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Research article

Parallel surfaces with constant mean curvature in a Galilean 3-space

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Abstract: In this paper, we investigate the parallel surfaces of all three types of surfaces of revolution in the Galilean 3-space G^3 under the condition that the mean curvature of the parallel surface is constant. For each type of revolution surface, we explicitly compute the mean curvature of the corresponding parallel surface and derive the ordinary differential equations governing the profile curves. By solving these equations, we obtain explicit expressions for the functions defining these surfaces. Furthermore, several representative examples are presented, and the resulting surfaces are visualized to illustrate the influence of the parallel surface transformation on the geometry of revolution surfaces in the Galilean setting.

Keywords: parallel surface; revolution surface; mean curvature; Galilean space
Mathematics Subject Classification: 53A10, 53A35, 53B30

1. Introduction

The theory of surfaces is a fundamental branch of differential geometry with wide applications in mathematics, physics, and engineering. Classical works such as [1, 2] provide a comprehensive framework for studying the geometric properties of curves and surfaces, particularly through curvature invariants and their analytical representations. Among various geometric settings, the Galilean 3-space G^3 plays a distinctive role due to its degenerate metric structure and its strong connection with kinematics. This space has attracted increasing attention in recent years, and numerous studies have focused on curves and surfaces within this framework. In particular, position vectors of curves in G^3 and several classes of surfaces with constant curvature have been investigated in [3–6], revealing substantial differences compared to their Euclidean counterparts.

Galilean space is one of the Cayley–Klein spaces with a degenerate metric structure. Therefore, several geometric notions such as distance, angle, normal vector, and curvature differ essentially from those in Euclidean space. In particular, a parallel surface in G^3 should be interpreted with respect to the Galilean normal direction and the corresponding signed offset distance. Hence, although the methodology employed in this paper is inspired by classical Euclidean theory, the study of revolution surfaces and their parallel surfaces in G^3 leads to substantially different geometric interpretations due to the degeneracy of the Galilean metric. This distinctive geometric framework gives rise to phenomena that do not appear in the Euclidean case.

In the literature, various geometric investigations have been carried out for specific curve families on surfaces. For example, minimal curves on ruled surfaces were studied in [7]. Surfaces of revolution with zero Gaussian curvature were investigated in [8], and those with constant mean curvature were studied in [9]. In [10], revolution surfaces and their focal surfaces were defined, and certain results were obtained regarding the conditions under which these surfaces become flat or minimal. In [11], surfaces of revolution in the three-dimensional pseudo-Galilean space G_1^3 were studied.

Parallel surfaces constitute another fundamental concept in differential geometry, arising from offset constructions along the normal direction of a given surface. In Galilean space, parallel surfaces were introduced by [12], where their fundamental geometric properties together with curvature relations were established. However, these results were obtained in a general framework and do not include the case of surfaces of revolution, which motivates the present study. Special classes of ruled surfaces satisfying constant mean curvature conditions have been examined in [13]. In addition, [14] studied parallel hypersurfaces in the four-dimensional Euclidean space E^4 and derived necessary and sufficient conditions for these hypersurfaces to be flat or minimal, showing that the theory of parallel hypersurfaces naturally extends to higher-dimensional settings.

In this paper, we address this gap by investigating parallel surfaces of revolution surfaces in G^3 and analyzing in detail the constant mean curvature condition on these surfaces. For all three types of revolution surfaces in G^3, the mean curvature of the corresponding parallel surfaces is derived explicitly. By imposing the constant mean curvature condition, we obtain differential equations for the generating functions of the profile curves and construct explicit solution families. Finally, graphical visualizations of the obtained surfaces are presented to illustrate the geometric effects of the parallel surface transformation. Beyond their theoretical significance, parallel surfaces with constant mean curvature also admit a natural interpretation in architectural and structural contexts. From an architectural perspective, real-world surfaces are not idealized single geometric objects but are instead realized as shell structures with finite thickness. In this setting, the inner and outer boundaries of a shell can be represented as parallel surfaces located at a constant distance along the normal direction, in accordance with offset surface constructions in architectural geometry. Because constant mean curvature surfaces are classically associated with equilibrium configurations of membranes and shell structures, the parallel surfaces studied here provide a geometric framework for modeling shell thickness and structural continuity [15, 16].

2. Preliminaries

In this section, we briefly recall the fundamental concepts of Galilean geometry that will be used throughout the paper. For further details, the reader may refer to standard works in Galilean differential geometry [5, 6, 8, 12, 17].

A point in G^3 is given by $P(x, y, z)$. The Galilean distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is defined as

$$d(P_1, P_2) = \begin{cases} |x_2 - x_1|, & \text{if } x_1 \neq x_2 \\ \sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2}, & \text{if } x_1 = x_2. \end{cases}$$

For vectors $\mathbf{a} = (x, y, z)$ and $\mathbf{b} = (x_1, y_1, z_1)$, the Galilean scalar product is defined by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \begin{cases} xx_1 & \text{if } x \neq 0 \text{ or } x_1 \neq 0, \\ yy_1 + zz_1 & \text{if } x = x_1 = 0. \end{cases}$$

This scalar product reflects the degenerate nature of the Galilean metric. In particular, the x-direction plays a distinguished role in G^3, leading to geometric notions that differ essentially from those of Euclidean space.

Let $\varphi(v^1, v^2) = \left(x(v^1, v^2),\ y(v^1, v^2),\ z(v^1, v^2)\right)$ be a parametric surface in G^3. The first-order partial derivatives are denoted by

$$\varphi_{,i} = \frac{\partial \varphi}{\partial v^i}(v^1, v^2),\ i \in \{1, 2\}.$$

The unit normal vector field is given by

$$N = \frac{\varphi_{,1} \wedge \varphi_{,2}}{w},$$

where $w = \left\| \varphi_{,1} \wedge \varphi_{,2} \right\|$. Because the metric of G^3 is degenerate, the normal vector should be interpreted with respect to the Galilean metric structure.

In [8, 18], the coefficients of the second fundamental form are given by

$$L_{ij} = \left\langle \frac{\varphi_{,ij} x_{,1} - x_{,ij} \varphi_{,1}}{x_{,1}},\ N \right\rangle = \left\langle \frac{\varphi_{,ij} x_{,2} - x_{,ij} \varphi_{,2}}{x_{,2}},\ N \right\rangle,$$

and the mean curvature H of the surface is defined as

$$2H = \sum_{i,j=1}^{2} g^{ij} L_{ij},$$

where

$$g^{ij} = g^i g^j \text{ for } i, j \in \{1, 2\},\ \text{and } g^1 = \frac{x_{,2}}{w},\ g^2 = -\frac{x_{,1}}{w}.$$

These coefficients arise from the induced Galilean structure and are used in the computation of the mean curvature of admissible surfaces in G^3.

