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Research article

On the P-intersection graphs of acts over semigroups: applications to connectivity and monophonic convexity with clique structure

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Abstract: In this paper, we present the Ψ-intersection graph $\Psi_{\mathcal{S}}(\mathcal{B})$ associated with an S-act over topological semigroups. We investigate how the internal structure of S-acts, particularly the presence and distribution of simple subacts, determines fundamental graph-theoretic properties of $\Psi_{\mathcal{S}}(\mathcal{B})$. Characterizations of connectivity, bipartiteness, girth, and diameter are established in terms of decomposition properties and chain conditions on subacts. We further analyze the clique structure of $\Psi_{\mathcal{S}}(\mathcal{B})$, showing that every clique contains at most one simple subact and describing maximal cliques through families of non-simple subacts intersecting via simple components. Domination-type parameters are also studied by introducing indecomposable controlling sets and deriving bounds for the control number of $\Psi_{\mathcal{S}}(\mathcal{B})$. Finally, by using monophonic convexity and the m-topology on graphs, we examine induced path structures and their interaction with algebraic paths in S-acts, revealing connections between geodesic behavior and semigroup actions. These results provide a unified algebraic, topological, and graph-theoretic framework and open new combinatorial and metric approaches to the study of topological path semigroups.

Keywords: semigroup action; topological semigroup; intersection graph; clique number; monophonic path
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1. Introduction

Graphs associated with algebraic structures have become an important tool for linking algebraic and combinatorial methods. In such constructions, algebraic substructures are represented as vertices, while adjacency is defined through algebraically meaningful relationships. Among these constructions, intersection graphs have received considerable attention because they provide an effective framework for translating algebraic characteristics, such as simplicity, decomposition properties, and chain conditions, into graph-theoretic invariants including connectivity, diameter, and clique number.

The theory of intersection graphs has been extensively developed in a variety of algebraic settings. The subgroup intersection graph of a finite group was first introduced by Csakany and Pollak [1], where vertices represent proper nontrivial subgroups and adjacency is determined by nontrivial intersections. Zelinka [2] subsequently investigated analogous constructions for finite abelian groups. The concept was later extended to ring theory by Mukherjee and Sen [3], who introduced the intersection graph of ideals. Additional investigations of subgroup intersection graphs were conducted by Shen [4], who established several structural, combinatorial, and metric properties of these graphs. Connectivity has been a central theme in this line of research. Moghaddamfar et al. [5] studied prime graphs of finite simple groups via vertex degrees. Kayacan [6] extended these results to broader classes of finite groups. Aprose and Fathima [7] investigated further structural properties of the intersection power graph of finite groups. These results were further strengthened by Beheshtipour and Jafarian [8], who determined clique numbers for wider classes of finite groups, and Damag et al. [9], introduced the concept of upper α-graphical topological spaces and investigated their properties. Intersection graph theory was subsequently extended to modules by Ahmed and Moh'd [10], opening a new direction in algebraic graph theory. Asir et al. [11] introduced the co-intersection graph of ideals and studied its structural and graph-theoretic properties. Related developments for ideals of commutative rings were introduced by Moh'd and Ahmed [12], who proposed the notion of simple intersection graphs of rings. The theoretical background for extending intersection graph methods to more general algebraic structures is provided by semigroup theory; see the classical monographs of [13–16]. Recent works have also emphasized the interaction between graph theory and topology. For instance, Damag et al. [17] investigated directed topological structures arising from graph-based models, while Damag et al. [18] studied monophonic sets and rough directed topological spaces with applications to directed networks. In the context of semigroup actions, chain conditions in semigroup-like structures were examined by Davvaz and Nazemian [19] for commutative monoids and by Khosravi and Roueentan [20] for Rees congruences of $\mathcal{S}$-acts. Intersection graph theory for $\mathcal{S}$-acts was later developed by Rasouli and Tehranian [21] and further generalized by Delfan et al. [22]. More recently, Liang et al. [23] introduced simple intersection graphs of $\mathcal{S}$-acts, extending the concept of simple intersection from rings to semigroup actions.

The study of intersection graphs arising from algebraic structures provides an effective bridge between algebra and graph theory, allowing structural properties of algebraic systems to be analyzed through graph invariants. Motivated by this interaction, we introduce the Ψ-intersection graph $\Psi_{\mathcal{S}}(\mathcal{B})$ associated with an $\mathcal{S}$-act over a topological semigroup and investigate how the distribution of simple and non-simple subacts influences its graph-theoretic structure. We establish results concerning structural and global properties of $\Psi_{\mathcal{S}}(\mathcal{B})$, including girth, bipartiteness, connectivity, diameter, and clique structure and relate these invariants to algebraic conditions such as the Artinian property of the underlying $\mathcal{S}$-act. We also study domination-type parameters by introducing indecomposable controlling sets and deriving bounds for the control number. Furthermore, monophonic properties of $\Psi_{\mathcal{S}}(\mathcal{B})$ are examined through the concept of m-topology on graphs, revealing connections between induced path structures and the distribution of simple subacts.

It is important to emphasize that the proposed Ψ-intersection graph differs fundamentally from classical intersection graph constructions arising from groups, rings, modules, and previously studied S-acts. In classical intersection graphs, two vertices are adjacent whenever their intersection is nontrivial. In contrast, adjacency in $\Psi_S(B)$ is determined by the stronger condition that the intersection must be a simple subact. Therefore, the proposed graph focuses on the interaction between minimal invariant components rather than general intersections. This distinction leads to different structural properties concerning connectivity, clique formation, diameter, and path behavior, providing a new perspective for studying topological semigroup actions through their simple subact structure.

The paper is organized as follows. Section 2 introduces the Ψ-intersection graph together with illustrative examples and investigates structural properties such as girth and bipartiteness. Section 3 is devoted to connectivity, diameter, and clique structure. Section 4 studies controlling sets and domination-type invariants, while Section 5 explores monophonic properties via the m-topology.

It should be noted that the topological assumptions are essential for the construction of the path semigroup framework used throughout this work. The compact-open topology guarantees that the path semigroups S_p and B_p inherit compatible topological semigroup structures, while continuity of the action ensures that B_p becomes a well-defined topological S-act. Consequently, the Ψ-intersection graph is not constructed from arbitrary algebraic semigroup actions, but from continuous actions on path spaces arising from topological semigroups.

Throughout this paper, a graph $\mathcal{G}$ consists of a vertex set $\mathcal{V}(\mathcal{G})$ and an edge set $\mathcal{E}(\mathcal{G})$. The order of $\mathcal{G}$, denoted by $O(\mathcal{G})$, is the number of vertices of $\mathcal{G}$. The degree of a vertex $v \in \mathcal{V}(\mathcal{G})$, denoted by ∂_v, is the number of vertices adjacent to v. A path in $\mathcal{G}$ is a finite sequence of distinct vertices such that every two consecutive vertices are adjacent. The graph $\mathcal{G}$ is said to be connected if for every pair of vertices $u, v \in \mathcal{V}(\mathcal{G})$, there exists a path joining them. The distance between two vertices u and v, denoted by $\partial(u, v)$, is the length of a shortest path between them; if no such path exists, we set $\partial(u, v) = \infty$. The diameter of $\mathcal{G}$ is defined by

$$\mathrm{diam}(\mathcal{G}) = \sup\{\partial(u, v) : u, v \in \mathcal{V}(\mathcal{G})\}.$$

A graph is called simple if it has no loops and no multiple edges. For $n \geq 3$, the cycle graph C_n is a graph consisting of a single cycle in which every vertex has degree two. A complete graph K_n is a graph in which every pair of distinct vertices is adjacent. A complete bipartite graph $K_{n,m}$ has its vertex set partitioned into two disjoint subsets V_1 and V_2 such that every edge joins a vertex of V_1 to a vertex of V_2. The wheel graph W_n ($n \geq 4$) is obtained by joining a single vertex to every vertex of the cycle C_{n-1}. The girth of $\mathcal{G}$, denoted by $g(\mathcal{G})$, is the length of the shortest cycle in $\mathcal{G}$ and is taken to be ∞ if $\mathcal{G}$ contains no cycles. A clique of $\mathcal{G}$ is a complete subgraph, and the clique number of $\mathcal{G}$, denoted by $\omega(\mathcal{G})$, is the maximum cardinality of a clique in $\mathcal{G}$. A path component of $\mathcal{G}$ is a maximal connected subgraph. A nonempty subset $A \subseteq \mathcal{V}(\mathcal{G})$ is called a controlling set if every vertex of $\mathcal{V}(\mathcal{G}) \setminus A$ is adjacent to at least one vertex of A. The family of all controlling sets is denoted by $\mathrm{Con}(\mathcal{G})$, and the control number of $\mathcal{G}$ is defined by

$$\Delta(\mathcal{G}) = \inf\{O(A) : A \in \mathrm{Con}(\mathcal{G})\}.$$

A controlling set is called indecomposable if removing any of its vertices destroys the controlling property. Finally, an edge joining two nonconsecutive vertices of a path P is called a chord, and a monophonic path is a path containing no chords.

Let $\mathcal{S}$ be a semigroup acting on a nonempty set $\mathcal{B}$ by a map $(s, a) \mapsto sa$. This map is called action, which is assumed associative in the sense that $(st)a = s(ta)$ for all $s, t \in \mathcal{S}$ and $a \in \mathcal{B}$. This action will satisfy $1a = a$ for every $a \in \mathcal{B}$ if $\mathcal{S}$ is a monoid and has the identity 1. The pair $(\mathcal{S}, \mathcal{B})$ will be called an $\mathcal{S}$-act. A subset $\mathcal{Z} \subseteq \mathcal{B}$ will be called invariant under the action of $\mathcal{S}$ if $s\mathcal{Z} \subseteq \mathcal{Z}$ for all $s \in \mathcal{S}$.

Theses subsets are exactly the subacts of $\mathcal{B}$. An element $x \in \mathcal{S}$ is called a left zero if $xs = x$ for all $s \in \mathcal{S}$. An $\mathcal{S}$-act $\mathcal{B}$ is called simple if it doesn't contain any proper invariant subset. The number of simple subacts of $\mathcal{B}$ is denoted by $\mathbb{S}(\mathcal{B})$. For proper subacts, a minimal subact has no proper subact in it, and a maximal subact has no larger subact containing it. If $\{\mathcal{B}_k \mid k \in \Delta\}$ is a collection of $\mathcal{S}$-acts, their coproduct is the disjoint union of each $\mathcal{B}_k$ indexed by k, with $\mathcal{S}$-action defined componentwise, i.e., $s(a,k) = (sa,k)$ for all$s \in \mathcal{S}$ and $a \in \mathcal{B}_k$. An $\mathcal{S}$-act is said to be completely reducible if it is isomorphic to a coproduct of simple subacts. The $\mathcal{S}$-act $\mathcal{B}$ is called semisimple if it is exhausted by its simple components. The union of all simple $\mathcal{S}$-actsubacts forms the socle, denoted by $\mathrm{Soc}(\mathcal{B})$, and one has $\mathcal{B} = \mathrm{Soc}(\mathcal{B})$ precisely in the semisimple case. Descending chains of $\mathcal{S}$-subacts in $\mathcal{B}$ are said to stabilize if they become constant after finitely many steps. When this condition holds for all such chains, the $\mathcal{S}$-act $\mathcal{B}$ is called Artinian. An element $a \in \mathcal{B}$ is called a fixed if $sa = a$ for all $s \in \mathcal{S}$.

2. The Ψ-intersection graph $\Psi_{\mathcal{S}}(\mathcal{B})$

For more details about the definitions and facts concerning the topological spaces and continuous functions, refer to [24]. A topological semigroup $\mathcal{S}$ is a semigroup endowed with a topology such that the multiplication $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$ is a continuous mapping. Let $I = [0,1]$ be the closed interval in the usual topological space $\mathbb{R}$. The path semigroup $\mathcal{S}_p$ of a topological semigroup $\mathcal{S}$ is defined as the set of all paths, that is, all continuous functions $\alpha : I \to \mathcal{S}$, with multiplication $\mathcal{S}_p \times \mathcal{S}_p \to \mathcal{S}_p$ given by $(\alpha_1\alpha_2)(t) = \alpha_1(t)\alpha_2(t)$ for all $\alpha_1, \alpha_2 \in \mathcal{S}_p$ and $t \in I$. Note that $\mathcal{S}_p$ is a topological semigroup when it is taken with the compact-open topology; see [24]. For $s \in \mathcal{S}$, define $\widehat{s} : I \to \mathcal{S}$ by $\widehat{s}(t) = s$ for all $t \in I$, which is called a constant path. Note that the constant path semigroup $\widehat{\mathcal{S}_p}$ is a topological semigroup in $\mathcal{S}_p$, where $\widehat{\mathcal{S}_p} := \{\widehat{s} \in \mathcal{S}_p : s \in \mathcal{S}\}$. In the theory of topological semigroups, action mappings are assumed to be continuous. Let $\mathcal{S}$ and $\mathcal{B}$ be topological semigroups. If $\mathcal{B}$ is an $\mathcal{S}$-act with a continuous action $\mathcal{S} \times \mathcal{B} \to \mathcal{B} : (s,a) \to sa$, then the path semigroup $\mathcal{B}_p$ is an $\mathcal{S}$-act under the continuous action $\mathcal{S} \times \mathcal{B}_p \to \mathcal{B}_p : (s,\alpha) \to s\alpha$, where $(s\alpha)(t) = s\alpha(t)$ for all $t \in I$.

Definition 2.1. Let $\mathcal{S}$ and $\mathcal{B}$ be two topological semigroups and let $\widehat{\mathcal{B}_p}$ be an $\mathcal{S}$-act. The Ψ-intersection graph $\Psi_{\mathcal{S}}(\mathcal{B})$ of $\widehat{\mathcal{B}_p}$ is the simple undirected graph whose vertex set consists of all nonempty subacts of $\widehat{\mathcal{B}_p}$. Two distinct vertices $\mathcal{X}$ and $\mathcal{Y}$ are adjacent if $\mathcal{X} \cap \mathcal{Y}$ is a simple subact of $\widehat{\mathcal{B}_p}$.

Example 2.2. Let $\mathcal{S} = \{1, \varepsilon\}$ be a topological monoid with the discrete topology, where 1 is the identity and $\varepsilon^2 = \varepsilon$. Let $\mathcal{B} = \{a,b,c,d\}$ be equipped with the discrete topology, and define a continuous action of $\mathcal{S}$ on $\mathcal{B}$ by $1x = x$ and $\varepsilon x = d$ for all $x \in \mathcal{B}$. Then $\widehat{\mathcal{B}_p}$ is an $\mathcal{S}$-act. Since the action of ε sends every element to d, a nonempty subset of $\widehat{\mathcal{B}_p}$ is a subact if and only if it contains $\widehat{d}$. Hence

$$\mathcal{V}[\Psi_{\mathcal{S}}(\mathcal{B})] = \{\{\widehat{d}\},\ \{\widehat{a},\widehat{d}\},\ \{\widehat{b},\widehat{d}\},\ \{\widehat{c},\widehat{d}\}, \{\widehat{a},\widehat{b},\widehat{d}\},\ \{\widehat{a},\widehat{c},\widehat{d}\},\ \{\widehat{b},\widehat{c},\widehat{d}\},\ \{\widehat{a},\widehat{b},\widehat{c},\widehat{d}\}\}.$$

Hence $\Psi_{\mathcal{S}}(\mathcal{B})$ has a nontrivial clique structure together with several induced triangles, as illustrated in Figure 1a.

Example 2.3. Let $\mathcal{S} = \{1, \varepsilon\}$ be a topological monoid with the discrete topology, where 1 is the identity and $\varepsilon^2 = \varepsilon$. Let $\mathcal{B} = \{1', 2', 3', 4'\}$ with the discrete topology, and define the action of $\mathcal{S}$ on $\mathcal{B}$ by

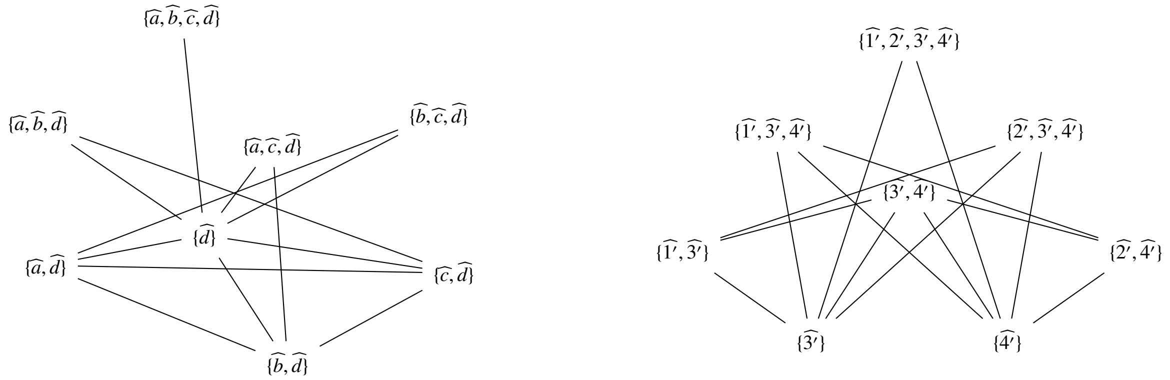
$$1x = x \quad \text{for all } x \in \mathcal{B},$$

$$\varepsilon 1' = 3', \quad \varepsilon 2' = 4', \quad \varepsilon 3' = 3', \quad \varepsilon 4' = 4'.$$

Then $\widehat{\mathcal{B}}_p$ is an $\mathcal{S}$-act. In this case, a nonempty subset of $\widehat{\mathcal{B}}_p$ is a subact precisely when it is closed under the action of ε. Since $\varepsilon(\widehat{1'}) = \widehat{3'}$ and $\varepsilon(\widehat{2'}) = \widehat{4'}$, any subact containing $\widehat{1'}$ must also contain $\widehat{3'}$, and any subact containing $\widehat{2'}$ must also contain $\widehat{4'}$. Hence

$$\mathcal{V}[\Psi_{\mathcal{S}}(\mathcal{B})] = \{\{\widehat{3'}\}, \{\widehat{4'}\}, \{\widehat{3'}, \widehat{4'}\}, \{\widehat{1'}, \widehat{3'}\}, \{\widehat{2'}, \widehat{4'}\}, \{\widehat{1'}, \widehat{3'}, \widehat{4'}\}, \{\widehat{2'}, \widehat{3'}, \widehat{4'}\}, \{\widehat{1'}, \widehat{2'}, \widehat{3'}, \widehat{4'}\}\}.$$

The corresponding graph is shown in Figure 1b.



