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Research article

A study of operator classes on Hilbert spaces

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Abstract: This paper introduces and studies a new class of operators on Hilbert spaces, called polynomially quasi-m-complex Helton operators, which extend classical Helton-type operator classes through polynomial perturbations and higher-order structural relations. Motivated by recent developments in operator theory, an operator $\mathcal{A} \in \mathcal{L}(\mathcal{K})$ is said to belong to this class if there exists a nonconstant polynomial p such that $p(\mathcal{A})^* \,\Theta(\mathcal{B}, \mathcal{A}, C)^m(I)\, p(\mathcal{A}) = 0$, where $\Theta(\mathcal{B}, \mathcal{A}, C)$ represents the corresponding m-complex Helton-type operator expression. We establish fundamental properties of this class and investigate its structural behavior, including stability under direct sums, invariance under closed invariant subspaces, and stability under powers of operators. In particular, we show that, under suitable conditions, the order of the class may increase from m to higher levels of the hierarchy. Several equivalent characterizations are also obtained in terms of kernels of operator polynomials, providing deeper insight into the underlying algebraic structure.

Keywords: Helton operators; isometry; quasi-m-isometries; mathematical transformations
Mathematics Subject Classification: 47B48

1. Introduction

In recent years, the theory of operator classes defined via higher-order identities has attracted considerable attention in functional analysis, particularly in the context of Hilbert space operators. A central role is played by m-isometric operators, introduced by Agler and Stankus [1–3], which generalize classical isometries through higher-order norm-preserving conditions. This line of research has been further developed through various extensions, including quasi-m-isometric operators and (m, C)-symmetric operators, as studied for instance by Mahmoud et al. [4, 5] and Chō et al. [6], where conjugation-based structures and perturbation techniques are incorporated.

More recently, polynomial modifications of these classes have emerged as a natural direction, leading to the notion of polynomially quasi-m-isometric operators, which capture stability under polynomial functional calculus. In parallel, the study of operators that become normal under polynomial transformations, so-called polynomially normal operators, has been investigated by Cho et al. [7], highlighting deep connections between algebraic relations and spectral properties.

These developments motivate the introduction and systematic study of new operator classes defined via polynomial perturbations and higher-order identities, unifying and extending several strands of modern operator theory. Let $\mathcal{K}$ be a complex Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$, which is linear in the first variable and conjugate-linear in the second variable. Let $\mathcal{L}(\mathcal{K})$ denote the algebra of all bounded linear operators on $\mathcal{K}$. An operator $\mathcal{A} \in \mathcal{L}(\mathcal{K})$ is called an *m-isometric operator* if

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \mathcal{A}^{*k} \mathcal{A}^k = 0. \tag{1.1}$$

This definition extends the classical notion of isometries, which corresponds to the case $m = 1$. The class of m-isometric operators has been widely studied due to its rich structural and spectral properties. A further generalization is given by the class of *n-quasi-m-isometric operators*. An operator $\mathcal{A} \in \mathcal{L}(\mathcal{K})$ is said to be n-quasi-m-isometric [8] if

$$\mathcal{A}^{*n} \left(\sum_{k=0}^{m} (-1)^k \binom{m}{k} \mathcal{A}^{*k} \mathcal{A}^k \right) \mathcal{A}^n = 0. \tag{1.2}$$

This condition weakens the m-isometric property by requiring it to hold on a subspace determined by the powers of $\mathcal{A}$. A natural extension of these notions is obtained by incorporating polynomial perturbations. An operator $\mathcal{A} \in \mathcal{L}(\mathcal{K})$ is called *polynomially m-isometric* ([9]) if there exists a nonconstant polynomial p such that

$$p(\mathcal{A})^* \left(\sum_{k=0}^{m} (-1)^k \binom{m}{k} \mathcal{A}^{*k} \mathcal{A}^k \right) p(\mathcal{A}) = 0. \tag{1.3}$$

This definition extends the class of m-isometric operators by requiring the m-isometric identity to hold after a polynomial perturbation acting on both sides, thereby enlarging the framework and allowing for a richer class of operators with similar structural properties.

Let C be a conjugation on $\mathcal{K}$, that is, an antilinear, isometric involution satisfying $C^2 = I$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{K}$ [10]. We denote by $C[\mathcal{K}]$ the set of all conjugations on $\mathcal{K}$; that is

$$C[\mathcal{K}] = \{C : \mathcal{K} \to \mathcal{K} \mid C \text{ is antilinear, isometric, and } C^2 = I\}.$$

An operator $\mathcal{A} \in \mathcal{L}(\mathcal{K})$ is called an *(m, C)-isometric operator* ([6, 11]) if

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \mathcal{A}^{*k} C \mathcal{A}^k C = 0. \tag{1.4}$$

This class provides a natural extension of m-isometric operators by incorporating a conjugation symmetry, and at the same time generalizes the framework of complex symmetric operators. In this setting, the operator $\mathcal{A}$ interacts with its adjoint through the involution C, leading to a richer structure that reflects both metric and symmetry properties.

Combining this conjugation-based structure with the quasi-framework, an operator $\mathcal{A} \in \mathcal{L}(\mathcal{K})$ is said to be *n-quasi-(m, C)-isometric* [4] if

$$\mathcal{A}^{*n}\left(\sum_{k=0}^{m}(-1)^k\binom{m}{k}\mathcal{A}^{*k}C\mathcal{A}^kC\right)\mathcal{A}^n = 0. \tag{1.5}$$

This class unifies higher-order isometric behavior with conjugation symmetry in a weakened (quasi) sense, where the defining identity holds asymptotically on the range of $\mathcal{A}^n$. It thus provides a flexible framework that captures both structural and asymptotic properties of operators.

Operators belonging to these classes have attracted considerable attention due to their stability under perturbations, their rich spectral behavior, and their relevance in the study of invariant subspaces and decomposition theory. Moreover, they serve as a natural bridge between classical isometric-type operators and symmetry-driven operator classes, thereby offering a unified perspective in modern operator theory.

Let $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{K})$. Following [12], define the linear map

$$\Theta(\mathcal{B}, \mathcal{A}) : \mathcal{L}(\mathcal{K}) \to \mathcal{L}(\mathcal{K}) \tag{1.6}$$

by

$$\Theta(\mathcal{B}, \mathcal{A})(X) = \mathcal{A}X - X\mathcal{B}, \quad X \in \mathcal{L}(\mathcal{K}).$$

By induction, the kth iterate of $\Theta(\mathcal{B}, \mathcal{A})$ applied to the identity operator I is given by

$$\Theta(\mathcal{B}, \mathcal{A})^k(I) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\mathcal{B}^j\mathcal{A}^{k-j}. \tag{1.7}$$

An operator $\mathcal{A} \in \mathcal{L}(\mathcal{K})$ is said to belong to the *mth Helton class* of $\mathcal{B} \in \mathcal{L}(\mathcal{K})$ if

$$\Theta(\mathcal{B}, \mathcal{A})^m(I) = 0. \tag{1.8}$$

For more details, see [13–15].

A natural extension of the Helton class is obtained by introducing a quasi-structure. We say that $\mathcal{A}$ belongs to the *n-quasi-Helton class* of $\mathcal{B}$ (see [16]) if

$$\mathcal{A}^{*n}\,\Theta(\mathcal{B}, \mathcal{A})^m(I)\,\mathcal{A}^n = 0. \tag{1.9}$$

This condition weakens the Helton identity by requiring it to hold on the range of $\mathcal{A}^n$ rather than on the whole space.

For $C \in C[\mathcal{K}]$, an operator $\mathcal{A} \in \mathcal{L}(\mathcal{K})$ is said to belong to the *m-complex Helton class* of $\mathcal{B} \in \mathcal{L}(\mathcal{K})$ (see [17]) if

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\mathcal{B}^jC\,\mathcal{A}^{m-j}C = 0. \tag{1.10}$$

This formulation incorporates conjugation symmetry into the Helton framework and connects naturally with complex symmetric operator theory.

Combining the quasi-structure with conjugation, we say that $\mathcal{A} \in \mathcal{L}(\mathcal{K})$ belongs to the *n-quasi-complex Helton class* (see [17]) of $\mathcal{B}$ if

$$\mathcal{A}^{*n}\left(\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\mathcal{B}^{j}C\,\mathcal{A}^{m-j}C\right)\mathcal{A}^{n}=0. \tag{1.11}$$

This class provides a unified framework that simultaneously captures higher-order commutation, conjugation symmetry, and asymptotic (quasi) behavior. A further generalization is obtained via polynomial perturbations. Let p be a nonzero complex polynomial. We say that $\mathcal{A}$ belongs to the polynomially quasi-Helton class of $\mathcal{B}$ if

$$p(\mathcal{A})^{*}\,\Theta(\mathcal{B},\mathcal{A})^{m}(I)\,p(\mathcal{A})=0. \tag{1.12}$$

In this case, we write $\mathcal{A} \in [p\text{-QH}_m(\mathcal{B})]$. These classes have attracted significant attention due to their stability properties, spectral behavior, and applications in invariant subspace theory and perturbation analysis. They offer a natural bridge between commutator-type identities, symmetry-driven operator theory, and polynomial functional calculus. For further details on the results cited above, we refer the reader to [18–20].

The study of a new class of operators represents a natural extension of operator theory and contributes to a deeper understanding of its algebraic and analytical structure. It also opens new research perspectives for those interested in this branch of mathematics. From this standpoint, the idea of this work arose in order to expand this field, develop some of its aspects, and enrich its results. We now introduce the main contribution of this paper, which provides a unified generalization of the previously discussed classes. More generally, let $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{K})$, let C be a conjugation on $\mathcal{K}$, and let p be a nonconstant complex polynomial. We say that $\mathcal{A}$ belongs to the *polynomially n-quasi-complex Helton class* of $\mathcal{B}$ if

$$p(\mathcal{A})^{*}\left(\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\mathcal{B}^{j}C\,\mathcal{A}^{m-j}C\right)p(\mathcal{A})=0. \tag{1.13}$$

This definition simultaneously incorporates three fundamental features: polynomial perturbations via the functional calculus, quasi-structure through the powers of $\mathcal{A}$, and conjugation symmetry induced by C. It therefore unifies and extends the classical Helton class, the n-quasi-Helton class, and the complex Helton class within a single operator-theoretic framework.

The introduction of this class provides a flexible and robust setting for the analysis of higher-order operator identities, allowing for a deeper investigation of structural and spectral properties. In particular, it enables the transfer of techniques from polynomial operator theory and complex symmetric operator theory to the study of Helton-type commutation relations.

This paper is devoted to the study of a new class of operators, called polynomially quasi-m-complex Helton $[p\text{-}QCH_m]$ operators, defined through polynomial functional relations acting on bounded linear operators in a Hilbert space. The main goal is to investigate their structural, stability, and spectral properties, and to clarify how these operators behave under natural algebraic and topological operations.

In Section 2, we begin by establishing several equivalent characterizations of the class in terms of operator polynomials and kernel-range relations. These characterizations provide a more tractable framework for analyzing the defining conditions of $[p\text{-}QCH_m]$ operators. We also present a matrix representation for elements of this class, which allows us to describe their internal structure in a concrete operator-theoretic form. Based on this representation, we derive a number of consequences that clarify how the polynomial relations are reflected at the level of block operator matrices.

A central part of Section 2 is devoted to stability properties. We prove that the class of $[p\text{-}QCH_m]$ operators is stable under direct sums and under restriction to closed invariant subspaces. In addition, we show that the class is closed with respect to norm limits, thereby confirming its robustness in the operator norm topology. We also investigate how the defining property behaves under changes of the order parameter m, and we establish results showing that the order of the class can be increased while preserving membership, which leads to a clearer hierarchical structure among different levels of the class. Section 3 focuses on spectral aspects of $[p\text{-}QCH_m]$ operators. In particular, we examine the relationship between the polynomial structure and spectral behavior. A key question addressed in Section 4 is whether the $[p\text{-}QCH_m]$ property is preserved under taking powers of an operator. We provide partial positive answers under suitable conditions and identify cases where invariance holds. These results highlight the interaction between polynomial identities and spectral stability, and they show that the behavior of powers plays an important role in understanding the global structure of the class.