In the construction of surfaces of revolution in G^3, analogous to the Euclidean case, two kinds of rotations are employed, which are defined as follows:

A Euclidean rotation about the non-isotropic x-axis is given by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where θ_1 is the Euclidean angle, and an isotropic rotation is given by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \theta_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} c\theta_2 \\ \frac{c}{2}(\theta_2)^2 \\ 0 \end{pmatrix},$$

where θ_2 is the isotropic angle, and $c \in \mathbb{R}$.

Given that a Euclidean plane is composed exclusively of isotropic vectors, whereas an isotropic plane encompasses both isotropic and non-isotropic vectors, it follows that three distinct classes of surfaces of revolution can be characterized within G^3.

For this aim, we consider the isotropic rotations. By rotating the isotropic curve $\alpha(t) = (0, f(t), g(t))$ about the yz- plane by isotropic rotation, we obtain the parametrization of the surface of revolution of Type I as

$$\varphi(s,t) = \left(cs,\ f(t) + \frac{cs^2}{2},\ g(t) \right), \tag{1}$$

where f and g are smooth functions, and $c \neq 0 \in \mathbb{R}$.

Next, we assume again without loss of generality, that the profile curve $\alpha(t) = (f(t), g(t), 0)$ lies in the isotropic xy- plane and is parameterized by

$$\varphi(s,t) = \left(f(t) + cs,\ g(t),\ sf(t) + \frac{cs^2}{2} \right), \tag{2}$$

where f and g are smooth functions, and $c \neq 0 \in \mathbb{R}$. The surface (2) is called the surface of revolution of Type II.

Finally, for the profile curve $\alpha(t) = (f(t), g(t), 0)$ in the isotropic xy-plane, a Type III surface of revolution in G^3 is parameterized by

$$\varphi(s,t) = (f(t),\ g(t)\cos s,\ -g(t)\sin s)\,. \tag{3}$$

In order to construct the parallel surfaces of the three types of revolution surfaces and to compute the mean curvature of these parallel surfaces, we introduce the following definitions:

Let M and M^λ be parallel surfaces in Galilean space. We define the parallel surface M^λ to the base surface M at distance λ as

$$\varphi^\lambda(s,t) = \varphi(s,t) + \lambda N, \tag{4}$$

where N is the normal vector of the base surface. In the Galilean setting, the parameter λ represents the signed Galilean offset distance measured along the chosen normal direction. Therefore, λ is assumed to be independent of the surface parameters.

The relation between the mean curvatures of two parallel surfaces is

$$H^\lambda = \frac{H}{1 - 2\lambda H}. \tag{5}$$

2.1. Parallel surfaces of Type I revolution surfaces with constant mean curvature in G^3

We consider the Type I revolution surface M in G^3 given by the parametrization (1), where f and g are smooth functions, and $c \neq 0$ is a constant. Without loss of generality, we assume $c > 0$. Hence, $sgn(c) = 1$. For convenience of notation, we set

$$W = f'(t)^2 + g'(t)^2.$$

A direct computation yields the unit normal vector field

$$N = \frac{(0,\ -g'(t),\ f'(t))}{\sqrt{W}}.$$

It follows from (4) that the parametrization of the parallel surface M^λ is given by

$$\varphi^\lambda(s,t) = \left(cs,\ f(t) + \frac{c}{2}s^2 - \lambda\frac{g'(t)}{\sqrt{W}},\ g(t) + \lambda\frac{f'(t)}{\sqrt{W}}\right). \tag{6}$$

The mean curvature of a Type I revolution surface in G^3 was obtained in [8] as

$$H = sgn(c)\frac{f'g'' - f''g'}{2(f'^2 + g'^2)^{\frac{3}{2}}},$$

where the assumption $c > 0$ has been used. Substituting this expression into the parallel surface formula (5), the mean curvature of the parallel surface M^λ is obtained as

$$H^\lambda = \frac{1}{2}\frac{(f'(t)g''(t) - f''(t)g'(t))}{W^{\frac{3}{2}} - \lambda\left(f'(t)g''(t) - f''(t)g'(t)\right)}.$$

We assume that the denominator in the above expression does not vanish; that is,

$$W^{\frac{3}{2}} - \lambda\left(f'(t)g''(t) - f''(t)g'(t)\right) \neq 0,$$

so that the parallel surface is regular. We further assume that the parallel surface has constant mean curvature, namely

$$H^\lambda = \frac{A}{2},$$

where A is a prescribed constant. Recall that for a plane curve $(f(t),\ g(t))$, the classical curvature is defined by

$$\kappa = \frac{f'(t)g''(t) - f''(t)g'(t)}{W^{\frac{3}{2}}}.$$

Using this definition, the constant mean curvature condition reduces to

$$\frac{\kappa}{1 - \lambda\kappa} = A.$$

At this stage, we assume $1 - \lambda\kappa \neq 0$, so the above expression is well-defined. Solving the equation for κ, we obtain

$$\kappa = A(1 - \lambda\kappa),$$

and hence,

$$\kappa + A\lambda\kappa = A,$$

and therefore,

$$\kappa = \frac{A}{1 + A\lambda},$$

provided that $1 + A\lambda \neq 0$. Therefore, the curvature of the generating plane curve (f, g) is constant. Equivalently, the functions f and g satisfy the following differential equation:

$$f'(t)g''(t) - f''(t)g'(t) = \frac{A}{1 + A\lambda} W^{\frac{3}{2}},$$

which characterizes plane curves of constant curvature.

Reparametrizing the generating curve (f, g) by its arc-length parameter u, we obtain

$$f_u^2 + g_u^2 = 1;$$

that is, $W = 1$ with respect to the arc-length parameter. In this case, the curvature simplifies to

$$(f_u g_{uu} - f_{uu} g_u) = \kappa,$$

where κ is constant. From a geometric point of view, this means that the curve $(f(u),\ g(u))$ is a plane curve of constant curvature. Consequently, it is either a straight line when $\kappa = 0$ or arc of a circle when $\kappa \neq 0$.

