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Research article

Complete description of the monotonicity of a class of sequences with a parameter

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Abstract: A complete description of the monotonicity character of the convex combinations of the sequences

$$x_n = \sum_{j=1}^{n} \frac{1}{j^\alpha} - \frac{n^{1-\alpha}}{1-\alpha} \quad \text{and} \quad y_n = \sum_{j=1}^{n} \frac{1}{j^\alpha} - \frac{(n+1)^{1-\alpha}}{1-\alpha}, \quad n \in \mathbb{N},$$

when $\alpha \in (0,1)$, on the whole domain $\mathbb{N}$, is given, considerably extending some special cases in the literature, and solving a highly nontrivial open problem completely. An important part in the study plays a few analytic inequalities that have been proved and used in the proof of the result that characterizes the monotonicity character of the sequences.

Keywords: real sequences; monotonicity character of sequences; convex combinations
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1. Introduction

Let $\mathbb{N}$ be the set $\{1,2,3,\ldots\}$, $\mathbb{Z}$ the set of whole numbers, $\mathbb{N}_s = \{i \in \mathbb{Z} : i \geq s\}$ where $s \in \mathbb{Z}$, $\mathbb{R}_+ = [0,+\infty)$, and let

$$C_j^\beta = \frac{\beta(\beta-1)\ldots(\beta-j+1)}{j!},$$

where $\beta \in \mathbb{R}$ and $j \in \mathbb{N}_0$, that is, the generalized binomial coefficients [17, 38].

One of the basic problems connected to real sequences is investigating their monotonicity. If a real sequence is monotone, then, in many cases, its long-term behavior is easier for investigating. For some basic facts on such sequences see, for instance, [12, 18, 38]. Some basic methods for dealing with such sequences can be found, for instance, in [2, 13, 14]. Somewhat harder problems can be found in [1,25,26]. For some recursive relations producing monotone sequences see, for instance, [18,22,27]. For showing the existence of monotone solutions of some recursive relations see, for instance, [3,10,29] (see also [31], which considered the existence of eventually nontrivial solutions of a class of recursive relations of higher order).

It is a rare situation that the monotonicity of a real sequence is described on its whole domain, which is usually a set of the form $\mathbb{N}_s$. This is essentially connected to the fact that mathematicians are usually interested only in the long-term behavior of real sequences such, as the boundedness, convergence, grow rate, etc.

There are many real sequences which are investigated or used in pairs. Some examples can be found, for example, in [11, 18, 36], some appear in comparison theorems [3, 28, 30] (for an old method, see the proofs of Problems 173 and 174 in [24, Chapter 1]), whereas some appear in studying of recursive relations [9, 21, 23].

Many books contain the well-known pair

$$a_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad b_n = \left(1 + \frac{1}{n}\right)^{n+1}, \quad n \in \mathbb{N},$$

(see, e.g., [18, 24, 38]; for some generalizations see [6]), the pair

$$\widehat{a}_n = \sum_{j=n+1}^{2n} \frac{1}{j} \quad \text{and} \quad \widehat{b}_n = \sum_{j=n}^{2n} \frac{1}{j}, \quad n \in \mathbb{N}, \tag{1.1}$$

(see, e.g., [11, 18, 24]; for some generalizations see [4, 36, 37]), the pair

$$\widetilde{\alpha}_n = \sum_{j=1}^{n} \frac{1}{j} - \ln n \quad \text{and} \quad \widetilde{\beta}_n = \sum_{j=1}^{n} \frac{1}{j} - \ln(n+1), \quad n \in \mathbb{N}, \tag{1.2}$$

(see, e.g., [11, 14, 18], the original source [5], as well as some applications [19, 20]), or the pair

$$\alpha_n = \sum_{j=1}^{n} \frac{1}{\sqrt{j}} - 2\sqrt{n} \quad \text{and} \quad \beta_n = \sum_{j=1}^{n} \frac{1}{\sqrt{j}} - 2\sqrt{n+1}, \quad n \in \mathbb{N}, \tag{1.3}$$

(see, e.g., [11, 18, 24]), and it is shown that the sequences are monotone and convergent.

Some related sequences or pairs of sequences can be found in [15] and [24]. The problem in [36] motivated us to conduct an investigation of the monotonicity character of a class of sequences in [33], where we can find two solutions to the problem, an elementary which was also given in [37], and one which is based on the Hermite-Hadamard inequalities [7, 8, 16]. Beside this, in [33] we showed that it is possible to describe the monotonicity character of the sequence

$$c_n^{(\alpha)} = (1-\alpha)\widehat{a}_n + \alpha\widehat{b}_n, \quad n \in \mathbb{N},$$

for each $\alpha \in [0, 1]$, on the whole domain $\mathbb{N}$, and the same fact was proved for the convex combinations of the sequences

$$\widetilde{a}_n = \sum_{j=n+1}^{3n} \frac{1}{j} \quad \text{and} \quad \widetilde{b}_n = \sum_{j=n}^{3n} \frac{1}{j}, \quad n \in \mathbb{N}, \tag{1.4}$$

but in a more technical way.

The investigations [33] suggested us to consider the monotonicity character of the convex combinations of some other pairs of real sequences appearing in the literature. Using the idea, in [34] we studied the convex combinations of the sequences in (1.2), whereas in [32] we studied the convex combinations of the sequences in (1.3), completely describing the monotonicity character of these sequences.

