\right\|_{L^{p(\cdot)}} \le \|\psi\|_{\dot{\mathbf{K}}^{\alpha(\cdot)}_{p(\cdot),q}}.$$

Far Field / Residual Part: For $\tau \in \mathcal{R}_l \subset \mathbb{R}^n \setminus \mathbf{B}_1$, $l \ge 2$, and $\varepsilon \in \mathbf{B}_0$, we have $|\varepsilon - \tau| \gtrsim 2^l$. Hence,

$$\mathcal{V}^R_{\alpha,s}(\psi\, \zeta_{\mathbb{R}^n \setminus \mathbf{B}_1})(\varepsilon) \lesssim \sum_{l\ge 2} \left(2^{l(\alpha s - n)} \int_{\mathcal{R}_l} |\psi(\tau)|\, d\tau\right)^{\frac{1}{s-1}}.$$

Applying variable-exponent Hölder and the characteristic function estimate,

$$\int_{\mathcal{R}_l} |\psi(\tau)|\, d\tau \lesssim 2^{\frac{ln}{p'_\infty}} \left\|\psi\, \zeta_{\mathcal{R}_l}\right\|_{L^{p(\cdot)}}.$$

Thus, the far-field term satisfies

$$\left\|\mathcal{V}^R_{\alpha,s}(\psi\, \zeta_{\mathbb{R}^n \setminus \mathbf{B}_1})\, \zeta_{\mathbf{B}_0}\right\|_{L^{q(\cdot)}} \lesssim \sum_{l\ge 2} \left(2^{l(\alpha s - n)} \cdot 2^{\frac{ln}{p'_\infty}} \left\|\psi\, \zeta_{\mathcal{R}_l}\right\|_{L^{p(\cdot)}}\right)^{\frac{1}{s-1}}.$$

Combining all parts gives the truncated Havin–Maz'ya Herz estimate:

$$\left\|\mathcal{V}^R_{\alpha,s}(\psi)\, \zeta_{\mathbf{B}_0}\right\|_{L^{q(\cdot)}} \lesssim \|\psi\|_{\dot{\mathbf{K}}^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)}.$$

□

3.3. Recoverability and relation to prior work

Under specific parameter choices, the present results recover and extend earlier contributions. Table 3 summarizes these instances.

Table 3. Havin–Maz'ya-style presentation: Selected instances of Theorem 3.1 demonstrating recoverability and generalization of previous results.

Parameter Configuration	Recovered / Specialized Result	Reference
$\mathfrak{a}(\cdot) = \mathfrak{a},\ \nu = 2$	Theorem 3.3 (special case)	[62]
$\mathfrak{p}(\cdot) = \mathfrak{p},\ \mathfrak{a}(\cdot) = \mathfrak{a}, \nu = 2$	Theorem 2.1 (specialization)	[63]
$\nu = 2$	Theorem 4.2 (recovered)	[64]

Theorem 3.2. *Let* $0 < \alpha < \mathrm{s}$ *and let* $\mathfrak{p}(\cdot)$ *be a variable exponent such that*

$$\mathfrak{p} \in \mathcal{P}_0^{\log}(\mathbb{R}^{\mathrm{s}}) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^{\mathrm{s}}) \quad \textit{and} \quad 1 < \mathfrak{p}^- \leq \mathfrak{p}^+ < \frac{\mathrm{s}}{\alpha}.$$

Let $\mathfrak{q}(\cdot) \in \mathcal{P}_0(\mathbb{R}^{\mathrm{s}})$*, and let* $\mathfrak{a}(\cdot) \in \mathrm{L}^\infty(\mathbb{R}^{\mathrm{s}})$ *satisfy the log-Hölder continuity conditions with*

$$\alpha - \frac{\mathrm{s}}{\mathfrak{p}^-} < \mathfrak{a}^- \leq \mathfrak{a}^+ < \frac{\mathrm{s}}{\mathfrak{p}'^+}.$$

Moreover, assume the following convergence condition holds:

$$\boxed{\mathfrak{a}^+ + \frac{\mathrm{s}}{\mathfrak{p}^-} + \frac{\alpha\mathrm{s} - \mathrm{s}}{\mathrm{s} - 1} < 0}.$$

For $\psi \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{\mathrm{s}})$ *and a truncation radius* $\mathrm{R} > 0$*, define the truncated Havin–Maz'ya potential as*

$$\mathcal{V}^{\mathrm{R}}_{\alpha,\mathrm{s}}(\psi)(\varepsilon) := \int_0^{\mathrm{R}} \left(\mathrm{r}^{\alpha\mathrm{s}} \frac{1}{|\mathcal{B}(\varepsilon, \mathrm{r})|} \int_{\mathcal{B}(\varepsilon,\mathrm{r})} |\psi(\tau)|\, d\tau \right)^{\frac{1}{\mathrm{s}-1}} \frac{d\mathrm{r}}{\mathrm{r}}.$$

Assume that $\mathcal{V}^{\mathrm{R}}_{\alpha,\mathrm{s}}$ *is bounded from*

$$\mathrm{L}^{\mathfrak{p}(\cdot)}(\mathbb{R}^{\mathrm{s}}) \quad \textit{into} \quad \mathrm{L}^{\mathfrak{p}^*(\cdot)}(\mathbb{R}^{\mathrm{s}}),$$

where

$$\frac{1}{\mathfrak{p}^*(\varepsilon)} = \frac{1}{\mathfrak{p}(\varepsilon)} - \frac{\alpha}{\mathrm{s}}.$$

Then $\mathcal{V}^{\mathrm{R}}_{\alpha,\mathrm{s}}$ *is also bounded as*

$$\mathcal{V}^{\mathrm{R}}_{\alpha,\mathrm{s}} : \dot{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathfrak{p}(\cdot),\mathfrak{q}(\cdot)}(\mathbb{R}^{\mathrm{s}}) \longrightarrow \dot{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathfrak{p}^*(\cdot),\mathfrak{q}(\cdot)}(\mathbb{R}^{\mathrm{s}}).$$

Proof. In view of Proposition 2.1, fix an arbitrary point $\varepsilon \in \mathcal{A}_\gamma$ for some index $\gamma \in \mathcal{Z}$. To facilitate the analysis of the truncated Havin–Maz'ya potential, we partition the ambient space $\mathbb{R}^{\mathrm{s}}$ into three geometrically distinct and mutually disjoint regions, namely,

$$\mathbb{R}^{\mathrm{s}} = \mathbf{B}_{\gamma-2} \cup \widetilde{\mathbf{B}}_\gamma \cup (\mathbb{R}^{\mathrm{s}} \setminus \mathbf{B}_{\gamma+2}).$$

Here, the inner region $\mathbf{B}_{\gamma-2}$ represents a neighborhood close to the origin, the intermediate region $\widetilde{\mathbf{B}}_\gamma$ captures the local shell surrounding the annulus indexed by γ, while the remaining set $\mathbb{R}^s \setminus \mathbf{B}_{\gamma+2}$ corresponds to the far-field contribution.

More precisely, the above sets are defined by

$$\mathbf{B}_r := \{\tau \in \mathbb{R}^s : |\tau| < 2^r\}, \qquad \widetilde{\mathbf{B}}_\gamma := \{\tau \in \mathbb{R}^s : 2^{\gamma-2} \le |\tau| < 2^{\gamma+2}\}.$$

Since the truncated Havin–Maz'ya potential $\mathcal{V}^R_{\alpha,s}$ is subadditive on positive functions, i.e.,

$$\mathcal{V}^R_{\alpha,s}(\psi_1 + \psi_2) \le \mathcal{V}^R_{\alpha,s}(\psi_1) + \mathcal{V}^R_{\alpha,s}(\psi_2),$$

we may estimate

$$\begin{aligned} \left|\mathcal{V}^R_{\alpha,s}\psi(\varepsilon)\right| &= \left|\mathcal{V}^R_{\alpha,s}\Big(\psi \cdot \chi_{\mathbf{B}_{\gamma-2}} + \psi \cdot \chi_{\widetilde{\mathbf{B}}_\gamma} + \psi \cdot \chi_{\mathbb{R}^s\setminus\mathbf{B}_{\gamma+2}}\Big)(\varepsilon)\right| \\ &\le \left|\mathcal{V}^R_{\alpha,s}(\psi \cdot \chi_{\mathbf{B}_{\gamma-2}})(\varepsilon) + \mathcal{V}^R_{\alpha,s}(\psi \cdot \chi_{\widetilde{\mathbf{B}}_\gamma})(\varepsilon) + \mathcal{V}^R_{\alpha,s}(\psi \cdot \chi_{\mathbb{R}^s\setminus\mathbf{B}_{\gamma+2}})(\varepsilon)\right|. \end{aligned}$$

We first concentrate on the contribution coming from the inner ball $\mathbf{B}_{\gamma-2}$. Note that we may write this ball as a disjoint union of dyadic annuli:

$$\mathbf{B}_{\gamma-2} = \bigcup_{\upsilon=-\infty}^{\gamma-2} \mathcal{A}_\upsilon, \qquad \mathcal{A}_\upsilon := \{\tau \in \mathbb{R}^s : 2^\upsilon \le |\tau| < 2^{\upsilon+1}\}.$$