(a) Ψ-intersection graph $\Psi_{\mathcal{S}}(\mathcal{B})$ in Example 2.2

(b) Ψ-intersection graph $\Psi_{\mathcal{S}}(\mathcal{B})$ in Example 2.3

Figure 1. Ψ-intersection graphs.

Example 2.4. Let $\mathcal{S} = \{1, \varepsilon\}$ be the monoid of Example 2.2 and let $\mathcal{B} = \mathbb{N} \cup \{0\}$ with the action $1x = x$ and $\varepsilon x = 0$ for all $x \in \mathcal{B}$. Then $\mathcal{B}$ is an infinite $\mathcal{S}$-act, and $\{0\}$ is a simple subact. Every subact must contain 0, yielding infinitely many vertices in $\Psi_{\mathcal{S}}(\mathcal{B})$. This example shows that the construction naturally extends to infinite $\mathcal{S}$-acts, although several finiteness properties of the graph may no longer hold.

Theorem 2.5. Let $\mathcal{S}$ and $\mathcal{B}$ be two topological semigroups and let $\widehat{\mathcal{B}}_p$ be an $\mathcal{S}$-act. Then $\widehat{\mathcal{B}}_p$ is simple, or $\widehat{\mathcal{B}}_p$ contains no simple subacts if and only if $\Psi_{\mathcal{S}}(\mathcal{B})$ is edgeless.

Proof. Assume $\Psi_{\mathcal{S}}(\mathcal{B})$ is edgeless. Suppose $\widehat{\mathcal{B}}_p$ is not simple and contains a simple subact $\mathcal{Z}$. Since $\widehat{\mathcal{B}}_p \cap \mathcal{Z} = \mathcal{Z}$ and $\mathcal{Z}$ is simple, the vertices $\widehat{\mathcal{B}}_p$ and $\mathcal{Z}$ are adjacent in $\Psi_{\mathcal{S}}(\mathcal{B})$, a contradiction. Hence either $\widehat{\mathcal{B}}_p$ is simple or it has no simple subacts.

Conversely, if $\widehat{\mathcal{B}}_p$ is simple, then the only nonempty subact is $\widehat{\mathcal{B}}_p$ itself, so the graph has one vertex and no edges. If $\widehat{\mathcal{B}}_p$ contains no simple subacts, then for any distinct vertices $\mathcal{B}_1, \mathcal{B}_2$, the intersection $\mathcal{B}_1 \cap \mathcal{B}_2$ is not simple, so they are not adjacent. Hence $\Psi_{\mathcal{S}}(\mathcal{B})$ is edgeless. □

Theorem 2.6. Let $\widehat{\mathcal{B}}_p$ be a semisimple $\mathcal{S}$-act with decomposition $\widehat{\mathcal{B}}_p = \bigsqcup_{i=1}^{n} \mathcal{B}_i$. Then $\deg(\widehat{\mathcal{B}}_p) = n$, $\deg(\mathcal{B}_i) = 2^{n-1} - 1$ for $i = 1, \ldots, n$, and if $\mathcal{Z}$ is a subact with m components ($2 \leq m \leq n$), then $\deg(\mathcal{Z}) = m\, 2^{n-m}$.

Proof. Since $\widehat{\mathcal{B}}_p$ is semisimple, every subact is a coproduct of some of the simple components $\mathcal{B}_1, \ldots, \mathcal{B}_n$.

First, a vertex C is adjacent to $\widehat{\mathcal{B}_p}$ if and only if $\widehat{\mathcal{B}_p} \cap C = C$ is simple, which occurs exactly when $C = \mathcal{B}_i$ for some i. Hence $\deg(\widehat{\mathcal{B}_p}) = n$.

Next, fix i. A vertex $C \neq \mathcal{B}_i$ is adjacent to $\mathcal{B}_i$ if and only if $\mathcal{B}_i \cap C = \mathcal{B}_i$, equivalently C contains $\mathcal{B}_i$. Thus $C = \mathcal{B}_i \sqcup \bigsqcup_{j \in J} \mathcal{B}_j$, where $J \subseteq \{1, \ldots, n\} \setminus \{i\}$ is nonempty. The number of such choices is $2^{n-1} - 1$. Hence $\deg(\mathcal{B}_i) = 2^{n-1} - 1$.

Finally, let $\mathcal{Z} = \bigsqcup_{r=1}^{m} \mathcal{B}_{i_r}$. A vertex C is adjacent to $\mathcal{Z}$ if and only if $\mathcal{Z} \cap C$ is exactly one of the components $\mathcal{B}_{i_r}$. For each fixed component $\mathcal{B}_{i_r}$, the vertex C must be $\mathcal{B}_{i_r} \sqcup \bigsqcup_{j \in J} \mathcal{B}_j$, where $J \subseteq \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$. There are 2^{n-m} such vertices. Summing over the m possible components gives $\deg(\mathcal{Z}) = m\, 2^{n-m}$. □

Theorem 2.7. Let $\widehat{\mathcal{B}_p} = \bigsqcup_{i=1}^{n} \mathcal{B}_i$ be semisimple. Then $\Psi_{\mathcal{S}}(\mathcal{B})$ has no Euler circuit. If $n \geq 3$, it has no Euler path, while if $n = 2$, it has a unique Euler path of length 2 joining $\mathcal{B}_1$ and $\mathcal{B}_2$ through $\widehat{\mathcal{B}_p}$.

Proof. By Theorem 2.6, $\deg(\mathcal{B}_i) = 2^{n-1} - 1$, which is odd. If $n \geq 3$, then there are at least three vertices of odd degree, so the graph has neither an Euler circuit nor an Euler path. If $n = 2$, the only vertices are $\mathcal{B}_1$, $\mathcal{B}_2$ and $\widehat{\mathcal{B}_p}$. The only edges are $\mathcal{B}_1\widehat{\mathcal{B}_p}$, and $\widehat{\mathcal{B}_p}\mathcal{B}_2$. Hence the graph is the path $\mathcal{B}_1 - \widehat{\mathcal{B}_p} - \mathcal{B}_2$, which has a unique Euler path of length 2. □

Proposition 2.8. $\Psi_{\mathcal{S}}(\mathcal{B})$ contains a 3-cycle if and only if it contains two adjacent non-simple vertices.

Proof. Suppose $\Psi_{\mathcal{S}}(\mathcal{B})$ contains a 3-cycle $\mathcal{B}_1 - \mathcal{B}_2 - \mathcal{B}_3 - \mathcal{B}_1$. Two simple subacts cannot be adjacent, so at most one vertex is simple. Hence at least two vertices are non-simple and adjacent.

Conversely, suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are adjacent non-simple vertices. Then $\mathcal{B}_1 \cap \mathcal{B}_2$ is simple and is adjacent to both $\mathcal{B}_1$ and $\mathcal{B}_2$. Hence $\mathcal{B}_1 - \mathcal{B}_2 - (\mathcal{B}_1 \cap \mathcal{B}_2) - \mathcal{B}_1$ is a 3-cycle. □

Theorem 2.9. If $3 < g(\Psi_{\mathcal{S}}(\mathcal{B})) < \infty$, then $\Psi_{\mathcal{S}}(\mathcal{B})$ is bipartite.

Proof. Since the girth is greater than 3, by Proposition 2.8, no two adjacent vertices are both non-simple. Also, two simple subacts cannot be adjacent. Hence every edge joins a simple vertex to a non-simple vertex.

Let $\mathcal{V}$ be the set of simple subacts and $\mathcal{V}'$ the set of non-simple subacts. Then there are no edges inside $\mathcal{V}$ and no edges inside $\mathcal{V}'$. Therefore $(\mathcal{V}, \mathcal{V}')$ is a bipartition, and $\Psi_{\mathcal{S}}(\mathcal{B})$ is bipartite. □

Theorem 2.10. Let $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_n$ be a semisimple $\mathcal{S}$-act with $n \geq 2$ and fix $i \in \{1, \ldots, n\}$. Let Γ_i be the induced subgraph of $\Psi_{\mathcal{S}}(\mathcal{B})$ on the set $N(\mathcal{B}_i) \setminus \{\widehat{\mathcal{B}_p}\}$. Then Γ_i is isomorphic to the graph whose vertices are the nonempty subsets of $\{1, \ldots, n\} \setminus \{i\}$, where two vertices are adjacent if and only if the corresponding subsets are disjoint.

Proof. Since $\widehat{\mathcal{B}_p}$ is semisimple, every subact of $\widehat{\mathcal{B}_p}$ is a coproduct of some of the simple components. A vertex C belongs to $N(\mathcal{B}_i) \setminus \{\widehat{\mathcal{B}_p}\}$ if and only if $C \neq \widehat{\mathcal{B}_p}$ and $\mathcal{B}_i \cap C$ is simple. Because $\mathcal{B}_i$ is simple, this means exactly that $\mathcal{B}_i \subseteq C$. Therefore every such C can be written uniquely in the form $C = \mathcal{B}_i \sqcup \left(\bigsqcup_{j \in A} \mathcal{B}_j\right)$ for some nonempty subset $A \subseteq \{1, \ldots, n\} \setminus \{i\}$, and conversely every such subset determines a unique vertex of $N(\mathcal{B}_i) \setminus \{\widehat{\mathcal{B}_p}\}$. Define $\varphi : \mathcal{V}(\Gamma_i) \longrightarrow \{A \subseteq \{1, \ldots, n\} \setminus \{i\} : A \neq \emptyset\}$ by

$$\varphi\left(\mathcal{B}_i \sqcup \Big(\bigsqcup_{j \in A} \mathcal{B}_j\Big)\right) = A.$$

By the preceding paragraph, φ is a bijection. It remains to compare adjacency. Let $C_A = \mathcal{B}_i \sqcup \left(\bigsqcup_{j\in A} \mathcal{B}_j\right)$ and $C_B = \mathcal{B}_i \sqcup \left(\bigsqcup_{j\in B} \mathcal{B}_j\right)$ be two vertices of Γ_i. Then

$$C_A \cap C_B = \mathcal{B}_i \sqcup \Big(\bigsqcup_{j\in A\cap B} \mathcal{B}_j \Big).$$

Hence C_A and C_B are adjacent in $\Psi_{\mathcal{S}}(\mathcal{B})$ if and only if their intersection is simple, and this happens if and only if $A \cap B = \varnothing$. Thus C_A and C_B are adjacent in Γ_i if and only if $\varphi(C_A)$ and $\varphi(C_B)$ are disjoint subsets. Therefore φ is a graph isomorphism. □

Theorem 2.11. Let $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_n$ be a semisimple $\mathcal{S}$-act with $n \geq 3$. Then every simple vertex $\mathcal{B}_i$ of $\Psi_{\mathcal{S}}(\mathcal{B})$ lies in exactly $\frac{1}{2}(3^{n-1} - 2^n + 1)$ distinct triangles.

Proof. Fix a simple vertex $\mathcal{B}_i$. By Theorem 2.10, the vertices of the induced graph Γ_i correspond bijectively to the nonempty subsets of $\{1, \ldots, n\} \setminus \{i\}$, and adjacency in Γ_i corresponds to disjointness of subsets. A triangle of $\Psi_{\mathcal{S}}(\mathcal{B})$ containing $\mathcal{B}_i$ is obtained by choosing two distinct vertices of $N(\mathcal{B}_i) \setminus \{\widehat{\mathcal{B}_p}\}$ that are adjacent to each other. Under the correspondence of Theorem 2.10, this is equivalent to choosing an unordered pair $\{A, B\}$ of nonempty disjoint subsets of $\{1, \ldots, n\} \setminus \{i\}$. We now count such pairs. For each element of the $(n-1)$-element set $\{1, \ldots, n\} \setminus \{i\}$ there are three possibilities: It belongs to A, or it belongs to B, or it belongs to neither set. Hence the total number of ordered pairs (A, B) of disjoint subsets is 3^{n-1}. Among these, the pairs with $A = \varnothing$ are counted by 2^{n-1}, and similarly the pairs with $B = \varnothing$ are counted by 2^{n-1}. The pair $(\varnothing, \varnothing)$ has been subtracted twice, so we add it back once. Therefore the number of ordered pairs (A, B) of nonempty disjoint subsets is $3^{n-1} - 2 \cdot 2^{n-1} + 1 = 3^{n-1} - 2^n + 1$. Since each unordered pair $\{A, B\}$ is counted exactly twice as an ordered pair, the number of unordered pairs is $\frac{1}{2}(3^{n-1} - 2^n + 1)$. This is exactly the number of triangles containing $\mathcal{B}_i$. □

Corollary 2.12. Let $\widehat{\mathcal{B}_p}$ be a semisimple $\mathcal{S}$-act with $n \geq 3$ simple components. Then every simple vertex of $\Psi_{\mathcal{S}}(\mathcal{B})$ lies in a triangle.

Proof. By Theorem 2.11, every simple vertex $\mathcal{B}_i$ lies in exactly $\frac{1}{2}(3^{n-1} - 2^n + 1)$ triangles. For $n \geq 3$, this number is positive. Therefore every simple vertex lies in at least one triangle. □

Theorem 2.13. Let $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_n$ be a semisimple $\mathcal{S}$-act with $n \geq 3$ simple components. Then every proper vertex of $\Psi_{\mathcal{S}}(\mathcal{B})$ lies on a triangle, whereas the vertex $\widehat{\mathcal{B}_p}$ lies on no triangle.

Proof. Let $\mathcal{Z}$ be a proper vertex of $\Psi_{\mathcal{S}}(\mathcal{B})$. Since $\widehat{\mathcal{B}_p}$ is semisimple, $\mathcal{Z}$ is the coproduct of a nonempty proper subset of the simple components. First, suppose that $\mathcal{Z}$ is simple, say $\mathcal{Z} = \mathcal{B}_i$. Since $n \geq 3$, Corollary 2.12 implies that $\mathcal{B}_i$ lies on a triangle. Now suppose that $\mathcal{Z}$ is non-simple. Choose a simple component $\mathcal{B}_i \subseteq \mathcal{Z}$. Since $\mathcal{Z}$ is proper, there exists a component $\mathcal{B}_j$ not contained in $\mathcal{Z}$. Consider the vertex $\mathcal{B}_i \sqcup \mathcal{B}_j$. Then $\mathcal{Z} \cap \mathcal{B}_i = \mathcal{B}_i$ is simple, so $\mathcal{Z}$ is adjacent to $\mathcal{B}_i$. Also, $\mathcal{B}_i \cap (\mathcal{B}_i \sqcup \mathcal{B}_j) = \mathcal{B}_i$ is simple, so $\mathcal{B}_i$ is adjacent to $\mathcal{B}_i \sqcup \mathcal{B}_j$. Finally, $\mathcal{Z} \cap (\mathcal{B}_i \sqcup \mathcal{B}_j) = \mathcal{B}_i$, because $\mathcal{B}_j \nsubseteq \mathcal{Z}$. Hence $\mathcal{Z}$ is adjacent to $\mathcal{B}_i \sqcup \mathcal{B}_j$, and thus $\mathcal{Z} - \mathcal{B}_i - (\mathcal{B}_i \sqcup \mathcal{B}_j) - \mathcal{Z}$ is a triangle. It remains to show that $\widehat{\mathcal{B}_p}$ lies on no triangle. Since $\widehat{\mathcal{B}_p}$ is adjacent only to simple vertices, any triangle containing $\widehat{\mathcal{B}_p}$ would require two distinct simple vertices adjacent to each other. But distinct simple vertices are never adjacent. Therefore $\widehat{\mathcal{B}_p}$ lies on no triangle. □

Corollary 2.14. Let $\widehat{\mathcal{B}_p}$ be a semisimple $\mathcal{S}$-act with $n \geq 3$ simple components. Then $\widehat{\mathcal{B}_p}$ is the unique vertex of $\Psi_{\mathcal{S}}(\mathcal{B})$ that does not lie on a triangle.

Proof. By Theorem 2.13, every proper vertex lies on a triangle, while the vertex $\widehat{\mathcal{B}_p}$ lies on no triangle. Hence $\widehat{\mathcal{B}_p}$ is the unique vertex with this property. □

3. Connectivity and clique structure

In this section, we prove that the connectedness of $\Psi_{\mathcal{S}}(\mathcal{B})$ is equivalent to the Artinian property of the underlying $\mathcal{S}$-act and derive explicit distance bounds in the connected case. We investigate clique behavior in $\Psi_{\mathcal{S}}(\mathcal{B})$, showing that cliques are tightly governed by simple subacts where a clique can contain at most one simple vertex and larger cliques are built by families of non-simple subacts intersecting through specified simple subacts.