A natural problem is to study the monotonicity character of the convex combinations of the sequences

$$x_n = 1 + \frac{1}{2^\alpha} + \cdots + \frac{1}{n^\alpha} - \frac{n^{1-\alpha}}{1-\alpha}, \tag{1.5}$$

$$y_n = 1 + \frac{1}{2^\alpha} + \cdots + \frac{1}{n^\alpha} - \frac{(n+1)^{1-\alpha}}{1-\alpha}, \tag{1.6}$$

for $n \in \mathbb{N}$, where $\alpha \in (0, 1)$, which are natural generalizations of the sequences in [32], where the case $\alpha = 1/2$ was considered.

In our recent paper [35], we have completely described the monotonicity character of the convex combinations of the sequences (1.5) and (1.6), that is, on the whole domain of the index, in the case $\alpha = 2/3$. However, the proofs of two auxiliary results given in [32] and [35] (i.e., Lemma 1 in [32] and [35]), which are some of the main ingredients of the proofs of the main results therein, cannot be modified to be applicable for other values of the parameter $\alpha \in (0, 1)$. The reason for this lies in the fact that in [32] and [35], there were some tricks and calculations used which can be employed only for power functions with rational powers. Moreover, it is also difficult to modify the tricks and the calculations therein so that such a modified proof holds for the rational powers in the interval $(0, 1)$ different from 1/2 and 2/3. Hence, some other methods must be employed to get the corresponding auxiliary result in the case $\alpha \in (0, 1) \setminus \{1/2, 2/3\}$.

Note that

$$x_n - y_n = \frac{(n+1)^{1-\alpha}}{1-\alpha} - \frac{n^{1-\alpha}}{1-\alpha} > 0,$$

for $n \in \mathbb{N}$,

$$\begin{aligned} x_{n+1} - x_n =& \frac{n^{1-\alpha}}{1-\alpha} - \frac{(n+1)^{1-\alpha}}{1-\alpha} + \frac{1}{(n+1)^\alpha} \\ =& \frac{1}{(n+1)^\alpha} - \frac{1}{\zeta_n^\alpha}, \end{aligned}$$

for $n \in \mathbb{N}$, and $\zeta_n \in (n, n+1)$ such that

$$(n+1)^{1-\alpha} - n^{1-\alpha} = (1-\alpha)\zeta_n^{-\alpha}, \tag{1.7}$$

from which it follows that

$$x_{n+1} < x_n, \quad \text{for} \quad n \in \mathbb{N},$$

and

$$\begin{aligned} y_{n+1} - y_n =& \frac{(n+1)^{1-\alpha}}{1-\alpha} - \frac{(n+2)^{1-\alpha}}{1-\alpha} + \frac{1}{(n+1)^\alpha} \\ =& \frac{1}{(n+1)^\alpha} - \frac{1}{\zeta_{n+1}^\alpha}, \end{aligned}$$

for $n \in \mathbb{N}$, from which it follows that

$$y_{n+1} > y_n, \quad \text{for} \quad n \in \mathbb{N}.$$

Hence,

$$y_1 < y_2 < \cdots < y_n < x_n < \cdots < x_2 < x_1,$$

for $n \in \mathbb{N}$, and, consequently, x_n and y_n are convergent.

Note that

$$x_n - y_n = \frac{(n+1)^{1-\alpha}}{1-\alpha} - \frac{n^{1-\alpha}}{1-\alpha} = \frac{1}{\zeta_n^\alpha}, \tag{1.8}$$

where $\zeta_n \in (n, n+1)$ is defined in (1.7).

By letting $n \to +\infty$ in (1.8), we get

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} y_n := \gamma(\alpha).$$

Now consider the sequence

$$c_n^{(\lambda)} := \lambda x_n + (1-\lambda) y_n, \quad n \in \mathbb{N}, \tag{1.9}$$

where $\lambda \in (0, 1)$.

Obviously $(c_n^{(\lambda)})_{n\in\mathbb{N}}$ converges to $\gamma(\alpha)$ as $n \to +\infty$, and

$$y_1 < y_2 < \cdots < y_n < \gamma(\alpha) < x_n < \cdots < x_2 < x_1$$

for $n \in \mathbb{N}$ (see, e.g., [1, 11, 16]).

Motivated by abovementioned studies, especially by the studies in [32, 33, 35], here we describe the monotonicity character of the sequences defined in (1.9), for each value $\alpha \in (0, 1)$, on the whole domain $\mathbb{N}$, considerably extending the special cases considered in the literature and solving the monotonicity problem for the sequences completely.

Before this, let us describe the monotonicity character of the sequences defined in (1.9) for sufficiently large values of the indices. Employing the formula

$$(1+x)^\beta = 1 + C_1^\beta x + C_2^\beta x^2 + C_3^\beta x^3 + O(x^4)$$

as $x \to 0$ ([14, 38]), we have

$$c_{n+1}^{(\lambda)} - c_n^{(\lambda)} = \lambda(x_{n+1} - x_n) + (1-\lambda)(y_{n+1} - y_n)$$