Overall, the results of this paper contribute to the development of polynomially defined operator classes in Hilbert space operator theory. They demonstrate how algebraic identities involving operator polynomials can be used to obtain stability, structural decomposition, and spectral invariance results. The study further emphasizes the interplay between polynomial functional calculus, operator matrix techniques, and spectral theory, providing a foundation for future investigations in generalized Helton-type operator frameworks.

2. Polynomially quasi-m-complex Helton class

In this section, we consider the class of *polynomially quasi-m-complex Helton operators*, which combines polynomial and quasi-Helton structures with conjugation symmetry.

Definition 2.1. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{K})$, let $C \in C[\mathcal{K}]$, and let $p \in \mathbb{C}[z] \setminus \{0\}$. We say that $\mathcal{A}$ belongs to the* polynomially quasi-complex m-Helton class *of $\mathcal{B}$ if*

$$p(\mathcal{A})^* \left(\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \mathcal{B}^j C \mathcal{A}^{m-j} C \right) p(\mathcal{A}) = 0.$$

In this case, we write $\mathcal{A} \in [p\text{-}\mathrm{QCH}_m(\mathcal{B})]$.

This class generalizes the n-quasi-complex Helton class by replacing the power $\mathcal{A}^n$ with an arbitrary polynomial $p(\mathcal{A})$. It provides a flexible framework that incorporates both polynomial functional calculus and conjugation symmetry within the Helton operator setting.

Let $\mathcal{A}, \mathcal{B}, C \in \mathcal{L}(\mathcal{K})$. For each integer $l \geq 0$, define

$$\Theta(\mathcal{B}, \mathcal{A}; C)^l(I) := \sum_{k=0}^{l} (-1)^{l-k} \mathcal{B}^k C \mathcal{A}^{l-k} C. \tag{2.1}$$

Then, in view of the above definition, $\mathcal{A} \in [p\text{-}\mathrm{QCH}_m(\mathcal{B})]$ if and only if

$$p(\mathcal{A})^* \Theta(\mathcal{B}, \mathcal{A}; C)^m(I)\, p(\mathcal{A}) = 0. \tag{2.2}$$

Remark 2.2. *(i) Note that for $p(t) = t^n$, Definition 2.1 reduces to [17, Definition 2.7].*

(ii) The class of m-complex Helton operators is a proper subset of $[p\text{-QCH}_m(\cdot)]$ for every polynomial $p \in \mathbb{C}[X]$, as illustrated by the following example.

Example 2.3. *Let $\mathcal{K} = \mathbb{C}^3$, and define the operators*

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Then, $\mathcal{A}^3 = 0$, $\mathcal{A}^2 \neq 0$, and $\mathcal{B}^3 = 0$, $\mathcal{B}^2 \neq 0$. Moreover, $\mathcal{B} \neq \mathcal{A}^k$ for any integer $k \geq 1$. Define a conjugation C: $\mathbb{C}^3 \to \mathbb{C}^3$ by $C(x_1, x_2, x_3) = (\overline{x_3}, \overline{x_2}, \overline{x_1})$. Then, $C \in C[\mathbb{C}^3]$ is a conjugation on $\mathcal{K}$. Consider the polynomial $p(t) = t^2(t-1) = t^3 - t^2$, which is not of the form t^k. Because $\mathcal{A}^3 = 0$, we have

$$p(\mathcal{A}) = \mathcal{A}^2(\mathcal{A} - I) = -\mathcal{A}^2 \neq 0, \quad p(\mathcal{A})^* = -(\mathcal{A}^*)^2.$$

Now, using the definition

$$\Theta(\mathcal{B}, \mathcal{A}; C)^2(I) = \sum_{k=0}^{2} (-1)^{2-k} \mathcal{B}^k C \mathcal{A}^{2-k} C,$$

and the fact that $C\mathcal{A}C = \mathcal{A}$, we obtain

$$\Theta(\mathcal{B}, \mathcal{A}; C)^2(I) = \mathcal{A}^2 - \mathcal{B}\mathcal{A} + \mathcal{B}^2.$$

Hence, $\Theta(\mathcal{B}, \mathcal{A}; C)^2(I) \neq 0$. Finally, because $\mathcal{A}^3 = 0$, we get

$$p(\mathcal{A})^* \Theta(\mathcal{B}, \mathcal{A}; C)^2(I)\, p(\mathcal{A}) = (\mathcal{A}^*)^2 \Theta(\mathcal{B}, \mathcal{A}; C)^2(I)\, \mathcal{A}^2 = 0.$$

Therefore, $\mathcal{A} \in [p\text{-QCH}_2(\mathcal{B})]$, and $\Theta(\mathcal{B}, \mathcal{A}; C)^2(I) \neq 0$.

Proposition 2.4. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{K})$, let $C \in C[\mathcal{K}]$, and let $p \in \mathbb{C}[z] \setminus \{0\}$. Then,*

(i) If $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$ and $q = \mu p$ for some $\mu \neq 0$, then $\mathcal{A} \in [q\text{-QCH}_m(\mathcal{B})]$.

(ii) If $p(\mathcal{A})$ is invertible, and $\mathcal{A} \in [p\text{-}QCH_m(\mathcal{B})]$, then

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \mathcal{B}^j C \mathcal{A}^{m-j} C = 0.$$

(iii) If $p(\mathcal{A}) = \mathcal{A}^n$, then $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$ reduces to the n-quasi-complex m-Helton condition.

Proof. (i) Suppose that $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$. Then, by Definition 2.1, we have

$$p(\mathcal{A})^* \Theta(\mathcal{B}, \mathcal{A}; C)^m(I)\, p(\mathcal{A}) = 0.$$

Let $q = \mu p$ with $\mu \neq 0$. Then, $q(\mathcal{A}) = \mu\, p(\mathcal{A})$, and hence,

$$q(\mathcal{A})^* = \overline{\mu}\, p(\mathcal{A})^*.$$

Therefore,
$$q(\mathcal{A})^* \, \Theta(\mathcal{B}, \mathcal{A}; C)^m(I) \, q(\mathcal{A}) = |\mu|^2 \, p(\mathcal{A})^* \, \Theta(\mathcal{B}, \mathcal{A}; C)^m(I) \, p(\mathcal{A}) = 0.$$
Thus, $\mathcal{A} \in [q\text{-QCH}_m(\mathcal{B})]$.

(ii) Assume that $p(\mathcal{A})$ is invertible and that $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$. Then,
$$p(\mathcal{A})^* \, \Theta(\mathcal{B}, \mathcal{A}; C)^m(I) \, p(\mathcal{A}) = 0.$$
Multiplying on the left by $\left(p(\mathcal{A})^*\right)^{-1}$ and on the right by $p(\mathcal{A})^{-1}$, we obtain
$$\Theta(\mathcal{B}, \mathcal{A}; C)^m(I) = 0.$$
Using the definition of $\Theta(\mathcal{B}, \mathcal{A}; C)^m(I)$, this gives
$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \mathcal{B}^j \, C \, \mathcal{A}^{m-j} \, C = 0.$$

(iii) If $p(\mathcal{A}) = \mathcal{A}^n$, then, the defining condition becomes
$$(\mathcal{A}^n)^* \, \Theta(\mathcal{B}, \mathcal{A}; C)^m(I) \, \mathcal{A}^n = 0,$$
that is,
$$(\mathcal{A}^*)^n \, \Theta(\mathcal{B}, \mathcal{A}; C)^m(I) \, \mathcal{A}^n = 0.$$
This is precisely the defining condition of the n-quasi-complex m-Helton class. Hence, the result follows. □

Proposition 2.5. *If $\mathcal{B} = \mathcal{A}^*$, then $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$ if and only if*
$$p(\mathcal{A})^* \left(\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \mathcal{A}^{*j} \, C \, \mathcal{A}^{m-j} \, C \right) p(\mathcal{A}) = 0.$$

Proof. By definition, we have $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$ if and only if
$$p(\mathcal{A})^* \, \Theta(\mathcal{B}, \mathcal{A}; C)^m(I) \, p(\mathcal{A}) = 0.$$
Taking $\mathcal{B} = \mathcal{A}^*$ in the expression of $\Theta(\mathcal{B}, \mathcal{A}; C)^m(I)$, we obtain
$$\Theta(\mathcal{A}^*, \mathcal{A}; C)^m(I) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \mathcal{A}^{*j} \, C \, \mathcal{A}^{m-j} \, C.$$
Substituting this into the defining condition yields
$$p(\mathcal{A})^* \left(\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \mathcal{A}^{*j} \, C \, \mathcal{A}^{m-j} \, C \right) p(\mathcal{A}) = 0,$$
which completes the proof. □

Remark 2.6. *In the particular case $\mathcal{B} = \mathcal{A}^*$, the class $[p\text{-QCH}_m(\mathcal{A}^*)]$ reduces to the class of polynomially quasi m-symmetric operators. Indeed, the defining condition becomes*

$$p(\mathcal{A})^* \left(\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \mathcal{A}^{*j}\, C\, \mathcal{A}^{m-j}\, C \right) p(\mathcal{A}) = 0,$$

which is precisely the condition for $\mathcal{A}$ to be a polynomially quasi-m-complex symmetric operator (see [21]).

Proposition 2.7. *(Commutative case) Let $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{K})$, let $C \in C[\mathcal{K}]$, and let $p \in \mathbb{C}[z] \setminus \{0\}$. Assume that $\mathcal{A}$ commutes with C (i.e., $\mathcal{A}C = C\mathcal{A}$). Then,*

$$\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})] \quad \Longleftrightarrow \quad \mathcal{A} \in [p\text{-QH}_m(\mathcal{B})].$$

Proof. By Definition 2.1,we have

$$\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})] \quad \Longleftrightarrow \quad p(\mathcal{A})^* \left(\sum_{j=0}^{m} (-1)^{m-j} \mathcal{B}^j C \mathcal{A}^{m-j} C \right) p(\mathcal{A}) = 0.$$

Since $\mathcal{A}C = C\mathcal{A}$, we have $C\mathcal{A}^{m-j}C = \mathcal{A}^{m-j}$. Therefore, the defining condition reduces to

$$p(\mathcal{A})^* \left(\sum_{j=0}^{m} (-1)^{m-j} \mathcal{B}^j \mathcal{A}^{m-j} \right) p(\mathcal{A}) = 0,$$

which is precisely the condition for $\mathcal{A} \in [p\text{-QH}_m(\mathcal{B})]$.