Now fix a point $\varepsilon \in \mathcal{A}_\gamma$. For small radii $0 < r \le 2^{\gamma-1}$, we clearly have

$$\mathcal{B}(\varepsilon, r) \cap \mathbf{B}_{\gamma-2} = \emptyset,$$

since the ball around ε of radius less than $2^{\gamma-1}$ does not intersect the inner region. Consequently, for such r, the inner integral vanishes identically. Hence, the only relevant contribution comes from larger radii, i.e.,

$$r \gtrsim 2^\gamma.$$

For these r, we estimate

$$\begin{aligned} \int_{\mathcal{B}(\varepsilon,r)\cap\mathbf{B}_{\gamma-2}} |\psi(\sigma)|\, d\sigma &\le \int_{\mathbf{B}_{\gamma-2}} |\psi(\sigma)|\, d\sigma \\ &= \sum_{\upsilon=-\infty}^{\gamma-2} \int_{\mathcal{A}_\upsilon} |\psi(\sigma)|\, d\sigma. \end{aligned}$$

Plugging into the truncated Havin–Maz'ya potential, by definition,

$$\mathcal{V}^R_{\alpha,s}(\psi \cdot \chi_{\mathbf{B}_{\gamma-2}})(\varepsilon) = \int_0^R \left(r^{\alpha s} \frac{1}{|\mathcal{B}(\varepsilon, r)|} \int_{\mathcal{B}(\varepsilon,r)\cap\mathbf{B}_{\gamma-2}} |\psi(\sigma)|\, d\sigma\right)^{\frac{1}{s-1}} \frac{dr}{r}.$$

Restricting the domain of integration in r to the effective range $r \geq c\,2^{\gamma}$, and inserting the previous bound, we arrive at

$$\mathcal{V}^{R}_{\alpha,s}(\psi \cdot \chi_{\mathbf{B}_{\gamma-2}})(\varepsilon) \lesssim \int_{c\,2^{\gamma}}^{R} \left(r^{\alpha s - s} \sum_{\upsilon=-\infty}^{\gamma-2} \int_{\mathcal{A}_{\upsilon}} |\psi(\sigma)|\, d\sigma \right)^{\frac{1}{s-1}} \frac{dr}{r}.$$

Simplification of the radial factor: Since $s \in (1, \frac{s}{\alpha})$, the exponent of r in the integrand is negative. Thus,

$$\int_{c\,2^{\gamma}}^{R} r^{\frac{\alpha s - s}{s-1}} \frac{dr}{r} \approx 2^{-\gamma \frac{s-\alpha s}{s-1}}.$$

Therefore,

$$\mathcal{V}^{R}_{\alpha,s}(\psi \cdot \chi_{\mathbf{B}_{\gamma-2}})(\varepsilon) \lesssim 2^{-\gamma \frac{s-\alpha s}{s-1}} \left(\sum_{\upsilon=-\infty}^{\gamma-2} \int_{\mathcal{A}_{\upsilon}} |\psi(\sigma)|\, d\sigma \right)^{\frac{1}{s-1}}.$$

For each annulus $\mathcal{A}_{\upsilon}$, by the variable-exponent Hölder inequality, we obtain

$$\int_{\mathcal{A}_{\upsilon}} |\psi| \leq \|\psi\, \chi_{\upsilon}\|_{L^{p(\cdot)}} \cdot \|\chi_{\upsilon}\|_{L^{p'(\cdot)}}.$$

By Lemma 2.2, the second factor admits the bound

$$\|\chi_{\upsilon}\|_{L^{p'(\cdot)}} \approx 2^{\frac{\upsilon s}{p'_{\infty}}}.$$

Thus,

$$\int_{\mathcal{A}_{\upsilon}} |\psi| \lesssim \|\psi\, \chi_{\upsilon}\|_{L^{p(\cdot)}}\, 2^{\frac{\upsilon s}{p'_{\infty}}}.$$

Substituting this bound into the preceding estimate for the truncated Havin–Maz'ya potential, we obtain the following pointwise control for the inner region:

$$\mathcal{V}^{R}_{\alpha,s}(\psi\, \chi_{\mathbf{B}_{\gamma-2}})(\varepsilon) \lesssim 2^{-\gamma \frac{s-\alpha s}{s-1}} \left(\sum_{\upsilon=-\infty}^{\gamma-2} 2^{\frac{\upsilon s}{p'_{\infty}}} \|\psi\, \chi_{\upsilon}\|_{L^{p(\cdot)}} \right)^{\frac{1}{s-1}}.$$

Next, we take the $L^{p^*(\cdot)}$-norm of the above expression over the annulus $\varepsilon \in \mathcal{A}_{\gamma}$. Recalling the standard norm equivalence for characteristic functions in variable exponent spaces, namely,

$$\|\chi_{\gamma}\|_{L^{p^*(\cdot)}} \approx 2^{\frac{\gamma s}{p^*_{\infty}}},$$

we obtain

$$\begin{aligned} &2^{\gamma a_{\infty}} \left\| \chi_{\gamma}\, \mathcal{V}^{R}_{\alpha,s}(\psi \cdot \chi_{\mathbf{B}_{\gamma-2}}) \right\|_{L^{p^*(\cdot)}} \\ &\quad \lesssim 2^{\gamma\left(a_{\infty} + \frac{s}{p^*_{\infty}} - \frac{s-\alpha s}{s-1}\right)} \left(\sum_{\upsilon=-\infty}^{\gamma-2} 2^{\frac{\upsilon s}{p'_{\infty}}} \|\psi\, \chi_{\upsilon}\|_{L^{p(\cdot)}} \right)^{\frac{1}{s-1}}. \end{aligned}$$

Under the standing assumptions

$$1 < \mathfrak{p}^- \le \mathfrak{p}^+ < \frac{s}{\alpha} \quad \text{and} \quad \alpha - \frac{s}{\mathfrak{p}^-} < \mathfrak{a}^- \le \mathfrak{a}^+ < \frac{s}{\mathfrak{p}'^+},$$

the associated geometric weight is summable as $\upsilon \to -\infty$. Consequently, after taking the $\ell^{\mathfrak{q}}$-quasi-norm with respect to the index γ, we deduce that the contribution stemming from the inner region is dominated by the Herz quasi-norm, namely,

$$\left\{ \sum_{\gamma<0} 2^{\gamma \mathfrak{a}_\infty \mathfrak{q}} \left\| \chi_\gamma \mathcal{V}^{R}_{\alpha,s}(\psi \cdot \chi_{\mathbf{B}_{\gamma-2}}) \right\|^{\mathfrak{q}}_{L^{\mathfrak{p}^*(\cdot)}} \right\}^{1/\mathfrak{q}} \lesssim \|\psi\|_{\dot{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathfrak{p}(\cdot),\mathfrak{q}(\cdot)}}.$$

Next, we recall a fundamental property of variable exponent spaces. Let $\varepsilon_\upsilon \in \mathcal{A}_\upsilon$ and $\varepsilon_\gamma \in \mathcal{A}_\gamma$, with $\upsilon \le \gamma - 2$. By the continuity assumption on the exponent $\mathfrak{p}(\cdot)$, there exists a constant $\mathcal{M} > 0$ such that

$$\frac{\left|\mathcal{A}_\gamma\right|^{\frac{1}{\mathfrak{p}(\varepsilon_\gamma)}}}{\left|\mathcal{A}_\upsilon\right|^{\frac{1}{\mathfrak{p}(\varepsilon_\upsilon)}}} \le \mathcal{M} 2^{(\gamma-\upsilon)\frac{s}{\mathfrak{p}^-}}.$$

We now turn to estimating the local contribution inside the inner ball $\mathbf{B}_{\gamma-2}$. For each $\varepsilon \in \mathcal{A}_\gamma$, the truncated Havin–Maz'ya potential admits the following bound:

$$\mathcal{V}^{R}_{\alpha,s}\left(\psi \cdot \chi_{\mathbf{B}_{\gamma-2}}\right)(\varepsilon) \le \sum_{\upsilon=-\infty}^{\gamma-2} \int_0^{c2^\gamma} \left(r^{\alpha s} \frac{1}{|\mathcal{B}(\varepsilon, r)|} \int_{\mathcal{A}_\upsilon \cap \mathcal{B}(\varepsilon,r)} |\psi(\tau)|\, d\tau \right)^{\frac{1}{s-1}} \frac{dr}{r},$$

where we restrict the radius to $r \lesssim 2^\gamma$ since ψ is supported in $\mathbf{B}_{\gamma-2}$.