$$
\begin{aligned}
=&\frac{1}{(n+1)^{\alpha}}-\frac{1}{1-\alpha}\Big(\lambda(n+1)^{1-\alpha}+(1-\lambda)(n+2)^{1-\alpha}-\lambda n^{1-\alpha}-(1-\lambda)(n+1)^{1-\alpha}\Big)\\
=&\frac{1}{n^{\alpha}}\Big(1+\frac{1}{n}\Big)^{-\alpha}-\frac{n^{1-\alpha}}{1-\alpha}\Big(\lambda\Big(1+\frac{1}{n}\Big)^{1-\alpha}+(1-\lambda)\Big(1+\frac{2}{n}\Big)^{1-\alpha}-\lambda-(1-\lambda)\Big(1+\frac{1}{n}\Big)^{1-\alpha}\Big)\\
=&\frac{1}{n^{\alpha}}\Big(1-\frac{\alpha}{n}+\frac{\alpha(\alpha+1)}{2n^2}+O\Big(\frac{1}{n^3}\Big)\Big)-\frac{n^{1-\alpha}}{1-\alpha}\Big(\lambda\Big(1+\frac{1-\alpha}{n}-\frac{(1-\alpha)\alpha}{2n^2}+O\Big(\frac{1}{n^3}\Big)\Big)\\
&+(1-\lambda)\Big(1+\frac{2(1-\alpha)}{n}-\frac{2(1-\alpha)\alpha}{n^2}+O\Big(\frac{1}{n^3}\Big)\Big)-\lambda-(1-\lambda)\Big(1+\frac{1-\alpha}{n}-\frac{(1-\alpha)\alpha}{2n^2}+O\Big(\frac{1}{n^3}\Big)\Big)\Big)\\
=&\frac{\alpha(\frac{1}{2}-\lambda)}{n^{\alpha+1}}+O\Big(\frac{1}{n^{\alpha+2}}\Big).
\end{aligned}\tag{1.10}
$$

From (1.10) we immediately get that $c_n^{(\lambda)}$ is eventually increasing for $\lambda\in[0,\frac{1}{2})$, and eventually decreasing if $\lambda\in(\frac{1}{2},1]$.

Further, we have

$$
c_{n+1}^{(1/2)}-c_n^{(1/2)}=-\frac{\alpha(\alpha+1)}{6n^{\alpha+2}}+O\Big(\frac{1}{n^{\alpha+3}}\Big),
$$

so $c_n^{(1/2)}$ is eventually decreasing.

Hence, $c_n^{(\lambda)}$ is eventually decreasing for $\lambda\in[1/2,1]$, and is eventually increasing for $\lambda\in[0,1/2)$.

2. Main results

First, we prove a lemma which is necessary for proving the result that characterizes the monotonicity character of the sequences. The lemma is highly nontrivial, so it can be also characterized as one of the main results in the paper, although it is on another topic (it is essentially on analytic inequalities).

Lemma 2.1. *Let* $\alpha\in(0,1)$ *and*

$$
g_\alpha(t):=\frac{(1-\alpha)(t+1)^{-\alpha}-(t+2)^{1-\alpha}+(t+1)^{1-\alpha}}{2(t+1)^{1-\alpha}-(t+2)^{1-\alpha}-t^{1-\alpha}}.\tag{2.1}
$$

Then,

$$
\lim_{t\to+\infty}g_\alpha(t)=\frac{1}{2},\tag{2.2}
$$

and the function g_α *is positive and strictly increasing on* $\mathbb{R}_+$.

Proof. Since the function $t^{1-\alpha}$ is (strictly) concave on $\mathbb{R}_+$ when $\alpha\in(0,1)$, then we have

$$
2(t+1)^{1-\alpha}-(t+2)^{1-\alpha}-t^{1-\alpha}>0,\tag{2.3}
$$

for every $t\in\mathbb{R}_+$.

Now note that

$$
g_\alpha(0)=\frac{2-\alpha-2^{1-\alpha}}{2-2^{1-\alpha}}=\frac{2^{\alpha+1}-\alpha2^{\alpha}-2}{2^{\alpha+1}-2}.\tag{2.4}
$$

Let

$$
f(t)=2^{t+1}-t2^{t}-2.
$$

Then, $f(0) = f(1) = 0$ and

$$\begin{aligned} f'(t) =& 2^{t+1}\ln 2 - 2^t - t2^t \ln 2 \\ =& 2^t\Big(\ln\frac{4}{e} - t\ln 2\Big). \end{aligned}$$

Hence, $f'(t) = 0$ for

$$t_0 = \frac{\ln\frac{4}{e}}{\ln 2}.$$

It is not difficult to see that

$$0 = \ln 1 < \ln\frac{4}{e} < \ln 2,$$

so $t_0 \in (0, 1)$. Since $f'(t) > 0$ for $t \in (0, t_0)$ and $f'(t) < 0$ for $t \in (t_0, 1)$, we have $f(t) > 0$ for $t \in (0, 1)$.

From this and (2.4), we have

$$g_\alpha(0) > 0 \quad \text{for} \quad \alpha \in (0, 1). \tag{2.5}$$

Further, we have

$$\begin{aligned} &(1-\alpha)(t+1)^{-\alpha} - (t+2)^{1-\alpha} + (t+1)^{1-\alpha} \\ =&\frac{1-\alpha}{t^\alpha}\Big(1+\frac{1}{t}\Big)^{-\alpha} - t^{1-\alpha}\Big(\Big(1+\frac{2}{t}\Big)^{1-\alpha} - \Big(1+\frac{1}{t}\Big)^{1-\alpha}\Big) \\ =&\frac{1-\alpha}{t^\alpha}\Big(1-\frac{\alpha}{t}+O\Big(\frac{1}{t^2}\Big)\Big) - t^{1-\alpha}\Big(1+\frac{2(1-\alpha)}{t} - \frac{2(1-\alpha)\alpha}{t^2} + O\Big(\frac{1}{t^3}\Big)\Big) \\ &+ t^{1-\alpha}\Big(1+\frac{1-\alpha}{t} - \frac{(1-\alpha)\alpha}{2t^2} + O\Big(\frac{1}{t^3}\Big)\Big) \\ =&\frac{(1-\alpha)\alpha}{2t^{\alpha+1}} + O\Big(\frac{1}{t^{\alpha+2}}\Big) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &2(t+1)^{1-\alpha} - (t+2)^{1-\alpha} - t^{1-\alpha} \\ =&t^{1-\alpha}\Big(2\Big(1+\frac{1}{t}\Big)^{1-\alpha} - \Big(1+\frac{2}{t}\Big)^{1-\alpha} - 1\Big) \\ =&2t^{1-\alpha}\Big(1+\frac{1-\alpha}{t} - \frac{(1-\alpha)\alpha}{2t^2} + O\Big(\frac{1}{t^3}\Big)\Big) - t^{1-\alpha}\Big(1+\frac{2(1-\alpha)}{t} - \frac{2(1-\alpha)\alpha}{t^2} + O\Big(\frac{1}{t^3}\Big)\Big) - t^{1-\alpha} \\ =&\frac{(1-\alpha)\alpha}{t^{\alpha+1}} + O\Big(\frac{1}{t^{\alpha+2}}\Big), \end{aligned} \tag{2.7}$$