The results of this section are obtained under different structural assumptions on the underlying S-act. The connectivity and distance estimates of Theorem 3.2 are derived under the Artinian hypothesis, while Theorem 3.5 establishes a general diameter bound for the nontrivial connected component whenever edges exist. Stronger and sharp conclusions are obtained in the semisimple setting, where explicit descriptions of clique structures, diameter, and coloring properties become available. The reader may therefore view the subsequent results as progressing from general S-acts to Artinian and finally semisimple S-acts.

Remark 3.1. The semisimplicity assumption plays a fundamental role in several results of this section. In particular, the decomposition $\mathcal{B}_p^b = \coprod_{i=1}^n \mathcal{B}_i$ into simple components is used to describe adjacency, degree formulas, clique structures, and distance properties of $\Psi_{\mathcal{B}}(\mathcal{B})$. For general $\mathcal{S}$-acts, such a decomposition may not exist, and consequently the corresponding conclusions need not remain valid. Therefore, the semisimple hypothesis should be regarded as an essential assumption for the results obtained in this section.

Theorem 3.2. Let $\mathcal{S}$ and $\mathcal{B}$ be two topological semigroups. Let $\widehat{\mathcal{B}_p}$ be an $\mathcal{S}$-act. If $\widehat{\mathcal{B}_p}$ is an Artinian $\mathcal{S}$-act, then $\Psi_{\mathcal{S}}(\mathcal{B})$ is connected.

Proof. Let $\mathcal{Z}$ and C be nonempty subacts of $\widehat{\mathcal{B}_p}$. Since $\widehat{\mathcal{B}_p}$ is Artinian, every nonempty subact contains a minimal subact, and every minimal subact is simple. Hence there exist simple subacts $\mathcal{Z}_1 \subseteq \mathcal{Z}$ and $C_1 \subseteq C$. If $\mathcal{Z}_1 = C_1$, then $\mathcal{Z} - \mathcal{Z}_1 - C$ is a path. If $\mathcal{Z}_1 \neq C_1$, then $\mathcal{Z} - \mathcal{Z}_1 - (\mathcal{Z}_1 \sqcup C_1) - C_1 - C$ is a path. Hence any two vertices are connected by a path, and therefore $\Psi_{\mathcal{S}}(\mathcal{B})$ is connected. □

Proposition 3.3. Let $\widehat{\mathcal{B}_p}$ be an Artinian $\mathcal{S}$-act. Then for any two vertices $\mathcal{Z}$ and C of $\Psi_{\mathcal{S}}(\mathcal{B})$, $1 \leq \partial(\mathcal{Z}, C) \leq 4$.

Proof. Since $\widehat{\mathcal{B}_p}$ is Artinian, Theorem 3.2 implies that $\Psi_{\mathcal{S}}(\mathcal{B})$ is connected, hence $\partial(\mathcal{Z}, C) \geq 1$. If $\mathcal{Z} \cap C$ is simple, then $\mathcal{Z} - C$ and $\partial(\mathcal{Z}, C) = 1$. If $\mathcal{Z} \cap C$ is nonempty non-simple, it contains a simple subact K, and $\mathcal{Z} - K - C$ is a path, hence $\partial(\mathcal{Z}, C) \leq 2$. If $\mathcal{Z} \cap C = \varnothing$, choose simple subacts $\mathcal{B}_1 \subseteq \mathcal{Z}$ and $\mathcal{B}_2 \subseteq C$. Then $\mathcal{Z} - \mathcal{B}_1 - (\mathcal{B}_1 \sqcup \mathcal{B}_2) - \mathcal{B}_2 - C$ is a path of length 4. Hence $\partial(\mathcal{Z}, C) \leq 4$. □

Theorem 3.4. Let $\widehat{\mathcal{B}_p}$ be an $\mathcal{S}$-act. Every vertex of $\Psi_{\mathcal{S}}(\mathcal{B})$ having positive degree lies in the same path component as $\widehat{\mathcal{B}_p}$, and every other component consists of a single isolated vertex.

Proof. Let $\mathcal{Z}$ be a vertex with $\deg(\mathcal{Z}) > 0$. If $\mathcal{Z}$ is simple, then $\widehat{\mathcal{B}_p} \cap \mathcal{Z} = \mathcal{Z}$ is simple, and $\mathcal{Z} - \widehat{\mathcal{B}_p}$. If $\mathcal{Z}$ is non-simple, then there exists C such that $\mathcal{Z} - C$. If C is simple, then $\mathcal{Z} - C - \widehat{\mathcal{B}_p}$. If C is non-simple, then $\mathcal{Z} \cap C$ is simple and $\mathcal{Z} - (\mathcal{Z} \cap C) - \widehat{\mathcal{B}_p}$. Thus every vertex of positive degree lies in the component containing $\widehat{\mathcal{B}_p}$, and all other components consist of isolated vertices. □

Theorem 3.5. Let $\widehat{\mathcal{B}_p}$ be an $\mathcal{S}$-act. Then $\Psi_{\mathcal{S}}(\mathcal{B})$ is complete if and only if $\widehat{\mathcal{B}_p}$ has exactly one nonempty proper subact.

Proof. Suppose $\Psi_{\mathcal{S}}(\mathcal{B})$ is complete and assume there exist two distinct proper subacts $\mathcal{Z}$ and C. If both are simple then they are not adjacent. If one is non-simple, then it is not adjacent to $\widehat{\mathcal{B}_p}$. In both cases, the graph cannot be complete.

Conversely, suppose $\widehat{\mathcal{B}_p}$ has a unique nonempty proper subact $\mathcal{Z}$. Then $\mathcal{Z}$ must be simple, and the vertex set of the graph is $\{\widehat{\mathcal{B}_p}, \mathcal{Z}\}$. Since $\widehat{\mathcal{B}_p} \cap \mathcal{Z} = \mathcal{Z}$ is simple, the two vertices are adjacent, and the graph is K_2. □

Theorem 3.6. Let $\widehat{\mathcal{B}_p}$ be an $\mathcal{S}$-act. If $\Psi_{\mathcal{S}}(\mathcal{B})$ has at least one edge, then every two vertices of positive degree in $\Psi_{\mathcal{S}}(\mathcal{B})$ are a distance at most 2. Equivalently, the unique nontrivial connected component of $\Psi_{\mathcal{S}}(\mathcal{B})$ has a diameter at most 2.

Proof. Let $\mathcal{Z}$ and C be two vertices of $\Psi_{\mathcal{S}}(\mathcal{B})$ having positive degree. If $\mathcal{Z}$ and C are adjacent, then $\partial(\mathcal{Z}, C) = 1$. Assume that $\mathcal{Z}$ and C are not adjacent. Then $\mathcal{Z} \cap C$ is either empty or a non-simple subact. First suppose that $\mathcal{Z} \cap C \neq \emptyset$. Since $\mathcal{Z}$ and C are not adjacent, the subact $\mathcal{Z} \cap C$ is non-simple. Because $\mathcal{Z}$ has positive degree, there exists a vertex $\mathcal{D}$ adjacent to $\mathcal{Z}$, and hence $L := \mathcal{Z} \cap \mathcal{D}$ is a simple subact of $\mathcal{Z}$. Since $\mathcal{Z} \cap C$ is a nonempty subact of $\mathcal{Z}$, it contains a simple subact K. Then $\mathcal{Z} \cap K = K$, and $C \cap K = K$ are simple, so $\mathcal{Z} - K - C$ is a path of length 2, and hence $\partial(\mathcal{Z}, C) \leq 2$. Now suppose that $\mathcal{Z} \cap C = \emptyset$. Since $\mathcal{Z}$ has positive degree, there exists a vertex $\mathcal{D}$ such that $\mathcal{Z} - \mathcal{D}$ and therefore $L := \mathcal{Z} \cap \mathcal{D}$ is a simple subact. Similarly, since C has positive degree, there exists a vertex $\mathcal{E}$ such that $\mathcal{E} - C$ and hence $M := C \cap \mathcal{E}$ is a simple subact. If $L = M$, then $L \subseteq \mathcal{Z} \cap C$, contradicting $\mathcal{Z} \cap C = \emptyset$. Hence $L \neq M$. Define $F = L \sqcup M$. Then $\mathcal{Z} \cap F = L$ and $C \cap F = M$ are simple subacts. Hence $\mathcal{Z} - F - C$ is a path of length 2, and therefore $\partial(\mathcal{Z}, C) \leq 2$. Thus every two vertices of positive degree are at distance at most 2, which means that the nontrivial connected component of $\Psi_{\mathcal{S}}(\mathcal{B})$ has a diameter at most 2. □

Theorem 3.7. Let $\widehat{\mathcal{B}_p}$ be an $\mathcal{S}$-act. Every clique in $\Psi_{\mathcal{S}}(\mathcal{B})$ contains at most one simple subact.

Proof. Suppose that Θ is a clique of $\Psi_{\mathcal{S}}(\mathcal{B})$ containing two distinct simple subacts $\mathcal{Z}$ and C. Since Θ is a clique, the vertices $\mathcal{Z}$ and C must be adjacent. Hence $\mathcal{Z} \cap C$ is a simple subact. However, $\mathcal{Z}$ and C are simple. Therefore any subact of $\mathcal{Z}$ is either $\emptyset$ or $\mathcal{Z}$ and similarly any subact of C is either $\emptyset$ or C. Since $\mathcal{Z} \cap C$ is nonempty and simple, we must have $\mathcal{Z} \cap C = \mathcal{Z}$ and $\mathcal{Z} \cap C = C$. Hence $\mathcal{Z} = C$, contradicting the assumption that they are distinct. Therefore a clique of $\Psi_{\mathcal{S}}(\mathcal{B})$ cannot contain two distinct simple subacts. □

Example 3.8. Let $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \mathcal{B}_3 \sqcup \mathcal{B}_4$ be a semisimple $\mathcal{S}$-act with four simple components.
(1) A clique with no simple vertices. Consider the non-simple subacts $X_2 = \mathcal{B}_1 \sqcup \mathcal{B}_2$, $X_3 = \mathcal{B}_1 \sqcup \mathcal{B}_3$, $X_4 = \mathcal{B}_1 \sqcup \mathcal{B}_4$. Then $X_2 \cap X_3 = \mathcal{B}_1$, $X_2 \cap X_4 = \mathcal{B}_1$, $X_3 \cap X_4 = \mathcal{B}_1$. Since $\mathcal{B}_1$ is simple, every pair among X_2, X_3, X_4 is adjacent in $\Psi_{\mathcal{S}}(\mathcal{B})$. Hence $\{X_2, X_3, X_4\}$ forms a clique whose vertices are all

non-simple.
(2) A clique containing exactly one simple vertex. Let $\Theta = \{\mathcal{B}_1, X_2, X_3, X_4\}$. Then $\mathcal{B}_1 \cap X_i = \mathcal{B}_1$ for $i = 2, 3, 4$, so $\mathcal{B}_1$ is adjacent to each X_i, and by part (1), the vertices X_2, X_3, X_4 are pairwise adjacent. Hence Θ is a clique containing exactly one simple subact.

Theorem 3.9. Let $\widehat{\mathcal{B}_p}$ be an $\mathcal{S}$-act and Θ be a clique in $\Psi_{\mathcal{S}}(\mathcal{B})$. Then Θ contains exactly one simple subact $\mathcal{Z}$ if and only if $\Theta = \{\mathcal{Z}\} \cup \mathcal{R}$, where $\mathcal{R} \subseteq N(\mathcal{Z})$ and every two vertices of $\mathcal{R}$ are $\mathcal{Z}$-adjacent.

Proof. Suppose Θ is a clique containing the unique simple subact $\mathcal{Z}$. Let $C, \mathcal{D} \in \Theta$ with $C \neq \mathcal{D}$ and $C, \mathcal{D} \neq \mathcal{Z}$. Since Θ is complete, C and $\mathcal{D}$ are adjacent and $C \cap \mathcal{D}$ is simple. Because $\mathcal{Z}$ is the only simple vertex in Θ, we obtain $C \cap \mathcal{D} = \mathcal{Z}$. Hence $C, \mathcal{D} \in N(\mathcal{Z})$ and the result follows.

Conversely, if $\Theta = \{\mathcal{Z}\} \cup \mathcal{R}$ with $\mathcal{R} \subseteq N(\mathcal{Z})$ and any two vertices of $\mathcal{R}$ intersect in $\mathcal{Z}$, then every pair of vertices in Θ is adjacent and Θ is a clique. □

Theorem 3.10. Let $\widehat{\mathcal{B}_p}$ be an $\mathcal{S}$-act and let Θ be a clique of $\Psi_{\mathcal{S}}(\mathcal{B})$ containing a unique simple subact $\mathcal{Z}$. Then $|\mathcal{V}(\Theta)| > 2$ if and only if $\widehat{\mathcal{B}_p} \notin \mathcal{V}(\Theta)$ and Θ contains at least two proper non-simple subacts that are pairwise $\mathcal{Z}$-adjacent.

Proof. Suppose $|\mathcal{V}(\Theta)| > 2$ and $\mathcal{Z}$ is the unique simple subact in Θ. Then Θ contains at least two vertices distinct from $\mathcal{Z}$. Since Θ is a clique, these vertices must be adjacent to $\mathcal{Z}$ and also adjacent to each other. Hence if $C, \mathcal{D} \in \Theta \setminus \{\mathcal{Z}\}$, then $C \cap \mathcal{D} = \mathcal{Z}$. Thus C and $\mathcal{D}$ are non-simple subacts containing $\mathcal{Z}$, and they are $\mathcal{Z}$-adjacent. Moreover $\widehat{\mathcal{B}_p}$ cannot belong to Θ, since $\widehat{\mathcal{B}_p}$ is adjacent only to simple subacts, while C and $\mathcal{D}$ are non-simple. Hence $\widehat{\mathcal{B}_p} \notin \mathcal{V}(\Theta)$.

Conversely, suppose $\widehat{\mathcal{B}_p} \notin \mathcal{V}(\Theta)$ and Θ contains two distinct proper non-simple subacts C and $\mathcal{D}$ such that $C \cap \mathcal{D} = \mathcal{Z}$. Then C and $\mathcal{D}$ are adjacent and both are adjacent to $\mathcal{Z}$. Hence $\{\mathcal{Z}, C, \mathcal{D}\}$ forms a clique of size at least 3. Therefore $|\mathcal{V}(\Theta)| > 2$. □

Example 3.11. Let $\widehat{\mathcal{B}_p} = \mathcal{Z} \sqcup C$ be a semisimple $\mathcal{S}$-act with exactly two simple components. Then the only maximal clique of $\Psi_{\mathcal{S}}(\mathcal{B})$ containing the simple subact $\mathcal{Z}$ is the trivial maximal clique $\{\mathcal{Z}, \widehat{\mathcal{B}_p}\}$.

Example 3.12. Let $\mathcal{S}$ be a finite topological semigroup with the discrete topology and suppose that $\mathcal{S}$, viewed as an $\mathcal{S}$-act over itself, has a unique minimal subact I. Then the only maximal clique of $\Psi_{\mathcal{S}}(\mathcal{S})$ containing the simple subact I is the trivial maximal clique $\{I, \mathcal{S}\}$.

Theorem 3.13. Let $\widehat{\mathcal{B}_p}$ be an $\mathcal{S}$-act and X a nonempty collection of simple subacts of $\widehat{\mathcal{B}_p}$ that induces cliques in $\Psi_{\mathcal{S}}(\mathcal{B})$. Then there exists a maximal clique in $\Psi_{\mathcal{S}}(\mathcal{B})$ induced by X.

Proof. Let Γ be the family of all cliques induced by X and ordered by inclusion. For any chain $\Theta_1 \subseteq \Theta_2 \subseteq \cdots$ in Γ, the union $\bigcup_i \Theta_i$ is again a clique induced by X. Hence every chain has an upper bound. By Zorn's lemma, Γ contains a maximal element, which is a maximal clique induced by X. □

Theorem 3.14. Let $\widehat{\mathcal{B}_p}$ be an $\mathcal{S}$-act. If Θ is a clique of $\Psi_{\mathcal{S}}(\mathcal{B})$, then there exists a unique nonempty set X of simple subacts such that Θ is induced by X.

Proof. Let X be the set of all simple subacts obtained as intersections of pairs of vertices of Θ. Since Θ is a clique, these intersections are simple and nonempty. Hence every adjacency in Θ occurs through an element of X, so Θ is induced by X.

Conversely, if Y is another set inducing Θ, then every simple intersection must belong to Y, hence $X \subseteq Y$. Similarly, every element of Y arises as an intersection of vertices of Θ, so $Y \subseteq X$. Therefore $X = Y$ and the set is unique. $\square$

Theorem 3.15. Let $\widehat{\mathcal{B}_p}$ be a semisimple non-simple $\mathcal{S}$-act. Then the diameter of $\Psi_{\mathcal{S}}(\mathcal{B})$ is equal to 2.

