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*Research article*

# New inequalities for the A-joint weighted numerical radius with applications

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**Abstract:** Let $(\mathcal{K}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Let $\mathbf{A}$ be a positive (semidefinite) bounded linear operator on $\mathcal{K}$. We introduce the Euclidean-arithmetic mean $\mathbf{A}$-numerical radius for a pair of $\mathbf{A}$-bounded operators $(\mathbf{U},\mathbf{V})$, defined by $\omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V}) := \sup\limits_{\eta\in\Sigma_1^{\mathbf{A}}}\left((1-\varepsilon)|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}|^2 + \varepsilon|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}|^2\right)^{\frac{1}{2}}$, where $\varepsilon\in[0,1]$, $\langle\eta_1,\eta_2\rangle_{\mathbf{A}} = \langle\mathbf{A}\eta_1,\eta_2\rangle$ for all $\eta_1,\eta_2\in\mathcal{K}$, and $\Sigma_1^{\mathbf{A}}$ denotes the unit $\mathbf{A}$-sphere. Furthermore, we define the integral Euclidean numerical radius and present several Hermite-Hadamard-type inequalities. Finally, we provide significant applications of our framework by deriving bounds for the Cartesian decomposition of operators and generalizing the Davis-Wielandt radius to a $p$-arithmetic mean context with $p\geq 1$.

**Keywords:** semi-Hilbertian space; $\mathbf{A}$-numerical radius; Euclidean-arithmetic mean numerical radius; Davis-Wielandt radius; operator inequalities; positive operator
**Mathematics Subject Classification:** 47A12, 47A30, 47A63, 47B65

## 1. Introduction

The geometry and spectral properties of bounded linear operators represent a cornerstone of modern functional analysis, providing fundamental tools for a wide range of applications, particularly in theoretical physics. We refer the reader to the book by Elin, Reich, and Shoikhet [1] and the references cited therein for further details. Building on this rich tradition, in recent years, the study of operators within semi-Hilbertian spaces has seen a significant resurgence of interest due to its rich potential within functional analysis [2]. Specifically, foundational work on metric properties of projections and partial isometries has been established by Arias et al. in [3, 4]. Furthermore, joint normality and closed operators in these spaces were deeply explored in [5,6], while metric operators and their general spectral properties were discussed in [7, 8]. This mathematical framework offers a broader perspective for defining operators that correspond to physical observables in quantum mechanics. Traditionally, the physical states of a quantum system are modeled using a standard Hilbert space $\mathcal{K}$ equipped with a conventional inner product $\langle\cdot,\cdot\rangle$. Within this classical model, operators representing physical observables are strictly required to be self-adjoint relative to this specific inner product. However, this conventional requirement can be overly restrictive. To overcome this limitation, the theory of non-Hermitian quantum mechanics introduces a more flexible approach by establishing a novel inner product under which the relevant operators become self-adjoint.

This alternative inner product is constructed using a specific metric operator $\mathbf{A}$, defined such that $\langle\eta_1,\eta_2\rangle_{\mathbf{A}} = \langle\mathbf{A}\eta_1,\eta_2\rangle$ for any elements $\eta_1,\eta_2 \in \mathcal{K}$. The properties of this metric operator vary depending on the specific quantum framework being utilized. For instance, in quasi-Hermitian quantum mechanics [9,10] and pseudo-Hermitian quantum mechanics [11], the metric operator $\mathbf{A}$ must be invertible, self-adjoint, and positive with respect to the original reference inner product. Conversely, within indefinite metric quantum mechanics [12], the underlying operator is unitary and self-adjoint, but it is not strictly required to be positive. From a purely mathematical perspective, the operator $\mathbf{A}$ is treated as self-adjoint and positive concerning the standard inner product of $\mathcal{K}$; nevertheless, it is generally considered to be non-invertible [2, 13]. This distinct mathematical approach ultimately motivates the need to establish appropriate invertibility criteria with respect to the metric operator $\mathbf{A}$.

Norm and numerical radius inequalities have garnered significant attention in this context, providing critical insights into operator geometry. For instance, the properties of $\mathbf{A}$-normal operators and their numerical radii were developed in [14, 15]. General improvements and bounds for the seminorm and numerical radius in semi-Hilbertian spaces were subsequently derived in [16, 17]. Furthermore, the Euclidean operator radius inequalities for operator pairs were generalized to standard and semi-Hilbert spaces in [18] and [16], respectively. Structural results concerning block matrices and $n \times n$ operator matrices were also heavily investigated in [19–21].