Assume first that $\kappa \neq 0$. Let

$$T(u) = (f_u(u),\ g_u(u))$$

denote the unit tangent vector of the curve. Writing it in polar form as

$$T(u) = (\cos\theta(u),\ \sin\theta(u)),$$

the Frenet formula yields

$$\theta'(u) = \kappa.$$

Because κ is constant, we obtain

$$\theta(u) = \kappa u + \varphi,$$

where φ is an integration constant. Consequently,

$$f_u(u) = \cos(\kappa u + \varphi),\ g_u(u) = \sin(\kappa u + \varphi). \tag{7}$$

Integrating Eq (7), we obtain

$$f(u) = \frac{1}{\kappa}\sin(\kappa u + \varphi) + C_1,\ g(u) = -\frac{1}{\kappa}\cos(\kappa u + \varphi) + C_2.$$

Denoting $R = 1/|\kappa|$ and setting $(x_0, y_0) = (C_1,\ C_2)$, the generating curve can be written as

$$f(u) = x_0 + R\sin(\kappa u + \varphi),\ g(u) = y_0 - R\cos(\kappa u + \varphi),$$

which represents a circle of radius R in the plane. In the special case $\kappa = 0$, we have $\theta(u) = \varphi$, and here,

$$f(u) = u\cos\varphi + C_1,\ g(u) = u\sin\varphi + C_2.$$

Therefore, the constant mean curvature condition for the parallel surface of a Type I revolution surface in G^3 forces the generating plane curve (f, g) to be a curve of constant curvature, namely a straight line or a circular arc.

Example 2.1. Consider the Type I revolution surface M in G^3 parametrized by (1) and its parallel surface M^λ given by (6). For the parameters

$$A = -6,\ \kappa = 3,\ c = 0.6,\ R = \frac{1}{3},\ x_0 = y_0 = 0,\ \varphi = \frac{\pi}{6},$$

the generating functions are given by

$$f(u) = \frac{1}{3}\sin(3u + \frac{\pi}{6}),\ g(u) = -\frac{1}{3}\cos\left(3u + \frac{\pi}{6}\right).$$

Hence, the surface M admits the parametrization

$$\varphi(s,u) = \left(0.6s,\ \frac{1}{3}\sin(3u + \frac{\pi}{6}) + 0.3s^2,\ \ -\frac{1}{3}\cos\left(3u + \frac{\pi}{6}\right)\right),$$

and the corresponding parametrization of the parallel surface M^λ for $\lambda = 1/2$ is given by

$$\varphi^{\frac{1}{2}}(s,u) = \left(0.6s,\ 0.3s^2 - \frac{1}{6}\sin(3u + \frac{\pi}{6}),\ \frac{1}{6}\cos\left(3u + \frac{\pi}{6}\right)\right).$$

For visualization purposes, the parameters are chosen as

$$s \in [-2.5, 2.5]\,,\ u \in [-\pi, \pi]\,.$$

Here, it is important to note that under the constant mean curvature condition, the parallel distance λ is given by $\lambda = (A - \kappa)/A\kappa$. For $A = -6$ and $\kappa = 3$, we obtain $\lambda = 1/2$. Thus, the value $\lambda = 1/2$ is not arbitrary; it is determined by the constant mean curvature condition. Geometrically, this means that $M^{1/2}$ is obtained from M by moving each point along the normal direction in G^3 by the same signed distance $\lambda = 1/2$. Although the surfaces are visualized in Euclidean coordinates, their parallelism should be understood with respect to the Galilean metric structure. Hence, the revolution surface M together with its corresponding parallel surface $M^{1/2}$ is shown in Figure 1.

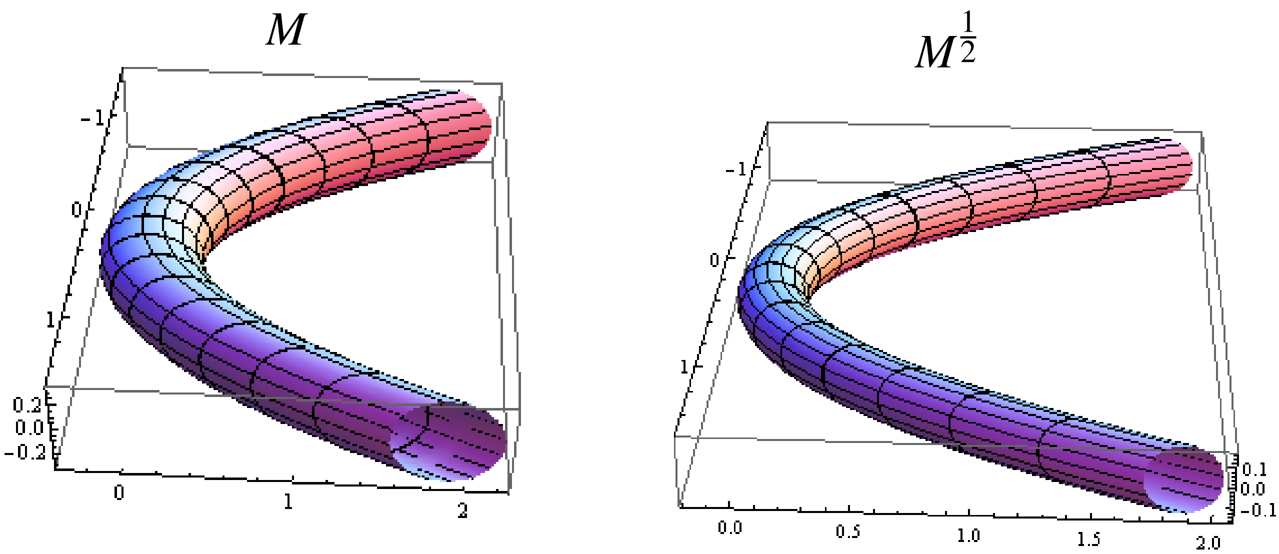

Figure 1. A Type I revolution surface M and its parallel surface $M^{1/2}$.

2.2. Parallel surfaces of Type II revolution surfaces with constant mean curvature in G^3

We consider a Type II revolution surface M given by the parametrization (2), where f and g are smooth functions of the parameter t, and $c \neq 0$ is a constant. A unit normal vector field of M can be chosen as

$$N = \frac{(0,\ f'(t)f(t),\ cg'(t))}{\sqrt{f'(t)^2 f(t)^2 + c^2 g'(t)^2}},$$

where the orientation is fixed throughout the paper. From the definition of parallel surfaces and the expressions of φ and N, Eq (4) yields the parametrization of M^λ in the form

$$\varphi^\lambda(s,t) = \left(f(t) + cs,\ g(t) + \lambda \frac{f'(t)f(t)}{\sqrt{f'(t)^2 f(t)^2 + c^2 g'(t)^2}},\ sf(t) + \frac{c}{2}s^2 + \lambda \frac{cg'(t)}{\sqrt{f'(t)^2 f(t)^2 + c^2 g'(t)^2}} \right). \quad (8)$$