Proof. Write $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_n$ with $n \geq 2$, and each $\mathcal{B}_i$ is simple. Since $\widehat{\mathcal{B}_p}$ is semisimple, it is Artinian. Hence, by Theorem 3.2, the graph $\Psi_{\mathcal{S}}(\mathcal{B})$ is connected. By Theorem 3.6, every two vertices in the connected component of $\widehat{\mathcal{B}_p}$ are at a distance at most 2. Since the graph is connected, it follows that $(\Psi_{\mathcal{S}}(\mathcal{B})) \leq 2$. It remains to show that the diameter is not equal to 1. Consider two distinct simple components $\mathcal{B}_1$ and $\mathcal{B}_2$. Since the decomposition is a coproduct, $\mathcal{B}_1 \cap \mathcal{B}_2 = \varnothing$. Therefore $\mathcal{B}_1$ and $\mathcal{B}_2$ are not adjacent in $\Psi_{\mathcal{S}}(\mathcal{B})$. On the other hand, $\widehat{\mathcal{B}_p} \cap \mathcal{B}_1 = \mathcal{B}_1$ and $\widehat{\mathcal{B}_p} \cap \mathcal{B}_2 = \mathcal{B}_2$ are simple, so both $\mathcal{B}_1$ and $\mathcal{B}_2$ are adjacent to $\widehat{\mathcal{B}_p}$. Hence $\mathcal{B}_1 - \widehat{\mathcal{B}_p} - \mathcal{B}_2$ is a path of length 2, and consequently $\partial(\mathcal{B}_1, \mathcal{B}_2) = 2$. Thus $(\Psi_{\mathcal{S}}(\mathcal{B})) = 2$. $\square$

Theorem 3.16. Let $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_n$ be a semisimple $\mathcal{S}$-act with $n \geq 2$ simple components and let $\mathcal{B}_i$ be fixed. For every partition $\Pi = \{U_1, U_2, \ldots, U_t\}$ of $\{1, \ldots, n\} \setminus \{i\}$ into nonempty pairwise disjoint subsets, define

$$\Theta_i(\Pi) = \{\mathcal{B}_i\} \cup \Big\{\mathcal{B}_i \sqcup \big(\bigsqcup_{j \in U_r} \mathcal{B}_j\big) : 1 \leq r \leq t\Big\}.$$

Then $\Theta_i(\Pi)$ is a maximal clique of $\Psi_{\mathcal{S}}(\mathcal{B})$ containing the simple vertex $\mathcal{B}_i$. Conversely, every maximal clique of $\Psi_{\mathcal{S}}(\mathcal{B})$ containing $\mathcal{B}_i$ is of the form $\Theta_i(\Pi)$ for a unique partition Π of $\{1, \ldots, n\} \setminus \{i\}$.

Proof. Fix a partition $\Pi = \{U_1, U_2, \ldots, U_t\}$ of $\{1, \ldots, n\} \setminus \{i\}$. For each r, let $C_r = \mathcal{B}_i \sqcup (\bigsqcup_{j \in U_r} \mathcal{B}_j)$. Then C_r is a non-simple subact containing $\mathcal{B}_i$. Hence $\mathcal{B}_i \cap C_r = \mathcal{B}_i$ is simple, so $\mathcal{B}_i$ is adjacent to each C_r. Now let $r \neq s$. Since U_r and U_s are disjoint, we have $C_r \cap C_s = \mathcal{B}_i$. Therefore C_r and C_s are adjacent. It follows that every two distinct vertices of $\Theta_i(\Pi)$ are adjacent, so $\Theta_i(\Pi)$ is a clique. To prove maximality, let $\mathcal{D}$ be a vertex adjacent to every vertex of $\Theta_i(\Pi)$. Since $\mathcal{D}$ is adjacent to $\mathcal{B}_i$, the intersection $\mathcal{D} \cap \mathcal{B}_i$ is simple. As $\mathcal{B}_i$ is simple, this implies $\mathcal{B}_i \subseteq \mathcal{D}$. Hence $\mathcal{D}$ has the form $\mathcal{D} = \mathcal{B}_i \sqcup (\bigsqcup_{j \in W} \mathcal{B}_j)$ for some subset $W \subseteq \{1, \ldots, n\} \setminus \{i\}$. Since $\mathcal{D}$ is adjacent to every C_r, we must have $\mathcal{D} \cap C_r = \mathcal{B}_i$ for each r. This means that $W \cap U_r = \varnothing$ for all r. But Π is a partition of $\{1, \ldots, n\} \setminus \{i\}$, so $\bigcup_{r=1}^{t} U_r = \{1, \ldots, n\} \setminus \{i\}$. Therefore $W = \varnothing$, and hence $\mathcal{D} = \mathcal{B}_i \in \Theta_i(\Pi)$. Thus no vertex outside $\Theta_i(\Pi)$ can be added to it, so $\Theta_i(\Pi)$ is maximal.

Conversely, let Θ be a maximal clique containing $\mathcal{B}_i$. By Theorem 3.9, every vertex of $\Theta \setminus \{\mathcal{B}_i\}$ is a non-simple subact adjacent to $\mathcal{B}_i$, and every two such vertices are $\mathcal{B}_i$-adjacent. Hence each vertex $C \in \Theta \setminus \{\mathcal{B}_i\}$ can be written uniquely in the form $C = \mathcal{B}_i \sqcup (\bigsqcup_{j \in U(C)} \mathcal{B}_j)$, where $U(C)$ is a nonempty subset of $\{1, \ldots, n\} \setminus \{i\}$. If $C, \mathcal{D} \in \Theta \setminus \{\mathcal{B}_i\}$ are distinct, then $C \cap \mathcal{D} = \mathcal{B}_i$ because they are $\mathcal{B}_i$-adjacent. Therefore $U(C) \cap U(\mathcal{D}) = \varnothing$. Thus the family $\Pi := \{U(C) : C \in \Theta \setminus \{\mathcal{B}_i\}\}$ consists of nonempty pairwise disjoint subsets of $\{1, \ldots, n\} \setminus \{i\}$. We claim that Π covers all of $\{1, \ldots, n\} \setminus \{i\}$. Suppose not. Then there exists $k \neq i$ such that $k \notin U(C)$ for every $C \in \Theta \setminus \{\mathcal{B}_i\}$. Consider the vertex $\mathcal{E} = \mathcal{B}_i \sqcup \mathcal{B}_k$. For every $C \in \Theta \setminus \{\mathcal{B}_i\}$, we have $U(C) \cap \{k\} = \varnothing$, and thus $\mathcal{E} \cap C = \mathcal{B}_i$. Also, $\mathcal{E} \cap \mathcal{B}_i = \mathcal{B}_i$. Hence $\mathcal{E}$ is adjacent to every vertex of Θ, contradicting the maximality of Θ. Therefore Π is a partition of $\{1, \ldots, n\} \setminus \{i\}$, and by construction $\Theta = \Theta_i(\Pi)$. The uniqueness of Π follows from the uniqueness of the subsets $U(C)$ attached to the vertices $C \in \Theta \setminus \{\mathcal{B}_i\}$. $\square$

Theorem 3.17. Let $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_n$ be a semisimple $\mathcal{S}$-act with $n \geq 2$ simple components. Then the clique number of $\Psi_{\mathcal{S}}(\mathcal{B})$ is equal to n.

Proof. We first show that $\omega(\Psi_{\mathcal{S}}(\mathcal{B})) \geq n$. Fix the simple component $\mathcal{B}_1$ and consider the set $\Theta = \{\mathcal{B}_1, \mathcal{B}_1 \sqcup \mathcal{B}_2, \mathcal{B}_1 \sqcup \mathcal{B}_3, \ldots, \mathcal{B}_1 \sqcup \mathcal{B}_n\}$. This set has n vertices. The vertex $\mathcal{B}_1$ is adjacent to each $\mathcal{B}_1 \sqcup \mathcal{B}_j$ because their intersection is $\mathcal{B}_1$. For $j \neq k$, we have $(\mathcal{B}_1 \sqcup \mathcal{B}_j) \cap (\mathcal{B}_1 \sqcup \mathcal{B}_k) = \mathcal{B}_1$, so any two distinct non-simple vertices in Θ are also adjacent. Therefore Θ is a clique, and hence $\omega(\Psi_{\mathcal{S}}(\mathcal{B})) \geq n$. It remains to prove the reverse inequality. Let Θ be an arbitrary clique in $\Psi_{\mathcal{S}}(\mathcal{B})$. By Theorem 3.7, Θ contains at most one simple vertex. We split it into two cases.

Case 1: Θ contains a simple vertex, say $\mathcal{B}_i$. By Theorem 3.9, every other vertex of Θ is adjacent to $\mathcal{B}_i$, and any two such vertices are $\mathcal{B}_i$-adjacent. Hence every vertex $C \in \Theta \setminus \{\mathcal{B}_i\}$ has the form $C = \mathcal{B}_i \sqcup (\bigsqcup_{j \in U(C)} \mathcal{B}_j)$, where $U(C)$ is a nonempty subset of $\{1, \ldots, n\} \setminus \{i\}$, and for distinct $C, \mathcal{D} \in \Theta \setminus \{\mathcal{B}_i\}$, we must have $U(C) \cap U(\mathcal{D}) = \varnothing$. Therefore the family $\{U(C) : C \in \Theta \setminus \{\mathcal{B}_i\}\}$ consists of pairwise disjoint nonempty subsets of an $(n-1)$-element set. Hence there are at most $n-1$ such subsets. It follows that $|\Theta| \leq 1 + (n-1) = n$.

Case 2: Θ contains no simple vertex. Choose any $C_0 \in \Theta$. Since C_0 is non-simple, it contains at least one simple component; denote one such component by $\mathcal{B}_i$. For every other vertex $\mathcal{D} \in \Theta$, the intersection $C_0 \cap \mathcal{D}$ is simple because Θ is a clique. In particular, $C_0 \cap \mathcal{D}$ must be one of the simple components contained in C_0. Hence distinct vertices of $\Theta \setminus \{C_0\}$ yield simple intersections with C_0 coming from the finite set of simple components contained in C_0. More concretely, each $\mathcal{D} \in \Theta$ may be written as $\mathcal{D} = \bigsqcup_{j \in W(\mathcal{D})} \mathcal{B}_j$ for a nonempty subset $W(\mathcal{D}) \subseteq \{1, \ldots, n\}$, and the condition that Θ is a clique means that for distinct $\mathcal{D}_1, \mathcal{D}_2 \in \Theta$, the intersection $W(\mathcal{D}_1) \cap W(\mathcal{D}_2)$ has cardinality 1. Fix C_0, and choose $i \in W(C_0)$. Then every other $W(\mathcal{D})$ can share at most one index with $W(C_0)$, and different vertices of Θ give rise to different choices compatible with this condition. Consequently, one cannot have more than n vertices in Θ. Thus $|\Theta| \leq n$ in this case as well. Since every clique has size at most n and there exists a clique of size n, we conclude that $\omega(\Psi_{\mathcal{S}}(\mathcal{B})) = n$. □

Corollary 3.18. Let $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_n$ be a semisimple $\mathcal{S}$-act with $n \geq 2$ simple components. Then a clique of $\Psi_{\mathcal{S}}(\mathcal{B})$ has maximum size if and only if it has the form $\{\mathcal{B}_i, \mathcal{B}_i \sqcup \mathcal{B}_j : j \neq i\}$ for some $i \in \{1, \ldots, n\}$.

Proof. Let Θ be a clique of maximum size. By Theorem 3.17, $|\Theta| = n$. By Theorem 3.7, Θ contains at most one simple vertex. If Θ contained no simple vertex, then the proof of Theorem 3.17 shows that its size could not exceed $n-1$ when all pairwise simple intersections are accounted for through the available simple components. Therefore Θ must contain exactly one simple vertex, say $\mathcal{B}_i$. By Theorem 3.16, the maximal cliques containing $\mathcal{B}_i$ are exactly those arising from partitions of $\{1, \ldots, n\} \setminus \{i\}$. The size of such a clique is $1 + t$, where t is the number of blocks of the partition. Hence $1 + t = n$ if and only if $t = n - 1$, and this happens exactly when every block of the partition is a singleton. Therefore $\Theta = \{\mathcal{B}_i, \mathcal{B}_i \sqcup \mathcal{B}_j : j \neq i\}$.

Conversely, for any fixed i, the set $\{\mathcal{B}_i, \mathcal{B}_i \sqcup \mathcal{B}_j : j \neq i\}$ is a clique of size n, so by Theorem 3.17, it is a maximum clique. □

Theorem 3.19. Let $\widehat{\mathcal{B}_p}$ be a semisimple $\mathcal{S}$-act with $n \geq 2$ simple components. Then every maximal clique of $\Psi_{\mathcal{S}}(\mathcal{B})$ contains at most one simple subact. Moreover, every maximal clique containing a simple subact contains exactly one simple subact.

Proof. By Theorem 3.7, every clique of $\Psi_{\mathcal{S}}(\mathcal{B})$ contains at most one simple subact. Hence every maximal clique also contains at most one simple subact. Now let Θ be a maximal clique containing a simple subact $\mathcal{Z}$. Since Θ contains a simple subact, it is enough to show that it cannot contain any besides $\mathcal{Z}$. But this follows immediately from Theorem 3.7, which states that no clique can contain two distinct simple subacts. Therefore $\mathcal{Z}$ is the unique simple subact of Θ. Hence every maximal clique of $\Psi_{\mathcal{S}}(\mathcal{B})$ contains at most one simple subact, and any maximal clique that contains a simple subact contains exactly one simple subact. □

Theorem 3.20. Let $\widehat{\mathcal{B}_p}$ be a semisimple $\mathcal{S}$-act with simple components $\mathcal{B}_1, \ldots, \mathcal{B}_n$, where $n \geq 2$. Then every vertex of $\Psi_{\mathcal{S}}(\mathcal{B})$ belongs to a maximal clique.

Proof. Let C be a vertex of $\Psi_{\mathcal{S}}(\mathcal{B})$. If C is simple, say $C = \mathcal{B}_i$, then by Theorem 3.16, there exists a maximal clique containing $\mathcal{B}_i$. Suppose C is non-simple. Then C contains a simple component $\mathcal{B}_i$. Hence C is adjacent to $\mathcal{B}_i$. Extend the clique $\{\mathcal{B}_i, C\}$ to a maximal clique using Zorn's lemma. Thus every vertex belongs to some maximal clique. □

Theorem 3.21. Let $\widehat{\mathcal{B}_p}$ be a semisimple $\mathcal{S}$-act with $n \geq 2$ simple components. Then $\Psi_{\mathcal{S}}(\mathcal{B})$ is the union of the maximal cliques determined by the simple subacts of $\widehat{\mathcal{B}_p}$.

Proof. Let $\mathcal{Z}$ be a simple subact. By Theorem 3.19, every maximal clique contains exactly one simple subact. Hence every maximal clique is associated with a unique simple subact. For each simple subact $\mathcal{Z}$, consider the maximal clique consisting of all vertices adjacent to $\mathcal{Z}$ that intersect pairwise through $\mathcal{Z}$. By Theorem 3.20, every vertex belongs to some maximal clique. Hence the entire graph is obtained as the union of maximal cliques associated with the simple subacts. □

Theorem 3.22. Let $\widehat{\mathcal{B}_p}$ be a semisimple $\mathcal{S}$-act with $n \geq 2$ simple components. Then the number of maximal cliques of $\Psi_{\mathcal{S}}(\mathcal{B})$ is at least n.

Proof. Each simple component $\mathcal{B}_i$ determines a maximal clique containing $\mathcal{B}_i$ (Theorem 3.16). If $\mathcal{B}_i \neq \mathcal{B}_j$, then the maximal cliques determined by them are distinct because each maximal clique contains exactly one simple vertex (Theorem 3.19). Therefore the graph has at least n maximal cliques. □

Theorem 3.23. Let $\widehat{\mathcal{B}_p}$ be a semisimple $\mathcal{S}$-act with $n \geq 2$ simple components. Then $\chi(\Psi_{\mathcal{S}}(\mathcal{B})) \geq n$. In particular, if $n = 2$, then $\chi(\Psi_{\mathcal{S}}(\mathcal{B})) = 2$.

Proof. By Theorem 3.17, the clique number of $\Psi_{\mathcal{S}}(\mathcal{B})$ is n. Hence $\chi(\Psi_{\mathcal{S}}(\mathcal{B})) \geq \omega(\Psi_{\mathcal{S}}(\mathcal{B})) = n$. If $n = 2$, then $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2$, and the vertices of $\Psi_{\mathcal{S}}(\mathcal{B})$ are exactly $\mathcal{B}_1, \mathcal{B}_2, \widehat{\mathcal{B}_p}$. The only edges are $\mathcal{B}_1 - \widehat{\mathcal{B}_p}$ and $\widehat{\mathcal{B}_p} - \mathcal{B}_2$, so $\Psi_{\mathcal{S}}(\mathcal{B})$ is a path on three vertices. Therefore $\chi(\Psi_{\mathcal{S}}(\mathcal{B})) = 2$. □

Remark 3.24. The equality $\chi(\Psi_{\mathcal{S}}(\mathcal{B})) = n$ does not hold in general for $n \geq 3$. The following example shows that more colors may be required.