Further developments include bounds for the $\mathbf{A}$-joint numerical radius and $\mathbf{A}$-numerical radius of operator matrices [8,22], refined $\mathbf{A}$-numerical radius inequalities [23–25], and studies on $2\times2$ operator matrices [21, 26]. Orthogonality and norm-parallelism in the $\mathbf{A}$-numerical radius were investigated in [27]. Additional bounds and generalizations have been broadly categorized: translatable radii and related inequalities were established in [28–30], while specific extensions of the $\mathbf{A}$-numerical radius were explored by Qiao et al. [17] and Zamani in [31]. Structured operator matrices were further addressed in [19,20,32], and $(\gamma,\delta)$-norm inequalities with lifting properties were developed in [33–35].

Despite the extensive literature on the $\mathbf{A}$-numerical radius, a significant gap remains regarding the balanced, weighted combinations of multi-operator tuples. Traditional joint numerical radii treat operators symmetrically, which falls short when analyzing quantum observables or multi-parameter systems where components possess unequal physical weights or analytical significance. To fill this gap, this paper introduces novel inequalities for the Euclidean-arithmetic mean numerical radius in semi-Hilbert spaces, building on foundational metric properties [3, 5], joint spectral radius formulations [8,36], and generalized Euclidean operator bounds [16,18]. By introducing the parameter $\varepsilon \in [0, 1]$, we offer a continuous transition and a tunable arithmetic mean approach to the joint numerical radius. This generalization is highly important because it not only unifies various existing symmetric inequalities into a single parametric framework, but it also provides significantly sharper, context-dependent bounds that are essential for precise spectral analysis in non-standard quantum mechanics.

The remainder of this paper is organized as follows. Section 2 provides the necessary preliminaries and notations regarding semi-Hilbertian spaces, **A**-adjoint operators, and the standard **A**-joint numerical radius. Section 3 is dedicated to our main results, where we establish several novel upper and lower bounds for the newly defined Euclidean-arithmetic mean **A**-numerical radius and its associated integral numerical radius. Section 4 explores direct applications of our framework, yielding specialized inequalities for a single operator's Cartesian decomposition and the generalized **A**-Davis-Wielandt radius. Finally, Section 5 concludes the paper by summarizing the new inequalities established for the A-joint weighted numerical radius and discussing possible directions for future research and further applications.

## 2. Preliminaries

To establish a rigorous foundation for our main results, this section outlines the fundamental algebraic and topological structures that govern semi-Hilbertian spaces. We begin by reviewing the standard notations, the construction of semi-inner products, and the essential properties of **A**-adjoint operators, before formally detailing the baseline joint numerical radii frameworks upon which our new generalizations are built.

We use the following notations: $(\mathcal{K}, \langle\cdot,\cdot\rangle)$ denotes a complex Hilbert space with norm $\|\cdot\|$ induced by the inner product. The $C^*$-algebra of bounded linear operators on $\mathcal{K}$ with identity $\mathbf{I}$ is denoted $\mathcal{L}(\mathcal{K})$. The sets $\mathbb{N}_0$ and $\mathbb{N}$ represent the nonnegative and positive integers, respectively. Unless specified, an operator refers to an element of $\mathcal{L}(\mathcal{K})$.

An operator $\mathbf{T} \in \mathcal{L}(\mathcal{K})$ is positive, denoted $\mathbf{T} \geq 0$, if $\langle \mathbf{T}\eta, \eta\rangle \geq 0$ for all $\eta \in \mathcal{K}$. For a positive operator $\mathbf{T}$, its square root is denoted $\mathbf{T}^{\frac{1}{2}}$. We assume $\mathbf{A}$ is a nonzero positive operator, inducing the semi-inner product

$$\langle\cdot,\cdot\rangle_{\mathbf{A}} : \mathcal{K} \times \mathcal{K} \longrightarrow \mathbb{C},\ (\eta, \mathbf{y}) \longmapsto \langle \eta, \mathbf{y}\rangle_{\mathbf{A}} := \langle \mathbf{A}\eta, \mathbf{y}\rangle.$$