The converse follows by reversing the argument. □

Remark 2.8. *In the commutative case, that is when $\mathcal{A}$ commutes with the conjugation C, the class $[p\text{-QCH}_m(\mathcal{B})]$ coincides with the class of polynomially quasi-Helton operators of order m, namely $[p\text{-QH}_m(\mathcal{B})]$.*

Proposition 2.9. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{K})$ and let $C \in C[\mathcal{K}]$. Then,*

$$\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})] \quad \Longleftrightarrow \quad C\mathcal{A}C \in [p\text{-QH}_m(C\mathcal{B}C)].$$

Proof. Recall that $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$ means that

$$p(\mathcal{A})^* \left(\sum_{j=0}^{m} (-1)^{m-j} \mathcal{B}^j C \mathcal{A}^{m-j} C \right) p(\mathcal{A}) = 0.$$

Because C is a conjugation, we have $CXC^k = CX^kC$ $(CXC)^* = CX^*C$, and in particular,

$$C\mathcal{A}^k C = (C\mathcal{A}C)^k, \quad C\mathcal{B}^j C = (C\mathcal{B}C)^j.$$

Moreover, using the identity $Cp(\mathcal{A})C = p(C\mathcal{A}C)$, we obtain

$$p(\mathcal{A}) = C\, p(C\mathcal{A}C)\, C, \quad p(\mathcal{A})^* = C\, p(C\mathcal{A}C)^*\, C.$$

Applying the conjugation $C(\cdot)C$ to the defining relation of $[p\text{-QCH}_m(.)]$, we get

$$p(C\mathcal{A}C)^* \left(\sum_{j=0}^{m} (-1)^{m-j} (C\mathcal{B}C)^j \, C \, (C\mathcal{A}C)^{m-j} \, C \right) p(C\mathcal{A}C) = 0.$$

Hence,

$$C\mathcal{A}C \in [p\text{-QH}_m(C\mathcal{B}C)].$$

The converse follows by applying the same argument to $C\mathcal{A}C$ and $C\mathcal{B}C$, as $C^2 = I$. □

Theorem 2.10. *Let $X \in \mathcal{B}(\mathcal{K})$ be an invertible operator, $C \in C[\mathcal{K}]$, and let $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{K})$ satisfy*

$$\mathcal{A}(X^*X) = (X^*X)\mathcal{A}, \quad \mathcal{B}(X^*X) = (X^*X)\mathcal{B}, \quad [C, X] = 0.$$

Then $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$ if and only if $X\mathcal{A}X^{-1} \in [p\text{-QCH}_m(X\mathcal{B}X^{-1})]$ for every $p \in \mathbb{C}[z] \setminus \{0\}$.

Proof. Let $p \in \mathbb{C}[z] \setminus \{0\}$, and assume that $X \in \mathcal{L}(\mathcal{K})$ is invertible with

$$\mathcal{A}(X^*X) = (X^*X)\mathcal{A}, \quad \mathcal{B}(X^*X) = (X^*X)\mathcal{B}, \quad [C, X] = 0.$$

Define

$$\widetilde{\mathcal{A}} = X\mathcal{A}X^{-1}, \quad \widetilde{\mathcal{B}} = X\mathcal{B}X^{-1}.$$

(⇒) Assume that $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$, that is

$$p(\mathcal{A})^* \, \Theta(\mathcal{B}, \mathcal{A}; C)^m(I) \, p(\mathcal{A}) = 0.$$

Since similarity preserves polynomial functional calculus, we have

$$p(\widetilde{\mathcal{A}}) = Xp(\mathcal{A})X^{-1}, \qquad p(\widetilde{\mathcal{A}})^* = X^{-1*}p(\mathcal{A})^*X^*.$$

Moreover, using $[C, X] = 0$, we obtain for all $k \geq 0$,

$$C\widetilde{\mathcal{A}}^kC = C(X\mathcal{A}^kX^{-1})C = X(C\mathcal{A}^kC)X^{-1},$$

and similarly,

$$C\widetilde{\mathcal{B}}^kC = X(C\mathcal{B}^kC)X^{-1}.$$

Hence,

$$\Theta(\widetilde{\mathcal{B}}, \widetilde{\mathcal{A}}; C)^m(I) = X \, \Theta(\mathcal{B}, \mathcal{A}; C)^m(I) \, X^{-1}.$$

Therefore,

$$\begin{aligned} p(\widetilde{\mathcal{A}})^*\Theta(\widetilde{\mathcal{B}}, \widetilde{\mathcal{A}}; C)^m(I)p(\widetilde{\mathcal{A}}) &= (X^{-1*}p(\mathcal{A})^*X^*) \, (X\Theta(\mathcal{B}, \mathcal{A}; C)^m(I)X^{-1}) \cdot (Xp(\mathcal{A})X^{-1}) \\ &= X^{-1*} \, p(\mathcal{A})^*\Theta(\mathcal{B}, \mathcal{A}; C)^m(I) \, p(\mathcal{A}) \, X^{-1} \\ &= 0. \end{aligned}$$

Thus,

$$X\mathcal{A}X^{-1} \in [p\text{-QCH}_m(X\mathcal{B}X^{-1})].$$

(⇐) The converse follows by applying the same argument to X^{-1} in place of X, as similarity is invertible.

Hence,

$$\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})] \quad \Longleftrightarrow \quad X\mathcal{A}X^{-1} \in [p\text{-QCH}_m(X\mathcal{B}X^{-1})].$$

This completes the proof. □

The following result characterizes the class via the kernels of operator polynomials, providing an equivalent formulation in terms of polynomial annihilation.

Theorem 2.11. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{K})$, and let $C \in C[\mathcal{K}]$. Let $p, q \in \mathbb{C}[z] \setminus \{0\}$ satisfying*

$$\ker(p(\mathcal{A})^*) = \ker(q(\mathcal{A})^*).$$

Then the following assertions are equivalent:

(i) $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$*;*
(ii) $\mathcal{A} \in [q\text{-QCH}_m(\mathcal{B})]$.

Proof. Let

$$X = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \mathcal{B}^j C\, \mathcal{A}^{m-j} C.$$

Assume that

$$\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})],$$

so that

$$p(\mathcal{A})^* X\, p(\mathcal{A}) = 0.$$

From this and the definition, we obtain

$$p(\mathcal{A})^* X\, p(\mathcal{A}) = 0 \;\Rightarrow\; \operatorname{Ran}(X p(\mathcal{A})) \subseteq \ker(p(\mathcal{A})^*).$$

Hence,

$$p(\mathcal{A})^* X p(\mathcal{A}) = 0 \;\Rightarrow\; q(\mathcal{A})^* X p(\mathcal{A}) = 0,$$

since

$$\ker(p(\mathcal{A})^*) = \ker(q(\mathcal{A})^*).$$

Taking adjoints yields

$$p(\mathcal{A})^* X^* q(\mathcal{A}) = 0,$$

which implies

$$\operatorname{Ran}(X^* q(\mathcal{A})) \subseteq \ker(p(\mathcal{A})^*) = \ker(q(\mathcal{A})^*).$$

Thus,

$$q(\mathcal{A})^* X^* q(\mathcal{A}) = 0.$$

Taking adjoints again gives

$$q(\mathcal{A})^* X q(\mathcal{A}) = 0,$$

hence

$$\mathcal{A} \in [q\text{-QCH}_m(\mathcal{B})].$$

The reverse implication follows by symmetry, interchanging p and q. Therefore,

$$\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})] \quad \Longleftrightarrow \quad \mathcal{A} \in [q\text{-QCH}_m(\mathcal{B})].$$

This completes the proof. □

The following proposition establishes that, under suitable conditions, the class associated with $t^n q(t)$ is contained in the class associated with $tq(t)$, and conversely.

Proposition 2.12. *Let* $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{K})$ *and* $C \in C[\mathcal{K}]$. *Let* $p(t) = t^n q(t)$ *with* $n \geq 2$ *and* $q \in \mathbb{C}[t] \setminus \{0\}$. *Assume that* $\ker(\mathcal{A}^*) = \ker(\mathcal{A}^{*2})$.

Then, the following are equivalent:

(i) $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$;
(ii) $\mathcal{A} \in [(tq)\text{-QCH}_m(\mathcal{B})]$.

Proof. Set

$$T := \Theta(\mathcal{B}, \mathcal{A}; C)^m(I).$$

(i) ⇒ (ii). Assume that $\mathcal{A} \in p\text{-QCH}_m(\mathcal{B})$. Then,

$$p(\mathcal{A})^*\, T\, p(\mathcal{A}) = 0.$$

Since $p(t) = t^n q(t)$, we obtain

$$q(\mathcal{A})^*(\mathcal{A}^*)^n T \mathcal{A}^n q(\mathcal{A}) = 0.$$

Let $y = \mathcal{A}^n q(\mathcal{A})x$. Then,

$$(\mathcal{A}^*)^n q(\mathcal{A})^* T y = 0.$$

Using the assumption $\ker(\mathcal{A}^*) = \ker(\mathcal{A}^{*2})$, we can reduce the power of $\mathcal{A}^*$ step by step, which yields

$$\mathcal{A}^* q(\mathcal{A})^* T y = 0.$$

Hence,

$$\mathcal{A}^* q(\mathcal{A})^* T \mathcal{A}^n q(\mathcal{A}) = 0.$$

Taking adjoints gives

$$(tq)(\mathcal{A})^*\, T\, (tq)(\mathcal{A}) = 0,$$

so $\mathcal{A} \in (tq)\text{-QCH}_m(\mathcal{B})$.

(ii) ⇒ (i). Assume that

$$q(\mathcal{A})^* \mathcal{A}^* T \mathcal{A} q(\mathcal{A}) = 0.$$

Multiplying on the left by $\mathcal{A}^{*n-1}$ and on the right by $\mathcal{A}^{n-1}$, we get

$$q(\mathcal{A})^*(\mathcal{A}^*)^n T \mathcal{A}^n q(\mathcal{A}) = 0.$$

Because $p(\mathcal{A}) = \mathcal{A}^n q(\mathcal{A})$, this is equivalent to

$$p(\mathcal{A})^* T p(\mathcal{A}) = 0.$$

Thus, $\mathcal{A} \in p\text{-QCH}_m(\mathcal{B})$. □

Remark 2.13. *When $p(t) = t^n$ (i.e., $q(t) = 1$), a similar characterization was proved for the class of n-quasi-m-isometric operators (see [18, Theorem 2.9]).*

Proposition 2.14. *(Restriction to invariant subspaces) Let $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{K})$, let $C \in C[\mathcal{K}]$, and let $p \in \mathbb{C}[z] \setminus \{0\}$. If $\mathcal{M} \subseteq \mathcal{K}$ is a closed subspace invariant under $\mathcal{A}$, $\mathcal{B}$, and C, then*

$$\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})] \implies \mathcal{A}|_{\mathcal{M}} \in [p\text{-QCH}_m(\mathcal{B}|_{\mathcal{M}})].$$

Proof. Assume that

$$\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})],$$

that is,

$$p(\mathcal{A})^* \, \Theta(\mathcal{B}, \mathcal{A}; C)^m(I) \, p(\mathcal{A}) = 0.$$

Because $\mathcal{M}$ is invariant under $\mathcal{A}$ and $\mathcal{B}$, it follows that

$$p(\mathcal{A})(\mathcal{M}) \subseteq \mathcal{M},$$

and $\mathcal{A}|_{\mathcal{M}}, \mathcal{B}|_{\mathcal{M}}$ are well-defined bounded operators on $\mathcal{M}$.

Moreover, because $\mathcal{M}$ is invariant under C, the restriction $C|_{\mathcal{M}}$ is a conjugation on $\mathcal{M}$.