Example 3.25. Let $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \mathcal{B}_3$ be a semisimple $\mathcal{S}$-act with three simple components. Consider the vertices $\mathcal{B}_1 \sqcup \mathcal{B}_2$, $\mathcal{B}_1 \sqcup \mathcal{B}_3$, and $\mathcal{B}_2 \sqcup \mathcal{B}_3$. For each pair of these vertices, the intersection is exactly one simple component, namely

$$(\mathcal{B}_1 \sqcup \mathcal{B}_2) \cap (\mathcal{B}_1 \sqcup \mathcal{B}_3) = \mathcal{B}_1,$$

$$(\mathcal{B}_1 \sqcup \mathcal{B}_2) \cap (\mathcal{B}_2 \sqcup \mathcal{B}_3) = \mathcal{B}_2,$$

and

$$(\mathcal{B}_1 \sqcup \mathcal{B}_3) \cap (\mathcal{B}_2 \sqcup \mathcal{B}_3) = \mathcal{B}_3.$$

Hence these three vertices are pairwise adjacent and form a triangle in $\Psi_{\mathcal{S}}(\mathcal{B})$. Consequently, they require three distinct colors in any proper coloring. Each simple vertex $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ is adjacent to two of these vertices and therefore must receive the remaining color among the three. Thus the vertices $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ together use all three colors. However, the vertex $\widehat{\mathcal{B}_p}$ is adjacent to each of $\mathcal{B}_1, \mathcal{B}_2$, and $\mathcal{B}_3$, so it cannot receive any of these three colors. Therefore a fourth color is required. This shows that $\chi(\Psi_{\mathcal{S}}(\mathcal{B})) \geq 4$ when $n = 3$.

Theorem 3.26. Let $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n$ be a semisimple $\mathcal{S}$-act with $n \geq 2$ simple components. Then $\Psi_{\mathcal{S}}(\mathcal{B})$ has exactly n maximum cliques.

Proof. By Corollary 3.18, every maximum clique of $\Psi_{\mathcal{S}}(\mathcal{B})$ has the form $\Theta_i = \{\mathcal{B}_i, \mathcal{B}_i \sqcup \mathcal{B}_j : j \neq i\}$ for some $i \in \{1, \ldots, n\}$. Hence there are at most n maximum cliques. On the other hand, for each $i \in \{1, \ldots, n\}$, the set Θ_i is a clique of size n by Corollary 3.18, and therefore it is a maximum clique. Since Θ_i contains the simple vertex $\mathcal{B}_i$ and no other simple vertex, the cliques $\Theta_1, \ldots, \Theta_n$ are pairwise distinct. Thus $\Psi_{\mathcal{S}}(\mathcal{B})$ has exactly n maximum cliques. □

Theorem 3.27. Let $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n$ be a semisimple $\mathcal{S}$-act with $n \geq 2$ simple components. Then each simple vertex $\mathcal{B}_i$ belongs to a unique maximum clique of $\Psi_{\mathcal{S}}(\mathcal{B})$.

Proof. Fix $i \in \{1, \ldots, n\}$. By Corollary 3.18, there exists a maximum clique $\Theta_i = \{\mathcal{B}_i, \mathcal{B}_i \sqcup \mathcal{B}_j : j \neq i\}$ containing $\mathcal{B}_i$. Now let Θ be any maximum clique containing $\mathcal{B}_i$. Again by Corollary 3.18, Θ has the form $\Theta = \{\mathcal{B}_k, \mathcal{B}_k \sqcup \mathcal{B}_j : j \neq k\}$ for some $k \in \{1, \ldots, n\}$. Since Θ contains the simple vertex $\mathcal{B}_i$ and a maximum clique contains exactly one simple vertex, we must have $k = i$. Hence $\Theta = \Theta_i$. Therefore $\mathcal{B}_i$ belongs to a unique maximum clique. □

Theorem 3.28. Let $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n$ be a semisimple $\mathcal{S}$-act with $n \geq 2$ simple components and let C be a non-simple vertex of $\Psi_{\mathcal{S}}(\mathcal{B})$. Then C belongs to a maximum clique if and only if C has exactly two simple components.

Proof. Assume that C belongs to a maximum clique Θ. By Corollary 3.18, there exists i such that $\Theta = \{\mathcal{B}_i, \mathcal{B}_i \sqcup \mathcal{B}_j : j \neq i\}$. Since C is non-simple and belongs to Θ, it must be of the form $\mathcal{B}_i \sqcup \mathcal{B}_j$ for some $j \neq i$. Hence C has exactly two simple components.

Conversely, assume that C has exactly two simple components, say $C = \mathcal{B}_i \sqcup \mathcal{B}_j$ with $i \neq j$. Then, by Corollary 3.18, the maximum clique $\Theta_i = \{\mathcal{B}_i, \mathcal{B}_i \sqcup \mathcal{B}_k : k \neq i\}$ contains the vertex $\mathcal{B}_i \sqcup \mathcal{B}_j = C$. Therefore C belongs to a maximum clique. □

Corollary 3.29. Let $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n$ be a semisimple $\mathcal{S}$-act with $n \geq 2$ simple components. If $C = \mathcal{B}_i \sqcup \mathcal{B}_j$ with $i \neq j$, then C belongs to exactly two maximum cliques, namely the unique maximum clique containing $\mathcal{B}_i$ and the unique maximum clique containing $\mathcal{B}_j$.

Proof. By Theorem 3.27, the simple vertices $\mathcal{B}_i$ and $\mathcal{B}_j$ belong to unique maximum cliques, say Θ_i and Θ_j. By Corollary 3.18, the vertex $\mathcal{B}_i \sqcup \mathcal{B}_j$ belongs to both Θ_i and Θ_j. Now let Θ be any maximum clique containing $C = \mathcal{B}_i \sqcup \mathcal{B}_j$. By Corollary 3.18, Θ has the form $\Theta = \{\mathcal{B}_k, \mathcal{B}_k \sqcup \mathcal{B}_\ell : \ell \neq k\}$ for some k. Since C belongs to Θ, we must have either $k = i$ or $k = j$. Thus Θ is either Θ_i or Θ_j. Therefore C belongs to exactly two maximum cliques. □

4. Indecomposable controlling families

In this section, we study domination-type parameters for $\Psi_{\mathcal{S}}(\mathcal{B})$ via controlling sets. We introduce a canonical controlling family formed from simple subacts together with those non-simple subacts that contain no simple subacts and show that it yields an indecomposable controlling set. We also obtain bounds for the control number $\Delta(\Psi_{\mathcal{S}}(\mathcal{B}))$, especially for semisimple $\mathcal{S}$-acts with finitely many components. By $\mathcal{J}(\widehat{\mathcal{B}_p})$, we mean the family of all simple subacts of $\widehat{\mathcal{B}_p}$ and all non-simple subacts of $\widehat{\mathcal{B}_p}$ that contain no simple subacts.

The notion of a controlling set considered here is closely related to domination-type parameters that have been investigated in several graph-theoretic settings. In the present framework, the control number measures the minimum size of a family of vertices whose adjacency relations collectively influence all remaining vertices of the Ψ-intersection graph.

Lemma 4.1. Let $\mathcal{Z}$ be a vertex of $\Psi_{\mathcal{S}}(\mathcal{B})$. Then $\mathcal{Z}$ is isolated if and only if $\mathcal{Z}$ is non-simple and contains no simple subact.

Proof. If $\mathcal{Z}$ is non-simple and contains no simple subact, then for every vertex C of $\Psi_{\mathcal{S}}(\mathcal{B})$, the intersection $\mathcal{Z} \cap C$ is a subact of $\mathcal{Z}$ and hence cannot be simple. Therefore $\mathcal{Z}$ is adjacent to no vertex, so it is isolated.

Conversely, suppose that $\mathcal{Z}$ is isolated. Then $\mathcal{Z}$ cannot be simple, because every simple subact is adjacent to $\widehat{\mathcal{B}_p}$. If $\mathcal{Z}$ contained a simple subact $\mathcal{K}$, then $\mathcal{Z} \cap \mathcal{K} = \mathcal{K}$ would be simple, so $\mathcal{Z}$ would be adjacent to $\mathcal{K}$, a contradiction. Hence $\mathcal{Z}$ is non-simple and contains no simple subact. $\square$

Theorem 4.2. Let $\mathcal{S}$ and $\mathcal{B}$ be any two topological semigroups. Let $\widehat{\mathcal{B}_p}$ be an $\mathcal{S}$-act. Then $\mathcal{J}(\widehat{\mathcal{B}_p})$ is an indecomposable controlling set of $\Psi_{\mathcal{S}}(\mathcal{B})$.

Proof. Let $\mathcal{Z}$ be a vertex of $\Psi_{\mathcal{S}}(\mathcal{B})$ with $\mathcal{Z} \notin \mathcal{J}(\widehat{\mathcal{B}_p})$. Then $\mathcal{Z}$ is non-simple and contains a simple subact, say $\mathcal{Z}_1$. Since $\mathcal{Z} \cap \mathcal{Z}_1 = \mathcal{Z}_1$ is simple, the vertices $\mathcal{Z}$ and $\mathcal{Z}_1$ are adjacent. As $\mathcal{Z}_1 \in \mathcal{J}(\widehat{\mathcal{B}_p})$, every vertex outside $\mathcal{J}(\widehat{\mathcal{B}_p})$ is adjacent to some vertex of $\mathcal{J}(\widehat{\mathcal{B}_p})$. Hence $\mathcal{J}(\widehat{\mathcal{B}_p})$ is a controlling set. To prove indecomposability, let $\mathcal{Y} \in \mathcal{J}(\widehat{\mathcal{B}_p})$. If $\mathcal{Y}$ is a non-simple subact containing no simple subact, then $\mathcal{Y}$ is isolated by Lemma 4.1, so $\mathcal{Y}$ belongs to every controlling set. If $\mathcal{Y}$ is simple, then $\mathcal{Y}$ is not adjacent to any other vertex of $\mathcal{J}(\widehat{\mathcal{B}_p})$: Indeed, two distinct simple subacts are never adjacent, and a non-simple vertex of $\mathcal{J}(\widehat{\mathcal{B}_p})$ contains no simple subact, so it cannot be adjacent to $\mathcal{Y}$. Thus, after removing $\mathcal{Y}$ from $\mathcal{J}(\widehat{\mathcal{B}_p})$, the vertex $\mathcal{Y}$ is not controlled. Therefore $\mathcal{J}(\widehat{\mathcal{B}_p})$ is indecomposable. $\square$

Proposition 4.3. Let $\mathcal{S}$ and $\mathcal{B}$ be any two topological semigroups. Let $\widehat{\mathcal{B}_p}$ be an $\mathcal{S}$-act. Then $\Delta(\Psi_{\mathcal{S}}(\mathcal{B})) \leq O(\mathcal{J}(\widehat{\mathcal{B}_p}))$, and if $\Psi_{\mathcal{S}}(\mathcal{B})$ is connected, then $\mathcal{J}(\widehat{\mathcal{B}_p})$ consists only of simple subacts of $\widehat{\mathcal{B}_p}$.

Proof. By Theorem 4.2, $\mathcal{J}(\widehat{\mathcal{B}_p})$ is a controlling set of $\Psi_{\mathcal{S}}(\mathcal{B})$. Hence, by the definition of the control number, $\Delta(\Psi_{\mathcal{S}}(\mathcal{B})) \leq O(\mathcal{J}(\widehat{\mathcal{B}_p}))$. If $\Psi_{\mathcal{S}}(\mathcal{B})$ is connected, then it has no isolated vertices. By Lemma 4.1, no non-simple subact of $\widehat{\mathcal{B}_p}$ can contain no simple subact. Therefore $\mathcal{J}(\widehat{\mathcal{B}_p})$ consists only of simple subacts. $\square$

Note that $\mathcal{J}(\widehat{\mathcal{B}_p})$ is not the only type of controlling set of $\Psi_{\mathcal{S}}(\mathcal{B})$. We now show that every controlling set contains an indecomposable controlling subset and that isolated vertices must belong to every controlling set.

Remark 4.4. If $\mathbb{A}$ is a controlling set of $\Psi_{\mathcal{S}}(\mathcal{B})$, then every isolated vertex of $\Psi_{\mathcal{S}}(\mathcal{B})$ belongs to $\mathbb{A}$.

Proof. Let $\mathcal{Z}$ be an isolated vertex of $\Psi_{\mathcal{S}}(\mathcal{B})$. If $\mathcal{Z} \notin \mathbb{A}$, then $\mathcal{Z}$ is a vertex outside $\mathbb{A}$ not adjacent to any vertex of $\mathbb{A}$, contradicting the fact that $\mathbb{A}$ is controlling. Hence $\mathcal{Z} \in \mathbb{A}$. □

Proposition 4.5. Let $\widehat{\mathcal{B}_p}$ be an $\mathcal{S}$-act. If $\mathbb{A}$ is a controlling set of $\Psi_{\mathcal{S}}(\mathcal{B})$, then there exists an indecomposable controlling set $\mathcal{R}$ such that $\mathcal{R} \subseteq \mathbb{A}$ and $O(\mathcal{R}) \leq O(\mathbb{A})$.

Proof. Let $\mathfrak{F}$ be the family of all controlling subsets of $\mathbb{A}$. Since $\mathbb{A} \in \mathfrak{F}$, the family $\mathfrak{F}$ is nonempty. Choose $\mathcal{R} \in \mathfrak{F}$ of minimum cardinality. Then $\mathcal{R} \subseteq \mathbb{A}$ and $O(\mathcal{R}) \leq O(\mathbb{A})$. If $\mathcal{R}$ were decomposable, then a proper subset of $\mathcal{R}$ would still be controlling, contradicting the minimality of $\mathcal{R}$. Therefore $\mathcal{R}$ is indecomposable. □

Proposition 4.6. Let $\widehat{\mathcal{B}_p}$ be a non-simple $\mathcal{S}$-act. Suppose that $\mathbb{A}$ is an indecomposable controlling set of $\Psi_{\mathcal{S}}(\mathcal{B})$. If $\widehat{\mathcal{B}_p} \notin \mathbb{A}$, then $\mathbb{A}$ contains at least one simple subact of $\widehat{\mathcal{B}_p}$.

Proof. Since $\widehat{\mathcal{B}_p} \notin \mathbb{A}$ and $\mathbb{A}$ is controlling, the vertex $\widehat{\mathcal{B}_p}$ must be adjacent to some vertex $C \in \mathbb{A}$. By the definition of adjacency, $\widehat{\mathcal{B}_p} \cap C = C$ is simple. Hence C is a simple subact of $\widehat{\mathcal{B}_p}$, so $\mathbb{A}$ contains at least one simple subact. □

Example 4.7. Let $\widehat{\mathcal{B}_p} = \mathcal{Z} \sqcup C \sqcup \mathcal{A}$ be a semisimple $\mathcal{S}$-act, where $\mathcal{Z}, C, \mathcal{A}$ are simple subacts. Consider the controlling sets $\mathbb{A}_1 = \{\mathcal{Z}, C \sqcup \mathcal{A}, \mathcal{A}\}$ and $\mathbb{A}_2 = \{\mathcal{Z} \sqcup C, \mathcal{A}\}$. Then $\mathbb{A}_1$ is decomposable, since $\{\mathcal{Z}, C \sqcup \mathcal{A}\}$ is already a controlling set. On the other hand, $\mathbb{A}_2$ is indecomposable: If $\mathcal{Z} \sqcup C$ is removed, then the vertices $\mathcal{Z}$ and C are not controlled, and if $\mathcal{A}$ is removed, then the vertex $\mathcal{A}$ is not controlled.

In the next proposition, we show that any controlling set of $\Psi_{\mathcal{S}}(\mathcal{B})$ can be replaced by an indecomposable controlling set consisting of isolated vertices together with vertices that contain simple subacts.

Proposition 4.8. Let $\widehat{\mathcal{B}_p}$ be an $\mathcal{S}$-act. Let $\mathbb{A}$ be a controlling set of $\Psi_{\mathcal{S}}(\mathcal{B})$. Then there exists an indecomposable controlling set $\mathcal{R}$ such that $O(\mathcal{R}) \leq O(\mathbb{A})$. Moreover, every isolated vertex of $\Psi_{\mathcal{S}}(\mathcal{B})$ belongs to $\mathcal{R}$, and if $\widehat{\mathcal{B}_p} \notin \mathcal{R}$, then $\mathcal{R}$ contains at least one simple subact of $\widehat{\mathcal{B}_p}$.

Proof. By Proposition 4.5, the controlling set $\mathbb{A}$ contains an indecomposable controlling subset $\mathcal{R}$ with $O(\mathcal{R}) \leq O(\mathbb{A})$. By Remark 4.4, every isolated vertex belongs to every controlling set, hence to $\mathcal{R}$. If $\widehat{\mathcal{B}_p} \notin \mathcal{R}$, then Proposition 4.6 implies that $\mathcal{R}$ contains at least one simple subact of $\widehat{\mathcal{B}_p}$. □

Example 4.9. Let $\widehat{\mathcal{B}_p} = \mathcal{Z} \sqcup C \sqcup \mathcal{A} \sqcup \mathcal{E}$ be a semisimple $\mathcal{S}$-act, where $\mathcal{Z}, C, \mathcal{A}, \mathcal{E}$ are simple subacts. Then the set $\mathcal{R} = \{\mathcal{Z}, C, \mathcal{A} \sqcup \mathcal{E}\}$ is an indecomposable controlling set of $\Psi_{\mathcal{S}}(\mathcal{B})$.

Example 4.10. Let $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \mathcal{B}_3 \sqcup \mathcal{B}_4 \sqcup \mathcal{B}_5$ be a semisimple $\mathcal{S}$-act with five simple components. Then the set $\mathcal{R} = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4 \sqcup \mathcal{B}_5\}$ is an indecomposable controlling set of $\Psi_{\mathcal{S}}(\mathcal{B})$.

Proposition 4.11. Let $\widehat{\mathcal{B}_p}$ be a semisimple $\mathcal{S}$-act with n components, where $n > 1$. Then $\Delta(\Psi_{\mathcal{S}}(\mathcal{B})) \leq n - 1$.