The mean curvature of a Type II revolution surface in G^3 was obtained in [8] as

$$H = \frac{c^2}{2\left((ff')^2 + c^2 g'^2\right)^{\frac{3}{2}}} \left[f(f'g'' - f''g') - f'^2 g' \right].$$

Substituting this expression into the parallel surface formula (5), the mean curvature of the parallel surface M^λ is obtained as

$$H^\lambda = \frac{1}{2} \frac{c^2 \left\{ f\,(f'g'' - f''g') - f'^2 g' \right\}}{\left((ff')^2 + c^2 g'^2 \right)^{\frac{3}{2}} - \lambda c^2 \left\{ f\,(f'g'' - f''g') - f'^2 g' \right\}}.$$

We impose the constant mean curvature condition $H^\lambda = A/2$, where A is a nonzero constant. For convenience, we introduce the auxiliary functions

$$Q = f\,(f'g'' - f''g') - f'^2 g',\ S = (ff')^2 + c^2 g'^2.$$

The constant mean curvature condition is then equivalent to

$$\frac{c^2 Q}{S^{\frac{3}{2}} - \lambda c^2 Q} = A,$$

which can be rewritten as

$$c^2 Q\,(1 + A\lambda) = A S^{\frac{3}{2}}.$$

Assuming $A \neq 0$, $1 + A\lambda \neq 0$, we obtain the fundamental relation

$$Q = \alpha S^{\frac{3}{2}}, \quad (9)$$

where the constant α is defined by

$$\alpha = \frac{A}{c^2\,(1 + A\lambda)}.$$

Because Eq (9) is an ordinary differential equation with smooth coefficients on the domain where ($S > 0$), the classical existence and uniqueness theorem guarantees a unique local solution for any prescribed initial data. Therefore, the generating functions are locally determined by their initial conditions, and the following explicit solution families describe all admissible profiles satisfying the constant mean curvature condition. We now determine explicit generating functions satisfying Eq (9).

Case 1: Let us first prescribe the function g by choosing

$$g(t) = t,\ g'(t) = 1,\ g''(t) = 0.$$

Then, $Q = -(ff')'$, $S = (ff')^2 + c^2$, and Eq (9) becomes

$$-(ff')' = \alpha\left((ff')^2 + c^2\right)^{\frac{3}{2}}.$$

Setting $u = ff'$, this equation reduces to the following first-order ordinary differential equation:

$$u' = -\alpha\left(u^2 + c^2\right)^{\frac{3}{2}}. \tag{10}$$

Equation (10) can be integrated by using the substitution $u = c\tan\theta$. Equivalently,

$$\int \frac{du}{(u^2 + c^2)^{\frac{3}{2}}} = -\alpha \int dt.$$

Because

$$\int \frac{du}{(u^2 + c^2)^{\frac{3}{2}}} = \frac{u}{c^2\sqrt{u^2 + c^2}},$$

we get

$$\frac{u}{c^2\sqrt{u^2 + c^2}} = -\alpha t + C_1.$$

Defining

$$k(t) = -\alpha t + C_1,$$

we have

$$k(t) = \frac{u}{c^2\sqrt{u^2 + c^2}}.$$

Solving this relation for u gives

$$u(t) = \frac{c^3 k(t)}{\sqrt{1 - c^4 k(t)^2}}.$$

Because $u = ff'$, we get

$$ff' = \frac{c^3 k(t)}{\sqrt{1 - c^4 k(t)^2}}.$$

On the other hand, because $(f^2)' = 2ff'$, it follows that

$$(f^2)' = \frac{2c^3 k(t)}{\sqrt{1 - c^4 k(t)^2}}.$$

Using $k'(t) = -\alpha$, we obtain

$$f^2(t) = \frac{2}{\alpha c}\sqrt{1 - c^4 k(t)^2} + C_2.$$

Consequently,

$$f(t) = \pm\sqrt{\frac{2}{\alpha c}\sqrt{1 - c^4 k(t)^2} + C_2},$$

where $1 - c^4 k(t)^2 > 0$, $k(t) = -\alpha t + C_1$.

Now, from

$$\alpha = \frac{A}{c^2(1 + A\lambda)},$$

we obtain

$$\lambda = \frac{1}{\alpha c^2} - \frac{1}{A}.$$

Therefore, the same distance can equivalently be written as

$$\lambda = -\left(\frac{1}{c^2 k'(t)} + \frac{1}{A}\right),$$

where $k^{'}(t) = -\alpha$ is a constant. Hence, λ is independent of t and defines a fixed signed parallel distance. Geometrically, this means that the parallel surface is obtained from the original surface by moving each point along the Galilean normal direction by the same signed distance λ, rather than by a parameter-dependent deformation.

Example 2.2. Consider the Type II revolution surface M in G^3 parametrized by (2) and its parallel surface M given by (8). For the choice of the parameters

$$A = \frac{-3}{17},\ c = 1,\ C_2 = 10,\ k(t) = \frac{3}{10}t,$$

we have $k^{'}(t) = 3/10$. Because $k(t) = -\alpha t + C_1$ in the general solution, this choice corresponds to $\alpha = -3/10$ and $C_1 = 0$. Therefore, the generating functions are given by

$$f(t) = \sqrt{10 - \frac{2}{3}\sqrt{100 - 9t^2}},\ g(t) = t.$$

Hence, the surface M admits the parametrization

$$\varphi(s,t) = \left(\sqrt{10 - \frac{2}{3}\sqrt{100 - 9t^2}} + s,\ t,\ s\sqrt{10 - \frac{2}{3}\sqrt{100 - 9t^2}} + \frac{1}{2}s^2\right).$$

Because λ denotes the fixed distance between the surface and its parallel surface, it must be independent of the parameter t. In the present example,

$$\lambda = -\left(\frac{1}{c^2 k'(t)} + \frac{1}{A}\right)$$

gives

$$\lambda = -\left(\frac{10}{3} - \frac{17}{3}\right) = \frac{7}{3}.$$

Therefore, the corresponding parametrization of the parallel surface M^{λ} for $\lambda = 7/3$ is given by

$$\varphi^{\frac{7}{3}}(s,t) = \left(\sqrt{10 - \frac{2}{3}\sqrt{100 - 9t^2}} + s,\ \frac{17}{10}t,\ s\sqrt{10 - \frac{2}{3}\sqrt{100 - 9t^2}} + \frac{1}{2}s^2 + \frac{7}{30}\sqrt{100 - 9t^2} \right).$$

For visualization purposes, the parameters are chosen as

$$s \in [-2, 2]\,,\ t \in [-2, 2]\,.$$

Hence, the revolution surface M together with its corresponding parallel surface $M^{7/3}$ is shown in Figure 2.