for sufficiently large t.

Using (2.6) and (2.7), we have

$$\begin{aligned} \lim_{t\to+\infty} g_\alpha(t) =& \lim_{t\to+\infty}\frac{(1-\alpha)(t+1)^{-\alpha} - (t+2)^{1-\alpha} + (t+1)^{1-\alpha}}{2(t+1)^{1-\alpha} - (t+2)^{1-\alpha} - t^{1-\alpha}} \\ =& \lim_{t\to+\infty}\frac{\frac{(1-\alpha)\alpha}{2t^{\alpha+1}} + O(\frac{1}{t^{\alpha+2}})}{\frac{(1-\alpha)\alpha}{t^{\alpha+1}} + O(\frac{1}{t^{\alpha+2}})} = \frac{1}{2}, \end{aligned} \tag{2.8}$$

proving the relation in (2.2).

We have

$$\begin{aligned}
g'_\alpha(t) =& \left(\frac{(1-\alpha)(t+1)^{-\alpha} - (t+2)^{1-\alpha} + (t+1)^{1-\alpha}}{2(t+1)^{1-\alpha} - (t+2)^{1-\alpha} - t^{1-\alpha}}\right)' \\
=& \frac{(-\alpha(t+1)^{-\alpha-1} - (t+2)^{-\alpha} + (t+1)^{-\alpha})(2(t+1)^{1-\alpha} - (t+2)^{1-\alpha} - t^{1-\alpha})}{(1-\alpha)^{-1}(2(t+1)^{1-\alpha} - (t+2)^{1-\alpha} - t^{1-\alpha})^2} \\
&- \frac{((1-\alpha)(t+1)^{-\alpha} - (t+2)^{1-\alpha} + (t+1)^{1-\alpha})(2(t+1)^{-\alpha} - (t+2)^{-\alpha} - t^{-\alpha})}{(1-\alpha)^{-1}(2(t+1)^{1-\alpha} - (t+2)^{1-\alpha} - t^{1-\alpha})^2} \\
=& \frac{-2\alpha(t+1)^{-2\alpha} + \alpha(t+1)^{-\alpha-1}(t+2)^{1-\alpha} + \alpha(t+1)^{-\alpha-1}t^{1-\alpha}}{(1-\alpha)^{-1}(2(t+1)^{1-\alpha} - (t+2)^{1-\alpha} - t^{1-\alpha})^2} \\
&+ \frac{-2(t+2)^{-\alpha}(t+1)^{1-\alpha} + (t+2)^{1-2\alpha} + (t+2)^{-\alpha}t^{1-\alpha}}{(1-\alpha)^{-1}(2(t+1)^{1-\alpha} - (t+2)^{1-\alpha} - t^{1-\alpha})^2} \\
&+ \frac{2(t+1)^{1-2\alpha} - (t+1)^{-\alpha}(t+2)^{1-\alpha} - (t+1)^{-\alpha}t^{1-\alpha}}{(1-\alpha)^{-1}(2(t+1)^{1-\alpha} - (t+2)^{1-\alpha} - t^{1-\alpha})^2} \\
&+ \frac{-2(1-\alpha)(t+1)^{-2\alpha} + (1-\alpha)(t+1)^{-\alpha}(t+2)^{-\alpha} + (1-\alpha)(t+1)^{-\alpha}t^{-\alpha}}{(1-\alpha)^{-1}(2(t+1)^{1-\alpha} - (t+2)^{1-\alpha} - t^{1-\alpha})^2} \\
&+ \frac{2(t+2)^{1-\alpha}(t+1)^{-\alpha} - (t+2)^{1-2\alpha} - (t+2)^{1-\alpha}t^{-\alpha}}{(1-\alpha)^{-1}(2(t+1)^{1-\alpha} - (t+2)^{1-\alpha} - t^{1-\alpha})^2} \\
&+ \frac{-2(t+1)^{1-2\alpha} + (t+1)^{1-\alpha}(t+2)^{-\alpha} + (t+1)^{1-\alpha}t^{-\alpha}}{(1-\alpha)^{-1}(2(t+1)^{1-\alpha} - (t+2)^{1-\alpha} - t^{1-\alpha})^2} \\
=& \frac{h_\alpha(t)}{(2(t+1)^{1-\alpha} - (t+2)^{1-\alpha} - t^{1-\alpha})^2},