Proof. Write $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_n$. If $n = 2$, then $\{\widehat{\mathcal{B}_p}\}$ is a controlling set, so $\Delta(\Psi_{\mathcal{S}}(\mathcal{B})) \leq 1$. Assume $n > 2$ and set $\mathbb{A} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{n-2}, \mathcal{B}_{n-1} \sqcup \mathcal{B}_n\}$. Let $\mathcal{Z}$ be a vertex not belonging to $\mathbb{A}$. If $\mathcal{Z}$ contains some component $\mathcal{B}_j$ with $1 \leq j \leq n-2$, then $\mathcal{Z}$ is adjacent to $\mathcal{B}_j \in \mathbb{A}$. Otherwise, $\mathcal{Z}$ contains no component among $\mathcal{B}_1, \ldots, \mathcal{B}_{n-2}$. Since $\mathcal{Z} \notin \mathbb{A}$, it is either $\mathcal{B}_{n-1}$ or $\mathcal{B}_n$, and in both cases, it is adjacent to $\mathcal{B}_{n-1} \sqcup \mathcal{B}_n \in \mathbb{A}$. Therefore $\mathbb{A}$ is a controlling set of cardinality $n-1$, and so $\Delta(\Psi_{\mathcal{S}}(\mathcal{B})) \leq n - 1$. □

Theorem 4.12. Let $\widehat{\mathcal{B}_p}$ be a semisimple $\mathcal{S}$-act with n components, where $n > 1$. Then $\Delta(\Psi_{\mathcal{S}}(\mathcal{B})) = 1$ when $n = 2$.

Proof. If $n = 2$, then $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2$ and the vertices of $\Psi_{\mathcal{S}}(\mathcal{B})$ are exactly $\mathcal{B}_1, \mathcal{B}_2, \widehat{\mathcal{B}_p}$. The graph is the path $\mathcal{B}_1 - \widehat{\mathcal{B}_p} - \mathcal{B}_2$. The singleton $\{\widehat{\mathcal{B}_p}\}$ is a controlling set, so $\Delta(\Psi_{\mathcal{S}}(\mathcal{B})) \leq 1$. Since no graph has control number 0, we obtain $\Delta(\Psi_{\mathcal{S}}(\mathcal{B})) = 1$. □

Theorem 4.13. Let $\widehat{\mathcal{B}_p}$ be a semisimple $\mathcal{S}$-act with components $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_n$, where $n > 1$. If $\mathbb{A}$ is a controlling set of $\Psi_{\mathcal{S}}(\mathcal{B})$, then either $\widehat{\mathcal{B}_p} \in \mathbb{A}$ or $\mathbb{A}$ contains at least one simple subact of $\widehat{\mathcal{B}_p}$.

Proof. Let $\mathbb{A}$ be a controlling set of $\Psi_{\mathcal{S}}(\mathcal{B})$. If $\widehat{\mathcal{B}_p} \in \mathbb{A}$, then the claim holds. Suppose that $\widehat{\mathcal{B}_p} \notin \mathbb{A}$. Since $\mathbb{A}$ is a controlling set, the vertex $\widehat{\mathcal{B}_p}$ must be adjacent to some vertex $C \in \mathbb{A}$. Hence $\widehat{\mathcal{B}_p} \cap C$ is a simple subact. But $\widehat{\mathcal{B}_p} \cap C = C$, so C is simple. Therefore $\mathbb{A}$ contains a simple subact of $\widehat{\mathcal{B}_p}$. □

Proposition 4.14. Let $\widehat{\mathcal{B}_p}$ be a semisimple $\mathcal{S}$-act with n simple components. Then the set $\mathbb{J} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n\}$ is a minimal controlling set of $\Psi_{\mathcal{S}}(\mathcal{B})$.

Proof. Let $\mathcal{Z}$ be a vertex of $\Psi_{\mathcal{S}}(\mathcal{B})$ with $\mathcal{Z} \notin \mathbb{J}$. Since $\widehat{\mathcal{B}_p}$ is semisimple, $\mathcal{Z}$ is the coproduct of a nonempty family of simple components. If $\mathcal{Z} \neq \widehat{\mathcal{B}_p}$, then $\mathcal{Z}$ contains some simple component $\mathcal{B}_i$, and $\mathcal{Z} \cap \mathcal{B}_i = \mathcal{B}_i$ is simple. Hence $\mathcal{Z}$ is adjacent to $\mathcal{B}_i \in \mathbb{J}$. If $\mathcal{Z} = \widehat{\mathcal{B}_p}$, then $\widehat{\mathcal{B}_p}$ is adjacent to every simple component $\mathcal{B}_i$. Thus every vertex outside $\mathbb{J}$ is adjacent to some vertex of $\mathbb{J}$, and therefore $\mathbb{J}$ is a controlling set.

To prove minimality, let $\mathcal{B}_i \in \mathbb{J}$. If $\mathcal{B}_i$ is removed from $\mathbb{J}$, then the vertex $\mathcal{B}_i$ is no longer controlled. Indeed, distinct simple vertices are not adjacent and $\widehat{\mathcal{B}_p} \notin \mathbb{J}$. Hence $\mathbb{J}$ is a minimal controlling set. □

Theorem 4.15. Let $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_n$ be a semisimple $\mathcal{S}$-act with $n > 1$ simple components. Then the set $\mathbb{A}_{ij} := \{\mathcal{B}_k : k \neq i, j\} \cup \{\mathcal{B}_i \sqcup \mathcal{B}_j\}$ is an indecomposable controlling set of $\Psi_{\mathcal{S}}(\mathcal{B})$ for every distinct $i, j \in \{1, \ldots, n\}$.

Proof. Fix distinct indices i and j and consider $\mathbb{A}_{ij} = \{\mathcal{B}_k : k \neq i, j\} \cup \{\mathcal{B}_i \sqcup \mathcal{B}_j\}$. We first show that $\mathbb{A}_{ij}$ is a controlling set. Let $\mathcal{Z}$ be any vertex of $\Psi_{\mathcal{S}}(\mathcal{B})$ not belonging to $\mathbb{A}_{ij}$. Since $\widehat{\mathcal{B}_p}$ is semisimple, $\mathcal{Z}$ is the coproduct of a nonempty family of simple components. If $\mathcal{Z}$ contains some component $\mathcal{B}_k$ with $k \neq i, j$, then $\mathcal{B}_k \in \mathbb{A}_{ij}$ and $\mathcal{Z} \cap \mathcal{B}_k = \mathcal{B}_k$ is simple. Hence $\mathcal{Z}$ is adjacent to $\mathcal{B}_k$. Suppose now that $\mathcal{Z}$ contains none of the components $\mathcal{B}_k$ with $k \neq i, j$. Then $\mathcal{Z}$ is a nonempty subact of $\mathcal{B}_i \sqcup \mathcal{B}_j$. Since $\mathcal{Z} \notin \mathbb{A}_{ij}$, it follows that $\mathcal{Z}$ is either $\mathcal{B}_i$ or $\mathcal{B}_j$. In either case, $\mathcal{Z}$ is adjacent to the vertex $\mathcal{B}_i \sqcup \mathcal{B}_j \in \mathbb{A}_{ij}$ because the intersection is simple. Therefore every vertex outside $\mathbb{A}_{ij}$ is adjacent to some vertex of $\mathbb{A}_{ij}$ and so $\mathbb{A}_{ij}$ is a controlling set.

It remains to prove that $\mathbb{A}_{ij}$ is indecomposable. Let $\mathcal{Y} \in \mathbb{A}_{ij}$. If $\mathcal{Y} = \mathcal{B}_k$ for some $k \neq i, j$, then after removing $\mathcal{B}_k$ from $\mathbb{A}_{ij}$, the vertex $\mathcal{B}_k$ is no longer controlled. Indeed, $\mathcal{B}_k$ is adjacent only to vertices containing $\mathcal{B}_k$, but the only vertex of $\mathbb{A}_{ij}$ containing $\mathcal{B}_k$ is $\mathcal{B}_k$ itself. If $\mathcal{Y} = \mathcal{B}_i \sqcup \mathcal{B}_j$, then after removing it from $\mathbb{A}_{ij}$, neither $\mathcal{B}_i$ nor $\mathcal{B}_j$ is controlled. For example, $\mathcal{B}_i$ is not adjacent to any $\mathcal{B}_k$ with $k \neq i, j$, since $\mathcal{B}_i \cap \mathcal{B}_k = \varnothing$. Thus removing any vertex from $\mathbb{A}_{ij}$ destroys the controlling property. Therefore $\mathbb{A}_{ij}$ is indecomposable. □

Theorem 4.16. Let $\widehat{\mathcal{B}_p}$ be a semisimple $\mathcal{S}$-act with decomposition $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_n$, where $n > 2$. Then minimal controlling sets of $\Psi_{\mathcal{S}}(\mathcal{B})$ need not all have the same cardinality. In particular, there exists a minimal controlling set of cardinality n and a controlling set of cardinality $n - 1$.

Proof. By Proposition 4.14, the set $\mathbb{J} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n\}$ is a minimal controlling set of $\Psi_{\mathcal{S}}(\mathcal{B})$. Hence there exists a minimal controlling set of cardinality n. Now choose distinct indices i and j and define $\mathbb{A}_{ij} = \{\mathcal{B}_k : k \neq i, j\} \cup \{\mathcal{B}_i \sqcup \mathcal{B}_j\}$. By Theorem 4.15, the set $\mathbb{A}_{ij}$ is an indecomposable controlling set. Its cardinality is $(n-2)+1 = n-1$. Therefore $\Psi_{\mathcal{S}}(\mathcal{B})$ admits controlling sets of different cardinalities, and hence minimal controlling sets need not all have the same cardinality. □

Corollary 4.17. Let $\widehat{\mathcal{B}_p}$ be a semisimple $\mathcal{S}$-act with $n > 1$ simple components. Then $\Psi_{\mathcal{S}}(\mathcal{B})$ has at least $\binom{n}{2}$ distinct indecomposable controlling sets of cardinality $n-1$.

Proof. For each unordered pair $\{i, j\}$ with $1 \leq i < j \leq n$, Theorem 4.15 yields an indecomposable controlling set $\mathbb{A}_{ij} = \{\mathcal{B}_k : k \neq i, j\} \cup \{\mathcal{B}_i \sqcup \mathcal{B}_j\}$ of cardinality $n-1$. If $\{i, j\} \neq \{r, s\}$, then $\mathbb{A}_{ij} \neq \mathbb{A}_{rs}$ because the unique non-simple vertex in $\mathbb{A}_{ij}$ is $\mathcal{B}_i \sqcup \mathcal{B}_j$, whereas the unique non-simple vertex in $\mathbb{A}_{rs}$ is $\mathcal{B}_r \sqcup \mathcal{B}_s$. Hence the sets $\mathbb{A}_{ij}$ are pairwise distinct. Since there are exactly $\binom{n}{2}$ unordered pairs $\{i, j\}$, the graph $\Psi_{\mathcal{S}}(\mathcal{B})$ has at least $\binom{n}{2}$ distinct indecomposable controlling sets of cardinality $n-1$. □

Theorem 4.18. Let $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_n$ be a semisimple $\mathcal{S}$-act with $n > 1$ simple components. Then the set of all simple vertices $\mathbb{S} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n\}$ is a controlling set of $\Psi_{\mathcal{S}}(\mathcal{B})$. Moreover, it is the unique controlling set consisting entirely of simple subacts.

Proof. Let $\mathcal{Z}$ be any vertex of $\Psi_{\mathcal{S}}(\mathcal{B})$. Since $\widehat{\mathcal{B}_p}$ is semisimple, $\mathcal{Z}$ contains at least one simple component, say $\mathcal{B}_i$. Then $\mathcal{B}_i \in \mathbb{S}$ and $\mathcal{Z} \cap \mathcal{B}_i = \mathcal{B}_i$ is simple. Hence $\mathcal{Z}$ is adjacent to $\mathcal{B}_i$ unless $\mathcal{Z} = \mathcal{B}_i$ itself, and in that case, $\mathcal{Z} \in \mathbb{S}$. Thus every vertex outside $\mathbb{S}$ is adjacent to some vertex of $\mathbb{S}$, so $\mathbb{S}$ is a controlling set.

Now let $\mathbb{T}$ be any controlling set consisting entirely of simple subacts. We show that $\mathbb{T} = \mathbb{S}$. Let $\mathcal{B}_i$ be any simple component. If $\mathcal{B}_i \notin \mathbb{T}$, then since $\mathbb{T}$ consists only of simple vertices, every vertex of $\mathbb{T}$ is a simple component distinct from $\mathcal{B}_i$. But two distinct simple components are never adjacent, because their intersection is empty. Therefore $\mathcal{B}_i$ would not be controlled by $\mathbb{T}$, contradicting the assumption that $\mathbb{T}$ is a controlling set. Hence $\mathcal{B}_i \in \mathbb{T}$ for every i, and so $\mathbb{S} \subseteq \mathbb{T}$. Since every element of $\mathbb{T}$ is a simple subact, it must be one of the $\mathcal{B}_i$, and thus $\mathbb{T} \subseteq \mathbb{S}$. Therefore $\mathbb{T} = \mathbb{S}$. □

Theorem 4.19. Let $\widehat{\mathcal{B}_p} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_n$ be a semisimple $\mathcal{S}$-act with $n > 1$ simple components. Let $\mathbb{A}$ be a controlling set of cardinality $n-1$ containing exactly one non-simple vertex. Then $\mathbb{A}$ is of the form $\mathbb{A}_{ij} = \{\mathcal{B}_k : k \neq i, j\} \cup \{\mathcal{B}_i \sqcup \mathcal{B}_j\}$ for some distinct $i, j \in \{1, \ldots, n\}$.

Proof. Let $\mathbb{A}$ be a controlling set of cardinality $n-1$ containing exactly one non-simple vertex, say C. Since all remaining $n-2$ vertices of $\mathbb{A}$ are simple, they must be simple components of $\widehat{\mathcal{B}_p}$. Hence there exist distinct indices i and j such that $\{\mathcal{B}_k : k \neq i, j\} \subseteq \mathbb{A}$, and the only component not represented by a simple vertex of $\mathbb{A}$ are $\mathcal{B}_i$ and $\mathcal{B}_j$. We claim that $C = \mathcal{B}_i \sqcup \mathcal{B}_j$. Since $\mathbb{A}$ is a controlling set, the simple vertex $\mathcal{B}_i$ must be controlled by some element of $\mathbb{A}$. It cannot be controlled by any simple vertex $\mathcal{B}_k$ with $k \neq i$, because distinct simple vertices are not adjacent. Therefore $\mathcal{B}_i$ must be adjacent to C, and hence $\mathcal{B}_i \subseteq C$. Similarly, $\mathcal{B}_j$ must also be adjacent to C, so $\mathcal{B}_j \subseteq C$.

Suppose that C contains some further component $\mathcal{B}_k$ with $k \neq i, j$. Then $C \cap \mathcal{B}_k = \mathcal{B}_k$ is simple, so C is adjacent to $\mathcal{B}_k$. This is not impossible by itself, but it shows that C is larger than necessary. We now use the cardinality hypothesis to show that no such extra component may occur. Since $\mathbb{A}$ has cardinality $n-1$, its simple vertices are exactly $\mathcal{B}_k$ for $k \neq i, j$. If C contained an extra component $\mathcal{B}_k$, then removing $\mathcal{B}_k$ from $\mathbb{A}$ would still leave a controlling set of cardinality $n-2$, because every vertex formerly controlled through $\mathcal{B}_k$ would still be controlled through C. This contradicts the fact that all $n-1$ vertices of $\mathbb{A}$ are needed to account for the n simple components with only one non-simple substitute. Hence C contains no component other than $\mathcal{B}_i$ and $\mathcal{B}_j$. Therefore $C = \mathcal{B}_i \sqcup \mathcal{B}_j$, and so $\mathbb{A} = \{\mathcal{B}_k : k \neq i, j\} \cup \{\mathcal{B}_i \sqcup \mathcal{B}_j\} = \mathbb{A}_{ij}$. □

Corollary 4.20. Let $\widehat{\mathcal{B}_p}$ be a semisimple $\mathcal{S}$-act with $n > 1$ simple components. Then every controlling set of cardinality $n-1$ containing exactly one non-simple vertex is indecomposable.

Proof. Let $\mathbb{A}$ be a controlling set of cardinality $n-1$ containing exactly one non-simple vertex. By Theorem 4.19, we have $\mathbb{A} = \mathbb{A}_{ij}$ for some distinct i, j. By Theorem 4.15, every set $\mathbb{A}_{ij}$ is indecomposable. Hence $\mathbb{A}$ is indecomposable. □

Theorem 4.21. Let $\widehat{\mathcal{B}_p}$ be a semisimple $\mathcal{S}$-act with $n > 1$ simple components. Then the family of controlling sets of the form $\mathbb{A}_{ij}$ is closed under relabelling of the simple components. In particular, any permutation of the simple components induces a permutation of the $\binom{n}{2}$ indecomposable controlling sets described in Theorem 4.15.