Let $P_{\mathcal{M}}$ denote the orthogonal projection onto $\mathcal{M}$. Then, for any $x \in \mathcal{M}$, we have

$$p(\mathcal{A}|_{\mathcal{M}})x = p(\mathcal{A})x.$$

Set

$$T := \Theta(\mathcal{B}, \mathcal{A}; C)^m(I).$$

Then for every $x \in \mathcal{M}$,

$$\langle T \, p(\mathcal{A})x, \; p(\mathcal{A})x \rangle = 0.$$

Since $p(\mathcal{A})x \in \mathcal{M}$, this implies

$$\langle P_{\mathcal{M}} T P_{\mathcal{M}} \, p(\mathcal{A})x, \; p(\mathcal{A})x \rangle = 0.$$

Observe that

$$\Theta(\mathcal{B}|_{\mathcal{M}}, \mathcal{A}|_{\mathcal{M}}; C|_{\mathcal{M}})^m(I_{\mathcal{M}}) = P_{\mathcal{M}} \, \Theta(\mathcal{B}, \mathcal{A}; C)^m(I) \, P_{\mathcal{M}}.$$

Therefore, for all $x \in \mathcal{M}$,

$$\left\langle \Theta(\mathcal{B}|_{\mathcal{M}}, \mathcal{A}|_{\mathcal{M}}; C|_{\mathcal{M}})^m(I_{\mathcal{M}}) \, p(\mathcal{A}|_{\mathcal{M}})x, \; p(\mathcal{A}|_{\mathcal{M}})x \right\rangle = 0,$$

which is equivalent to

$$p(\mathcal{A}|_{\mathcal{M}})^* \, \Theta(\mathcal{B}|_{\mathcal{M}}, \mathcal{A}|_{\mathcal{M}}; C|_{\mathcal{M}})^m(I_{\mathcal{M}}) \, p(\mathcal{A}|_{\mathcal{M}}) = 0.$$

Hence,

$$\mathcal{A}|_{\mathcal{M}} \in [p\text{-QCH}_m(\mathcal{B}|_{\mathcal{M}})].$$

This completes the proof. □

Proposition 2.15. *Let $p \in \mathbb{C}[x] \setminus \{0\}$, and let $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{K})$. Suppose that $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$, and that $\operatorname{Ran}(p(\mathcal{A}))$ is dense in $\mathcal{K}$. Then, $\mathcal{A} \in [\mathrm{CH}_m(\mathcal{B})]$, where $[\mathrm{CH}_m(\mathcal{B})]$ denotes the class of m-complex Helton operators, that is*

$$\Theta(\mathcal{B}, \mathcal{A}; C)^m(I) = 0.$$

Proof. Set

$$T := \Theta(\mathcal{B}, \mathcal{A}; C)^m(I).$$

Because $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$, we have

$$p(\mathcal{A})^* \, T \, p(\mathcal{A}) = 0.$$

Hence, for every $x \in \mathcal{K}$,

$$\langle T \, p(\mathcal{A})x, \; p(\mathcal{A})x \rangle = 0.$$

Therefore,

$$\langle Ty, y \rangle = 0 \quad \text{for all } y \in \overline{\operatorname{Ran}(p(\mathcal{A}))}.$$

Because $\operatorname{Ran}(p(\mathcal{A}))$ is dense in $\mathcal{K}$, and T is bounded, it follows by continuity that

$$\langle Ty, y \rangle = 0 \quad \text{for all } y \in \mathcal{K}.$$

Hence, $T = 0$, that is,

$$\Theta(\mathcal{B}, \mathcal{A}; C)^m(I) = 0.$$

Therefore,

$$\mathcal{A} \in [\mathrm{CH}_m(\mathcal{B})].$$

This completes the proof. □

We now present the following result concerning the stability of the polynomially quasi-m-complex Helton class under operator-norm limits.

Theorem 2.16. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{K})$, and let $C \in C[\mathcal{K}]$. Let $p \in \mathbb{C}[z] \setminus \{0\}$, and let $\{\mathcal{A}_k\}_{k \in \mathbb{N}}, \{\mathcal{B}_k\}_{k \in \mathbb{N}} \subset \mathcal{L}(\mathcal{K})$ be sequences such that*

$$p(\mathcal{A}_k)^* \, \Theta(\mathcal{B}_k, \mathcal{A}_k, C)^m \, p(\mathcal{A}_k) = 0 \quad \textit{for all } k \in \mathbb{N}.$$

If

$$\|\mathcal{A}_k - \mathcal{A}\| \to 0, \quad \textit{and} \quad \|\mathcal{B}_k - \mathcal{B}\| \to 0,$$

then

$$p(\mathcal{A})^* \, \Theta(\mathcal{B}, \mathcal{A}, C)^m \, p(\mathcal{A}) = 0.$$

In particular, $\mathcal{A}$ belongs to the polynomially quasi-m-complex Helton-class associated with $\mathcal{B}$ and C.

Proof. Because $\mathcal{A}_k \to \mathcal{A}$ in operator norm, the polynomial functional calculus yields

$$p(\mathcal{A}_k) \to p(\mathcal{A}), \quad p(\mathcal{A}_k)^* \to p(\mathcal{A})^*.$$

Moreover, $\Theta(\mathcal{B}_k, \mathcal{A}_k, C)$ depends continuously on $(\mathcal{A}_k, \mathcal{B}_k)$ in the operator norm, hence,

$$\Theta(\mathcal{B}_k, \mathcal{A}_k, C)^m \to \Theta(\mathcal{B}, \mathcal{A}, C)^m.$$

By continuity of multiplication in $\mathcal{L}(\mathcal{K})$, we obtain

$$p(\mathcal{A}_k)^*\, \Theta(\mathcal{A}_k, \mathcal{B}_k, C)^m\, p(\mathcal{A}_k) \;\longrightarrow\; p(\mathcal{A})^*\, \Theta(, \mathcal{B}, \mathcal{A}, C)^m\, p(\mathcal{A}).$$

Because

$$p(\mathcal{A}_k)^*\, \Theta(\mathcal{B}_k, \mathcal{A}_k, C)^m\, p(\mathcal{A}_k) = 0 \quad \forall\ k,$$

passing to the limit gives

$$p(\mathcal{A})^*\, \Theta(\mathcal{B}, \mathcal{A}, C)^m\, p(\mathcal{A}) = 0.$$

Thus, $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B}, C)]$. □

The following result shows that the class of $[p\text{-QCH}_m]$ operators is stable under taking direct sums.

Proposition 2.17. *Let $\mathcal{A}_i, \mathcal{B}_i \in \mathcal{L}(\mathcal{K}_i)$, and let $C_i \in C[\mathcal{K}_i]$ for $i = 1, 2$. If $\mathcal{A}_i \in [p\text{-QCH}_m(\mathcal{B}_i)],\ i = 1, 2$, then, $\mathcal{A}_1 \oplus \mathcal{A}_2 \in [p\text{-QCH}_m(\mathcal{B}_1 \oplus \mathcal{B}_2)]$.*

Proof. Let $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, and define

$$\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2, \quad \mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2, \quad C = C_1 \oplus C_2.$$

Then, $C \in C[\mathcal{K}]$.

Because $\mathcal{A}_i \in [p\text{-QCH}_m(\mathcal{B}_i)]$ for $i = 1, 2$, we have

$$p(\mathcal{A}_i)^* \Theta(\mathcal{B}_i, \mathcal{A}_i, C_i)^m (I_{\mathcal{K}_i}) p(\mathcal{A}_i) = 0, \quad i = 1, 2.$$

We now compute $\Theta(\mathcal{B}, \mathcal{A}, C)^m$. Using Definition 2.1 and the standard identities for direct sums, we obtain

$$\begin{aligned}
\Theta(\mathcal{B}, \mathcal{A}, C)^m &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} (\mathcal{B}_1 \oplus \mathcal{B}_2)^j (C_1 \oplus C_2)(\mathcal{A}_1 \oplus \mathcal{A}_2)^{m-j} (C_1 \oplus C_2) \\
&= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \Big(\mathcal{B}_1^j C_1 \mathcal{A}_1^{m-j} C_1 \oplus \mathcal{B}_2^j C_2 \mathcal{A}_2^{m-j} C_2 \Big) \\
&= \Theta(\mathcal{B}_1, \mathcal{A}_1, C_1)^m \oplus \Theta(\mathcal{B}_2, \mathcal{A}_2, C_2)^m.
\end{aligned}$$

By induction, it follows that

$$\Theta(\mathcal{B}, \mathcal{A}, C)^m = \Theta(\mathcal{B}_1, \mathcal{A}_1, C_1)^m \oplus \Theta(\mathcal{B}_2, \mathcal{A}_2, C_2)^m.$$

Applying this to the identity operator $I = I_{\mathcal{K}_1} \oplus I_{\mathcal{K}_2}$, we get

$$\Theta(\mathcal{B}, \mathcal{A}, C)^m (I) = \Theta(\mathcal{B}_1, \mathcal{A}_1, C_1)^m (I_{\mathcal{K}_1}) \oplus \Theta(\mathcal{B}_2, \mathcal{A}_2, C_2)^m (I_{\mathcal{K}_2}).$$

Hence,

$$p(\mathcal{A})^*\Theta(\mathcal{B},\mathcal{A},C)^m(I)P(\mathcal{A}) = p(\mathcal{A}_1)^*\Theta(\mathcal{B}_1,\mathcal{A}_1,C_1)^m(I_{\mathcal{K}_1})p(\mathcal{A}_1)\oplus p(\mathcal{A}_2)^*\Theta(\mathcal{B}_2,\mathcal{A}_2,C_2)^m(I_{\mathcal{K}_2})p(\mathcal{A}_2),$$

which equivalent to

$$p(\mathcal{A})^*\Theta(\mathcal{B},\mathcal{A},C)^m(I)p(\mathcal{A}) = 0.$$

Consequently,

$$\mathcal{A}_1\oplus\mathcal{A}_2 \in [p\text{-QCH}_m(\mathcal{B}_1\oplus\mathcal{B}_2)].$$

This completes the proof. □

Lemma 2.18. *Let $\mathcal{A},\mathcal{B}\in\mathcal{L}(\mathcal{K})$, and let $C\in C[\mathcal{K}]$, and $k\in\mathbb{Z}_+$. Then,*

$$\Theta(\mathcal{B};\mathcal{A},C)^{k+1}(I) = \mathcal{B}\,\Theta(\mathcal{B};\mathcal{A},C)^k(I) - \Theta(\mathcal{B};\mathcal{A},C)^k(I)\,(C\mathcal{A}C).$$

Proof. We prove the identity by using the explicit formula of $\Theta(\mathcal{B},\mathcal{A},C)$.

Recall that

$$\Theta(\mathcal{B},\mathcal{A},C)^m(I) = \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\mathcal{B}^jC\mathcal{A}^{m-j}C.$$

We compute $\Theta(\mathcal{B},\mathcal{A},C)^{k+1}(I)$ using the binomial structure. By definition,

$$\Theta(\mathcal{B},\mathcal{A},C)^{k+1}(I) = \sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}\mathcal{B}^jC\mathcal{A}^{k+1-j}C.$$

Using the Pascal identity

$$\binom{k+1}{j} = \binom{k}{j}+\binom{k}{j-1},$$

we split the sum into two parts:

$$\Theta(\mathcal{B},\mathcal{A},C)^{k+1}(I) = \sum_{j=0}^{k+1}(-1)^{k+1-j}(\binom{k}{j}+\binom{k}{j-1})\mathcal{B}^jC\mathcal{A}^{k+1-j}C.$$

We treat each part separately.

First part:

$$\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k}{j}\mathcal{B}^jC\mathcal{A}^{k+1-j}C = \mathcal{B}\,\Theta(\mathcal{B},\mathcal{A},C)(I)^k,$$

since factoring one power of $\mathcal{B}$ shifts the index.

Second part: Reindexing $j\mapsto j+1$, we obtain

$$\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k}{j-1}\mathcal{B}^jC\mathcal{A}^{k+1-j}C = \Theta(\mathcal{B},\mathcal{A},C)^k(I)\,(C\mathcal{A}C).$$

Combining both parts yields

$$\Theta(\mathcal{B},\mathcal{A},C)^{k+1}(I) = \mathcal{B}\,\Theta(\mathcal{B},\mathcal{A},C)^k(I) - \Theta(\mathcal{B},\mathcal{A},C)^k(I)\,(C\mathcal{A}C),$$

which completes the proof. □

The following proposition shows a stability property of the class p-QCH$_m$ under additional commutativity assumptions, leading to an increase in the order from m to $m+1$.