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
\frac{h_\alpha(t)}{1-\alpha} =& \frac{\alpha(t+2)}{(t+1)^{\alpha+1}(t+2)^\alpha} - \frac{2}{(t+1)^{2\alpha}} + \frac{\alpha t}{t^\alpha(t+1)^{\alpha+1}} - \frac{2(t+1)}{(t+1)^\alpha(t+2)^\alpha} \\
&+ \frac{t}{t^\alpha(t+2)^\alpha} - \frac{t+2}{(t+1)^\alpha(t+2)^\alpha} - \frac{t}{t^\alpha(t+1)^\alpha} + \frac{1-\alpha}{(t+1)^\alpha(t+2)^\alpha} \\
&+ \frac{1-\alpha}{t^\alpha(t+1)^\alpha} + \frac{2(t+2)}{(t+1)^\alpha(t+2)^\alpha} - \frac{t+2}{t^\alpha(t+2)^\alpha} + \frac{t+1}{(t+1)^\alpha(t+2)^\alpha} + \frac{t+1}{t^\alpha(t+1)^\alpha} \\
=& \frac{\alpha(t+1)+\alpha}{(t+1)^{\alpha+1}(t+2)^\alpha} - \frac{2}{(t+1)^{2\alpha}} + \frac{\alpha(t+1)-\alpha}{t^\alpha(t+1)^{\alpha+1}} - \frac{2t+2}{(t+1)^\alpha(t+2)^\alpha} \\
&+ \frac{t}{t^\alpha(t+2)^\alpha} - \frac{t+2}{(t+1)^\alpha(t+2)^\alpha} - \frac{t}{t^\alpha(t+1)^\alpha} + \frac{1-\alpha}{(t+1)^\alpha(t+2)^\alpha} \\
&+ \frac{1-\alpha}{t^\alpha(t+1)^\alpha} + \frac{2t+4}{(t+1)^\alpha(t+2)^\alpha} - \frac{t+2}{t^\alpha(t+2)^\alpha} + \frac{t+1}{(t+1)^\alpha(t+2)^\alpha} + \frac{t+1}{t^\alpha(t+1)^\alpha} \\
=& \frac{\alpha}{(t+1)^\alpha(t+2)^\alpha} + \frac{\alpha}{(t+1)^{\alpha+1}(t+2)^\alpha} - \frac{2}{(t+1)^{2\alpha}} + \frac{\alpha}{t^\alpha(t+1)^\alpha} \\
&- \frac{\alpha}{t^\alpha(t+1)^{\alpha+1}} - \frac{2t+2}{(t+1)^\alpha(t+2)^\alpha} + \frac{t}{t^\alpha(t+2)^\alpha} - \frac{t+2}{(t+1)^\alpha(t+2)^\alpha} \\
&- \frac{t}{t^\alpha(t+1)^\alpha} + \frac{1-\alpha}{(t+1)^\alpha(t+2)^\alpha} + \frac{1-\alpha}{t^\alpha(t+1)^\alpha} + \frac{2t+4}{(t+1)^\alpha(t+2)^\alpha}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{t+2}{t^\alpha(t+2)^\alpha}+\frac{t+1}{(t+1)^\alpha(t+2)^\alpha}+\frac{t+1}{t^\alpha(t+1)^\alpha}\\
=&\frac{2}{(t+1)^\alpha(t+2)^\alpha}-\frac{2}{(t+1)^{2\alpha}}+\frac{\alpha}{(t+1)^{\alpha+1}(t+2)^\alpha}-\frac{\alpha}{t^\alpha(t+1)^{\alpha+1}}+\frac{2}{t^\alpha(t+1)^\alpha}-\frac{2}{t^\alpha(t+2)^\alpha}\\
=&\frac{2t^\alpha(t+1)^\alpha-2t^\alpha(t+2)^\alpha+2(t+1)^\alpha(t+2)^\alpha-2(t+1)^{2\alpha}+\alpha t^\alpha(t+1)^{\alpha-1}-\alpha(t+2)^\alpha(t+1)^{\alpha-1}}{t^\alpha(t+1)^{2\alpha}(t+2)^\alpha}\\
=&\frac{\widehat{h}_\alpha(t)}{t^\alpha(t+1)^{2\alpha}(t+2)^\alpha}.
\end{aligned}\tag{2.10}
$$