Proof. Let σ be a permutation of $\{1, \dots, n\}$. It induces a relabelling of the simple components by $\mathcal{B}_i \mapsto \mathcal{B}_{\sigma(i)}$. Under this relabelling, the set $\mathbb{A}_{ij} = \{\mathcal{B}_k : k \neq i, j\} \cup \{\mathcal{B}_i \sqcup \mathcal{B}_j\}$ is sent to $\{\mathcal{B}_{\sigma(k)} : \sigma(k) \neq \sigma(i), \sigma(j)\} \cup \{\mathcal{B}_{\sigma(i)} \sqcup \mathcal{B}_{\sigma(j)}\} = \mathbb{A}_{\sigma(i)\sigma(j)}$. By Theorem 4.15, $\mathbb{A}_{\sigma(i)\sigma(j)}$ is again an indecomposable controlling set. Hence the family $\{\mathbb{A}_{ij} : 1 \leq i < j \leq n\}$ is closed under relabelling. Since distinct pairs $\{i, j\}$ give distinct sets $\mathbb{A}_{ij}$ by Corollary 4.17, the permutation σ induces a permutation of this family. □

5. On monophonic relations in $\Psi_{\mathcal{S}}(\mathcal{B})$

This section turns to chordless paths and the corresponding monophonic notions in $\Psi_{\mathcal{S}}(\mathcal{B})$. Since monophonic paths capture a stricter form of connectivity than ordinary paths, they provide finer information about how simple intersections can occur without creating chords. We investigate how the action theory decomposition of $\mathcal{B}$ and the distribution of simple subacts influence the existence and behavior of monophonic paths, and we derive consequences for path based invariants in $\Psi_{\mathcal{S}}(\mathcal{B})$. Let $\mathcal{S}$ and $\mathcal{B}$ be any two topological semigroups. Let $\widehat{\mathcal{B}_p}$ be an $\mathcal{S}$-act. A subact $\mathcal{B}_1$ in $\Psi_{\mathcal{S}}(\mathcal{B})$ is called an m-subfact of a subfact $\mathcal{B}_2$ if the length of a longest monophonic path between $\mathcal{B}_1$ and $\mathcal{B}_2$ equals 2. For $\mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$, the m-open neighbor $\Omega^m(\mathcal{Z})$ of $\mathcal{Z}$ is a set of all m-subfacts of $\mathcal{Z}$, and the m-closed neighbor $\Omega^m[\mathcal{Z}]$ of $\mathcal{Z}$ is $\Omega^m[\mathcal{Z}] = \Omega^m(\mathcal{Z}) \cup \{\mathcal{Z}\}$. Define the collection of subsets of $\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))$ as follows:

$$\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B})) = \{\Omega^m[\mathcal{Z}] : \mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))\}.$$

The m-topological space (shortly, m-space) of $\Psi_{\mathcal{S}}(\mathcal{B})$ is a pair $(\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})), T_{\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))})$, where $T_{\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))}$ is generated by a subbasis $\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))$.

Example 5.1. Let $\mathcal{S} = \{1\}$ be a trivial topological semigroup and $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ with discrete topological space. It is clear that $\widehat{(\mathbb{Z}_4)_p}$ is an $\mathcal{S}$-act, where $1 \cdot \widehat{x} = \widehat{x}$ for all $\widehat{x} \in \widehat{(\mathbb{Z}_4)_p}$ and the non-trivial subacts are all nonempty proper subsets of $\widehat{(\mathbb{Z}_4)_p}$. That is,

$$\mathcal{V}[\Psi_{\mathcal{S}}(\mathbb{Z}_4)] = P\left(\widehat{(\mathbb{Z}_4)_p}\right) \setminus \left\{\emptyset, \widehat{(\mathbb{Z}_4)_p}\right\}.$$

The graph $\Psi_{\mathcal{S}}(\mathbb{Z}_4))$ in Figure 2 that has the subfacts set $\mathcal{V}(\Psi_{\mathcal{S}}(\mathbb{Z}_4))$ has the subbasis

$$\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathbb{Z}_4)) = P\left(\widehat{(\mathbb{Z}_4)}_p\right) \setminus \left\{\emptyset, \widehat{(\mathbb{Z}_4)}_p\right\}.$$

That is, $T_{\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathbb{Z}_4))}$ is discrete.

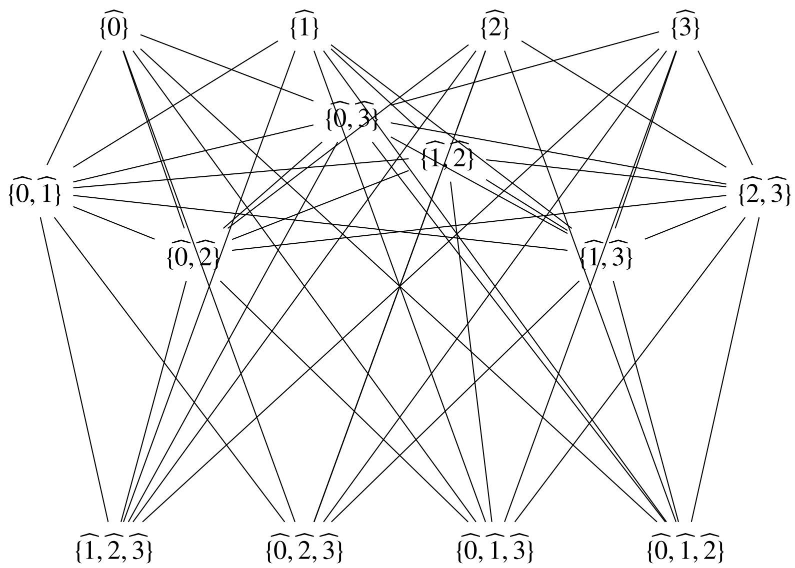


Figure 2. The Ψ-intersection graph $\Psi_{\mathcal{S}}(\mathbb{Z}_4)$.

Example 5.2. In Example 2.2, the graph $\Psi_{\mathcal{S}}(\mathbb{Z}_3)$ in Figure 1a has the subfacts set

$$\mathcal{V}[\Psi_{\mathcal{S}}(\mathbb{Z}_3)] = \{\{\widehat{0}\}, \{\widehat{1}\}, \{\widehat{2}\}, \{\widehat{0},\widehat{1}\}, \{\widehat{0},\widehat{2}\}, \{\widehat{1},\widehat{2}\}\}$$

and the subbasis

$$\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathbb{Z}_3)) = \Big\{\{\{\widehat{0}\}, \{\widehat{1}\}, \{\widehat{1},\widehat{2}\}\}, \{\{\widehat{1}\}, \{\widehat{2}\}, \{\widehat{0},\widehat{1}\}\}, \{\{\widehat{0}\}, \{\widehat{1}\}, \{\widehat{2}\}, \{\widehat{0},\widehat{2}\}\}, \{\{\widehat{1}\}, \{\widehat{0},\widehat{2}\}\},$$

$$\{\{\widehat{2}\}, \{\widehat{0},\widehat{1}\}\}, \{\{\widehat{0}\}, \{\widehat{1},\widehat{2}\}\}\Big\}.$$

In Example 2.6, the graph $\Psi_{\widehat{\mathcal{S}}_p}(\mathcal{S}_p)$ in Figure 2 that has the subfacts set $\mathcal{V}[\Psi_{\widehat{\mathcal{S}}_p}(\mathcal{S}_p)] = \{I_{\widehat{k}} : k = 2, 3, \ldots, n\}$ has the subbasis

$$\mathbb{N}^m(\Psi_{\widehat{\mathcal{S}}_p}(\mathcal{S}_p)) = \left\{\{I_{\widehat{n}}\}, \{I_{\widehat{k}} : k = 2, 3, \ldots, n-1\}\right\}.$$

That is,

$$T_{\mathbb{N}^m(\Psi_{\widehat{\mathcal{S}}_p}(\mathcal{S}_p))} = \left\{\emptyset, \mathcal{V}[\Psi_{\widehat{\mathcal{S}}_p}(\mathcal{S}_p)], \{I_{\widehat{n}}\}, \{I_{\widehat{k}} : k = 2, 3, \ldots, n-1\}\right\}.$$

Let $\Psi_{\mathcal{S}}(\mathcal{B})$ be any graph and any $\mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$ be an isolated subfact. Note that the set $\{\mathcal{Z}\}$ is open in $(\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})), T_{\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))})$. The sets $\{\mathcal{Z}\}$ and $\{\mathcal{Z}'\}$ are open sets for any isolated edge $\mathcal{Z}\mathcal{Z}' \in \mathcal{E}(\Psi_{\mathcal{S}}(\mathcal{B}))$. A topology in which intersection of any numbers of open sets is also an open set is called an Alexandroff topology. In the following results, we assume that $\Psi_{\mathcal{S}}(\mathcal{B})$ has locally finite property, that is, every vertex has finite degree.

Theorem 5.3. In $\Psi_{\mathcal{S}}(\mathcal{B}) = (\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})), \mathcal{E}(\Psi_{\mathcal{S}}(\mathcal{B})))$, the m-space $(\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})), T_{\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))})$ has an Alexandroff property.

Proof. Let $\mathcal{F} = \{\Omega^m[\mathcal{Z}] : \mathcal{Z} \in F \subseteq \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))\}$ be any family in $\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))$. We show that $\cap_{\mathcal{Z}\in F}\Omega^m[\mathcal{Z}]$ is an open set. If $\mathcal{Z}' \in \cap_{\mathcal{Z}\in F}\Omega^m[\mathcal{Z}]$ then, $\mathcal{Z}' \in \Omega^m[\mathcal{Z}]$ for all $\mathcal{Z} \in F$. Hence $\mathcal{Z} \in \Omega^m[\mathcal{Z}']$ for all $\mathcal{Z} \in F$. That is, $F \subseteq \Omega^m[\mathcal{Z}']$. By locally finite property of $\Psi_{\mathcal{S}}(\mathcal{B})$, we get $\cap_{\mathcal{Z}\in F}\Omega^m[\mathcal{Z}]$ is an open set. □

By using the theorem above, we can put $\mathbb{O}(\mathcal{Z})$ to denote the smallest open set where $\mathcal{Z} \in \mathbb{O}(\mathcal{Z})$. Note that if the subfact $\mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$ is an isolated subfact in a graph $\Psi_{\mathcal{S}}(\mathcal{B}) = (\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})), \mathcal{E}(\Psi_{\mathcal{S}}(\mathcal{B})))$, then $\mathbb{O}(\mathcal{Z}) = \{\mathcal{Z}\}$. Also, $\mathbb{O}(\mathcal{Z}) = \{\mathcal{Z}\}$ and $\mathbb{O}(\mathcal{Z}') = \{\mathcal{Z}'\}$ if the edge γ is an isolated edge with $J_\gamma = \{\mathcal{Z}, \mathcal{Z}'\}$.

Theorem 5.4. $\mathbb{O}(\mathcal{Z}) = \cap_{\mathcal{Z}'\in\Omega^m[\mathcal{Z}]}\Omega^m[\mathcal{Z}']$ for all $\mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$.

Proof. By the concept of $\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))$, $\Omega^m[\mathcal{Z}]$ is open set in $(\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})), T_{\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))})$ for all $\mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$. From Theorem 5.3, $\cap_{\mathcal{Z}'\in\Omega^m[\mathcal{Z}]}\Omega^m[\mathcal{Z}']$ is an open set. If $\mathcal{Z}' \in \Omega^m[\mathcal{Z}]$, then it is clear that $\mathcal{Z} \in \Omega^m[\mathcal{Z}']$. Hence $\cap_{\mathcal{Z}'\in\Omega^m[\mathcal{Z}]}\Omega^m[\mathcal{Z}']$ is an open set involving $\mathcal{Z}$. By the concept of $\mathbb{O}(\mathcal{Z})$, we get $\mathbb{O}(\mathcal{Z}) \subseteq \cap_{\mathcal{Z}'\in\Omega^m[\mathcal{Z}]}\Omega^m[\mathcal{Z}']$. Let $\mathbb{O}(\mathcal{Z}) = \cap_{\mathcal{Z}'\in F}\Omega^m[\mathcal{Z}']$ for some subsets F of $\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$. Then $\mathcal{Z} \in \Omega^m[\mathcal{Z}']$ for all $\mathcal{Z}' \in F$. This implies $\mathcal{Z}' \in \Omega^m[\mathcal{Z}]$ for all $\mathcal{Z}' \in F$. Then $F \subseteq \Omega^m[\mathcal{Z}]$. Therefore

$$\cap_{\mathcal{Z}'\in\Omega^m[\mathcal{Z}]}\Omega^m[\mathcal{Z}'] \subseteq \cap_{\mathcal{Z}'\in F}\Omega^m[\mathcal{Z}'] = \mathbb{O}(\mathcal{Z}).$$

So, $\mathbb{O}(\mathcal{Z}) = \cap_{\mathcal{Z}'\in\Omega^m[\mathcal{Z}]}\Omega^m[\mathcal{Z}']$. □

Theorem 5.5. Let $\mathcal{Z}, \mathcal{Z}' \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$. Then $\mathcal{Z} \in \mathbb{O}(\mathcal{Z}')$ if and only if $\Omega^m[\mathcal{Z}'] \subseteq \Omega^m[\mathcal{Z}]$.

Proof. Let $\mathcal{Z} \in \mathbb{O}(\mathcal{Z}')$. From theorem above, $\Omega^m[\mathcal{Z}'] = \cap_{\mathcal{Z}''\in\Omega^m[\mathcal{Z}']}\Omega^m[\mathcal{Z}'']$. Then

$$\mathcal{Z} \in \cap_{\mathcal{Z}''\in\Omega^m[\mathcal{Z}']}\Omega^m[\mathcal{Z}''].$$

That is, $\mathcal{Z} \in \Omega^m[\mathcal{Z}'']$ for all $\mathcal{Z}'' \in \Omega^m[\mathcal{Z}']$. Hence $\mathcal{Z}'' \in \Omega^m[\mathcal{Z}]$ for all $\mathcal{Z}'' \in \Omega^m[\mathcal{Z}']$. Then $\Omega^m[\mathcal{Z}'] \subseteq \Omega^m[\mathcal{Z}]$.

Conversely, let $\Omega^m[\mathcal{Z}'] \subseteq \Omega^m[\mathcal{Z}]$. Hence

$$\mathcal{Z} \in \cap_{\mathcal{Z}''\in\Omega^m[\mathcal{Z}]}\Omega^m[\mathcal{Z}''] \subseteq \cap_{\mathcal{Z}''\in\Omega^m[\mathcal{Z}']}\Omega^m[\mathcal{Z}''] = \mathbb{O}(\mathcal{Z}').$$

That is, $\mathcal{Z} \in \mathbb{O}(\mathcal{Z}')$. □

Theorem 5.6. In the graph $\Psi_{\mathcal{S}}(\mathcal{B})$, $T_{\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))}$ is a discrete topology if and only if $\Omega^m[\mathcal{Z}'] \nsubseteq \Omega^m[\mathcal{Z}]$ and $\Omega^m[\mathcal{Z}] \nsubseteq \Omega^m[\mathcal{Z}']$ for all $\mathcal{Z}, \mathcal{Z}' \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$.

Proof. Let $(\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})), T_{\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))})$ be a discrete. Hence $\Omega^m[\mathcal{Z}] = \{\mathcal{Z}\}$ for all $\mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$. Then for $\mathcal{Z} \neq \mathcal{Z}' \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$, $\mathcal{Z}' \notin \Omega^m[\mathcal{Z}]$ and $\mathcal{Z} \notin \Omega^m[\mathcal{Z}']$. Hence $\Omega^m[\mathcal{Z}'] \nsubseteq \Omega^m[\mathcal{Z}]$ and $\Omega^m[\mathcal{Z}] \nsubseteq \Omega^m[\mathcal{Z}']$. Conversely, let $\mathcal{Z}$ be any subfact in $\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$. Note that $\mathcal{Z} \in \Omega^m[\mathcal{Z}]$. If $\mathcal{Z}' \neq \mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$ and $\mathcal{Z}' \in \Omega^m[\mathcal{Z}]$, then by Theorem 5.5, $\Omega^m[\mathcal{Z}] \subseteq \Omega^m[\mathcal{Z}']$, and this contradicts the above hypothesis. Then $\mathbb{O}(\mathcal{Z}) = \{\mathcal{Z}\}$ for all $\mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$. That is, $(\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})), T_{\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))})$ is a discrete. □

Let $F \subseteq \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$. In $(\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})), T_{\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))})$, $\overline{F}$ denotes the closure set of F, the smallest closed sets involving F. Note that $\mathcal{Z} \in \overline{F}$ if and only if $H \cap F \neq \emptyset$ for any open H involving $\mathcal{Z}$. A set F is said to be a dense in A if $\overline{F} = A$, that is, for all open sets O, $F \cap O \neq \emptyset$.

Theorem 5.7. Let $\mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$. Then $\overline{\Omega^m[\mathcal{Z}]} \subseteq \overline{\Omega^m[\mathcal{Z}']}$ for all $\mathcal{Z}' \in \Omega^m[\mathcal{Z}]$.

Proof. Let $\mathcal{Z}'' \in \overline{\Omega^m[\mathcal{Z}]}$. Then for every open set F containing $\mathcal{Z}''$, $F \cap \Omega^m[\mathcal{Z}] \neq \emptyset$. Since $\Omega^m[\mathcal{Z}] \subseteq \Omega^m[\mathcal{Z}']$ for all $\mathcal{Z}' \in \Omega^m[\mathcal{Z}]$, then $F \cap \Omega^m[\mathcal{Z}'] \neq \emptyset$ for all $\mathcal{Z}' \in \Omega^m[\mathcal{Z}]$. Then $\mathcal{Z}'' \in \overline{\Omega^m[\mathcal{Z}']}$ for all $\mathcal{Z}' \in \Omega^m[\mathcal{Z}]$. Hence $\overline{\Omega^m[\mathcal{Z}]} \subseteq \overline{\Omega^m[\mathcal{Z}']}$ for all $\mathcal{Z}' \in \Omega^m[\mathcal{Z}]$. □

Corollary 5.8. Let $\mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$. Then $\overline{\{\mathcal{Z}\}} \subseteq \overline{\Omega^m[\mathcal{Z}']}$ for all $\mathcal{Z}' \in \Omega^m[\mathcal{Z}]$.