Let $\varepsilon \in \mathcal{A}_\gamma$ and $\tau \in \mathcal{A}_\upsilon$ with $\upsilon \le \gamma - 2$. In this configuration, the relative distance satisfies

$$|\varepsilon - \tau| \approx |\varepsilon| \approx 2^\gamma.$$

Consequently, for such indices, the kernel term in the truncated Havin–Maz'ya potential can be estimated as

$$r^{\alpha s} \frac{1}{|\mathcal{B}(\varepsilon, r)|} \int_{\mathcal{A}_\upsilon \cap \mathcal{B}(\varepsilon,r)} |\psi(\tau)|\, d\tau \;\lesssim\; 2^{\gamma(\alpha s - s)} \int_{\mathcal{A}_\upsilon} |\psi(\tau)|\, d\tau.$$

To estimate the integral over the annulus $\mathcal{A}_\upsilon$, we invoke Hölder's inequality together with the standard bound for characteristic functions:

$$\|\chi_\upsilon\|_{L^{\mathfrak{p}'(\cdot)}} \lesssim 2^{\frac{\upsilon s}{\mathfrak{p}'_\infty}}.$$

This yields

$$\int_{\mathcal{A}_\upsilon} |\psi(\tau)|\, d\tau \;\lesssim\; \|\psi \cdot \chi_\upsilon\|_{L^{\mathfrak{p}(\cdot)}}\, 2^{\frac{\upsilon s}{\mathfrak{p}'_\infty}}.$$

Combining the above estimates, we arrive at the following pointwise control for the inner contribution of the truncated Havin–Maz'ya potential:

$$\mathcal{V}^{R}_{\alpha,s}(\psi \cdot \chi_{\mathbf{B}_{\gamma-2}})(\varepsilon) \;\lesssim\; \sum_{\upsilon=-\infty}^{\gamma-2} 2^{\frac{\gamma(\alpha s - s)}{s-1}} \cdot 2^{\frac{\upsilon s}{(s-1)\mathfrak{p}'_\infty}}\, \|\psi \cdot \chi_\upsilon\|_{L^{\mathfrak{p}(\cdot)}}.$$

Finally, considering the $L^{p^*(\cdot)}$-norm over the annulus $\mathcal{A}_\gamma$, we deduce

$$\left\|\mathcal{V}^R_{\alpha,s}(\psi\cdot\chi_{\mathbf{B}_{\gamma-2}})\chi_\gamma\right\|_{L^{p^*(\cdot)}} \lesssim \sum_{\upsilon=-\infty}^{\gamma-2} 2^{\frac{\gamma(\alpha s-s)}{s-1}}\cdot 2^{\frac{\upsilon s}{(s-1)p'_\infty}} \times \|\psi\cdot\chi_\upsilon\|_{L^{p(\cdot)}}\,\|\chi_\gamma\|_{L^{p^*(\cdot)}}.$$

Using $\|\chi_\gamma\|_{L^{p^*(\cdot)}} \lesssim 2^{\frac{\gamma s}{p^*_\infty}}$, we obtain

$$\left\|\mathcal{V}^R_{\alpha,s}\left(\psi\cdot\chi_{\mathbf{B}_{\gamma-2}}\right)\chi_\gamma\right\|_{L^{p^*(\cdot)}} \lesssim 2^{\frac{\gamma s}{p^*_\infty}}\sum_{\upsilon=-\infty}^{\gamma-2} 2^{\frac{\gamma(\alpha s-s)}{s-1}}\cdot 2^{\frac{\upsilon s}{(s-1)p'_\infty}}\|\psi\cdot\chi_\upsilon\|_{L^{p(\cdot)}}.$$

From the local estimate obtained above, for $\varepsilon\in\mathcal{A}_\gamma$, we have

$$\left\|\mathcal{V}^R_{\alpha,s}(\psi\cdot\chi_{\mathbf{B}_{\gamma-2}})\chi_\gamma\right\|_{L^{p^*(\cdot)}} \lesssim 2^{\gamma\left(\frac{s}{p^*_\infty}+\frac{\alpha s-s}{s-1}\right)}\sum_{\upsilon=-\infty}^{\gamma-2} 2^{\frac{\upsilon s}{(s-1)p'_\infty}}\,\|\psi\cdot\chi_\upsilon\|_{L^{p(\cdot)}}.$$

Hence, the γ–series decays (and converges) under the standard condition

$$\boxed{\alpha\, s < s,} \qquad \text{and equivalently,} \qquad \boxed{\frac{s-\alpha\, s}{s-1} > -\frac{s}{p^*_\infty}}.$$

In particular, the standard hypothesis for the truncated Havin–Maz'ya potential

$$\boxed{\alpha < 1}$$

guarantees the required geometric decay in γ and thus convergence of the corresponding series.

The inner sum in υ is summable as $\upsilon\to-\infty$ since $\frac{s}{(s-1)p'_\infty}>0$.

To evaluate $2^{\gamma a_\infty}\mathcal{V}^R_{\alpha,s}(\psi\cdot\chi_{\mathbf{B}_{\gamma-2}})$ within the ℓ^q-norm, we follow a procedure analogous to the one employed previously, with $2^{\gamma a_\infty}$ taking the place of $2^{\gamma a(0)}$ throughout the derivation.

Recall that the truncated Havin–Maz'ya potential is

$$\mathcal{V}^R_{\alpha,s}(\psi)(\varepsilon) := \int_0^R\left(r^{\alpha s}\frac{1}{|\mathcal{B}(\varepsilon,r)|}\int_{\mathcal{B}(\varepsilon,r)}|\psi(\sigma)|\,d\sigma\right)^{\frac{1}{s-1}}\frac{dr}{r}.$$

Since $\operatorname{supp}(\psi\chi_{\mathbf{B}_{\gamma-2}})\subset\mathbf{B}_{\gamma-2}$ and $\varepsilon\in\mathcal{A}_\gamma$, for radii $r\lesssim 2^\gamma$ and $\upsilon\le\gamma-2$, we have $|\varepsilon-\sigma|\approx 2^\gamma$ whenever $\sigma\in\mathcal{A}_\upsilon$. Hence,

$$r^{\alpha s}\frac{1}{|\mathcal{B}(\varepsilon,r)|}\int_{\mathcal{A}_\upsilon\cap\mathcal{B}(\varepsilon,r)}|\psi(\sigma)|\,d\sigma \lesssim 2^{\gamma(\alpha s-s)}\int_{\mathcal{A}_\upsilon}|\psi(\sigma)|\,d\sigma,$$

and therefore

$$\mathcal{V}^R_{\alpha,s}(\psi\chi_{\mathbf{B}_{\gamma-2}})(\varepsilon) \lesssim \sum_{\upsilon=-\infty}^{\gamma-2} 2^{\frac{\gamma(\alpha s-s)}{s-1}}\left(\int_{\mathcal{A}_\upsilon}|\psi(\sigma)|\,d\sigma\right)^{\frac{1}{s-1}}.$$

By variable-exponent Hölder and Lemma 2.2,

$$\int_{\mathcal{A}_\upsilon}|\psi| \lesssim \|\psi\chi_\upsilon\|_{L^{p(\cdot)}}\,\|\chi_\upsilon\|_{L^{p'(\cdot)}} \approx \|\psi\chi_\upsilon\|_{L^{p(\cdot)}}\,2^{\frac{\upsilon s}{p'_\infty}}.$$

Using the log–Hölder property of $\mathfrak{a}$ and the standard ratio estimate for variable exponents (cf. Lemma 2.2),

$$\frac{\left|\mathcal{A}_\gamma\right|^{\frac{1}{\mathfrak{p}(\varepsilon_\gamma)}}}{\left|\mathcal{A}_\upsilon\right|^{\frac{1}{\mathfrak{p}(\varepsilon_\upsilon)}}} \lesssim 2^{(\gamma-\upsilon)\frac{s}{\mathfrak{p}^-}},$$

one has

$$\begin{aligned}
&\left\|2^{\gamma\mathfrak{a}_\infty}\cdot\mathcal{V}^{R}_{\alpha,s}\left(\psi\cdot\chi_{\mathbf{B}_{\gamma-2}}\right)\cdot\chi_\gamma\right\|_{L^{\mathfrak{p}^*(\cdot)}(\mathbb{R}^s)}\\
&\lesssim \sum_{\upsilon=-\infty}^{\gamma-2} 2^{(\gamma-\upsilon)\mathfrak{a}^+}\cdot 2^{\frac{\gamma(\alpha s-s)}{s-1}}\cdot 2^{(\gamma-\upsilon)\frac{s}{\mathfrak{p}^-}}\cdot 2^{\frac{\upsilon s}{(s-1)\mathfrak{p}'_\infty}}\left\|2^{\upsilon\mathfrak{a}(\cdot)}\psi\cdot\chi_\upsilon\right\|_{L^{\mathfrak{p}(\cdot)}}\\
&= \sum_{\upsilon=-\infty}^{\gamma-2} 2^{(\gamma-\upsilon)\left(\mathfrak{a}^++\frac{s}{\mathfrak{p}^-}\right)}\cdot 2^{\frac{\gamma(\alpha s-s)}{s-1}}\cdot 2^{\frac{\upsilon s}{(s-1)\mathfrak{p}'_\infty}}\left\|2^{\upsilon\mathfrak{a}(\cdot)}\psi\cdot\chi_\upsilon\right\|_{L^{\mathfrak{p}(\cdot)}}.
\end{aligned}$$

For every non-negative index γ, we apply Lemma 2.1, which establishes a generalized Minkowski-type inequality in the context of weighted $\ell^{\mathfrak{q}}$ spaces. This allows us to systematically decompose the summation and subsequently derive

$$\begin{aligned}
&\left\{\sum_{\gamma=0}^{\infty}\left\|2^{\gamma\mathfrak{a}_\infty}\cdot\mathcal{V}^{R}_{\alpha,s}\left(\psi\cdot\chi_{\mathbf{B}_{\gamma-2}}\right)\cdot\chi_\gamma\right\|^{\mathfrak{q}_\infty}_{L^{\mathfrak{p}^*(\cdot)}(\mathbb{R}^s)}\right\}^{\frac{1}{\mathfrak{q}_\infty}}\\
&\leq \mathcal{M}\left\{\sum_{\gamma=0}^{\infty}\left(2^{\gamma\left(\mathfrak{a}^++\frac{s}{\mathfrak{p}^-}+\frac{\alpha s-s}{s-1}\right)}\cdot\sum_{\upsilon=-\infty}^{0}2^{\upsilon\left(\frac{s}{(s-1)\mathfrak{p}'_\infty}\right)}\sup_{\upsilon\leq 0}\left\|2^{\upsilon\mathfrak{a}(\cdot)}\psi\cdot\chi_\upsilon\right\|_{L^{\mathfrak{p}(\cdot)}(\mathbb{R}^s)}\right)^{\mathfrak{q}_\infty}\right\}^{\frac{1}{\mathfrak{q}_\infty}}\\
&\quad+\mathcal{M}\left\{\sum_{\gamma=0}^{\infty}\left(\sum_{\upsilon=1}^{\gamma-2}2^{(\gamma-\upsilon)\left(\mathfrak{a}^++\frac{s}{\mathfrak{p}^-}+\frac{\alpha s-s}{s-1}\right)}\cdot\left\|2^{\upsilon\mathfrak{a}(\cdot)}\psi\cdot\chi_\upsilon\right\|_{L^{\mathfrak{p}(\cdot)}(\mathbb{R}^s)}\right)^{\mathfrak{q}_\infty}\right\}^{\frac{1}{\mathfrak{q}_\infty}}.
\end{aligned}$$