Proposition 2.19. *Let $p \in \mathbb{C}[z] \setminus \{0\}$, and let $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$ for some positive integer m. If $[\mathcal{A}^*, \mathcal{B}] = [\mathcal{A}, C\mathcal{A}C] = 0$, then $\mathcal{A} \in [p\text{-QCH}_{(m+1)}(\mathcal{B})]$.*

Proof. Let $p \in \mathbb{C}[z] \setminus \{0\}$, and assume that

$$\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})],$$

that is,

$$p(\mathcal{A})^*\Theta(\mathcal{B}, \mathcal{A}, C)^m(I)\, p(\mathcal{A}) = 0.$$

Because $[\mathcal{A}^*, \mathcal{B}] = 0$, we have

$$\mathcal{B}^j p(\mathcal{A})^* = p(\mathcal{A})^*\mathcal{B}^j, \quad \text{for all } \ j \geq 0.$$

Also, the condition $[\mathcal{A}, C\mathcal{A}] = 0$ implies that the conjugation intertwining is compatible with powers of $\mathcal{A}$, and hence,

$$C\mathcal{A}^k C\, \mathcal{A} = \mathcal{A}\, C\mathcal{A}^k C, \quad \text{for all } \ k \geq 0.$$

Using Lemma 2.18, we have the recursion identity

$$\Theta(\mathcal{B}; \mathcal{A}, C)^{m+1}(I) = \mathcal{B}\,\Theta(\mathcal{B}; \mathcal{A}, C)^m(I) - \Theta(\mathcal{B}; \mathcal{A}, C)^m(I)\,(C\mathcal{A}C).$$

Multiplying the defining identity of the p-QCH$_m(\cdot)$ class on the left by $\mathcal{B}$ and on the right by $C\mathcal{A}C$, and using the commutation relations, we obtain

$$p(\mathcal{A})^*\mathcal{B}\Theta(\mathcal{B}, \mathcal{A}, C)^m(I)p(\mathcal{A}) = p(\mathcal{A})^*\Theta(\mathcal{B}, \mathcal{A}, C)^m(I)(C\mathcal{A}C)p(\mathcal{A}).$$

Hence,

$$p(\mathcal{A})^*\Big(\mathcal{B}\Theta(\mathcal{B}, \mathcal{A}, C)^m(I) - \Theta(\mathcal{A}, \mathcal{B}, C)^m(I)(C\mathcal{A}C)\Big)p(\mathcal{A}) = 0.$$

By Lemma 2.18, the term in brackets is exactly

$$\Theta(\mathcal{B}, \mathcal{A}, C)^{m+1}(I).$$

Therefore,

$$p(\mathcal{A})^*\Theta(\mathcal{B}, \mathcal{A}, C)^{m+1}(I)\, p(\mathcal{A}) = 0,$$

which shows that

$$\mathcal{A} \in [p\text{-QCH}_{m+1}(\mathcal{B})].$$

This completes the proof. □

The next theorem provides a structural description of operators belonging to the polynomially quasi-m-QCH-class. It shows that, under a natural spectral-type assumption on the range of $p(\mathcal{A})$, the operator $\mathcal{A}$ admits a canonical upper triangular decomposition compatible with the reducing decomposition induced by $\overline{\text{Ran}(p(\mathcal{A}))}$. This representation reveals that the class is determined by its action on the invariant subspace $\overline{\text{Ran}(p(\mathcal{A}))}$, where it reduces to a classical m-complex Helton-type condition, and the complementary part is annihilated by the polynomial p.

Theorem 2.20. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{K})$ and $p \in \mathbb{C}[z] \setminus \{0\}$ and $C = C_1 \oplus C_2$, where $C_1 \in C[\overline{Ran(p(\mathcal{A}))}]$ and $C_2 \in C[\ker(p(\mathcal{A})^*)]$.*

If $p(\mathcal{A}) \neq 0$, $\overline{Ran(p(\mathcal{A}))} \neq \mathcal{K}$ and $\overline{Ran(p(\mathcal{A}))}$ reduces $\mathcal{B}$, then the following statements are equivalent:

(1) $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$.

(2) *$\mathcal{A}$ and $\mathcal{B}$ have upper triangular representations*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ 0 & \mathcal{A}_3 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \mathcal{B}_1 & 0 \\ 0 & \mathcal{B}_2 \end{pmatrix} \quad \text{on } \mathcal{K} = \overline{Ran(p(\mathcal{A}))} \oplus \ker(p(\mathcal{A})^*),$$

where $\mathcal{A}_1 \in [C_1 H_m(\mathcal{B}_1)]$ and $p(\mathcal{A}_3) = 0$.

Proof. Let $p \in \mathbb{C}[z] \setminus \{0\}$, and set

$$\mathcal{M} = \overline{\text{Ran}(p(\mathcal{A}))}, \quad \mathcal{N} = \ker(p(\mathcal{A})^*).$$

Then we have the orthogonal decomposition

$$\mathcal{K} = \mathcal{M} \oplus \mathcal{N}.$$

Because $\mathcal{M}$ reduces $\mathcal{B}$, it follows that both $\mathcal{M}$ and $\mathcal{N}$ are invariant under $\mathcal{B}$. Hence, $\mathcal{B}$ admits the block diagonal representation

$$\mathcal{B} = \begin{pmatrix} \mathcal{B}_1 & 0 \\ 0 & \mathcal{B}_2 \end{pmatrix} \quad \text{on } \mathcal{K} = \mathcal{M} \oplus \mathcal{N}.$$

Moreover, because $\mathcal{M} = \overline{\text{Ran}(p(\mathcal{A}))}$, it is invariant under $\mathcal{A}$, and therefore, $\mathcal{A}$ has an upper triangular matrix representation

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ 0 & \mathcal{A}_3 \end{pmatrix}.$$

We now prove the equivalence.

(1) $\Rightarrow$ (2)

Assume that $\mathcal{A} \in [P\text{-QCH}_m(\mathcal{B})]$. Then, by definition, there exists a conjugation $C = C_1 \oplus C_2$ such that

$$p(\mathcal{A})^* \left(\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \mathcal{B}^j C \mathcal{A}^{m-j} C \right) p(\mathcal{A}) = 0.$$

Let

$$\mathcal{M} = \overline{\text{Ran}(p(\mathcal{A}))}, \quad \mathcal{N} = \ker(p(\mathcal{A})^*),$$

so that $\mathcal{K} = \mathcal{M} \oplus \mathcal{N}$.

Because $\mathcal{M}$ reduces $\mathcal{B}$, we have

$$\mathcal{B} = \begin{pmatrix} \mathcal{B}_1 & 0 \\ 0 & \mathcal{B}_2 \end{pmatrix},$$

and $\mathcal{M}$ is invariant under $\mathcal{A}$, hence,

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ 0 & \mathcal{A}_3 \end{pmatrix}.$$

We now show that $\mathcal{A}_1 \in [\mathrm{CH}_m(\mathcal{B}_1)]$.

Let $x \in \mathcal{M} = \overline{\mathrm{Ran}(p(\mathcal{A}))}$. Then, there exists a sequence $(x_n) \subset \mathrm{Ran}(p(\mathcal{A}))$ such that

$$x_n \to x \quad \text{in } \mathcal{K}.$$

For each n, there exists $y_n \in \mathcal{K}$ such that $x_n = p(\mathcal{A})y_n$.

Using the defining identity, we obtain

$$p(\mathcal{A})^* \left(\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \mathcal{B}^j C \mathcal{A}^{m-j} C \right) x_n = 0.$$

Taking the inner product with y_n, we get

$$\left\langle \left(\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \mathcal{B}^j C \mathcal{A}^{m-j} C \right) x_n, x_n \right\rangle = 0.$$

Because $x_n \in \mathrm{Ran}(p(\mathcal{A})) \subset \mathcal{M}$, and $\mathcal{M}$ reduces $\mathcal{B}$, we have

$$\mathcal{B}^j x_n = \mathcal{B}_1^j x_n, \quad C x_n = C_1 x_n, \quad \mathcal{A}^{m-j} x_n = \mathcal{A}_1^{m-j} x_n.$$

Hence,

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \left\langle \mathcal{B}_1^j C_1 \mathcal{A}_1^{m-j} C_1 x_n, x_n \right\rangle = 0.$$

Now, using the continuity of $\mathcal{A}_1$, $\mathcal{B}_1$, and C_1, and passing to the limit as $n \to \infty$, we obtain

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \left\langle \mathcal{B}_1^j C_1 \mathcal{A}_1^{m-j} C_1 x, x \right\rangle = 0, \quad \forall x \in \mathcal{M}.$$

Therefore,

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \mathcal{B}_1^j C_1 \mathcal{A}_1^{m-j} C_1 = 0 \quad \text{on } \mathcal{M},$$

which shows that

$$\mathcal{A}_1 \in [\mathrm{CH}_m(\mathcal{B}_1)].$$

Finally, we show that $p(\mathcal{A}_3) = 0$.

Recall that

$$\mathcal{N} = \ker(p(\mathcal{A})^*).$$

Because $\mathcal{K} = \mathcal{M} \oplus \mathcal{N}$, the operator $p(\mathcal{A})$ admits a block upper triangular matrix representation

$$p(\mathcal{A}) = \begin{pmatrix} p(\mathcal{A}_1) & * \\ 0 & p(\mathcal{A}_3) \end{pmatrix} \quad \text{on } \mathcal{K} = \mathcal{M} \oplus \mathcal{N}.$$

Taking adjoints, we obtain

$$p(\mathcal{A})^* = \begin{pmatrix} p(\mathcal{A}_1)^* & 0 \\ * & p(\mathcal{A}_3)^* \end{pmatrix}.$$

Let $x \in \mathcal{N} = \ker(p(\mathcal{A})^*)$. Writing $x = 0 \oplus x_2$ with $x_2 \in \mathcal{N}$, we have

$$p(\mathcal{A})^* x = 0.$$

From the block form, this gives

$$p(\mathcal{A}_3)^* x_2 = 0, \quad \forall x_2 \in \mathcal{N}.$$

Thus,

$$p(\mathcal{A}_3)^* = 0 \quad \text{on } \mathcal{N},$$

which implies

$$p(\mathcal{A}_3) = 0.$$

Hence, the restriction $\mathcal{A}_3$ satisfies $p(\mathcal{A}_3) = 0$, completing the proof.

Thus, the representation in (2) holds.

(2) ⇒ (1)

Suppose that

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ 0 & \mathcal{A}_3 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \mathcal{B}_1 & 0 \\ 0 & \mathcal{B}_2 \end{pmatrix}$$

on

$$\mathcal{K} = \mathcal{M} \oplus \mathcal{N} = \overline{\operatorname{Ran}(p(\mathcal{A}))} \oplus \ker(p(\mathcal{A})^*),$$

where $p(\mathcal{A}_3) = 0$ and

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \mathcal{B}_1^j C_1 \mathcal{A}_1^{m-j} C_1 = 0.$$

A direct computation gives

$$p(\mathcal{A}) = \begin{pmatrix} p(\mathcal{A}_1) & X \\ 0 & p(\mathcal{A}_3) \end{pmatrix} = \begin{pmatrix} p(\mathcal{A}_1) & X \\ 0 & 0 \end{pmatrix},$$

and therefore

$$p(\mathcal{A})^* = \begin{pmatrix} p(\mathcal{A}_1)^* & 0 \\ X^* & 0 \end{pmatrix}.$$

Hence,

$$p(\mathcal{A})\, p(\mathcal{A})^* = \begin{pmatrix} p(\mathcal{A}_1) p(\mathcal{A}_1)^* + XX^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix},$$

where $F = F^* \geq 0$ on $\mathcal{M}$.