The space $(\mathcal{K}, \|\cdot\|_{\mathbf{A}})$ is a semi-Hilbert space, where $\|\eta\|_{\mathbf{A}} = \sqrt{\langle \eta, \eta\rangle_{\mathbf{A}}}$ is the seminorm for all $\eta \in \mathcal{K}$. We define the unit **A**-sphere in $\mathcal{K}$ as $\Sigma_1^{\mathbf{A}} := \{\eta \in \mathcal{K} : \|\eta\|_{\mathbf{A}} = 1\}$. When $\mathbf{A} = \mathbf{I}$, this reduces to the standard unit sphere $\Sigma_1$. Note that $(\mathcal{K}, \|\cdot\|_{\mathbf{A}})$ is neither a normed nor a complete space in general. However, it is a Hilbert space if and only if $\mathbf{A}$ is injective and $\overline{\mathcal{R}(\mathbf{A})} = \mathcal{R}(\mathbf{A})$, where $\mathcal{R}(\mathbf{A})$ is the range of $\mathbf{A}$ and $\overline{\mathcal{R}(\mathbf{A})}$ its closure with respect to the norm $\|\cdot\|$. For clarity, consider the Hilbert space $\ell^2$ of square-summable complex sequences $\eta = \{x_n\}$, with inner product $\langle \eta, \mathbf{y}\rangle = \sum_{k\in\mathbb{N}} x_k\overline{y_k}$. Define $\mathbf{A}$ as the diagonal operator on $\ell^2$ with $\mathbf{A}e_1 = 0$ and $\mathbf{A}e_n = \frac{e_n}{n}$ for $n \geq 2$, where $\{e_n\}$ is the canonical basis. Since $\mathbf{A}$ is neither injective nor satisfies $\overline{\mathcal{R}(\mathbf{A})} = \mathcal{R}(\mathbf{A})$, the semi-Hilbert space $(\ell^2, \langle\cdot,\cdot\rangle_{\mathbf{A}})$ is neither normed nor complete.

An operator $\mathbf{R} \in \mathcal{L}(\mathcal{K})$ is an $\mathbf{A}$-adjoint of $\mathbf{T}$ if $\langle \mathbf{T}\eta, \mathbf{y}\rangle_{\mathbf{A}} = \langle \eta, \mathbf{R}\mathbf{y}\rangle_{\mathbf{A}}$ for all $\eta, \mathbf{y} \in \mathcal{K}$, or equivalently, $\mathbf{AR} = \mathbf{T}^*\mathbf{A}$. Not every $\mathbf{T} \in \mathcal{L}(\mathcal{K})$ has an $\mathbf{A}$-adjoint, and when it does exist, it is not unique. This poses challenges in studying operators on semi-Hilbert spaces. The Douglas range inclusion theorem [37] is crucial here, stating that the equation $\mathbf{TD} = \mathbf{S}$ has a solution $\mathbf{D} \in \mathcal{L}(\mathcal{K})$ if and only if $\mathcal{R}(\mathbf{S}) \subseteq \mathcal{R}(\mathbf{T})$, or equivalently, there exists $\alpha > 0$ such that $\|\mathbf{S}^*\eta\| \leq \alpha\|\mathbf{T}^*\eta\|$ for all $\eta \in \mathcal{K}$. By Douglas' theorem, we can select an $\mathbf{A}$-adjoint operator $\mathbf{T}^{\sharp_{\mathbf{A}}} = \mathbf{A}^\dagger\mathbf{T}^*\mathbf{A}$, where $\mathbf{A}^\dagger$ is the Moore-Penrose pseudo-inverse of $\mathbf{A}$ (see [3]). The operator $\mathbf{T}^{\sharp_{\mathbf{A}}}$ shares some properties with $\mathbf{T}^*$, but not all, e.g., $(\mathbf{T}^{\sharp_{\mathbf{A}}})^{\sharp_{\mathbf{A}}} = \mathbf{T}$ holds if and only if $\mathcal{R}(\mathbf{T}) \subseteq \overline{\mathcal{R}(\mathbf{A})}$.