Consequently, if

$$\boxed{\mathfrak{a}^+ + \frac{s}{\mathfrak{p}^-} + \frac{\alpha s - s}{s-1} < 0} \quad \text{(and in particular } \alpha\,\mathfrak{p}^+ < 1 \text{ so that } \frac{\alpha s - s}{s-1} < 0\text{)},$$

then the geometric factors decay and we conclude

$$\left\{\sum_{\gamma=0}^{\infty}\left\|2^{\gamma\mathfrak{a}_\infty}\cdot\mathcal{V}^{R}_{\alpha,s}\left(\psi\cdot\chi_{\mathbf{B}_{\gamma-2}}\right)\cdot\chi_\gamma\right\|^{\mathfrak{q}_\infty}_{L^{\mathfrak{p}^*(\cdot)}(\mathbb{R}^s)}\right\}^{\frac{1}{\mathfrak{q}_\infty}} \leq \mathcal{M}\,\|\psi\|_{\mathbf{K}^{\mathfrak{a}(\cdot)}_{\mathfrak{p}(\cdot),\mathfrak{q}(\cdot)}(\mathbb{R}^s)}.$$

For the middle part $\mathcal{V}^{R}_{\alpha,s}\left(\psi\cdot\chi_{\widetilde{\mathbf{A}}_\gamma}\right)$, by using the boundedness $\mathcal{V}^{R}_{\alpha,s}: L^{\mathfrak{p}(\cdot)}(\mathbb{R}^s)\to L^{\mathfrak{p}^*(\cdot)}(\mathbb{R}^s)$, we obtain

$$\begin{aligned}
&\left\|\mathcal{V}^{R}_{\alpha,s}\left(\psi\cdot\chi_{\widetilde{\mathbf{A}}_\gamma}\right)\right\|_{\widetilde{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathfrak{p}^*(\cdot),\mathfrak{q}(\cdot)}(\mathbb{R}^s)}\\
&\leq \mathcal{M}\cdot\|\psi\|_{\widetilde{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathfrak{p}(\cdot),\mathfrak{q}(\cdot)}(\mathbb{R}^s)}.
\end{aligned}$$

For the far-field part, consider $\varepsilon\in\mathcal{A}_\gamma$ with $\gamma<0$:

$$\mathcal{V}^{R}_{\alpha,s}(\psi\cdot\chi_{\mathbb{R}^s\setminus\mathbf{B}_{\gamma+2}})(\varepsilon).$$

Using the decomposition $\mathbb{R}^s \setminus \mathbf{B}_{\gamma+2} = \bigcup_{\upsilon=\gamma+3}^{\infty} \mathcal{A}_\upsilon$, we get

$$2^{\gamma\mathfrak{a}(0)}\,\mathcal{V}^{R}_{\alpha,s}(\psi\chi_{\mathbb{R}^s\setminus\mathbf{B}_{\gamma+2}})(\varepsilon) \lesssim \sum_{\upsilon=\gamma+3}^{\infty} 2^{(\gamma-\upsilon)\mathfrak{a}^-}\, 2^{\frac{\upsilon(\alpha s-s)}{s-1}}\, 2^{\frac{\upsilon s}{(s-1)\mathfrak{p}'_\infty}} \left\|2^{\upsilon\mathfrak{a}(\cdot)}\psi\chi_\upsilon\right\|^{\frac{1}{s-1}}_{L^{\mathfrak{p}(\cdot)}}.$$

By taking the $L^{\mathfrak{p}^*(\cdot)}$-norm and noting that $\|\chi_\gamma\|_{L^{\mathfrak{p}^*(\cdot)}} \sim 2^{\frac{\gamma s}{\mathfrak{p}^*_\infty}}$, we deduce

$$\begin{aligned}&\left\|2^{\gamma\mathfrak{a}(0)}\,\mathcal{V}^{R}_{\alpha,s}(\psi\chi_{\mathbb{R}^s\setminus\mathbf{B}_{\gamma+2}})\chi_\gamma\right\|_{L^{\mathfrak{p}^*(\cdot)}}\\ &\qquad\lesssim 2^{\frac{\gamma s}{\mathfrak{p}^*_\infty}} \sum_{\upsilon=\gamma+3}^{\infty} 2^{(\gamma-\upsilon)\mathfrak{a}^-}\, 2^{\frac{\upsilon(\alpha s-s)}{s-1}}\, 2^{\frac{\upsilon s}{(s-1)\mathfrak{p}'_\infty}} \left\|2^{\upsilon\mathfrak{a}(\cdot)}\psi\chi_\upsilon\right\|^{\frac{1}{s-1}}_{L^{\mathfrak{p}(\cdot)}}.\end{aligned}$$

The geometric factor in γ decays as $2^{\gamma(\frac{s}{\mathfrak{p}^*_\infty}-\mathfrak{a}^-)}$, while the υ-exponent satisfies

$$-\mathfrak{a}^- + \frac{\alpha s - s}{s-1} + \frac{s}{(s-1)\mathfrak{p}'_\infty} < 0,$$

ensuring summability provided

$$\boxed{\alpha\, s^+ < s}.$$

Hence, after summation and taking the $\ell^{\mathfrak{q}_\infty}_{>}$-norm, we conclude

$$\Big(\sum_{\gamma=0}^{\infty}\left\|2^{\gamma\mathfrak{a}_\infty}\,\mathcal{V}^{R}_{\alpha,s}(\psi\chi_{\mathbb{R}^s\setminus\mathbf{B}_{\gamma+2}})\chi_\gamma\right\|^{\mathfrak{q}_\infty}_{L^{\mathfrak{p}^*(\cdot)}}\Big)^{1/\mathfrak{q}_\infty} \lesssim \|\psi\|_{\dot{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathfrak{p}(\cdot),\mathfrak{q}(\cdot)}(\mathbb{R}^s)}.$$

This establishes the result. □

3.4. Recoverability and connection to previous results

To illustrate the flexibility and refinement of the proposed variable-parameter Herz framework, Table 4 summarizes representative parameter configurations that recover or sharpen classical results.

Table 4. Representative instances of Theorem 3.2 demonstrating how the variable-parameter Herz–Havin–Maz'ya framework recovers or refines previous results under specific parameter regimes.

Parameter Configuration	Recovered / Specialized Result	Reference
$\mathfrak{a}(\cdot) = \mathfrak{a},\ \nu = 2,\ \mathfrak{p}(\cdot) = \mathfrak{p},\ \mathfrak{q}(\cdot) = \mathfrak{q}$	Recovers a refinement of the classical potential estimate in main result 2.3	[64]
$\alpha(\cdot) = 0,\ \mathfrak{p}(\cdot) = \mathfrak{q}(\cdot),\ \nu = 2$	Recovers and sharpens the result of Theorem 6 in [72]	[72]

4. Application of Havin–Maz'ya potentials to the Laplace equation via Dirichlet energy minimization

In this section, we investigate the existence, uniqueness, and regularity of weak solutions to the variable-exponent $\mathfrak{p}(\cdot)$-Laplace equation [28] via the Euler–Lagrange framework associated with a Dirichlet-type energy functional. Precisely, we study solutions $\mathfrak{u}$ satisfying the Euler–Lagrange equation

$$-\operatorname{div}(|\nabla \mathfrak{u}|^{\mathfrak{p}(\varepsilon)-2}\nabla \mathfrak{u}) = \psi \quad \text{in } \mathbf{R}^m,$$

where ψ is a given function in a suitable variable-exponent Herz space.

4.1. Functional framework and standing assumptions

In this sub-subsection, we develop the problem in the Euler–Lagrange framework with the use of a Dirichlet energy functional and develop the standing assumptions for existence and regularity of solutions.

Let $m \geq 2$ and $\Omega = \mathbf{R}^m$ (or a sufficiently regular unbounded domain). Let $\psi \in \dot{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathfrak{p}(\cdot),\mathfrak{q}(\cdot)}(\mathbf{R}^m)$ be given. Consider the Dirichlet-type energy functional

$$\mathcal{E}(\mathfrak{u}) := \int_{\mathbf{R}^m} \frac{1}{\mathfrak{p}(\varepsilon)}|\nabla \mathfrak{u}(\varepsilon)|^{\mathfrak{p}(\varepsilon)}\, d\varepsilon - \int_{\mathbf{R}^m} \mathfrak{u}(\varepsilon)\psi(\varepsilon)\, d\varepsilon,$$

defined on the Sobolev-type space

$$\mathcal{D}^{1,\mathfrak{p}(\cdot)}(\mathbf{R}^m) := \overline{C_0^\infty(\mathbf{R}^m)}^{\|\nabla(\cdot)\|_{L^{\mathfrak{p}(\cdot)}}}.$$

The associated Euler–Lagrange equation is the $\mathfrak{p}(\cdot)$-Laplace equation

$$-\operatorname{div}(|\nabla \mathfrak{u}|^{\mathfrak{p}(\varepsilon)-2}\nabla \mathfrak{u}) = \psi \quad \text{in } \mathbf{R}^m,$$

which characterizes weak solutions as critical points of $\mathcal{E}$.