Corollary 5.9. Let $\mathcal{Z}, \mathcal{Z}' \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$. Then $\mathcal{Z} \in \overline{\{\mathcal{Z}'\}}$ if and only if $\Omega^m[\mathcal{Z}] \subseteq \Omega^m[\mathcal{Z}']$.

Theorem 5.10. Let $F \subseteq \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$. Then $\overline{F} = \{\mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})) : \mathbb{O}(\mathcal{Z}) \cap F \neq \emptyset\} = \{\mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})) : \Omega^m[\mathcal{Z}] \subseteq \Omega^m[C] \text{ for some } C \in F\}$.

Proof. Let $\mathcal{Z} \in \overline{F}$. By the definition of closure, every open set containing $\mathcal{Z}$ meets F. In particular, the smallest open set $\mathbb{O}(\mathcal{Z})$ meets F. Hence $\mathbb{O}(\mathcal{Z}) \cap F \neq \emptyset$. Conversely, assume that $\mathbb{O}(\mathcal{Z}) \cap F \neq \emptyset$. Let $C \in \mathbb{O}(\mathcal{Z}) \cap F$. If H is any open set containing $\mathcal{Z}$, then by minimality of $\mathbb{O}(\mathcal{Z})$ we have $\mathbb{O}(\mathcal{Z}) \subseteq H$. Therefore $C \in H \cap F$, so $H \cap F \neq \emptyset$. Hence $\mathcal{Z} \in \overline{F}$. Now let $\mathcal{Z} \in \overline{F}$. By the first part there exists $C \in \mathbb{O}(\mathcal{Z}) \cap F$. By Theorem 5.5, the relation $C \in \mathbb{O}(\mathcal{Z})$ is equivalent to $\Omega^m[\mathcal{Z}] \subseteq \Omega^m[C]$. Thus $\overline{F} \subseteq \{\mathcal{Z} : \Omega^m[\mathcal{Z}] \subseteq \Omega^m[C] \text{ for some } C \in F\}$. Conversely, if $\Omega^m[\mathcal{Z}] \subseteq \Omega^m[C]$ for some $C \in F$, then by Theorem 5.5, we get $C \in \mathbb{O}(\mathcal{Z})$. Hence $\mathbb{O}(\mathcal{Z}) \cap F \neq \emptyset$ and therefore $\mathcal{Z} \in \overline{F}$. This proves the second equality. □

Theorem 5.11. Define a binary relation $\leq_m$ on $\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$ by $\mathcal{Z} \leq_m C \iff \mathcal{Z} \in \overline{\{C\}}$. Then $\leq_m$ is a preorder on $\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$, and $\mathcal{Z} \leq_m C \iff \Omega^m[\mathcal{Z}] \subseteq \Omega^m[C]$.

Proof. By Corollary 5.9, $\mathcal{Z} \in \overline{\{C\}} \iff \Omega^m[\mathcal{Z}] \subseteq \Omega^m[C]$, so the stated equivalence holds. To prove reflexivity, note that $\Omega^m[\mathcal{Z}] \subseteq \Omega^m[\mathcal{Z}]$ for every $\mathcal{Z}$. Hence $\mathcal{Z} \leq_m \mathcal{Z}$. To prove transitivity, suppose $\mathcal{Z} \leq_m C$ and $C \leq_m \mathcal{D}$. Then $\Omega^m[\mathcal{Z}] \subseteq \Omega^m[C]$ and $\Omega^m[C] \subseteq \Omega^m[\mathcal{D}]$. Therefore, $\Omega^m[\mathcal{Z}] \subseteq \Omega^m[\mathcal{D}]$ and again by Corollary 5.9, we obtain $\mathcal{Z} \leq_m \mathcal{D}$. Thus $\leq_m$ is a preorder. □

Corollary 5.12. For every $\mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$, $\overline{\{\mathcal{Z}\}} = \{C \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})) : C \leq_m \mathcal{Z}\} = \{C \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})) : \Omega^m[C] \subseteq \Omega^m[\mathcal{Z}]\}$.

Proof. The first equality is the definition of the preorder $\leq_m$ in Theorem 5.11. The second equality follows directly from the equivalence established in Theorem 5.11. □

Theorem 5.13. The topology $T_{\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))}$ is a T_0 topology if and only if $\Omega^m[\mathcal{Z}] \neq \Omega^m[C]$ for every two distinct vertices $\mathcal{Z}, C \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$.

Proof. Assume first that $(\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})), T_{\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))})$ is T_0. Suppose there exist distinct vertices $\mathcal{Z}$ and C such that $\Omega^m[\mathcal{Z}] = \Omega^m[C]$. Then by Theorem 5.5, we have $\mathcal{Z} \in \mathbb{O}(C)$ and $C \in \mathbb{O}(\mathcal{Z})$. Hence every open set containing one of them contains the other. This contradicts the T_0 property. Therefore $\Omega^m[\mathcal{Z}] \neq \Omega^m[C]$ whenever $\mathcal{Z} \neq C$. Conversely, assume that $\Omega^m[\mathcal{Z}] \neq \Omega^m[C]$ for all distinct $\mathcal{Z}, C$. Let $\mathcal{Z} \neq C$. Then either $\Omega^m[\mathcal{Z}] \nsubseteq \Omega^m[C]$ or $\Omega^m[C] \nsubseteq \Omega^m[\mathcal{Z}]$. In the first case, by Theorem 5.5, $\mathcal{Z} \notin \mathbb{O}(C)$, so the open set $\mathbb{O}(C)$ contains C but not $\mathcal{Z}$. In the second case, the open set $\mathbb{O}(\mathcal{Z})$ contains $\mathcal{Z}$ but not C. Therefore the space is T_0. □

Theorem 5.14. The following statements are equivalent:

(1) $(\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})), T_{\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))})$ is a T_1 space;
(2) $(\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})), T_{\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))})$ is discrete;
(3) for every two distinct vertices $\mathcal{Z}, C \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$ and neither $\Omega^m[\mathcal{Z}] \subseteq \Omega^m[C]$ nor $\Omega^m[C] \subseteq \Omega^m[\mathcal{Z}]$ holds.

Proof. Assume (1). Let $\mathcal{Z} \neq C$. Since the space is T_1, the singleton $\{C\}$ is closed, so $\mathcal{Z} \notin \overline{\{C\}}$. By Corollary 5.9, this means $\Omega^m[\mathcal{Z}] \nsubseteq \Omega^m[C]$. Interchanging the roles of $\mathcal{Z}$ and C gives $\Omega^m[C] \nsubseteq \Omega^m[\mathcal{Z}]$. Thus (3) holds. Assume (3). Then Theorem 5.6 implies that the topology is discrete. Hence (2) holds. Assume (2). Then every singleton is open and therefore also closed. Hence the space is T_1. Thus (1) holds. □

Theorem 5.15. A subset $F \subseteq \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$ is dense in $(\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})), T_{\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))})$ if and only if for every vertex $\mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$, there exists $C \in F$ such that $\Omega^m[\mathcal{Z}] \subseteq \Omega^m[C]$.

Proof. Assume that F is dense. Then $\overline{F} = \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$. Let $\mathcal{Z}$ be any vertex. Since $\mathcal{Z} \in \overline{F}$, Theorem 5.10 implies that there exists $C \in F$ such that $\Omega^m[\mathcal{Z}] \subseteq \Omega^m[C]$. Conversely, assume that for every $\mathcal{Z}$, there exists $C \in F$ with $\Omega^m[\mathcal{Z}] \subseteq \Omega^m[C]$. By Theorem 5.10, this implies $\mathcal{Z} \in \overline{F}$ for every vertex $\mathcal{Z}$. Hence $\overline{F} = \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$, so F is dense. □

Corollary 5.16. Let $F \subseteq \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$. Then F is dense if and only if $\mathbb{O}(\mathcal{Z}) \cap F \neq \emptyset$ for every $\mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$.

Proof. By Theorem 5.10, for each vertex $\mathcal{Z}$, we have $\mathcal{Z} \in \overline{F} \iff \mathbb{O}(\mathcal{Z}) \cap F \neq \emptyset$. Thus $\overline{F} = \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$ if and only if $\mathbb{O}(\mathcal{Z}) \cap F \neq \emptyset$ for every $\mathcal{Z}$. This is exactly the definition of density. □

Theorem 5.17. Let $F \subseteq \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$. Then F is closed in $(\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})), T_{\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))})$ if and only if whenever $\mathcal{Z} \in F$ and $\Omega^m[C] \subseteq \Omega^m[\mathcal{Z}]$, one has $C \in F$.

Proof. Assume first that F is closed. Let $\mathcal{Z} \in F$ and suppose that $\Omega^m[C] \subseteq \Omega^m[\mathcal{Z}]$. By Corollary 5.9, this implies $C \in \overline{\{\mathcal{Z}\}}$. Since $\{\mathcal{Z}\} \subseteq F$ and F is closed, we have $\overline{\{\mathcal{Z}\}} \subseteq F$. Therefore $C \in F$. Conversely, assume that F satisfies the stated condition. Let $C \in \overline{F}$. By Theorem 5.10, there exists $\mathcal{Z} \in F$ such that $\Omega^m[C] \subseteq \Omega^m[\mathcal{Z}]$. By the assumed condition, this implies $C \in F$. Therefore $\overline{F} \subseteq F$, and hence F is closed. □

Theorem 5.18. Let $F \subseteq \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$. Then the interior of F is given by $\mathrm{Int}(F) = \{\mathcal{Z} \in F : \mathbb{O}(\mathcal{Z}) \subseteq F\} = \{\mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})) : \Omega^m[C] \subseteq \Omega^m[\mathcal{Z}] \Rightarrow C \in F\}$.

Proof. Let $\mathcal{Z} \in \mathrm{Int}(F)$. Then there exists an open set H such that $\mathcal{Z} \in H \subseteq F$. Since $\mathbb{O}(\mathcal{Z})$ is the smallest open set containing $\mathcal{Z}$, we have $\mathbb{O}(\mathcal{Z}) \subseteq H \subseteq F$. Hence $\mathbb{O}(\mathcal{Z}) \subseteq F$. Conversely, if $\mathcal{Z} \in F$ and $\mathbb{O}(\mathcal{Z}) \subseteq F$, then $\mathbb{O}(\mathcal{Z})$ is an open set containing $\mathcal{Z}$ and contained in F. Therefore $\mathcal{Z} \in \mathrm{Int}(F)$. This proves the first equality. For the second equality, let $\mathcal{Z} \in \mathrm{Int}(F)$. By the first equality, $\mathbb{O}(\mathcal{Z}) \subseteq F$. If $\Omega^m[C] \subseteq \Omega^m[\mathcal{Z}]$, then Corollary 5.9 implies that $C \in \overline{\{\mathcal{Z}\}}$, and by Theorem 5.5, this is equivalent to $\mathcal{Z} \in \mathbb{O}(C)$. Since $C \in \mathbb{O}(\mathcal{Z})$, it follows that $C \in F$. Conversely, suppose that every vertex C satisfying $\Omega^m[C] \subseteq \Omega^m[\mathcal{Z}]$ belongs to F. Let $C \in \mathbb{O}(\mathcal{Z})$. Then by Theorem 5.5, we have $\Omega^m[C] \subseteq \Omega^m[\mathcal{Z}]$. Hence $C \in F$. Therefore $\mathbb{O}(\mathcal{Z}) \subseteq F$, and thus $\mathcal{Z} \in \mathrm{Int}(F)$. □

Theorem 5.19. A subset $U \subseteq \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$ is open if and only if for every $\mathcal{Z} \in U$ and every $C \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$ with $\Omega^m[\mathcal{Z}] \subseteq \Omega^m[C]$, one has $C \in U$.

Proof. Assume that U is open and let $\mathcal{Z} \in U$. Since $\mathbb{O}(\mathcal{Z})$ is the smallest open set containing $\mathcal{Z}$, we have $\mathbb{O}(\mathcal{Z}) \subseteq U$. If $\Omega^m[\mathcal{Z}] \subseteq \Omega^m[C]$, then by Theorem 5.5 we get $C \in \mathbb{O}(\mathcal{Z})$. Hence $C \in U$. Conversely, assume that U satisfies the stated condition. Let $\mathcal{Z} \in U$ and let $C \in \mathbb{O}(\mathcal{Z})$. By Theorem 5.5, $C \in \mathbb{O}(\mathcal{Z})$ implies $\Omega^m[\mathcal{Z}] \subseteq \Omega^m[C]$. Therefore $C \in U$. Thus $\mathbb{O}(\mathcal{Z}) \subseteq U$ for every $\mathcal{Z} \in U$. Since $U = \bigcup_{\mathcal{Z}\in U} \mathbb{O}(\mathcal{Z})$, it follows that U is open. □

Corollary 5.20. A subset $F \subseteq \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$ is closed if and only if for every $\mathcal{Z} \in F$ and every $C \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$ with $\Omega^m[C] \subseteq \Omega^m[\mathcal{Z}]$, one has $C \in F$.

Proof. This is exactly Theorem 5.17. □

Theorem 5.21. For every subset $F \subseteq \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$, the boundary of F is given by $\partial F = \{\mathcal{Z} \in \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})) : \mathbb{O}(\mathcal{Z}) \cap F \neq \emptyset$ and $\mathbb{O}(\mathcal{Z}) \cap (\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})) \setminus F) \neq \emptyset\}$.

Proof. By definition, $\partial F = \overline{F} \setminus \mathrm{Int}(F)$. Let $\mathcal{Z} \in \partial F$. Then $\mathcal{Z} \in \overline{F}$, so by Theorem 5.10, we have $\mathbb{O}(\mathcal{Z}) \cap F \neq \emptyset$. Also, $\mathcal{Z} \notin \mathrm{Int}(F)$, so by Theorem 5.18, we have $\mathbb{O}(\mathcal{Z}) \nsubseteq F$. Hence $\mathbb{O}(\mathcal{Z}) \cap (\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})) \setminus F) \neq \emptyset$. Conversely, suppose that $\mathbb{O}(\mathcal{Z})$ meets both F and its complement. Since $\mathbb{O}(\mathcal{Z}) \cap F \neq \emptyset$, Theorem 5.10 implies that $\mathcal{Z} \in \overline{F}$. Since $\mathbb{O}(\mathcal{Z})$ is not contained in F, Theorem 5.18 gives $\mathcal{Z} \notin \mathrm{Int}(F)$. Therefore $\mathcal{Z} \in \overline{F} \setminus \mathrm{Int}(F) = \partial F$. □

Theorem 5.22. The m-space $(\mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B})), T_{\mathbb{N}^m(\Psi_{\mathcal{S}}(\mathcal{B}))})$ is connected if and only if there is no nonempty proper subset $U \subsetneq \mathcal{V}(\Psi_{\mathcal{S}}(\mathcal{B}))$ such that $\Omega^m[\mathcal{Z}] \subseteq \Omega^m[C]$ and $\mathcal{Z} \in U$ imply $C \in U$, while $\Omega^m[C] \subseteq \Omega^m[\mathcal{Z}]$ and $\mathcal{Z} \in U$ imply $C \in U$.

Proof. Assume first that the m-space is disconnected. Then there exists a nonempty proper subset U that is both open and closed. Since U is open, Theorem 5.19 shows that whenever $\mathcal{Z} \in U$ and $\Omega^m[\mathcal{Z}] \subseteq \Omega^m[C]$, one has $C \in U$. Since U is closed, Corollary 5.20 shows that whenever $\mathcal{Z} \in U$ and $\Omega^m[C] \subseteq \Omega^m[\mathcal{Z}]$, one has $C \in U$. Hence U satisfies the stated condition. Conversely, suppose there exists a nonempty proper subset U satisfying the stated condition. The first condition and Theorem 5.19 imply that U is open. The second condition and Corollary 5.20 imply that U is closed. Hence U is a nonempty proper clopen subset, so the m-space is disconnected. □

Corollary 5.23. If the m-space is connected, then every nonempty proper open subset U has a nonempty boundary.

Proof. Let U be a nonempty proper open subset. If $\partial U = \emptyset$, then by Theorem 5.21, every minimal open neighborhood either lies entirely in U or entirely in its complement and in particular $\overline{U} = U$. Hence U is closed. Therefore U is a nonempty proper clopen subset, contradicting connectedness. Thus $\partial U \neq \emptyset$. □

6. Conclusions

In this paper, we introduced the Ψ-intersection graph $\Psi_{\mathcal{S}}(\mathcal{B})$ associated with an $\mathcal{S}$-act over a topological semigroup and examined its structural and combinatorial properties. We showed that the distribution of simple and non-simple subacts plays a key role in determining graph-theoretic features such as girth, bipartiteness, connectivity, diameter, and clique structure. We also studied domination-type parameters through controlling sets and indecomposable controlling families, obtaining bounds for the control number of $\Psi_{\mathcal{S}}(\mathcal{B})$. Furthermore, we explored monophonic properties via the m-topology on graphs and analyzed induced path structures in relation to algebraic paths in $\mathcal{S}$-acts. These results highlight a close connection between semigroup actions and algebraic graph theory, and suggest further study of metric and convexity properties of $\Psi_{\mathcal{S}}(\mathcal{B})$.

Use of Generative-AI tools declaration

The author declares that she has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

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