Define $\mathcal{L}_{\mathbf{A}}(\mathcal{K})$ and $\mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$ as the sets of operators admitting $\mathbf{A}$-adjoints and $\sqrt{\mathbf{A}}$-adjoints, respectively. By Douglas' theorem,

$$\mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K}) = \left\{\mathbf{T} \in \mathcal{L}(\mathcal{K}) : \exists\, c > 0 \text{ such that } \|\mathbf{T}\eta\|_{\mathbf{A}} \leq c\|\eta\|_{\mathbf{A}},\ \forall \eta \in \mathcal{K}\right\}.$$

Operators in $\mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$ are $\mathbf{A}$-bounded. The sets $\mathcal{L}_{\mathbf{A}}(\mathcal{K})$ and $\mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$ are subalgebras of $\mathcal{L}(\mathcal{K})$, generally neither closed nor dense, with inclusions $\mathcal{L}_{\mathbf{A}}(\mathcal{K}) \subseteq \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K}) \subseteq \mathcal{L}(\mathcal{K})$.

For a complex Hilbert space $(\mathcal{K}, \langle\cdot,\cdot\rangle)$ and a positive operator $\mathbf{A}$, the $\mathbf{A}$-joint numerical radius of $\mathbf{A}$-bounded operators $\mathbf{T}, \mathbf{S}$ is defined as [36]

$$\omega_{\mathbf{A},e}(\mathbf{T}, \mathbf{S}) = \sup_{\eta \in \Sigma_1^{\mathbf{A}}} \sqrt{|\langle \mathbf{T}\eta, \eta\rangle_{\mathbf{A}}|^2 + |\langle \mathbf{S}\eta, \eta\rangle_{\mathbf{A}}|^2}.$$

Feki [36] established the following upper bounds:

$$\omega_{\mathbf{A},e}(\mathbf{T}, \mathbf{S}) \leq \sqrt{\|\mathbf{T}\|_{\mathbf{A}}^4 + \|\mathbf{S}\|_{\mathbf{A}}^4 + 2\omega_{\mathbf{A}}^2(\mathbf{S}^{\sharp_{\mathbf{A}}}\mathbf{T})} \leq \|\mathbf{T}\|_{\mathbf{A}}^2 + \|\mathbf{S}\|_{\mathbf{A}}^2,$$

$$\omega_{\mathbf{A},e}(\mathbf{T}, \mathbf{S}) \leq \left[\omega_{\mathbf{A}}\left((\mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T})^2 + (\mathbf{S}^{\sharp_{\mathbf{A}}}\mathbf{S})^2\right) + 2\omega_{\mathbf{A}}^2(\mathbf{S}^{\sharp_{\mathbf{A}}}\mathbf{T})\right]^{\frac{1}{4}},$$

$$\begin{aligned}\omega_{\mathbf{A},e}(\mathbf{T}, \mathbf{S}) &\leq \frac{\sqrt{2}}{2}\sqrt{\|\mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T} + \mathbf{S}^{\sharp_{\mathbf{A}}}\mathbf{S}\|_{\mathbf{A}} + \|\mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T} - \mathbf{S}^{\sharp_{\mathbf{A}}}\mathbf{S}\|_{\mathbf{A}} + 2\omega_{\mathbf{A}}(\mathbf{S}^{\sharp_{\mathbf{A}}}\mathbf{T})} \\ &\leq \sqrt{\|\mathbf{T}\|_{\mathbf{A}}^2 + \|\mathbf{S}\|_{\mathbf{A}}^2 + \omega_{\mathbf{A}}(\mathbf{S}^{\sharp_{\mathbf{A}}}\mathbf{T})},\end{aligned}$$

and

$$\omega_{\mathbf{A},e}(\mathbf{T}, \mathbf{S}) \leq \sqrt{\max\left(\|\mathbf{T}\|_{\mathbf{A}}^2, \|\mathbf{S}\|_{\mathbf{A}}^2\right) + \omega_{\mathbf{A}}(\mathbf{S}^{\sharp_{\mathbf{A}}}\mathbf{T})}. \tag{2.1}$$

Inequality (2.1) is sharp. Additionally,

$$\omega_{\mathbf{A},e}(\mathbf{T}, \mathbf{S}) \leq \sqrt{\max\left\{\omega_{\mathbf{A}}(\mathbf{T}), \omega_{\mathbf{A}}(\mathbf{S})\right\}\sqrt{\|\mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T} + \mathbf{S}^{\sharp_{\mathbf{A}}}\mathbf{S}\|_{\mathbf{A}} + 2\omega_{\mathbf{A}}(\mathbf{S}^{\sharp_{\mathbf{A}}}\mathbf{T})}}.$$