Standing assumptions. To ensure well-posedness and applicability of the Havin–Maz'ya potential estimates, we assume:

- $0 < \beta < 1$ and $s = m$;
- $\mathfrak{p}(\cdot) \in \mathcal{P}_0^{\log}(\mathbf{R}^m) \cap \mathcal{P}_\infty^{\log}(\mathbf{R}^m)$ satisfying
$$1 < \mathfrak{p}^- \leq \mathfrak{p}^+ < \frac{m}{\beta};$$
- $\mathfrak{q}(\cdot) \in \mathcal{P}_0(\mathbf{R}^m)$;
- $\mathfrak{a}(\cdot) \in L^\infty(\mathbf{R}^m)$ is log-Hölder continuous and satisfies
$$\beta - \frac{m}{\mathfrak{p}^-} < \mathfrak{a}^- \leq \mathfrak{a}^+ < \frac{m}{(\mathfrak{p}')^+}.$$

These assumptions guarantee the coercivity, weak lower semicontinuity of $\mathcal{E}$, and reflexivity of $\mathcal{D}^{1,\mathfrak{p}(\cdot)}(\mathbf{R}^m)$, which are essential for proving existence, uniqueness, and Herz space regularity of weak solutions.

4.2. Finiteness of the linear term

We verify that for every $u \in \mathcal{D}^{1,p(\cdot)}(\mathbf{R}^m)$, the linear functional

$$\langle u, \psi \rangle := \int_{\mathbf{R}^m} u(\varepsilon)\psi(\varepsilon)\, d\varepsilon$$

is well-defined and finite whenever $\psi \in \dot{\mathbf{K}}^{a(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^m)$.

Let $\{\mathcal{A}_\kappa\}_{\kappa\in\mathbf{Z}}$ denote the standard dyadic annuli decomposition:

$$\mathcal{A}_\kappa := \{\varepsilon \in \mathbf{R}^m : 2^\kappa \le |\varepsilon| < 2^{\kappa+1}\}, \qquad \chi_\kappa := \chi_{\mathcal{A}_\kappa}.$$

Then we may write

$$\int_{\mathbf{R}^m} |u\psi| = \sum_{\kappa\in\mathbf{Z}} \int_{\mathcal{A}_\kappa} |u(\varepsilon)\psi(\varepsilon)|\, d\varepsilon.$$

Applying the variable-exponent Hölder inequality on each annulus gives

$$\int_{\mathcal{A}_\kappa} |u\psi| \le C_\kappa\, \|u\chi_\kappa\|_{L^{p^*(\cdot)}} \|\psi\chi_\kappa\|_{L^{p(\cdot)}},$$

where

$$\frac{1}{p^*(\varepsilon)} = \frac{1}{p(\varepsilon)} - \frac{\beta}{m}, \qquad C_\kappa = 2\max\{1, |\mathcal{A}_\kappa|\}.$$

Introducing the Herz weight $2^{\kappa a(\cdot)}$ and rearranging factors, we obtain

$$\int_{\mathcal{A}_\kappa} |u\psi| \le \|2^{-\kappa a(\cdot)}\chi_\kappa\|_{L^{r(\cdot)}} \|2^{\kappa a(\cdot)} u\chi_\kappa\|_{L^{p^*(\cdot)}} \|2^{\kappa a(\cdot)}\psi\chi_\kappa\|_{L^{p(\cdot)}},$$

where the auxiliary exponent $r(\cdot)$ satisfies

$$\frac{1}{r(\cdot)} = 1 - \frac{1}{p(\cdot)} - \frac{1}{p^*(\cdot)} = \frac{\beta}{m} > 0.$$

By standard estimates for characteristic functions (see Lemma 2.2), we have

$$\|2^{-\kappa a(\cdot)}\chi_\kappa\|_{L^{r(\cdot)}} \approx 2^{-\kappa a_\infty} |\mathcal{A}_\kappa|^{1/r_\infty} \approx 2^{\kappa(\beta - a_\infty)}.$$

Consequently,

$$\int_{\mathcal{A}_\kappa} |u\psi| \lesssim 2^{\kappa(\beta - a_\infty)} \|2^{\kappa a(\cdot)} u\chi_\kappa\|_{L^{p^*(\cdot)}} \|2^{\kappa a(\cdot)}\psi\chi_\kappa\|_{L^{p(\cdot)}}.$$

Summing over all $\kappa \in \mathbf{Z}$ and applying the discrete $\ell^{q(\cdot)}$-Hölder inequality yields the final estimate

$$\int_{\mathbf{R}^m} |u\psi| \lesssim \|u\|_{\dot{\mathbf{K}}^{a(\cdot)}_{p^*(\cdot),q(\cdot)}} \|\psi\|_{\dot{\mathbf{K}}^{a(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Since $\beta - a_\infty < 0$, the geometric series converges, and hence the linear functional $\langle u, \psi \rangle$ is finite for all $u \in \mathcal{D}^{1,p(\cdot)}(\mathbf{R}^m)$.

4.3. Coercivity and weak lower semicontinuity

We first establish coercivity of the Dirichlet-type functional $\mathcal{E} : \mathcal{D}^{1,\mathfrak{p}(\cdot)}(\mathbf{R}^{\mathrm{m}}) \to \mathbf{R}$. There exist constants $C_1, C_2 > 0$ such that for all $\mathfrak{u} \in \mathcal{D}^{1,\mathfrak{p}(\cdot)}(\mathbf{R}^{\mathrm{m}})$:

$$\mathcal{E}(\mathfrak{u}) \geq C_1 \|\nabla \mathfrak{u}\|_{\mathrm{L}^{\mathfrak{p}(\cdot)}}^{\mathfrak{p}^-} - C_2 \|\psi\|_{\dot{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathfrak{p}(\cdot),\mathfrak{q}(\cdot)}}^{\mathfrak{p}^+}.$$

Using the Peter–Paul inequality, we first estimate the pointwise product:

$$|\mathfrak{u}(\varepsilon)\,\psi(\varepsilon)| \leq \frac{\delta}{\mathfrak{p}^*(\varepsilon)}\,|\mathfrak{u}(\varepsilon)|^{\mathfrak{p}^*(\varepsilon)} + C(\delta, \mathfrak{p}(\varepsilon))\,|\psi(\varepsilon)|^{\mathfrak{p}(\varepsilon)},$$

for almost every $\varepsilon \in \mathbf{R}^{\mathrm{m}}$ and any $\delta > 0$. Here, $\mathfrak{p}^*(\varepsilon)$ is the Sobolev conjugate exponent defined by $\frac{1}{\mathfrak{p}^*(\varepsilon)} = \frac{1}{\mathfrak{p}(\varepsilon)} - \frac{\beta}{\mathrm{m}}$.

Next, we integrate both sides over $\mathbf{R}^{\mathrm{m}}$:

$$\int_{\mathbf{R}^{\mathrm{m}}} |\mathfrak{u}\,\psi|\,d\varepsilon \leq \delta \int_{\mathbf{R}^{\mathrm{m}}} \frac{1}{\mathfrak{p}^*(\varepsilon)} |\mathfrak{u}|^{\mathfrak{p}^*(\varepsilon)}\,d\varepsilon + \int_{\mathbf{R}^{\mathrm{m}}} C(\delta, \mathfrak{p}(\varepsilon))\,|\psi|^{\mathfrak{p}(\varepsilon)}\,d\varepsilon.$$

Applying the variable-exponent Sobolev inequality (see [68]), we can bound the first integral in terms of the gradient:

$$\int_{\mathbf{R}^{\mathrm{m}}} |\mathfrak{u}|^{\mathfrak{p}^*(\varepsilon)}\,d\varepsilon \leq C_{\mathrm{sob}}^{\mathfrak{p}^{*-}}\,\|\nabla \mathfrak{u}\|_{\mathrm{L}^{\mathfrak{p}(\cdot)}}^{\mathfrak{p}^{*-}},$$

where $\mathfrak{p}^{*-} := \frac{\mathrm{m}\,\mathfrak{p}^-}{\mathrm{m} - \beta\,\mathfrak{p}^-} > \mathfrak{p}^-$.

Thus, after integration, we obtain

$$\int_{\mathbf{R}^{\mathrm{m}}} |\mathfrak{u}\,\psi|\,d\varepsilon \leq \delta\, C_{\mathrm{sob}}^{\mathfrak{p}^{*-}} \|\nabla \mathfrak{u}\|_{\mathrm{L}^{\mathfrak{p}(\cdot)}}^{\mathfrak{p}^{*-}} + C_\delta \|\psi\|_{\mathrm{L}^{\mathfrak{p}(\cdot)}}^{\mathfrak{p}^+}.$$

Since $\mathfrak{p}^{*-} > \mathfrak{p}^-$, the first term involves a higher power of the gradient. By choosing $\delta > 0$ sufficiently small and applying the continuous embedding [70, Corollary 1], we have

$$\dot{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathfrak{p}(\cdot),\mathfrak{q}(\cdot)} \hookrightarrow \mathrm{L}^{\mathfrak{p}(\cdot)},$$

and we can absorb this term into the left-hand side of the energy inequality.