Now, consider the operator

$$\Theta := \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \mathcal{B}^j C \mathcal{A}^{m-j} C.$$

With respect to the decomposition $\mathcal{K} = \mathcal{M} \oplus \mathcal{N}$, we write

$$\Theta(\mathcal{B}, \mathcal{A}, C)^m = \begin{pmatrix} \Theta_1(\mathcal{B}_1, \mathcal{A}_1, C) & * \\ * & * \end{pmatrix},$$

where

$$\Theta_1 = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \mathcal{B}_1^j C_1 \mathcal{A}_1^{m-j} C_1 = 0.$$

We now compute

$$p(\mathcal{A})\, p(\mathcal{A})^* \,\Theta(\mathcal{B}, \mathcal{A}, C)^m \, p(\mathcal{A})\, p(\mathcal{A})^*.$$

Using block multiplication, we obtain

$$p(\mathcal{A})\, p(\mathcal{A})^* \,\Theta(\mathcal{B}, \mathcal{A}, C)^m \, p(\mathcal{A})\, p(\mathcal{A})^* = \begin{pmatrix} F\Theta_1 F & 0 \\ 0 & 0 \end{pmatrix}.$$

Because $\Theta_1(\mathcal{B}_1, \mathcal{A}_1, C)_1^m = 0$, it follows that

$$F\Theta_1(\mathcal{B}_1, \mathcal{A}_1, C)_1^m F = 0,$$

and hence,

$$p(\mathcal{A})\, p(\mathcal{A})^* \,\Theta(\mathcal{B}, \mathcal{A}, C)^m \, p(\mathcal{A})\, p(\mathcal{A})^* = 0.$$

Because

$$p(\mathcal{A})\, p(\mathcal{A})^* \Theta(\mathcal{B}, \mathcal{A}, C)^m p(\mathcal{A})\, p(\mathcal{A})^* = 0,$$

we have already shown that

$$p(\mathcal{A})\, p(\mathcal{A})^* \,\Theta(\mathcal{B}, \mathcal{A}, C)^m \, p(\mathcal{A})\, p(\mathcal{A})^* = 0.$$

Recall that

$$\overline{\operatorname{Ran}(p(\mathcal{A})^*)} = \ker(p(\mathcal{A}))^\perp, \quad \text{and} \quad \mathcal{K} = \ker(p(\mathcal{A})) \oplus \ker(p(\mathcal{A}))^\perp.$$

Let

$$y \in \overline{\operatorname{Ran}(p(\mathcal{A})^*)} = \ker(p(\mathcal{A}))^\perp.$$

Then, there exists a sequence $y_n = p(\mathcal{A})^* x_n$ such that $y_n \to y$. For each n, we have

$$\begin{aligned} \langle \Theta\, p(\mathcal{A}) y_n,\ p(\mathcal{A}) y_n \rangle &= \langle p(\mathcal{A})\, p(\mathcal{A})^* \Theta(\mathcal{B}, \mathcal{A}, C)^m p(\mathcal{A})\, p(\mathcal{A})^* x_n,\ x_n \rangle \\ &= 0. \end{aligned}$$

Hence,

$$p(\mathcal{A})^* \Theta(\mathcal{B}, \mathcal{A}, C)^m p(\mathcal{A})\, y_n = 0.$$

By continuity, passing to the limit yields

$$p(\mathcal{A})^* \Theta(\mathcal{B}, \mathcal{A}, C)^m p(\mathcal{A})\, y = 0 \quad \text{for all } \ y \in \ker(p(\mathcal{A}))^\perp.$$

On the other hand, if $y \in \ker(p(\mathcal{A}))$, then $p(\mathcal{A})y = 0$, so

$$p(\mathcal{A})^*\Theta(\mathcal{B}, \mathcal{A}, C)^m p(\mathcal{A})\, y = 0.$$

Consequently,

$$p(\mathcal{A})^*\Theta(\mathcal{B}, \mathcal{A}, C)^m p(\mathcal{A}) = 0 \quad \text{on} \;\; \mathcal{K} = \ker(p(\mathcal{A})) \oplus \ker(p(\mathcal{A}))^{\perp}.$$

Hence, by definition,

$$\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})].$$

Thus, (1) and (2) are equivalent. □

Corollary 2.21. *(Reduction of the class) Let $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{K})$ be as in Theorem 2.20, and assume that*

$$\mathcal{K} = \overline{\text{Ran}(p(\mathcal{A}))} \oplus \ker(p(\mathcal{A})^*)$$

with

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ 0 & \mathcal{A}_3 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \mathcal{B}_1 & 0 \\ 0 & \mathcal{B}_2 \end{pmatrix}, \quad p(\mathcal{A}_3) = 0.$$

Then

$$\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})] \quad \Longleftrightarrow \quad \mathcal{A}_1 \in [C_1 H_m(\mathcal{B}_1)].$$

In particular, the study of the class $[p\text{-QCH}_m(\mathcal{B})]$ reduces to the study of the classical class $[C_1 H_m(\mathcal{B}_1)]$ acting on $\overline{\text{Ran}(p(\mathcal{A}))}$.

Proof. Let

$$\mathcal{K} = \overline{\text{Ran}(p(\mathcal{A}))} \oplus \ker(p(\mathcal{A})^*),$$

and consider the block representations

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ 0 & \mathcal{A}_3 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \mathcal{B}_1 & 0 \\ 0 & \mathcal{B}_2 \end{pmatrix}, \quad C = C_1 \oplus C_2.$$

Assume that

$$\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})], \quad \text{that is} \quad p(\mathcal{A})^*\, \Theta(\mathcal{B}, \mathcal{A}, C)^m\, p(\mathcal{A}) = 0.$$

Because $\overline{\text{Ran}(p(\mathcal{A}))}$ is invariant under $\mathcal{A}$ and reduces $\mathcal{B}$, the operator

$$\Theta(\mathcal{B}, \mathcal{A}, C)^m = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \mathcal{B}^j C \mathcal{A}^{m-j} C$$

admits the block form

$$\Theta(\mathcal{B}, \mathcal{A}, C)^m = \begin{pmatrix} \Theta(\mathcal{B}_1, \mathcal{A}_1, C_1)^m & * \\ * & * \end{pmatrix}.$$

Moreover, because

$$p(\mathcal{A}) = \begin{pmatrix} p(\mathcal{A}_1) & * \\ 0 & p(\mathcal{A}_3) \end{pmatrix} \quad \text{with} \;\; p(\mathcal{A}_3) = 0,$$

we obtain

$$p(\mathcal{A})^* = \begin{pmatrix} p(\mathcal{A}_1)^* & 0 \\ * & 0 \end{pmatrix}.$$

Hence, when computing

$$p(\mathcal{A})^*\,\Theta(\mathcal{B},\mathcal{A},C)^m\,p(\mathcal{A}),$$

all terms involving the second component $\ker(p(\mathcal{A})^*)$ vanish, and the identity reduces to the compression on $\overline{\mathrm{Ran}(p(\mathcal{A}))}$:

$$p(\mathcal{A}_1)^*\,\Theta(\mathcal{B}_1,\mathcal{A}_1,C_1)^m\,p(\mathcal{A}_1) = 0.$$

Therefore,

$$\mathcal{A}_1 \in [[\mathrm{C}_1\mathrm{H}_m(\mathcal{B}_1)].$$

Conversely, if $\mathcal{A}_1 \in [C_1H_m(\mathcal{B}_1)]$, and $p(\mathcal{A}_3) = 0$, then the same block computation shows that the full operator satisfies

$$p(\mathcal{A})^*\,\Theta(\mathcal{B},\mathcal{A},C)^m\,p(\mathcal{A}) = 0,$$

so $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$.

This proves the equivalence. □

3. On the spectrum and orthogonality in the $[p\text{-QCH}_m(\cdot)]$-class

Proposition 3.1. *(Spectral decomposition) Let $\mathcal{A},\mathcal{B} \in \mathcal{L}(\mathcal{K})$ be as in Theorem 2.20, and assume that $\mathcal{A}$ admits the block representation*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ 0 & \mathcal{A}_3 \end{pmatrix} \quad on \;\; \mathcal{K} = \overline{\mathrm{Ran}(p(\mathcal{A}))} \oplus \ker(p(\mathcal{A})^*),$$

with $p(\mathcal{A}_3) = 0$. Then,

$$\sigma(\mathcal{A}) = \sigma(\mathcal{A}_1) \cup \sigma(\mathcal{A}_3),$$

and

$$\sigma(\mathcal{A}_3) \subseteq \{z \in \mathbb{C} : p(z) = 0\}.$$

Proof. Because $\mathcal{A}$ has an upper triangular block matrix representation with respect to the decomposition

$$\mathcal{K} = \overline{\mathrm{Ran}(p(\mathcal{A}))} \oplus \ker(p(\mathcal{A})^*),$$

it follows from standard spectral theory of triangular operator matrices that the spectrum of $\mathcal{A}$ is the union of the spectra of its diagonal entries, that is

$$\sigma(\mathcal{A}) = \sigma(\mathcal{A}_1) \cup \sigma(\mathcal{A}_3).$$

Next, since $p(\mathcal{A}_3) = 0$, the operator $\mathcal{A}_3$ is algebraic as annihilating polynomial p. Hence, every spectral value $\lambda \in \sigma(\mathcal{A}_3)$ must satisfy $p(\lambda) = 0$, which implies

$$\sigma(\mathcal{A}_3) \subseteq \{z \in \mathbb{C} : p(z) = 0\}.$$

This completes the proof. □

Theorem 3.2. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{K})$, and let $C \in C[\mathcal{K}]$. Let $p \in \mathbb{C}[z] \setminus \{0\}$, and set*

$$\Phi := \Theta(\mathcal{B}, \mathcal{A}, C)^m(I).$$

Assume that

$$\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})], \quad \langle \Phi x, x\rangle \geq 0 \quad \forall\, x \in \mathcal{K}, \quad \textit{and} \quad \ker(\Phi) = \{0\}.$$

Then:

(1) $\sigma_p(\mathcal{A}) \subseteq p^{-1}(0)$. In particular, if all zeros of p are real, then $\sigma_p(\mathcal{A}) \subseteq \mathbb{R}$.
(2) Let $a, b \in \mathbb{C}$ with $a \neq b$. If $\mathcal{A}x = ax$, and $\mathcal{A}y = by$, and

$$p(a) \neq 0, \quad \textit{and} \quad p(b) \neq 0,$$

then

$$\langle \Phi x, y\rangle = 0.$$

(3) Let $a, b \in \mathbb{C}$ with $a \neq b$, and let $\{x_n\}, \{y_n\}$ be sequences of unit vectors such that

$$\|(\mathcal{A} - a)x_n\| \to 0, \quad \|(\mathcal{A} - b)y_n\| \to 0.$$

If

$$p(a) \neq 0, \quad \textit{and} \quad p(b) \neq 0,$$

then

$$\langle \Phi x_n, y_n\rangle \to 0.$$

Proof. Set

$$\Phi := \Theta(\mathcal{B}, \mathcal{A}, C)^m(I).$$

By assumption,

$$p(\mathcal{A})^*\, \Phi\, p(\mathcal{A}) = 0, \quad \Phi \geq 0, \quad \ker(\Phi) = \{0\}.$$

(1) Description of the point spectrum.

Let $\lambda \in \sigma_p(\mathcal{A})$, and let $x \neq 0$ satisfy

$$\mathcal{A}x = \lambda x.$$

Then,

$$p(\mathcal{A})x = p(\lambda)x.$$

Hence,

$$0 = \langle p(\mathcal{A})^*\Phi p(\mathcal{A})x, x\rangle = \langle \Phi p(\mathcal{A})x, p(\mathcal{A})x\rangle = |p(\lambda)|^2\langle \Phi x, x\rangle.$$

Because $\Phi \geq 0$, and $\ker(\Phi) = \{0\}$, we have

$$\langle \Phi x, x\rangle > 0.$$

Therefore,

$$p(\lambda) = 0,$$

which yields
$$\sigma_p(\mathcal{A}) \subseteq p^{-1}(0).$$

If all zeros of p are real, it follows immediately that

$$\sigma_p(\mathcal{A}) \subseteq \mathbb{R}.$$

(2) Orthogonality of eigenvectors.

Let $a, b \in \mathbb{C}$ with $a \neq b$, and suppose

$$\mathcal{A}x = ax, \quad \mathcal{A}y = by, \quad x, y \neq 0,$$

with
$$p(a) \neq 0 \quad \text{and} \quad p(b) \neq 0.$$

Then,
$$p(\mathcal{A})x = p(a)x, \quad p(\mathcal{A})y = p(b)y.$$

Using the identity $p(\mathcal{A})^*\Phi p(\mathcal{A}) = 0$, we obtain

$$0 = \langle p(\mathcal{A})^*\Phi p(\mathcal{A})x, y\rangle = \langle \Phi p(\mathcal{A})x, p(\mathcal{A})y\rangle = p(a)\overline{p(b)}\,\langle \Phi x, y\rangle.$$

Because $p(a) \neq 0$ and $p(b) \neq 0$, we conclude that

$$\langle \Phi x, y\rangle = 0.$$

(3) Orthogonality for approximate eigenvectors.