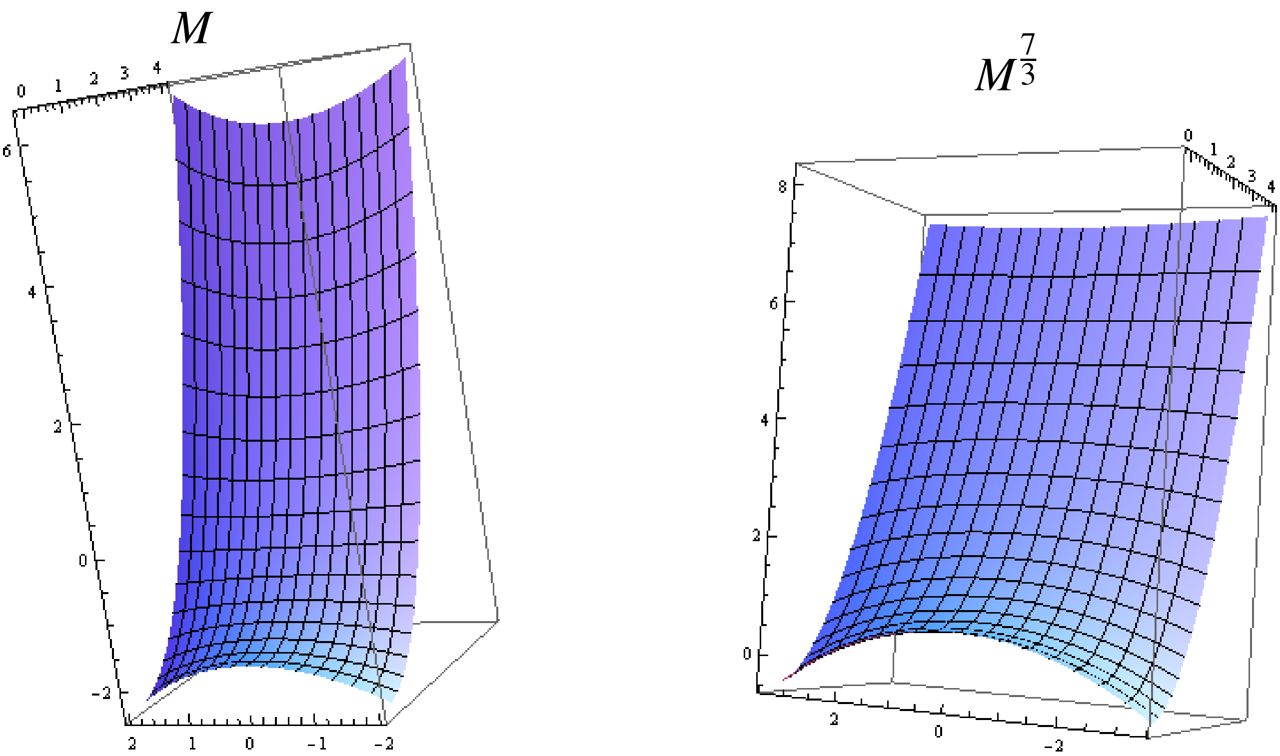

Figure 2. A Type II revolution surface M and its parallel surface $M^{7/3}$ for $g(t) = t$.

The figure illustrates the geometric effect of the parallel surface transformation on a Type II revolution surface generated by the choice $g(t) = t$. The parallel surface remains at a constant Galilean distance $\lambda = 7/3$ from the original surface while preserving its overall revolution structure and constant mean curvature property.

Case 2: Let us first prescribe the function f by choosing

$$f(t) = t,\ f'(t) = 1,\ f''(t) = 0.$$

Then, $Q = tg'' - g'$, $S = t^2 + c^2 (g')^2$, and Eq (9) reduces to

$$tg'' - g' = \alpha \left(t^2 + c^2 (g')^2 \right)^{\frac{3}{2}}.$$

Setting $p = g'$, the above equation becomes

$$tp' - p = \alpha \left(t^2 + c^2 p^2\right)^{\frac{3}{2}}. \tag{11}$$

Equation (11) admits implicit solutions of the form

$$\frac{p}{\sqrt{t^2 + c^2 p^2}} = \frac{\alpha}{2} t^2 + C_3.$$

Indeed, differentiating the left- hand side gives

$$\frac{d}{dt}\left(\frac{p}{\sqrt{t^2 + c^2 p^2}}\right) = \frac{t(tp' - p)}{(t^2 + c^2 p^2)^{\frac{3}{2}}}.$$

Using the differential equation $tp' - p = \alpha \left(t^2 + c^2 p^2\right)^{3/2}$, we get

$$\frac{d}{dt}\left(\frac{p}{\sqrt{t^2 + c^2 p^2}}\right) = \alpha t,$$

which yields

$$\frac{p}{\sqrt{t^2 + c^2 p^2}} = \frac{\alpha}{2} t^2 + C_3.$$

Introducing

$$\ell(t) = \frac{\alpha}{2} t^2 + C_3,$$

we have

$$\frac{p}{\sqrt{t^2 + c^2 p^2}} = \ell(t).$$

Solving this relation for the function p, we obtain

$$p(t) = \pm \frac{|t|\, \ell(t)}{\sqrt{1 - c^2 \ell(t)^2}}.$$

Indeed, substituting this expression into

$$S = t^2 + c^2 p(t)^2$$

gives

$$S = t^2 + c^2 \frac{|t|^2\, \ell(t)^2}{1 - c^2 \ell(t)^2} = \frac{t^2}{1 - c^2 \ell(t)^2}.$$

Hence,

$$\sqrt{S} = \frac{|t|}{\sqrt{1 - c^2 \ell(t)^2}}.$$

Therefore, the solution must be considered on a connected interval not containing $t = 0$. That is,

$$t \in I \subset (0,\ \infty) \text{ or } t \in I \subset (-\infty,\ 0).$$

On such an interval, $|t| = sgn(t)t$, where $sgn(t)$ is constant. Hence,

$$p(t) = \pm sgn(t)\frac{t\ell(t)}{\sqrt{1 - c^2\ell(t)^2}}.$$

Because $p = g'$, the generating function g is obtained as

$$g(t) = \pm sgn(t)\int \frac{t\ell(t)}{\sqrt{1 - c^2\ell(t)^2}}dt + C_4.$$

Using

$$\ell'(t) = \alpha t,$$

we get

$$tdt = \frac{d\ell}{\alpha}.$$

Hence,

$$g(t) = \pm\frac{sgn(t)}{\alpha}\int \frac{\ell}{\sqrt{1 - c^2\ell^2}}d\ell + C_4.$$

By integration,

$$\int \frac{\ell}{\sqrt{1 - c^2\ell^2}}d\ell = -\frac{1}{c^2}\sqrt{1 - c^2\ell^2}.$$

Therefore,

$$g(t) = \pm\frac{sgn(t)}{\alpha c^2}\sqrt{1 - c^2\ell(t)^2} + C_4,$$

where $1 - c^2\ell(t)^2 > 0$, and $t \neq 0$. Now, from

$$\alpha = \frac{A}{c^2\left(1 + A\lambda\right)},$$

we obtain

$$\lambda = \frac{1}{\alpha c^2} - \frac{1}{A},$$

so the same parallel distance can equivalently be written as

$$\lambda = \frac{1}{c^2\ell''(t)} - \frac{1}{A},$$

where $\ell''(t) = \alpha$ is a constant. Hence, λ is independent of t and defines a fixed signed parallel distance.