By using the change of variables $t=x-1$, we have

$$
\begin{aligned}
\widehat{h}_\alpha(x-1)=&2(x-1)^\alpha x^\alpha-2(x-1)^\alpha(x+1)^\alpha+2x^\alpha(x+1)^\alpha-2x^{2\alpha}\\
&+\alpha(x-1)^\alpha x^{\alpha-1}-\alpha(x+1)^\alpha x^{\alpha-1}\\
=&2(x^\alpha-(x-1)^\alpha)((x+1)^\alpha-x^\alpha)-\alpha x^{\alpha-1}((x+1)^\alpha-(x-1)^\alpha),
\end{aligned}\tag{2.11}
$$

where $x\geq 1$.

We show that

$$
2(x^\alpha-(x-1)^\alpha)((x+1)^\alpha-x^\alpha)-\alpha x^{\alpha-1}((x+1)^\alpha-(x-1)^\alpha)\geq 0,\tag{2.12}
$$

for $x\geq 1$, from which it will follow that the function $g_\alpha(t)$ is strictly increasing for $t\geq 0$.

To do this we use the change of variables $x=\frac{1}{t}$, and (2.12) is transformed to

$$
2(1-(1-t)^\alpha)((1+t)^\alpha-1)-\alpha t((1+t)^\alpha-(1-t)^\alpha)\geq 0,\tag{2.13}
$$

for $t\in[0,1]$. The inequality obviously holds for $t=0$. So, we may assume that $t\in(0,1]$. The inequality can be written in the form

$$
2\int_{1-t}^{1}u^{\alpha-1}du\int_{1}^{1+t}u^{\alpha-1}du\geq t\Big(\int_{1}^{1+t}u^{\alpha-1}du+\int_{1-t}^{1}u^{\alpha-1}du\Big)=:L(t).\tag{2.14}
$$

By using the Cauchy–Bunyakovsky inequality, and one of the Hadamard-Hermite inequalities to the concave function $u^{1-\alpha}$ ([7, 8, 16]), we have

$$
\begin{aligned}
L(t)=&\frac{1}{t}\Big(\int_{1-t}^{1}du\Big)^2\int_{1}^{1+t}u^{\alpha-1}du+\frac{1}{t}\Big(\int_{1}^{1+t}du\Big)^2\int_{1-t}^{1}u^{\alpha-1}du\\
\leq&\frac{1}{t}\Big(\int_{1-t}^{1}u^{1-\alpha}du\Big)\Big(\int_{1-t}^{1}u^{\alpha-1}du\Big)\Big(\int_{1}^{1+t}u^{\alpha-1}du\Big)\\
&+\frac{1}{t}\Big(\int_{1}^{1+t}u^{1-\alpha}du\Big)\Big(\int_{1}^{1+t}u^{\alpha-1}du\Big)\Big(\int_{1-t}^{1}u^{\alpha-1}du\Big)\\
=&2\frac{1}{2t}\Big(\int_{1-t}^{1+t}u^{1-\alpha}du\Big)\Big(\int_{1-t}^{1}u^{\alpha-1}du\Big)\Big(\int_{1}^{1+t}u^{\alpha-1}du\Big)\\
\leq&2\int_{1-t}^{1}u^{\alpha-1}du\int_{1}^{1+t}u^{\alpha-1}du,
\end{aligned}\tag{2.15}
$$

so the inequality (2.14) holds, that is, the inequality (2.13), and consequently the inequality (2.12) for $x \geq 1$. From (2.9)–(2.12), and since $\alpha \in (0,1)$, it follows that

$$g'_\alpha(t) > 0, \tag{2.16}$$

for $t > 0$.

From (2.16) we conclude that the function $g_\alpha(t)$ is strictly increasing on $\mathbb{R}_+$, as claimed. □

The following theorem completely describes the monotonicity character of the convex combinations of the sequences in (1.5) and (1.6).

Theorem 2.1. *Let $(c_n^{(\lambda)})_{n\in\mathbb{N}}$ be the sequence defined in* (1.9)*, where $\lambda \in [0,1]$, and*

$$\lambda_k := g_\alpha(k), \quad k \in \mathbb{N}_0, \tag{2.17}$$

where g_α is the function defined in (2.1)*.*

Then, the following statements hold:

(a) if $\lambda \in [1/2, 1]$, then the sequence $(c_n^{(\lambda)})_{n\in\mathbb{N}}$ is strictly decreasing;

(b) if $\lambda \in [0, \lambda_1)$, then the sequence $(c_n^{(\lambda)})_{n\in\mathbb{N}}$ is strictly increasing;

(c) if $\lambda = \lambda_1$, then the sequence $(c_n^{(\lambda_1)})_{n\in\mathbb{N}_2}$ is strictly increasing and $c_1^{(\lambda_1)} = c_2^{(\lambda_1)}$;

(d) if $\lambda \in (\lambda_k, \lambda_{k+1})$ for a fixed $k \in \mathbb{N}$, then the sequence is strictly increasing for $n \geq k+1$ and strictly decreasing for $1 \leq n \leq k+1$;

(e) if $\lambda = \lambda_k$ for a fixed $k \in \mathbb{N}_2$, then the sequence is strictly increasing for $n \geq k+1$, $c_k^{(\lambda_k)} = c_{k+1}^{(\lambda_k)}$, and it is strictly decreasing for $1 \leq n \leq k$.

Proof. (a) We have

$$\begin{aligned}
c_{n+1}^{(\lambda)} - c_n^{(\lambda)} =& \lambda(x_{n+1} - x_n) + (1-\lambda)(y_{n+1} - y_n) \\
=& \lambda\Big(\frac{n^{1-\alpha}}{1-\alpha} - \frac{(n+1)^{1-\alpha}}{1-\alpha} + \frac{1}{(n+1)^\alpha}\Big) \\
&+ (1-\lambda)\Big(\frac{(n+1)^{1-\alpha}}{1-\alpha} - \frac{(n+2)^{1-\alpha}}{1-\alpha} + \frac{1}{(n+1)^\alpha}\Big) \\
=& \frac{1}{(n+1)^\alpha} - \frac{1}{1-\alpha}\Big((n+2)^{1-\alpha} - (n+1)^{1-\alpha}\Big) \\
&- \frac{\lambda}{1-\alpha}\Big(2(n+1)^{1-\alpha} - (n+2)^{1-\alpha} - n^{1-\alpha}\Big) \\
=& \frac{1}{1-\alpha}\Big(2(n+1)^{1-\alpha} - (n+2)^{1-\alpha} - n^{1-\alpha}\Big)(g_\alpha(n) - \lambda),
\end{aligned} \tag{2.18}$$

for every $n \in \mathbb{N}$.

From (2.3), we have

$$2(n+1)^{1-\alpha} - (n+2)^{1-\alpha} - n^{1-\alpha} > 0, \tag{2.19}$$

for every $n \in \mathbb{N}_0$.

Since $g_\alpha(t) < 1/2$ for $t \in \mathbb{R}_+$, we have

$$g_\alpha(n) < 1/2 \leq \lambda, \tag{2.20}$$

for every $n \in \mathbb{N}$.

From (2.18)–(2.20), and since $\alpha \in (0,1)$, we get

$$c_{n+1}^{(\lambda)} < c_n^{(\lambda)},$$

for every $n \in \mathbb{N}$, that is, the sequence $(c_n^{(\lambda)})_{n\in\mathbb{N}}$ is strictly decreasing for each $\lambda \in [1/2, 1]$.