Let $a \neq b$, and let $\{x_n\}, \{y_n\}$ be sequences of unit vectors such that

$$\|(\mathcal{A} - a)x_n\| \to 0, \quad \|(\mathcal{A} - b)y_n\| \to 0,$$

with
$$p(a) \neq 0 \quad \text{and} \quad p(b) \neq 0.$$

By continuity of the polynomial functional calculus,

$$\|p(\mathcal{A})x_n - p(a)x_n\| \to 0, \quad \|p(\mathcal{A})y_n - p(b)y_n\| \to 0.$$

Using again the identity,
$$\langle \Phi p(\mathcal{A})x_n, p(\mathcal{A})y_n\rangle = 0 \quad \forall n.$$

Passing to the limit, we obtain
$$p(a)\overline{p(b)}\,\langle \Phi x_n, y_n\rangle \to 0.$$

Because $p(a) \neq 0$, and $p(b) \neq 0$, it follows that

$$\langle \Phi x_n, y_n\rangle \to 0.$$

This completes the proof. □

Proposition 3.3. *Let $p \in \mathbb{C}[z] \setminus \{0\}$, and suppose $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$. If $\lambda \in \sigma_p(\mathcal{A})$, and $p(\lambda) \neq 0$, then for every $x \in \ker(\mathcal{A} - \lambda I)$,*

$$\Theta(\mathcal{B}, \mathcal{A}, C)^m(I)x \in \ker(\mathcal{A} - \lambda I)^{\perp}.$$

Proof. Let $\lambda \in \sigma_p(\mathcal{A})$, and let $x \in \ker(\mathcal{A} - \lambda I)$, so that $\mathcal{A}x = \lambda x$.

Because $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$, we have

$$p(\mathcal{A})^* \Theta(\mathcal{B}, \mathcal{A}, C)^m(I)\, p(\mathcal{A}) = 0.$$

Taking the inner product with x, we obtain

$$\langle \Theta(\mathcal{B}, \mathcal{A}, C)^m(I)\, p(\mathcal{A})x,\ p(\mathcal{A})x \rangle = 0.$$

Since $\mathcal{A}x = \lambda x$, functional calculus gives

$$p(\mathcal{A})x = p(\lambda)x.$$

Hence,

$$0 = \langle \Theta(\mathcal{B}, \mathcal{A}, C)^m(I)\, p(\lambda)x,\ p(\lambda)x \rangle .$$

Using the inner product property

$$\langle \alpha u, \alpha v \rangle = |\alpha|^2 \langle u, v \rangle, \quad \alpha \in \mathbb{C},$$

we obtain

$$0 = |p(\lambda)|^2 \langle \Theta(\mathcal{B}, \mathcal{A}, C)^m x,\ x \rangle .$$

Because $p(\lambda) \neq 0$, it follows that

$$\langle \Theta(\mathcal{B}, \mathcal{A}, C)^m x,\ x \rangle = 0, \quad \forall x \in \ker(\mathcal{A} - \lambda I).$$

By the polarization identity, we extend the relation

$$\langle (\mathcal{B}, \mathcal{A}, C)^m x,\ x \rangle = 0, \quad \forall x \in \ker(\mathcal{A} - \lambda I),$$

to a bilinear form identity.

Indeed, let $x, y \in \ker(\mathcal{A} - \lambda I)$. Define the sesquilinear form

$$\varphi(x, y) := \langle \Theta(\mathcal{B}, \mathcal{A}, C)^m x,\ y \rangle .$$

Then, φ is sesquilinear on $\mathcal{K}$. Because

$$\varphi(x, x) = 0 \quad \forall x \in \ker(\mathcal{A} - \lambda I),$$

we apply the polarization identity:

$$\varphi(x, y) = \frac{1}{4} \sum_{k=0}^{3} i^k\, \varphi(x + i^k y,\ x + i^k y).$$

Because $\ker(\mathcal{A}-\lambda I)$ is a linear subspace, we have

$$x + i^k y \in \ker(\mathcal{A}-\lambda I) \quad \text{for all } \; k = 0, 1, 2, 3.$$

Hence,

$$\varphi(x + i^k y,\; x + i^k y) = 0, \quad \forall k = 0, 1, 2, 3.$$

Therefore,

$$\varphi(x, y) = 0, \quad \forall x, y \in \ker(\mathcal{A}-\lambda I),$$

that is,

$$\langle \Theta_m(\mathcal{B}, \mathcal{A}, C)x,\; y\rangle = 0, \quad \forall x, y \in \ker(\mathcal{A}-\lambda I).$$

Hence,

$$\Theta(\mathcal{B}, \mathcal{A}, C)^m(I)(\ker(\mathcal{A}-\lambda I)) \subseteq \ker(\mathcal{A}-\lambda I)^{\perp}.$$

Therefore,

$$\Theta(\mathcal{B}, \mathcal{A}, C)^m(I)x \in \ker(\mathcal{A}-\lambda I)^{\perp}, \quad \forall x \in \ker(\mathcal{A}-\lambda I),$$

which completes the proof. □

4. Stability of $[p\text{-QCH}_m(\cdot)]$ classes under operator powers

In this section, we are concerned with the following natural question: *Does the p-QCH_m property remain invariant under taking powers? More precisely, if $A \in [p\text{-QCH}_m(B)]$, under what conditions does one have $A^r \in [p\text{-QCH}_m(B^r)]$ for $r \in \mathbb{N}$?*

In the following, we present a partial answer to this question.

Lemma 4.1. *Let $p \in \mathbb{C}[x] \setminus \{0\}$. If $\mathcal{A} \in [p\text{-QCH}_2(\mathcal{B})]$ such that $[\mathcal{A}, C\mathcal{A}C] = 0,\; [\mathcal{A}, \mathcal{B}^*] = 0$, then,*

$$\begin{aligned} p(\mathcal{A})^*\,\Theta(B^{m+2}, \mathcal{A}^{m+2}, C)^2(I)\,p(\mathcal{A}) = {} & B\,p(\mathcal{A})^*\,\Theta(B^{m+1}, \mathcal{A}^{m+1}, C)^2(I)\,p(\mathcal{A})(CAC) \\ & - B^2\,p(\mathcal{A})^*\,\Theta(B^m, \mathcal{A}^m, C)^2(I)\,p(\mathcal{A})(C\mathcal{A}^2C). \end{aligned}$$

Proof. Because $\mathcal{A} \in [p\text{-QCH}_2(\mathcal{B}, C)]$, it follows that

$$p(\mathcal{A})^*(\mathcal{B}^2 - 2\mathcal{B}C\mathcal{A}C + C\mathcal{A}^2C)p(\mathcal{A}) = 0.$$

Hence, we obtain

$$\begin{aligned} p(\mathcal{A})^*\mathcal{B}^2 p(\mathcal{A}) &= p(\mathcal{A})^*(2\mathcal{B}C\mathcal{A}C - C\mathcal{A}^2C)p(\mathcal{A}), \\ p(\mathcal{A})^*C\mathcal{A}^2Cp(\mathcal{A}) &= p(\mathcal{A})^*(2\mathcal{B}C\mathcal{A}C - \mathcal{B}^2)p(\mathcal{A}). \end{aligned}$$

Now for $m = 1$, we have

$$\begin{aligned} p(\mathcal{A})^*\Phi(\mathcal{B}^3, \mathcal{A}^3, C)^2(I)p(\mathcal{A}) &= p(\mathcal{A})^*(\mathcal{B}^6 - 2\mathcal{B}^3C\mathcal{A}^3C + C\mathcal{A}^6C)p(\mathcal{A}) \\ &= p(\mathcal{A})^*\mathcal{B}^6 p(\mathcal{A}) - 2p(\mathcal{A})^*\mathcal{B}^3C\mathcal{A}^3Cp(\mathcal{A}) + p(\mathcal{A})^*C\mathcal{A}^6Cp(\mathcal{A}). \end{aligned}$$

Using the previous identities, we rewrite

$$p(\mathcal{A})^*\mathcal{B}^6 p(\mathcal{A}) = \mathcal{B}^4 p(\mathcal{A})^*\mathcal{B}^2 p(\mathcal{A}),$$

$$p(\mathcal{A})^* C\mathcal{A}^6 C p(\mathcal{A}) = p(\mathcal{A})^* C\mathcal{A}^2 C p(\mathcal{A})\, C\mathcal{A}^4 C.$$

Thus,

$$\begin{aligned} p(\mathcal{A})^* \Phi(\mathcal{B}^3, \mathcal{A}^3, C)^2(I) p(\mathcal{A}) = {}& 2\mathcal{B}\, p(\mathcal{A})^* \Theta(\mathcal{B}^2, \mathcal{A}^2, C)^2(I) p(\mathcal{A})(C\mathcal{A}C) \\ & - \mathcal{B}^2 p(\mathcal{A})^* \Theta(\mathcal{B}, \mathcal{A}, C)^2(I) p(\mathcal{A})(C\mathcal{A}^2 C). \end{aligned}$$

Inductive step. Assume that

$$\begin{aligned} p(\mathcal{A})^* \Phi(\mathcal{B}^{m+2}, \mathcal{A}^{m+2}, C)^2(I) p(\mathcal{A}) = {}& \mathcal{B}\, p(\mathcal{A})^* \Theta(\mathcal{B}^{m+1}, \mathcal{A}^{m+1}, C)^2(I) p(\mathcal{A})(C\mathcal{A}C) \\ & - \mathcal{B}^2 p(\mathcal{A})^* \Theta(\mathcal{B}^m, \mathcal{A}^m, C)^2(I) p(\mathcal{A})(C\mathcal{A}^2 C) \end{aligned}$$

holds for some $m \in \mathbb{N}$.

We prove it for $m + 1$.

$$p(\mathcal{A})^* \Phi(\mathcal{B}^{m+3}, \mathcal{A}^{m+3}, C)^2(I) p(\mathcal{A}) = p(\mathcal{A})^*(\mathcal{B}^{2m+6} - 2\mathcal{B}^{m+3} C\mathcal{A}^{m+3} C + C\mathcal{A}^{2m+6} C) p(\mathcal{A}).$$

Expanding,

$$\begin{aligned} p(\mathcal{A})^* \Phi(\mathcal{B}^{m+3}, \mathcal{A}^{m+3}, C)^2(I) p(\mathcal{A}) = {}& p(\mathcal{A})^* \mathcal{B}^{2m+6} p(\mathcal{A}) - 2p(\mathcal{A})^* \mathcal{B}^{m+3} C\mathcal{A}^{m+3} C p(\mathcal{A}) \\ & + p(\mathcal{A})^* C\mathcal{A}^{2m+6} C p(\mathcal{A}). \end{aligned}$$

Using

$$p(\mathcal{A})^*(\mathcal{B}^2 - 2\mathcal{B}C\mathcal{A}C + C\mathcal{A}^2 C) p(\mathcal{A}) = 0,$$

we obtain

$$\begin{aligned} p(\mathcal{A})^* \mathcal{B}^{2m+6} p(\mathcal{A}) &= \mathcal{B}^{2m+4} p(\mathcal{A})^* \mathcal{B}^2 p(\mathcal{A}), \\ p(\mathcal{A})^* C\mathcal{A}^{2m+6} C p(\mathcal{A}) &= p(\mathcal{A})^* C\mathcal{A}^2 C p(\mathcal{A})\, C\mathcal{A}^{2m+4} C. \end{aligned}$$