This shows that the Dirichlet energy functional is coercive:

$$\mathcal{E}(\mathfrak{u}) \geq C_1 \|\nabla \mathfrak{u}\|_{\mathrm{L}^{\mathfrak{p}(\cdot)}}^{\mathfrak{p}^-} - C_2 \|\psi\|_{\dot{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathfrak{p}(\cdot),\mathfrak{q}(\cdot)}}^{\mathfrak{p}^+},$$

for some positive constants $C_1, C_2 > 0$.

The functional

$$\mathfrak{u} \longmapsto \int_{\mathbf{R}^{\mathrm{m}}} \frac{1}{\mathfrak{p}(\varepsilon)} |\nabla \mathfrak{u}(\varepsilon)|^{\mathfrak{p}(\varepsilon)}\,d\varepsilon$$

is convex and strongly continuous, hence sequentially weakly lower semicontinuous. The linear functional

$$\mathfrak{u} \longmapsto -\int_{\mathbf{R}^{\mathrm{m}}} \mathfrak{u}(\varepsilon)\psi(\varepsilon)\,d\varepsilon$$

is continuous and therefore weakly continuous. Since the sum of a weakly lower semicontinuous functional and a weakly continuous functional remains weakly lower semicontinuous, we conclude that $\mathcal{E}$ is sequentially weakly lower semicontinuous on $\mathcal{D}^{1,\mathfrak{p}(\cdot)}(\mathbf{R}^{\mathrm{m}})$.

4.4. Existence of a minimizer and the Euler–Lagrange equation

We now establish the existence of a minimizer of the energy functional $\mathcal{E}$ and the corresponding Euler–Lagrange equation, while providing pointwise control using Havin–Maz'ya potentials.

Let $\{\mathfrak{u}_n\}_{n\in\mathbf{N}} \subset \mathcal{D}^{1,\mathfrak{p}(\cdot)}(\mathbf{R}^{\mathrm{m}})$ be a minimizing sequence, i.e.,

$$\lim_{n\to\infty} \mathcal{E}(\mathfrak{u}_n) = \inf_{\mathfrak{v}\in\mathcal{D}^{1,\mathfrak{p}(\cdot)}(\mathbf{R}^{\mathrm{m}})} \mathcal{E}(\mathfrak{v}).$$

By the coercivity of $\mathcal{E}$ (proved earlier), there exists a constant $C > 0$ such that

$$\|\nabla\mathfrak{u}_n\|_{\mathrm{L}^{\mathfrak{p}(\cdot)}(\mathbf{R}^{\mathrm{m}})} \le C, \quad \text{for all } n \in \mathbf{N}.$$

Hence, $\{\mathfrak{u}_n\}$ is bounded in $\mathcal{D}^{1,\mathfrak{p}(\cdot)}$.

Since $\mathfrak{p}^- > 1$, the space $\mathcal{D}^{1,\mathfrak{p}(\cdot)}(\mathbf{R}^{\mathrm{m}})$ is reflexive (see [69]). Therefore, there exists a subsequence (denoted by $\{\mathfrak{u}_n\}$) and a function $\mathfrak{u}_0 \in \mathcal{D}^{1,\mathfrak{p}(\cdot)}(\mathbf{R}^{\mathrm{m}})$ such that

$$\mathfrak{u}_n \rightharpoonup \mathfrak{u}_0 \quad \text{weakly in } \mathcal{D}^{1,\mathfrak{p}(\cdot)}.$$

By the weak lower semicontinuity of $\mathcal{E}$, we have

$$\mathcal{E}(\mathfrak{u}_0) \le \liminf_{n\to\infty} \mathcal{E}(\mathfrak{u}_n) = \inf_{\mathfrak{v}} \mathcal{E}(\mathfrak{v}),$$

showing that $\mathfrak{u}_0$ is a minimizer. Uniqueness follows from the strict convexity of $\mathfrak{u} \mapsto \int_{\mathbf{R}^{\mathrm{m}}} |\nabla\mathfrak{u}|^{\mathfrak{p}(\varepsilon)}\, d\varepsilon$.

The minimizer $\mathfrak{u}_0$ satisfies the Euler–Lagrange equation:

$$-\mathrm{div}\Big(|\nabla\mathfrak{u}_0|^{\mathfrak{p}(\varepsilon)-2}\nabla\mathfrak{u}_0\Big) = \psi \quad \text{in } \mathbf{R}^{\mathrm{m}}.$$

More precisely, in the weak sense, for all test functions $\varphi \in C_0^\infty(\mathbf{R}^{\mathrm{m}})$:

$$\int_{\mathbf{R}^{\mathrm{m}}} |\nabla\mathfrak{u}_0|^{\mathfrak{p}(\varepsilon)-2}\nabla\mathfrak{u}_0 \cdot \nabla\varphi\, d\varepsilon = \int_{\mathbf{R}^{\mathrm{m}}} \psi\, \varphi\, d\varepsilon.$$

Weak solutions of the $\mathfrak{p}(\cdot)$-Laplace equation admit pointwise bounds using nonlinear potential operators of Havin–Maz'ya–Wolff type:

$$|\mathfrak{u}(\varepsilon)| \le C\Big[\mathcal{V}^{\mathrm{R}}_{\beta,\mathrm{m}}(|\psi|^{\mathfrak{p}'(\cdot)-1})(\varepsilon) + \mathrm{Tail}(\mathfrak{u},\mathrm{R})(\varepsilon)\Big],$$

where $\beta = \frac{1}{\mathfrak{p}^-}$ and

$$\mathrm{Tail}(\mathfrak{u},\mathrm{R})(\varepsilon) = \mathrm{R}^{\beta\mathrm{m}}\left(\frac{1}{|\mathcal{B}(\varepsilon,\mathrm{R})|}\int_{\mathcal{B}(\varepsilon,\mathrm{R})} |\mathfrak{u}|^{\mathfrak{p}^--1}\, dy\right)^{\frac{1}{\mathfrak{p}^--1}}.$$

Since $\psi \in \dot{\mathbf{K}}^{\alpha(\cdot)}_{\mathfrak{p}(\cdot),\mathfrak{q}(\cdot)}$, one has

$$|\psi|^{\mathfrak{p}'(\cdot)-1} \in \dot{\mathbf{K}}^{\alpha(\cdot)}_{\mathfrak{p}'(\cdot),\mathfrak{q}(\cdot)}, \quad \Big\||\psi|^{\mathfrak{p}'(\cdot)-1}\Big\| \lesssim \|\psi\|^{\mathfrak{p}^+-1}_{\dot{\mathbf{K}}_{\mathfrak{p}(\cdot),\mathfrak{q}(\cdot)}}.$$

For each dyadic annulus $\mathcal{A}_\kappa$:

$$\left\|2^{\kappa\alpha(\cdot)}|\psi|^{p'(\cdot)-1}\chi_\kappa\right\|_{L^{p'(\cdot)}} = \left\|(2^{\kappa\alpha(\cdot)}\psi\chi_\kappa)^{p'(\cdot)-1}\right\|_{L^{p'(\cdot)}}.$$

Using modular estimates in variable exponent spaces and summing over κ, we obtain

$$\left\|\mathcal{V}^{R}_{\beta,m}(|\psi|^{p'(\cdot)-1})\right\|_{\dot{K}^{\alpha(\cdot)}_{(p^*(\cdot))',q(\cdot)}} \lesssim \|\psi\|^{p^+-1}_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

The tail is estimated by

$$\mathrm{Tail}(u,R)(\varepsilon) \lesssim R^{\beta-\frac{m}{p^*(\varepsilon)}}\|u\|_{L^{p^*(\cdot)}(\mathcal{B}(\varepsilon,R))}.$$

Since $\alpha^- > \beta - \frac{m}{p^-}$, the tail belongs to the same Herz space and is controlled by $\|\nabla u\|_{L^{p(\cdot)}}$. Combining the pointwise Havin–Maz'ya estimate and tail control gives

$$\|u\|_{\dot{K}^{\alpha(\cdot)}_{p^*(\cdot),q(\cdot)}} \lesssim \|\psi\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} + \|\psi\|^{p^+-1}_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

This shows that the weak solution u belongs to the expected Herz space, with explicit bounds in terms of the data ψ.

5. Conclusions and future directions

We worked on a new variational framework on Euler–Lagrange equations of Dirichlet-type energy functionals involving a nonlinear and nonlocal analysis through truncated Havin–Maz'ya–Wolff potentials in the present work. The presence and uniqueness of minimizers in nonstandard growth equations of the type of $p(\cdot)$–Laplace equations were ensured through the creation of coercivity and weak lower semicontinuity of the underlying functional. One of the most important contributions of this work was the investigation around the globe of truncated nonlinear potentials of Herz spaces that offer effective pointwise control on weak solutions, as well as the previous works on the nonlinear potentials. The generality and strength of the proposed framework was illustrated by numerous nontrivial examples and remarks, in which the classical results were obtained as the limiting cases with appropriate parameter choices.