Example 2.3. Consider the Type II revolution surface M in G^3 parametrized by (2) and its parallel surface M^λ given by (8). For the parameters

$$A = 2,\ c = \frac{1}{2},\ \ell(t) = \frac{1}{2}t^2 + 0.4,\ C_3 = 0.4,\ C_4 = 0,$$

the generating functions are given by

$$f(t) = t,\ g(t) = -4sgn(t)\sqrt{1 - \frac{1}{4}\left(\frac{1}{2}t^2 + 0.4\right)^2}.$$

Here, $t = 0$ is excluded. Indeed, by the result obtained in Case 2,

$$S = t^2 + c^2 \frac{|t|^2 \ell(t)^2}{1 - c^2\ell(t)^2} = \frac{t^2}{1 - c^2\ell(t)^2},$$

and hence,

$$\sqrt{S} = \frac{|t|}{\sqrt{1 - c^2\ell(t)^2}}.$$

Because $f(t) = t$, $f'(t) = 1$, the unit normal vector is given by

$$N = \left(0, \ \frac{t}{\sqrt{S}}, \ \frac{cg'(t)}{\sqrt{S}}\right).$$

Using the expression for $g'(t)$ obtained in Case 2,

$$g'(t) = \pm \frac{|t| \, \ell(t)}{\sqrt{1 - c^2\ell(t)^2}},$$

we obtain

$$\frac{t}{\sqrt{S}} = sgn(t)\sqrt{1 - c^2\ell(t)^2},$$

and

$$\frac{cg'(t)}{\sqrt{S}} = \pm c\ell(t).$$

Hence, the unit normal vector becomes

$$N = \left(0, sgn(t)\sqrt{1 - c^2\ell(t)^2}, \pm c\ell(t)\right).$$

Therefore, the second component of the unit normal vector contains the factor $sgn(t)$, which is undefined at $t = 0$. Consequently, the parametrization is regular only on connected intervals excluding $t = 0$. Accordingly, the surface M and its parallel surface M^λ must be considered separately on the domains $t < 0$ and $t > 0$. The surface M admits the parametrization

$$\varphi(s,t) = \left(t + \frac{1}{2}s, \ -4sgn(t)\sqrt{1 - \frac{1}{4}\left(\frac{1}{2}t^2 + 0.4\right)^2}, \ st + \frac{1}{4}s^2\right),$$

and the corresponding parametrization of the parallel surface M^λ for $\lambda = 7/2$ is given by

$$\varphi^{\frac{7}{2}}(s,t) = \left(t + \frac{1}{2}s, -\frac{1}{2}sgn(t)\sqrt{1 - \frac{1}{4}\left(\frac{1}{2}t^2 + 0.4\right)^2}, \ st + \frac{1}{4}s^2 + \frac{7}{8}t^2 + 0.7\right).$$

To preserve the regularity of the parametrization, the parameters are chosen as

$$s \in [-1.5, \ 1.5], \ t \in [-1.2, \ -0.1] \cup [0.1, \ 1.2].$$

Because $\ell(t) = (1/2)t^2 + 0.4$, we have $\ell''(t) = 1$. Using $\lambda = 1/(c^2\ell''(t)) - 1/A$ with $A = 2$ and $c = 1/2$, we obtain $\lambda = 7/2$. Therefore, the parallel surface is located at the fixed signed distance $\lambda = 7/2$ from

M. The surfaces M and $M^{7/2}$ are depicted in Figure 3.

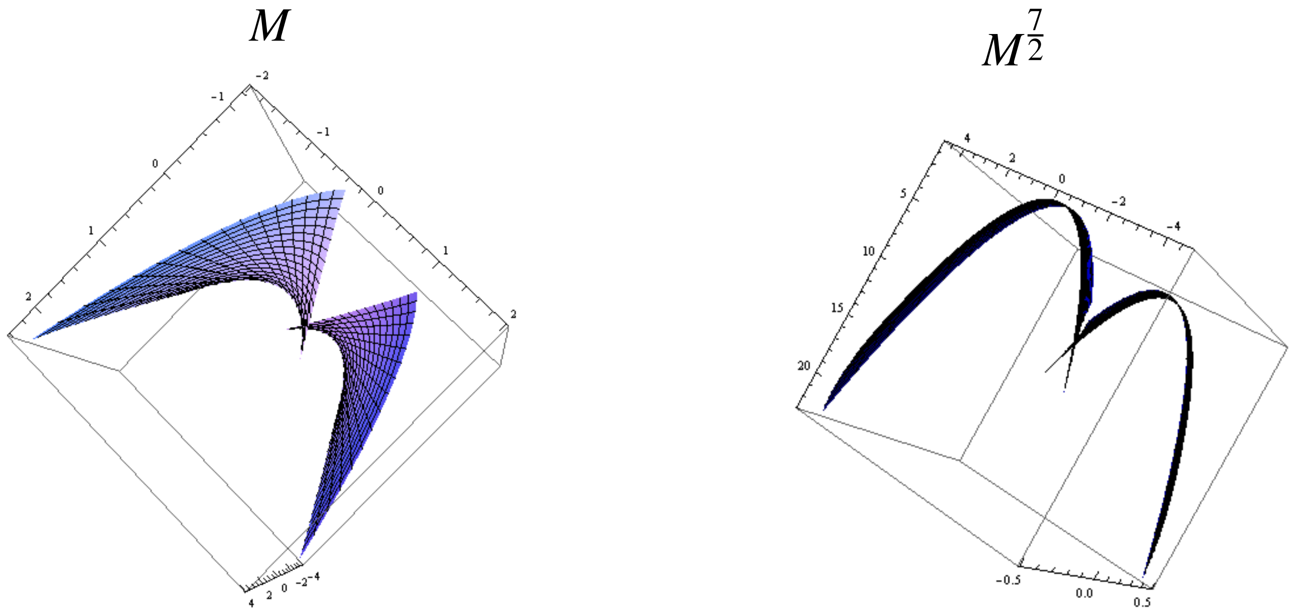

Figure 3. A Type II revolution surface M and its parallel surface $M^{7/2}$.