Thus,

$$\begin{aligned} p(\mathcal{A})^* \Theta(\mathcal{B}^{m+3}, \mathcal{A}^{m+3}, C)^2(I) p(\mathcal{A}) = {}& \mathcal{B}^{2m+4} p(\mathcal{A})^*(2\mathcal{B}C\mathcal{A}C - C\mathcal{A}^2 C) p(\mathcal{A}) \\ & - 2p(\mathcal{A})^* \mathcal{B}^{m+3} C\mathcal{A}^{m+3} C p(\mathcal{A}) \\ & + p(\mathcal{A})^*(2\mathcal{B}C\mathcal{A}C - \mathcal{B}^2) p(\mathcal{A})\, C\mathcal{A}^{2m+4} C. \end{aligned}$$

Rearranging terms yields

$$\begin{aligned} p(\mathcal{A})^* \Theta(\mathcal{B}^{m+3}, \mathcal{A}^{m+3}, C)^2(I) p(\mathcal{A}) = {}& \mathcal{B}\, p(\mathcal{A})^* \Theta(\mathcal{B}^{m+2}, \mathcal{A}^{m+2}, C)^2(I) p(\mathcal{A})(C\mathcal{A}C) \\ & - \mathcal{B}^2 p(\mathcal{A})^* \Theta(\mathcal{B}^{m+1}, \mathcal{A}^{m+1}, C)^2(I) p(\mathcal{A})(C\mathcal{A}^2 C). \end{aligned}$$

Hence the result follows by induction. □

Lemma 4.2. *Let $p \in \mathbb{C}[x] \setminus \{0\}$, and let $C \in C[\mathcal{K}]$. Let $\mathcal{A} \in [p\text{-QCH}_2(\mathcal{B})]$ such that $[\mathcal{A}, C\mathcal{A}C] = 0$, and $[\mathcal{A}, \mathcal{B}^*] = 0$. Assume that there exists $q \in \mathbb{C}[x] \setminus \{0\}$ such that $p(t)^2 = q(t^2)$, $\forall t \in \mathbb{C}$. Then $\mathcal{A}^2 \in [q\text{-QCH}_2(\mathcal{B}^2)]$.*

Proof. Because $\mathcal{A} \in [p\text{-QCH}_2(\mathcal{B})]$, we have

$$p(\mathcal{A})^*(\mathcal{B}^2 - 2\mathcal{B}C\mathcal{A}C + C\mathcal{A}^2C)p(\mathcal{A}) = 0. \tag{4.1}$$

On the other hand,

$$\begin{aligned} q(\mathcal{A}^2)^*\Theta(\mathcal{B}^2, \mathcal{A}^2, C)^2(I)q(\mathcal{A}^2) &= p(\mathcal{A})^{*2}\Theta(\mathcal{B}^2, \mathcal{A}^2, C)^2(I)p(\mathcal{A})^2 \\ &= p(\mathcal{A})^*p(\mathcal{A})^*\Big[(\mathcal{B}^4 - 2\mathcal{B}^2C\mathcal{A}^2C + C\mathcal{A}^4C)p(\mathcal{A})\Big]p(\mathcal{A}). \end{aligned}$$

Using the commutativity assumptions, we obtain

$$\mathcal{B}^4 = \mathcal{B}^2\mathcal{B}^2, \quad C\mathcal{A}^4C = (C\mathcal{A}^2C)(C\mathcal{A}^2C),$$

and hence

$$\begin{aligned} p(\mathcal{A})^*(\mathcal{B}^4 - 2\mathcal{B}^2C\mathcal{A}^2C + C\mathcal{A}^4C)p(\mathcal{A}) = \mathcal{B}^2p(\mathcal{A})^*\mathcal{B}^2p(\mathcal{A}) - 2p(\mathcal{A})^*\mathcal{B}^2C\mathcal{A}^2C\,p(\mathcal{A}) \\ + p(\mathcal{A})^*C\mathcal{A}^2C\,p(\mathcal{A})\,C\mathcal{A}^2C. \end{aligned}$$

From (4.1), we have

$$p(\mathcal{A})^*\mathcal{B}^2p(\mathcal{A}) = p(\mathcal{A})^*(2\mathcal{B}C\mathcal{A}C - C\mathcal{A}^2C)p(\mathcal{A})$$

and

$$p(\mathcal{A})^*C\mathcal{A}^2C\,p(\mathcal{A}) = p(\mathcal{A})^*(2\mathcal{B}C\mathcal{A}C - \mathcal{B}^2)p(\mathcal{A}).$$

Substituting these identities, we obtain

$$\begin{aligned} &p(\mathcal{A})^*(\mathcal{B}^4 - 2\mathcal{B}^2C\mathcal{A}^2C + C\mathcal{A}^4C)p(\mathcal{A}) \\ &= p(\mathcal{A})^*(2\mathcal{B}^3C\mathcal{A}C - 4\mathcal{B}^2C\mathcal{A}^2C + 2\mathcal{B}C\mathcal{A}^3C)p(\mathcal{A}) \\ &= 2\mathcal{B}\,p(\mathcal{A})^*(\mathcal{B}^2 - 2\mathcal{B}C\mathcal{A}C + C\mathcal{A}^2C)p(\mathcal{A})\,C\mathcal{A}C \\ &= 2\mathcal{B}\,p(\mathcal{A})^*\Theta(\mathcal{B}; \mathcal{A}, C)^2(I)p(\mathcal{A})\,C\mathcal{A}C \\ &= 0. \end{aligned}$$

Therefore,

$$q(\mathcal{A}^2)^*\Theta(\mathcal{B}^2, \mathcal{A}^2, C)^2(I)q(\mathcal{A}^2) = 0,$$

which shows that

$$\mathcal{A}^2 \in [q\text{-QCH}_2(\mathcal{B}^2)].$$

This completes the proof. □

Theorem 4.3. *Under the same hypotheses as in Lemma 4.2, we have*

$$\mathcal{A}^2 \in [q\text{-QCH}_m(\mathcal{B}^2)].$$

Proof. Because $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$, we have

$$p(\mathcal{A})^* \Theta(\mathcal{B}; \mathcal{A}, C)^m(I) p(\mathcal{A}) = 0. \tag{4.2}$$

Hence, for each $q = 0, 1, \ldots, m$, we get

$$\mathcal{B}^q \Big(p(\mathcal{A})^* \Theta(\mathcal{B}; \mathcal{A}, C)^m(I) p(\mathcal{A}) \Big) (C \mathcal{A}^{m-q} C) = 0.$$

Using the commutativity assumptions, this yields

$$p(\mathcal{A})^* \mathcal{B}^q \Theta(\mathcal{B}; \mathcal{A}, C)^m(I) C \mathcal{A}^{m-q} C p(\mathcal{A}) = 0, \quad q = 0, 1, \ldots, m.$$

Summing over q, we obtain

$$p(\mathcal{A})^{*2} \left(\sum_{0 \le q \le m} (-1)^s \binom{m}{s} \mathcal{B}^q \Theta(\mathcal{B}; \mathcal{A}, C)^m(I) C \mathcal{A}^{m-q} C \right) p(\mathcal{A})^2 = 0.$$

Expanding $\Theta(\mathcal{B}; \mathcal{A}, C)^m(I)$, we get

$$\begin{aligned} & p(\mathcal{A})^{*2} \left(\sum_{0 \le q \le m} \sum_{0 \le j \le m} (-1)^j \binom{m}{q} \binom{m}{j} \mathcal{B}^{s+j} C \mathcal{A}^{2m-(q+j)} C \right) p(\mathcal{A})^2 \\ & = p(\mathcal{A})^{*2} \left(\sum_{0 \le k \le 2m} \sum_{0 \le j \le k} (-1)^j \binom{m}{k-j} \binom{m}{j} \mathcal{B}^k C \mathcal{A}^{2m-k} C \right) p(\mathcal{A})^2. \end{aligned}$$

Using the combinatorial identity

$$\sum_{0 \le j \le k} (-1)^j \binom{m}{k-j} \binom{m}{j} = \begin{cases} 0, & k \text{ odd}, \\ (-1)^r \binom{m}{r}, & k = 2r, \end{cases}$$

we obtain

$$p(\mathcal{A})^{*2} \left(\sum_{0 \le r \le m} (-1)^r \binom{m}{r} \mathcal{B}^{2r} C \mathcal{A}^{2m-2r} C \right) p(\mathcal{A})^2 = 0.$$

Thus,

$$p(\mathcal{A})^{*2} \Theta(\mathcal{B}^2; \mathcal{A}^2, C)^m(I) p(\mathcal{A})^2 = 0.$$

Finally, Because $p(t)^2 = q(t^2)$, we have $p(\mathcal{A})^2 = q(\mathcal{A}^2)$, and hence

$$q(\mathcal{A}^2)^* \Phi_m(\mathcal{B}^2; \mathcal{A}^2, C)(I) q(\mathcal{A}^2) = 0.$$

Therefore,

$$\mathcal{A}^2 \in [q\text{-QCH}_m(\mathcal{B}^2)].$$

This completes the proof. □

Theorem 4.4. *Under the same hypotheses as in Lemma 4.2, there exists a sequence of polynomials* $\{q_r\}_{r \ge 1}$ *such that*

$$\mathcal{A}^{2^r} \in [q_r\text{-QCH}_m(\mathcal{B}^{2^r})], \quad r \ge 1.$$

Proof. We proceed by induction on r.

Step 1: $r = 1$.

By the Lemma 4.2, because $\mathcal{A} \in [p\text{-QCH}_m(\mathcal{B})]$, and

$$[\mathcal{A}, C\mathcal{A}C] = 0, \quad [\mathcal{A}, \mathcal{B}^*] = 0, \quad p(t)^2 = q(t^2),$$

there exists a polynomial $q_1 = q$ such that

$$\mathcal{A}^2 \in [q_1\text{-QCH}_m(\mathcal{B}^2)].$$

Step 2: Induction hypothesis.

Assume that for some $r \geq 1$, there exists a polynomial q_r such that

$$\mathcal{A}^{2^r} \in [q_r\text{-QCH}_m(\mathcal{B}^{2^r})].$$

Step 3: Stability of assumptions. Because

$$[\mathcal{A}, \mathcal{B}] = [\mathcal{A}, \mathcal{B}^*] = 0, \quad [\mathcal{A}, C\mathcal{A}C] = 0,$$

these relations are preserved under taking powers. Hence,

$$[\mathcal{A}^{2^r}, (\mathcal{B}^{2^r})^*] = 0, \quad [\mathcal{A}^{2^r}, C\mathcal{A}^{2^r}C] = 0.$$

Step 4: Application of Lemma 4.2.

Applying Lemma 4.2 to the pair $(\mathcal{A}^{2^r}, \mathcal{B}^{2^r})$, we obtain a polynomial q_{r+1} such that

$$(\mathcal{A}^{2^r})^2 \in [q_{r+1}\text{-QCH}_m((\mathcal{B}^{2^r})^2)].$$

Step 5: Identification of powers.

Because

$$(\mathcal{A}^{2^r})^2 = \mathcal{A}^{2^{r+1}}, \quad (\mathcal{B}^{2^r})^2 = \mathcal{B}^{2^{r+1}},$$

we conclude that

$$\mathcal{A}^{2^{r+1}} \in [q_{r+1}\text{-QCH}_m(\mathcal{B}^{2^{r+1}})].$$

This completes the induction. □

Remark 4.5. *If $p(t) = t^n$, then*

$$p(t)^2 = t^{2n} = p(t^2),$$

so the polynomial q in the theorem coincides with p. Hence,

$$\mathcal{A}^2 \in [p\text{-QCH}_2(\mathcal{B}^2)].$$

In this case, the above theorem reduces exactly to [17, Lemma 2.19].

5. Conclusions

In this paper, we introduced the class of polynomially quasi-m-complex Helton operators as a polynomial extension of the classical m-complex Helton class. We established several fundamental properties of this class, including its stability under direct sums, invariant subspaces, and powers of operators, together with several equivalent characterizations. These results provide a broader framework for studying higher-order polynomial operator identities and suggest several directions for future investigations, particularly in spectral theory and other generalized operator classes.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this manuscript.

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