The current research lays some good foundations for future research. Initially, it is possible to consider the generalization of the analysis to completely nonlinear and higher-order fractional derivatives, and it is possible to investigate even more sophisticated nonlocal and nonstandard growth processes. Second, the boundedness and regularity properties of truncated Havin–Maz'ya–Wolff potentials of other functional spaces (e.g., variable-exponent Morrey or Besov spaces) would be of interest for further expanding their range of application in nonlinear PDEs. Lastly, the Herz space framework would allow applications to controllability, optimization, and stability issues of physical models with nonlocal or fractional dynamics, exploiting the flexibility and total ordering available to the Herz space framework.

Author contributions

Z. A. Khan: Conceptualization, validation, writing–original draft; W. Afzal: Conceptualization, methodology, validation, visualization, writing–original draft, writing–review and editing; M. Abbas: Conceptualization; D. Breaz: Validation, visualization; L. I. Cotîrlă: Validation, visualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors express their gratitude to the Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2026R8), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare no conflict of interest.

References

1. O. P. Agrawal, Formulation of Euler–Lagrange equations for fractional variational problems, *J. Math. Anal. Appl.,* **272** (2002), 368–379. http://dx.doi.org/10.1016/S0022-247X(02)00180-4
2. C. W. Gear, B. Leimkuhler, G. K. Gupta, Automatic integration of Euler–Lagrange equations with constraints, *J. Comput. Appl. Math.,* **12** (1985), 77–90. http://dx.doi.org/10.1016/0377-0427(85)90008-1
3. M. Crampin, On the differential geometry of the Euler–Lagrange equations, and the inverse problem of Lagrangian dynamics, *J. Phys. A: Math. Gen.,* **14** (1981), 2567. http://dx.doi.org/10.1088/0305-4470/14/10/012
4. D. Baleanu, J. J. Trujillo, On exact solutions of a class of fractional Euler–Lagrange equations, *Nonlinear Dyn.,* **52** (2008), 331–335. http://dx.doi.org/10.1007/s11071-007-9281-7
5. W. Afzal, M. Abbas, W. Hamali, A. M. Mahnashi, M. D. L. Sen, Hermite–Hadamard-type inequalities via Caputo–Fabrizio fractional integral for h-Godunova–Levin and (h_1, h_2)-convex functions, *Fractal Fract.,* **7** (2023), 687. http://dx.doi.org/10.3390/fractalfract7090687
6. O. P. Agrawal, Generalized Euler–Lagrange equations and transversality conditions for FVPs in terms of the Caputo derivative, *J. Vib. Control,* **13** (2007), 1217–1237. https://doi.org/10.1177/1077546307077472
7. J. M. Ball, *Minimizers and the Euler-Lagrange equations*, In Trends and Applications of Pure Mathematics to Mechanics, Springer, Berlin, Heidelberg, 2005, 1–4. http://dx.doi.org/10.1007/3-540-12916-2_47
8. M. Israr, W. Afzal, S. Ahmad, D. Breaz, L. I. Cotîrlă, Enhancing deep image prior with multiscale attention, SVD pooling, and preconditioned optimizers for image processing, *Eur. J. Pure Appl. Math.,* **18** (2025), 6638. https://doi.org/10.29020/nybg.ejpam.v18i4.6638
9. P. J. Rabier, W. C. Rheinboldt, On the numerical solution of the Euler–Lagrange equations, *SIAM J. Numer. Anal.,* **32** (1995), 318–329. http://dx.doi.org/10.1137/0732014

10. A. Al-Omari, M. H. Alqahtani, Some operators in soft primal spaces, *AIMS Math.,* **9** (2024), 10756–10774. https://doi.org/10.3934/math.2024525

11. M. Klimek, Solutions of Euler–Lagrange equations in fractional mechanics, In *AIP Conf. Proc.,* **956** (2007), 73–78. https://doi.org/10.1063/1.2820982

12. R. Groll, S. Jakirlić, C. Tropea, Comparative study of Euler/Euler and Euler/Lagrange approaches simulating evaporation in a turbulent gas–liquid flow, *Int. J. Numer. Meth. Fluids,* **59** (2009), 873–906. http://dx.doi.org/10.1002/fld.1914

13. A. Adrees, W. Afzal, K. Shabbir, N. M. Dahshan, A. E. Abuzeid, R. A. Tahir, et al., On some properties and integral inequalities for modified (p, h)-convex stochastic processes, *Open J. Math. Sci.,* **10** (2026), 477–491. https://doi.org/10.30538/oms2026.0300

14. A. Sokolichin, G. Eigenberger, A. Lapin, A. Lübert, Dynamic numerical simulation of gas-liquid two-phase flows Euler/Euler versus Euler/Lagrange, *Chem. Eng. Sci.,* **52** (1997), 611–626. http://dx.doi.org/10.1016/S0009-2509(96)00480-8

15. M. A. Herzallah, D. Baleanu, Fractional-order Euler–Lagrange equations and formulation of Hamiltonian equations, *Nonlinear Dyn.,* **58** (2009), 385–391. http://dx.doi.org/10.1007/s11071-009-9561-5

16. Z. A. Khan, W. Afzal, M. Abbas, J. Ro, A. A. Zaagan, Some well known inequalities on two dimensional convex mappings by means of pseudo L-R interval order relations via fractional integral operators having non-singular kernel, *AIMS Math.,* **9** (2024), 16061–16092. https://doi.org/10.3934/math.2024778

17. Z. Y. Li, J. L. Fu, L. Q. Chen, Euler–Lagrange equation from nonlocal-in-time kinetic energy of nonconservative system, *Phys. Lett. A,* **374** (2009), 106–109. http://dx.doi.org/10.1016/j.physleta.2009.10.080

18. S. Balachandar, Lagrangian and Eulerian drag models that are consistent between Euler–Lagrange and Euler–Euler (two-fluid) approaches for homogeneous systems, *Phys. Rev. Fluids,* **5** (2020), 084302. http://dx.doi.org/10.1103/PhysRevFluids.5.084302

19. W. Afzal, Boundedness and regularity of the Navier–Stokes system in generalized Herz spaces via a novel fractional potential framework, *Chaos Soliton. Fract.,* **201** (2025), 117086. http://dx.doi.org/10.1016/j.chaos.2025.117086

20. I. A. Kogan, P. J. Olver, Invariant Euler–Lagrange equations and the invariant variational bicomplex, *Acta Appl. Math.,* **76** (2003), 137–193. http://dx.doi.org/10.1023/A:1021781304548

21. O. Alghamdi, A. Al-Omari, M. H. Alqahtani, Novel operators in the frame of primal topological spaces, *AIMS Math.,* **9** (2024), 25792–25808. https://doi.org/10.3934/math.20241260

22. C. Pérez, On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted L^p-spaces with different weights, *Proc. London Math. Soc.,* **71** (1995), 135–157. http://dx.doi.org/10.1112/plms/s3-71.1.135

23. W. Reichel, Characterization of balls by Riesz-potentials, *Ann. Mat. Pura Appl.,* **188** (2009), 235–245. http://dx.doi.org/10.1007/s10231-008-0073-6

24. W. Afzal, Regularity of parabolic Ornstein–Uhlenbeck equations via boundedness of fractional Muckenhoupt-type weighted singular operators in variable Herz spaces, *J. Pseudo-Differ. Oper. Appl.,* **17** (2026), 2. http://dx.doi.org/10.1007/s11868-025-00685-w

25. J. G. Bak, Sharp estimates for the Bochner-Riesz operator of negative order in $\mathbb{R}^2$, *Proc. Amer. Math. Soc.,* **125** (1997), 1977–1986. http://dx.doi.org/10.1090/S0002-9939-97-03723-4

26. W. Afzal, A note on the boundedness of the Wolff potential on complete noncompact manifolds in Zygmund spaces, *Open J. Math. Anal.,* **10** (2026), 215–219. https://doi.org/10.30538/psrp-oma2026.0199

27. A. Almeida, P. Harjulehto, P. Hästö, T. Lukkari, Riesz and Wolff potentials and elliptic equations in variable exponent weak Lebesgue spaces, *Ann. Mat. Pura Appl.,* **194** (2015), 405–424. http://dx.doi.org/10.1007/s10231-014-0398-9

28. M. Borowski, I. Chlebicka, B. Miasojedow, Boundedness of Wolff-type potentials and applications to PDEs, *Nonlinear Anal. Real World Appl.,* **76** (2024), 104025. http://dx.doi.org/10.1016/j.nonrwa.2023.104025

29. V. G. Maz'ya, V. P. Havin, A nonlinear potential theory, *Uspehi Mat. Nauk,* **27** (1972), 67–138.

30. D. Cardona, M. Ruzhansky, Boundedness of pseudo-differential operators in subelliptic Sobolev and Besov spaces on compact Lie groups, *Complex Var. Elliptic Equ.,* **69** (2024), 1049–1082. https://doi.org/10.1080/17476933.2023.2196416

31. R. Korte, T. Kuusi, A note on the Wolff potential estimate for solutions to elliptic equations involving measures, *Adv. Calc. Var.,* **3** (2010), 99–113. http://dx.doi.org/10.1515/ACV.2010.005

32. J. Malý, Wolff potential estimates of superminimizers of Orlicz type Dirichlet integrals, *Manuscripta Math.,* **110** (2003), 513–525. http://dx.doi.org/10.1007/s00229-003-0358-4

33. T. Lukkari, F. Y. Maeda, N. Marola, Wolff potential estimates for elliptic equations with nonstandard growth and applications, *Forum Math.,* **22** (2010), 1061–1087. http://dx.doi.org/10.1515/FORUM.2010.057