Figure 3 illustrates another family of Type II revolution surfaces corresponding to the choice $f(t) = t$. The associated parallel surface is located at the constant Galilean distance $\lambda = 7/2$ from the original surface. Comparing Figures 2 and 3 shows that different generating functions produce distinct surface geometries, although the parallel surface construction preserves the constant mean curvature property in both cases.

2.3. Parallel surfaces of Type III revolution surfaces with constant mean curvature in G^3

We consider a Type III surface of revolution in G^3 given by the parametrization (3), where f and g are smooth functions of the parameter t. This parametrization represents a revolution surface obtained by rotating the planar profile curve $(f(t), g(t), 0)$ about the x-axis. Throughout the discussion, we assume that $f'(t)g(t) \neq 0$, which guarantees the regularity of the surface.

Under this assumption, the unit normal vector field of the surface can be written as

$$N = \frac{(0, -f'(t)g(t)\cos s,\ f'(t)g(t)\sin s)}{|f'(t)g(t)|}.$$

For notational convenience, we introduce the auxiliary function

$$\varepsilon(t) = sgn(f'(t)g(t)),$$

which allows the normal vector to be expressed in the simplified form

$$N = (0,\ -\varepsilon(t)\cos s,\ \varepsilon(t)\sin s)\,.$$

From the definition of parallel surfaces and the expressions of φ and N, Eq (4) yields the parametrization of M^{λ} in the form

$$\varphi^{\lambda}(s,t) = (f(t),\ (g(t) - \lambda\varepsilon(t))\cos s, -(g(t) - \lambda\varepsilon(t))\sin s)\,. \tag{12}$$

Hence, the parallel surface M^λ is again a surface of revolution with the same axis of rotation as the original surface. Its radius function is shifted by the amount $\lambda\varepsilon(t)$. In order to avoid singularities, we additionally assume that $g(t) - \lambda\varepsilon(t) \neq 0$.

The mean curvature of a Type III revolution surface in G^3 was obtained in [8] as

$$H = \frac{sgn(f'g)}{2g}.$$

Substituting this expression into the parallel surface formula (5), the mean curvature of the parallel surface M^λ is obtained as

$$H^\lambda = \frac{\varepsilon(t)}{2\left(g(t) - \lambda\varepsilon(t)\right)}.$$

We now impose the constant mean curvature condition $H^\lambda = A/2$, where A is a prescribed constant. Then, the above equation can be rewritten as

$$\frac{\varepsilon(t)}{g(t) - \lambda\varepsilon(t)} = A.$$

Equivalently, this relation can be written as

$$(1 + A\lambda)\,\varepsilon(t) = Ag(t)\,.$$

Assuming $A \neq 0$, we immediately obtain

$$g(t) = \frac{1 + A\lambda}{A}\varepsilon(t)\,.$$

Because $\varepsilon(t) = sgn(f'(t)g(t))$ is a sign function, it is constant on each connected interval where the orientation is fixed. Consequently, the function $g(t)$ is locally constant and may take only the values

$$\pm\frac{1 + A\lambda}{A}.$$

If we denote $\rho = (1+A\lambda)/A > 0$, then the radius function can be written compactly as $g(t) = \rho\varepsilon(t)$. This is consistent with Theorem 4 in [9], which states that a Type III revolution surface in G^3 has constant mean curvature if and only if its radius function is a nonzero constant. In that case, the surface is flat and coincides with a cylindrical surface generated over a Euclidean circle. Geometrically, this phenomenon reflects a rigidity property of Type III revolution surfaces in the Galilean setting. Because the mean curvature of M^λ depends only on the shifted radius function $g(t) - \lambda\varepsilon(t)$, imposing the constant mean curvature condition forces this radius to be constant. Consequently, on each connected interval where the orientation is fixed, the corresponding parallel surface reduces to a cylindrical surface. Unlike Types I and II, where nontrivial families of profile curves arise from differential equations, the Type III case admits only this rigid geometric configuration.

Example 2.4. Consider the Type III revolution surface M in G^3 parametrized by (3) and its parallel surface M^λ given by (12). For the parameters

$$A = 2,\ \varepsilon(t) = 1,$$

the generating functions are given by

$$f(t) = \ln t,\ \ g(t) = \frac{3}{2}.$$

Hence, the surface M admits the parametrization

$$\varphi(s,t) = \left(\ln t,\ \frac{3}{2}\cos s,\ \ -\frac{3}{2}\sin s\right),$$

and the corresponding parametrization of the parallel surface M^{λ} for $\lambda = 1$ is given by

$$\varphi^{1}(s,t) = \left(\ln t,\ \frac{1}{2}\cos s, -\frac{1}{2}\sin s\right).$$

For visualization purposes, the parameters are chosen as

$$s \in [0, 2\pi]\,,\ \ t \in [1, 2]\,.$$

The surface M is a circular cylinder of radius $3/2$, whereas its parallel surface M^1 is a concentric circular cylinder of radius $1/2$. The surfaces M and M^1 are depicted in Figure 4.