(b) We have $\lambda \in [0, \lambda_1)$. Hence, by the monotonicity of the function g_α, we have

$$g_\alpha(n) > g_\alpha(1) = \lambda_1 > \lambda, \tag{2.21}$$

for each $n \in \mathbb{N}$.

From (2.18), (2.19), and (2.21), and since $\alpha \in (0,1)$, we get

$$c_{n+1}^{(\lambda)} > c_n^{(\lambda)},$$

for each $n \in \mathbb{N}$, that is, the sequence $(c_n^{(\lambda)})_{n\in\mathbb{N}}$ is strictly increasing.

(c) We have $\lambda = \lambda_1$. Then, from (2.18), (2.19), and (2.21), and since $\alpha \in (0,1)$, we have

$$c_{n+1}^{(\lambda)} > c_n^{(\lambda)},$$

for each $n \in \mathbb{N}_2$, that is, the subsequence $(c_n^{(\lambda_1)})_{n\in\mathbb{N}_2}$ is strictly increasing, and $c_2^{(\lambda_1)} = c_1^{(\lambda_2)}$.

(d) We have $\lambda \in (\lambda_k, \lambda_{k+1})$ for a fixed $k \in \mathbb{N}$. Hence, by the monotonicity of the function g_α, we have

$$g_\alpha(n) \geq g_\alpha(k+1) = \lambda_{k+1} > \lambda, \tag{2.22}$$

for $n \geq k+1$, and

$$g_\alpha(n) \leq g_\alpha(k) = \lambda_k < \lambda, \tag{2.23}$$

for $1 \leq n \leq k$.

From (2.18), (2.19), and (2.22), and since $\alpha \in (0,1)$, we have

$$c_{n+1}^{(\lambda)} > c_n^{(\lambda)},$$

for $n \geq k+1$, from which it follows that the subsequence $(c_n^{(\lambda)})_{n\in\mathbb{N}_{k+1}}$ is increasing.

From (2.18), (2.19), and (2.23), and since $\alpha \in (0,1)$, we have

$$c_{n+1}^{(\lambda)} < c_n^{(\lambda)},$$

for $1 \leq n \leq k$, from which it follows that the subsequence/string $(c_n^{(\lambda)})_{n=\overline{1,k+1}}$ is decreasing.

(e) If $\lambda = \lambda_k$, then from (2.18), (2.19), (2.22), and (2.23), and since $\alpha \in (0,1)$, we have

$$c_{n+1}^{(\lambda)} > c_n^{(\lambda)},$$

for $n \geq k+1$, and

$$c_{n+1}^{(\lambda)} < c_n^{(\lambda)},$$

for $n < k$, and $c_k^{(\lambda_k)} = c_{k+1}^{(\lambda_k)}$, that is, $(c_n^{(\lambda_k)})_{n\in\mathbb{N}_{k+1}}$ is increasing, the subsequence/string $(c_n^{(\lambda_k)})_{n=\overline{1,k}}$ is decreasing, and $c_k^{(\lambda_k)} = c_{k+1}^{(\lambda_k)}$, as claimed. □

Remark 2.1. *Since* $(\lambda_k)_{k\in\mathbb{N}_0}$ *is strictly increasing and* $\lim_{k\to+\infty}\lambda_k = 1/2$*, we have*

$$\bigcup_{k=1}^{\infty}(\lambda_k, \lambda_{k+1}] = \Big(\lambda_1, \frac{1}{2}\Big),$$

and consequently

$$[0, \lambda_1) \cup \bigcup_{k=1}^{\infty}(\lambda_k, \lambda_{k+1}] \cup [1/2, 1] = [0, 1],$$

which means that all possible cases are considered in Theorem 2.1.

3. Conclusions

Our considerable inspection of the literature on some pairs of sequences of real numbers has shown that several problems on the pairs of sequences are essentially connected to the investigation of their convex combinations. This observation has motivated us to investigate the convex combinations of several known pairs of sequences, which we have done recently in a series of papers. As far as we know, the problem has not been considered so far. Here we give a complete description of the monotonicity character of the convex combinations of the sequences defined in (1.5) and (1.6) when $\alpha \in (0, 1)$, on the whole domain $\mathbb{N}$, considerably extending some special cases in the literature, and solving a highly nontrivial open problem completely. To do this, we prove a few analytic inequalities and use them in the proof of the result that characterizes the monotonicity character of the sequences. The methods and ideas presented here could be modified and developed further in order to get some other results on the topic.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

References

1. D. Adamović, *Collected works*, Belgrade: University of Belgrade, Faculty of Mathematics, 2009.
2. M. I. Bashmakov, B. M. Bekker, V. M. Gol’hovoi, *Zadachi po matematike. Algebra i analiz*, (in Russian), Nauka, 1982.