Feki [36] also derived lower and upper bounds:

$$\frac{\sqrt{2}}{2}\max\left\{\omega_{\mathbf{A}}(\mathbf{T} + \mathbf{S}), \omega_{\mathbf{A}}(\mathbf{T} - \mathbf{S})\right\} \leq \omega_{\mathbf{A},e}(\mathbf{T}, \mathbf{S}) \leq \frac{\sqrt{2}}{2}\sqrt{\omega_{\mathbf{A}}^2(\mathbf{T} + \mathbf{S}) + \omega_{\mathbf{A}}^2(\mathbf{T} - \mathbf{S})}.$$

The constant $\frac{\sqrt{2}}{2}$ is sharp in both inequalities. Furthermore,

$$\frac{\sqrt{2}}{2}\sqrt{\omega_{\mathbf{A}}(\mathbf{T}^2+\mathbf{S}^2)} \leq \omega_{\mathbf{A},e}(\mathbf{T},\mathbf{S}) \leq \sqrt{\|\mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T}+\mathbf{S}^{\sharp_{\mathbf{A}}}\mathbf{S}\|_{\mathbf{A}}}$$

with $\frac{\sqrt{2}}{2}$ a sharp constant. In [36], inequalities involving the **A**-Davis-Wielandt radius and **A**-numerical radii of **A**-bounded operators were also obtained.

We introduce the following generalization of the joint numerical radius, which forms the central object of study in this paper.

**Definition 2.1.** *Let* $(\mathbf{U},\mathbf{V}) \in \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K}) \times \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$ *be a pair of operators, let* $\varepsilon \in [0,1]$, *and let* $p \geq 1$. *We define the p-arithmetic mean* **A**-*numerical radius of the pair* $(\mathbf{U},\mathbf{V})$ *as*

$$\omega_{\mathbf{A},p,\varepsilon}(\mathbf{U},\mathbf{V}) := \sup_{\eta\in\Sigma_1^{\mathbf{A}}} \left((1-\varepsilon)\left|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right|^p + \varepsilon\left|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}\right|^p\right)^{\frac{1}{p}}.$$

**Remark 2.1.** *This definition introduces a parameterized, weighted balance between the two operators. We note several immediate properties of this functional. First, it is evident that* $\omega_{\mathbf{A},p,\varepsilon}(\mathbf{U},\mathbf{V}) \geq 0$ *for any pair* $(\mathbf{U},\mathbf{V}) \in \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K}) \times \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$. *Furthermore,* $\omega_{\mathbf{A},p,\varepsilon}(\mathbf{U},\mathbf{V}) = 0$ *implies* $\mathbf{AU} = \mathbf{AV} = 0$ *(which strictly means* $(\mathbf{U},\mathbf{V}) = (0,0)$ *if* **A** *is injective). It is also absolutely homogeneous: For* $\alpha \in \mathbb{C}$, $\omega_{\mathbf{A},p,\varepsilon}(\alpha\mathbf{U},\alpha\mathbf{V}) = |\alpha|\,\omega_{\mathbf{A},p,\varepsilon}(\mathbf{U},\mathbf{V})$. *By the weighted Minkowski inequality,*

$$\omega_{\mathbf{A},e,\varepsilon}(\mathbf{U}+\mathbf{C},\mathbf{V}+\mathbf{D}) \leq \omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V}) + \omega_{\mathbf{A},e,\varepsilon}(\mathbf{C},\mathbf{D}),$$

*showing that* $\omega_{\mathbf{A},p,\varepsilon}(\cdot,\cdot)$ *is a seminorm on* $\mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})\times\mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$. *By the convexity of the function* $g(t) = t^p$, $p \geq 1$ *on* $[0,\infty)$, *we have*

$$\begin{aligned}\left((1-\varepsilon)\left|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right|^p + \varepsilon\left|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}\right|^p\right)^{\frac{1}{p}} &\geq (1-\varepsilon)\left|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right| + \varepsilon\left|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}\right| \\ &\geq \left|\langle\left[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right]\eta,\eta\rangle_{\mathbf{A}}\right|\end{aligned}$$

*for all* $\eta \in \mathcal{K}$. *Taking the supremum over* $\eta \in \Sigma_1^{\mathbf{A}}$ *yields the fundamental lower bound*

$$\omega_{\mathbf{A},p,\varepsilon}(\mathbf{U},\mathbf{V}) \geq \omega_{\mathbf{A}}\left[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right] \tag{2.2}$$

*for* $\varepsilon \in [0,1]$ *and* $p \geq 1$.