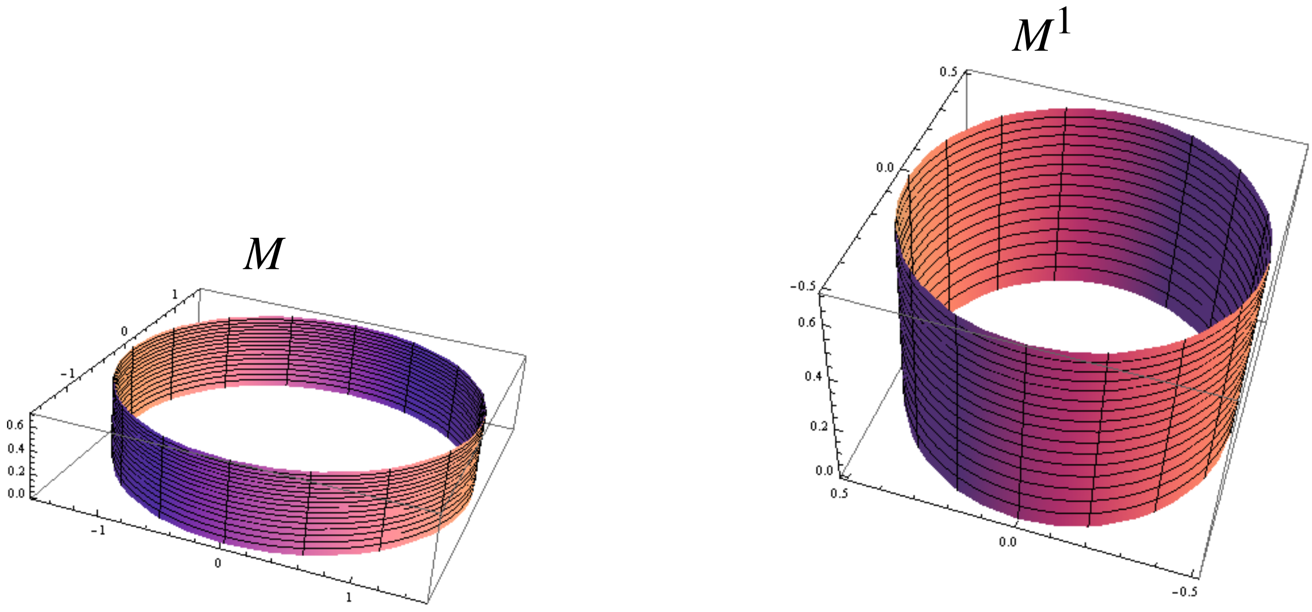

Figure 4. A Type III revolution surface M and its parallel surface M^1.

Figure 4 depicts a Type III revolution surface and its parallel surface. Because the radius function is constant, both surfaces are circular cylinders sharing the same axis of revolution. The parallel transformation changes only the radius of the cylinder, demonstrating the rigidity of the Type III case under the constant mean curvature condition.

3. Conclusions

In this paper, we investigated the parallel surfaces of revolution surfaces in G^3 under prescribed mean curvature conditions. By explicitly computing the mean curvature of the parallel surface, we derived the corresponding differential equations for the profile functions and obtained explicit families of revolution surfaces in G^3. The resulting geometric behavior varies significantly among the three types considered. Type I surfaces are generated by plane curves of constant curvature, Type II surfaces yield nontrivial explicit solution families arising from nonlinear differential equations, and Type III surfaces exhibit a rigidity phenomenon forcing the radius function to be constant.

To illustrate these results, we presented graphical representations of the obtained surfaces for particular choices of the parameters and solution functions. These visualizations provide additional insight into the geometry of parallel revolution surfaces and highlight the effect of the parallel transformation on their shape.

The results obtained in this study extend existing work on revolution and parallel surfaces in Galilean geometry and may serve as a basis for future investigations of more general curvature conditions and other classes of surfaces in non-Euclidean spaces.

Author contributions

İsmet Gölgeleyen: Conceptualization, formal analysis, investigation, methodology, project administration, software, supervision, validation, visualization, writing original draft, writing–review & editing. Elif Yaren Bulgan: Formal analysis, investigation, software, validation, visualization, writing–original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no competing interests.

References

1. B. A. Dubrovin, A. T. Fomenko, S. P. Novikov, *Modern geometry – Methods and applications: Part I. The geometry of surfaces, transformation groups, and fields*, New York: Springer, 1984. https://doi.org/10.1007/978-1-4684-9946-9
2. A. Gray, E. Abbena, S. Salamon, *Modern differential geometry of curves and surfaces with mathematica*, 3 Eds., New York: Chapman & Hall/CRC, 2006. https://doi.org/10.1201/9781315276038

3. A. T. Ali, Position vectors of curves in the Galilean space G^3, *Mat. Vesn.*, **64** (2012), 200–210.
4. M. E. Aydın, M. A. Kulahçı, A. O. Öğrenmiş, Constant curvature translation surfaces in Galilean 3-space, *Int. Electron. J. Geom.*, **12** (2019), 9–19. https://doi.org/10.36890/iejg.545741
5. Z. M. Šipuš, Ruled Weingarten surfaces in Galilean space, *Period. Math. Hung.*, **56** (2008), 213–225. https://doi.org/10.1007/s10998-008-6213-6
6. Ž. M. Šipuš, B. Divjak, Surfaces of constant curvature in the Pseudo-Galilean space, *Int. J. Math. Math. Sci.*, **2012** (2012), 375264. https://doi.org/10.1155/2012/375264
7. Y. Yaylı, I. Gölgeleyen, Minimal curves on ruled surfaces generated by Legendre curves, *J. New Theory*, **49** (2024), 43–52. https://doi.org/10.53570/jnt.1545097
8. M. Dede, C. Ekici, W. Goemans, Surfaces of revolution with vanishing curvature in Galilean 3-space, *J. Math. Phys. Anal. Geo.*, **14** (2018), 141–152. https://doi.org/10.15407/mag14.02.141
9. İ. Gölgeleyen, Y. Yaylı, E. Y. Bulgan, Surfaces of revolution with constant mean curvature in Galilean 3-space, *Mathematics*, **14** (2026), 1066. https://doi.org/10.3390/math14061066
10. İ. Kişi, A new approach to revolution surface with its focal surface in the Galilean 3-space G_3, *Hacet. J. Math. Stat.*, **50** (2021), 1722–1737. https://doi.org/10.15672/hujms.905636
11. D. W. Yoon, Surfaces of revolution in the three dimensional Pseudo-Galilean space, *Glas. Mat.*, **48** (2013), 415–428. https://doi.org/10.3336/gm.48.2.13
12. M. Dede, C. Ekici, A. C. Çöken, On the parallel surfaces in the Galilean space, *Hacet. J. Math. Stat.*, **42** (2013), 605–615.
13. İ. Gölgeleyen, Y. Yaylı, E. Y. Bulgan, Developable ruled surfaces with constant mean curvature along a curve, *Mathematics*, **14** (2026), 234. https://doi.org/10.3390/math14020234
14. S. Büyükkütük, I. Kişi, G. Öztürk, E. Kişi, Parallel hypersurfaces in E^4 and their applications to rotational hypersurfaces, *Mathematics*, **13** (2025), 3684. https://doi.org/10.3390/math13223684
15. U. Dierkes, S. Hildebrandt, A. Küster, O. Wohlrab, *Minimal surfaces I: Boundary value problems*, Berlin, Heidelberg: Springer, 1992. https://doi.org/10.1007/978-3-662-02791-2
16. S. Adriaenssens, P. Block, D. Veenendaal, C. Williams, *Shell structures for architecture: Form finding and optimization*, London: Routledge, 2014. https://doi.org/10.4324/9781315849270
17. M. Dede, Tubular surfaces in Galilean space, *Math. Commun.*, **18** (2013), 209–217.
18. O. Röschel, *Die geometrie des galileischen raumes*, Habilitationsschrift, Leoben, 1984.