3. L. Berg, On the asymptotics of nonlinear difference equations, *Z. Anal. Anwend.*, **21** (2002), 1061–1074. https://doi.org/10.4171/zaa/1127
4. A. Y. Dorogovtsev, *Matematika segodnya '90*, (in Russian), Kiev: Vishcha Shkola, 1990.
5. L. Eulero, De progressionibvs harmonicis observationes, *Comm. Acad. Sci. Petrop.*, **7** (1740), 150–161.
6. S. Golubev, R. Ushakov, Reshenie zadachi 485, *Kvant*, **11** (1978), 20–22.
7. J. Hadamard, Étude sur les propriétés des fonctions entiéres et en particulier d'une fonction considérée par Riemann, *J. Math. Pures Appl.*, **58** (1893), 171–215.
8. C. Hermite, Sur deux limites d'une intégrale définie, *Mathesis*, **3** (1883), 6.
9. B. D. Iričanin, Dynamics of a class of higher order difference equations, *Discrete Dyn. Nat. Soc.*, **2007** (2007), 073849.
10. C. M. Kent, Convergence of solutions in a nonhyperbolic case, *Nonlinear Anal.*, **47** (2001), 4651–4665. https://doi.org/10.1016/S0362-546X(01)00578-8
11. P. P. Korovkin, *Inequalities*, Moscow: Mir Publishers, 1975.
12. W. A. J. Kosmala, *A friendly introduction to analysis: Single and multivariable*, 2 Eds., Pearson Prentice Hall, 2004.
13. V. A. Krechmar, *A problem book in algebra*, Moscow: Mir Publishers, 1978.
14. I. I. Lyashko, A. K. Boyarchuk, Y. G. Gay, G. P. Golovach, *Matematicheskiy analiz v primerah i zadachah*, (in Russian), Vishcha Shkola, 1974.
15. T. Martynenko, R. Ushakov, Zadacha 485, *Kvant*, **1** (1978), 28–29.
16. D. S. Mitrinović, *Analytic inequalities*, Berlin: Springer-Verlag, 1970. https://doi.org/10.1007/978-3-642-99970-3
17. D. S. Mitrinović, *Matematička indukcija, binomna formula, kombinatorika*, (in Serbian), Beograd: Gradjevinska Knjiga, 1980.
18. D. S. Mitrinović, D. D. Adamović, *Nizovi i redovi: Definicije, stavovi, zadaci, problemi*, (in Serbian), Beograd: Naučna Knjiga, 1980.
19. D. S. Mitrinović, J. D. Kečkić, *Metodi izračunavanja konačnih zbirova*, (in Serbian), Beograd: Naučna Knjiga, 1984.
20. N. Nielsen, *Handbuch der theorie der gammafunktion*, (in German), Teubner, 1906.
21. G. Papaschinopoulos, C. J. Schinas, On a system of two nonlinear difference equations, *J. Math. Anal. Appl.*, **219** (1998), 415–426. https://doi.org/10.1006/jmaa.1997.5829
22. G. Papaschinopoulos, C. J. Schinas, G. Stefanidou, On the recursive sequence $x_{n+1} = A + (x_{n-1}^p / x_n^q)$, *Adv. Differ. Equ.*, **2009** (2009), 327649. https://doi.org/10.1155/2009/327649
23. G. Papaschinopoulos, C. J. Schinas, G. Stefanidou, Two modifications of the Beverton–Holt equation, *Int. J. Difference Equ.*, **4** (2009), 115–136.
24. G. Polya, G. Szegö, *Problems and theorems in analysis I*, Berlin: Springer, 1998. https://doi.org/10.1007/978-3-642-61983-0

25. V. A. Sadovnichiy, A. Grigoryan, S. Konyagin, *Zadachi studencheskih matematichsikh olimpiad*, (in Russian), Izdatel'stvo Moskovskogo Universiteta, 1987.

26. V. A. Sadovnichiy, A. S. Podkolzin, *Zadachi studencheskih olimpiad po matematike*, (in Russian), Moskva: Nauka, 1978.

27. S. Stević, On the recursive sequence $x_{n+1} = g(x_n, x_{n-1})/(A + x_n)$, *Appl. Math. Lett.*, **15** (2002), 305–308. https://doi.org/10.1016/S0893-9659(01)00135-5

28. S. Stević, Asymptotic behavior of a class of nonlinear difference equations, *Discrete Dyn. Nat. Soc.*, **2006** (2006), 47156. https://doi.org/10.1155/DDNS/2006/47156

29. S. Stević, On monotone solutions of some classes of difference equations, *Discrete Dyn. Nat. Soc.*, **2006** (2006), 53890. https://doi.org/10.1155/DDNS/2006/53890

30. S. Stević, Asymptotics of some classes of higher order difference equations, *Discrete Dyn. Nat. Soc.*, **2007** (2007), 56813. https://doi.org/10.1155/2007/56813

31. S. Stević, Nontrivial solutions of a higher-order rational difference equation, *Math. Notes*, **84** (2008), 718–724. https://doi.org/10.1134/S0001434608110138

32. S. Stević, Complete picture on the monotonicity character of a class of sequences with a parameter, *J. Math. Inequal.*, **19** (2025), 1093–1107. http://dx.doi.org/10.7153/jmi-2025-19-70

33. S. Stević, Convex combinations of some convergent sequences, *Math. Methods Appl. Sci.*, **48** (2025), 2819–2832. https://doi.org/10.1002/mma.10463

34. S. Stević, On some sequences whose limit is the Euler gamma constant, *J. Inequal. Appl.*, **2025** (2025), 75. https://doi.org/10.1186/s13660-025-03321-7

35. S. Stević, B. Iričanin, W. Kosmala, Z. Šmarda, Monotonicity character of convex combinations of two sequences generated by the square of the cube root function, *J. Inequal. Appl.*, **2025** (2025), 122. https://doi.org/10.1186/s13660-025-03375-7

36. R. P. Ushakov, Zadacha 3, In: *Matematika segodnya '90*, (in Russian), Kiev: Vishcha Shkola, 1990.

37. R. P. Ushakov, Reshenie Zadachi 3 iz Matematika Segodnya '90, In: *Matematika segodnya '94*, (in Russian), Kiev: Vishcha Shkola, 1994.

38. V. A. Zorich, *Mathematical analysis I*, 2 Eds., Berlin: Springer, 2015. https://doi.org/10.1007/978-3-662-48792-1