34. M. Abbas, W. Afzal, T. Botmart, A. M. Galal, Jensen, Ostrowski and Hermite–Hadamard type inequalities for h-convex stochastic processes by means of center-radius order relation, *AIMS Math.,* **8** (2023), 16013–16030. https://doi.org/10.3934/math.2023817

35. P. Baroni, Riesz potential estimates for a general class of quasilinear equations, *Calc. Var. Partial Dif.,* **53** (2015), 803–846. http://dx.doi.org/10.1007/s00526-014-0782-y

36. I. Chlebicka, Y. Youn, A. Zatorska-Goldstein, Wolff potentials and measure data vectorial problems with Orlicz growth, *Calc. Var. Partial Dif.,* **62** (2023), 64. http://dx.doi.org/10.1007/s00526-022-02402-7

37. G. Cupini, P. Marcellini, E. Mascolo, Local boundedness of weak solutions to elliptic equations with p, q-growth, *Math. Eng.,* **5** (2023), 1–28. http://dx.doi.org/10.3934/mine.2023065

38. G. Mingione, Gradient estimates below the duality exponent, *Math. Ann.,* **346** (2010), 571–627. http://dx.doi.org/10.1007/s00208-009-0421-2

39. R. O'Neil, Convolution operators and $L(p, q)$ spaces, *Duke Math. J.,* **30** (1963), 129–142. http://dx.doi.org/10.1215/S0012-7094-63-03015-1

40. G. Peetre, Espaces d'interpolation, généralisations, applications, *Rend. Sem. Mat. Fis. Milano,* **34** (1964), 133–164. https://doi.org/10.1007/BF02923402

41. W. Afzal, M. Abbas, M. Z. Meetei, S. Bourazza, Tensorial Maclaurin approximation bounds and structural properties for mixed-norm Orlicz–Zygmund spaces, *Mathematics,* **13** (2025), 917. https://doi.org/10.3390/math13060917

42. M. Kim, K. A. Lee, S. C. Lee, Wolff potential estimates and Wiener criterion for nonlocal equations with Orlicz growth, *J. Funct. Anal.,* **288** (2025), 110690. http://dx.doi.org/10.1016/j.jfa.2024.110690

43. A. M. Abd El-latif, M. H. Alqahtani, F. A. Gharib, Strictly wider class of soft sets via supra soft δ-closure operator, *Int. J. Anal. Appl.,* **22** (2024), 47. https://doi.org/10.28924/2291-8639-22-2024-47

44. J. Maas, J. van Neerven, Boundedness of Riesz transforms for elliptic operators on abstract Wiener spaces, *J. Funct. Anal.,* **257** (2009), 2410–2475. http://dx.doi.org/10.1016/j.jfa.2009.07.001

45. V. S. Guliyev, F. Deringoz, On the Riesz potential and its commutators on generalized Orlicz-Morrey spaces, *J. Funct. Spaces,* **2014** (2014), 617414. http://dx.doi.org/10.1155/2014/617414

46. J. Wu, Boundedness for Riesz-type potential operators on Herz-Morrey spaces with variable exponent, *Math. Inequal. Appl.,* **18** (2015), 471–484. http://dx.doi.org/10.7153/mia-18-35

47. H. Rafeiro, S. Samko, Riesz potential operator in continual variable exponents Herz spaces, *Math. Nachr.,* **288** (2015), 465–475. http://dx.doi.org/10.1002/mana.201300270

48. H. Ahmad, M. Tariq, A. Asghar, W. Afzal, M. Aphane, R. Efendiev, Some new notions of mathematical integral inequalities: Theory and applications, *Int. J. Anal. Appl.,* **24** (2026), 175. https://doi.org/10.28924/2291-8639-24-2026-175

49. Z. A. Ameen, M. H. Alqahtani, Baire category soft sets and their symmetric local properties, *Symmetry,* **15** (2023), 1810. https://doi.org/10.3390/sym15101810

50. W. Afzal, M. Abbas, D. Breaz, L. I. Cotîrlă, Fractional Hermite–Hadamard, Newton–Milne, and convexity involving arithmetic–geometric mean-type inequalities in Hilbert and mixed-norm Morrey spaces $\ell_{q(\cdot)}(M_{p(\cdot),v(\cdot)})$ with variable exponents, *Fractal Fract.,* **8** (2024), 518. http://dx.doi.org/10.3390/fractalfract8090518

51. Y. Sawano, T. Shimomura, Sobolev embeddings for Riesz potentials of functions in Musielak–Orlicz–Morrey spaces over non-doubling measure spaces, *Integr. Transf. Spec. F.,* **25** (2014), 976–991. http://dx.doi.org/10.1080/10652469.2014.955099

52. T. Ohno, T. Shimomura, Generalized Riesz potential operators on Musielak–Orlicz–Morrey spaces over unbounded metric measure spaces, *Anal. Math. Phys.,* **15** (2025), 24. http://dx.doi.org/10.1007/s13324-024-01004-2

53. H. Aimar, S. Hartzstein, B. Iaffei, B. Viviani, The Riesz potential as a multilinear operator into general BMO_β spaces, *J. Math. Sci.,* **173** (2011), 643–655. http://dx.doi.org/10.1007/s10958-011-0261-4

54. W. Afzal, M. Abbas, M. Tariq, J. E. Mac, H. Ahmad, A note on the boundedness of the multidimensional Katugampola operator in Campanato spaces, *Gulf J. Math.,* **23** (2026), 1. https://doi.org/10.56947/5e9waj50

55. J. J. Hasanov, R. Ayazoglu, S. Bayrakci, B-maximal commutators, commutators of B-singular integral operators and B-Riesz potentials on B-Morrey spaces, *Open Math.,* **18** (2020), 715–730. http://dx.doi.org/10.1515/math-2020-0033

56. V. I. Burenkov, Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces. I, *Eurasian Math. J.,* **3** (2012), 11–32.

57. A. Al-Omari, M. H. Alqahtani, Primal structure with closure operators and their applications, *Mathematics,* **11** (2023), 4946. https://doi.org/10.3390/math11244946

58. B. Sultan, F. Azmi, M. Sultan, M. Mehmood, N. Mlaiki, Boundedness of Riesz potential operator on grand Herz-Morrey spaces, *Axioms,* **11** (2022), 583. http://dx.doi.org/10.3390/axioms11110583

59. W. Afzal, M. Abbas, N. M. Aloraini, J. Ro, Resolution of open problems via Orlicz-Zygmund spaces and new geometric properties of Morrey spaces in the Besov sense with non-standard growth, *AIMS Math.,* **10** (2025), 13908–13940. http://dx.doi.org/10.3934/math.2025630

60. M. H. Alqahtani, A. M. Abd El-latif, Separation axioms via novel operators in the frame of topological spaces and applications, *AIMS Math.,* **9** (2024), 14213–14227. https://doi.org/10.3934/math.2024690

61. W. Afzal, M. Abbas, J. E. Macías-Díaz, A. Gallegos, Y. Almalki, Boundedness and Sobolev-type estimates for the exponentially damped Riesz potential with applications to the regularity theory of elliptic PDEs, *Fractal Fract.,* **9** (2025), 458. http://dx.doi.org/10.3390/fractalfract9070458

62. M. Izuki, Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization, *Anal. Math.,* **36** (2010), 33–50. http://dx.doi.org/10.2478/s11533-009-0057-9

63. X. Li, D. Yang, Boundedness of some sublinear operators on Herz spaces, *Illinois J. Math.,* **40** (1996), 484–501. http://dx.doi.org/10.1215/ijm/1255986021

64. A. Almeida, D. Drihem, Maximal, potential and singular type operators on Herz spaces with variable exponents, *J. Math. Anal. Appl.,* **394** (2012), 781–795. http://dx.doi.org/10.1016/j.jmaa.2012.03.059

65. W. Afzal, M. Abbas, M. Tariq, E. Hincal, W. M. Abdelfattah, Sharp boundedness criteria for the Wolff potential on complete Riemannian manifolds, *Int. J. Math. Comput. Sci.,* **21** (2026), 519–526. https://doi.org/10.69793/ijmcs/02.2026/khan

66. D. Drihem, F. Seghiri, Notes on the Herz-type Hardy spaces of variable smoothness and integrability, *Math. Inequal. Appl.,* **19** (2016), 145–165. http://dx.doi.org/10.7153/mia-19-11

67. R. A. Adams, J. J. F. Fournier, *Sobolev spaces,* 2nd ed., Academic Press, Amsterdam, 2003.

68. P. Harjulehto, P. Hästö, Sobolev inequalities for variable exponents attaining the values 1 and n, *Publ. Mat.,* **52** (2008), 347–363. http://dx.doi.org/10.5565/PUBLMAT_52108_17

69. L. Diening, P. Harjulehto, P. Hästö, M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents,* Springer, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-18363-8

70. M. R. Ebert, U. Kähler, I. Sabadini, J. Toft (Eds.), *New tools in mathematical analysis and applications: Proceedings of the 14th ISAAC Congress 2023*, Ribeirão Preto, Brazil, Springer Nature Switzerland, 2025.

71. W. Afzal, M. Abbas, O. M. Alsalami, Bounds of different integral operators in tensorial Hilbert and variable exponent function spaces, *Mathematics,* **12** (2024), 2464. http://dx.doi.org/10.3390/math12162464

72. M. Nasir, A. Raza, L. I. Cotîrlă, D. Breaz, Boundedness of Bessel–Riesz operator in variable Lebesgue measure spaces, *Mathematics,* **13** (2025), 410. http://dx.doi.org/10.3390/math13030410