*Let us highlight some important particular cases derived from specific choices of the parameters p and* $\varepsilon$:

- ***Symmetric case*** ($\varepsilon = \frac{1}{2}$)***:*** *The operators are weighted equally, yielding a scaled version of the standard joint numerical radius:*

$$\omega_{\mathbf{A},p,\frac{1}{2}}(\mathbf{U},\mathbf{V}) = \left(\frac{1}{2}\right)^{\frac{1}{p}}\omega_{\mathbf{A},p}(\mathbf{U},\mathbf{V}) \quad \text{for } p \geq 1.$$

- ***Linear case*** ($p = 1$)***:*** *The functional reduces to a weighted sum of absolute values:*

$$\omega_{\mathbf{A},\varepsilon}(\mathbf{U},\mathbf{V}) := \omega_{\mathbf{A},1,\varepsilon}(\mathbf{U},\mathbf{V}) = \sup_{\eta\in\Sigma_1^{\mathbf{A}}}\left((1-\varepsilon)\left|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right| + \varepsilon\left|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}\right|\right).$$

- ***Euclidean case** ($p = 2$): This serves as the primary metric constraint for generalized operator inequalities in much of the literature:*

$$\omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V}) := \omega_{\mathbf{A},2,\varepsilon}(\mathbf{U},\mathbf{V}) = \sup_{\eta\in\Sigma_1^{\mathbf{A}}}\left((1-\varepsilon)\left|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right|^2 + \varepsilon\left|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}\right|^2\right)^{\frac{1}{2}}.$$

Inspired by these results, this paper derives several upper and lower bounds for the joint $p$-arithmetic mean **A**-numerical radius of operator pairs and the integral numerical radius, with applications to related functionals.

## 3. Main results

In this section, we present our main results. Specifically, we establish several novel upper and lower bounds for the Euclidean-arithmetic mean **A**-numerical radius $\omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})$. By leveraging reverse Young and Jensen inequalities, we deduce generalized relations that capture the spectral behavior of the operator pair $(\mathbf{U},\mathbf{V})$. We also extend these evaluations to generalized $p$-arithmetic mean cases, providing a rich, parametric framework for operator inequalities in semi-Hilbertian spaces.

The following result provides a sharp upper bound for the Euclidean arithmetic mean **A**-numerical radius utilizing a refined reverse Young's inequality.

**Theorem 3.1.** *Let $\mathbf{U},\mathbf{V}\in\mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$. Let $\varepsilon\in[0,1]$. Then,*

$$\omega^2_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V}) \le R\omega^2_{\mathbf{A}}(\mathbf{U}-\mathbf{V}) + \omega^{2(1-\varepsilon)}_{\mathbf{A}}(\mathbf{U})\omega^{2\varepsilon}_{\mathbf{A}}(\mathbf{V}). \tag{3.1}$$

*In particular, for $\varepsilon = \frac{1}{2}$, we obtain*

$$\omega^2_{\mathbf{A},e}(\mathbf{U},\mathbf{V}) \le \omega^2_{\mathbf{A}}(\mathbf{U}-\mathbf{V}) + 2\omega_{\mathbf{A}}(\mathbf{U})\omega_{\mathbf{A}}(\mathbf{V}).$$

*Proof.* We apply the double inequality from Kittaneh and Manasrah [38,39], refining and providing an additive reverse for Young's inequality:

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 \le (1-\varepsilon)a + \varepsilon b - a^{1-\varepsilon}b^{\varepsilon} \le R\left(\sqrt{a}-\sqrt{b}\right)^2, \tag{3.2}$$

where $a,b\ge 0$, $\varepsilon\in[0,1]$, $r=\min\{1-\varepsilon,\varepsilon\}$, and $R=\max\{1-\varepsilon,\varepsilon\}$.

Setting $a = |\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}|^2$ and $b = |\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}|^2$ in (3.2), we obtain

$$\begin{aligned}
&(1-\varepsilon)\left|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right|^2 + \varepsilon\left|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}\right|^2\\
&\le \left|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right|^{2(1-\varepsilon)}\left|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}\right|^{2\varepsilon} + R\left(\left|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right| - \left|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}\right|\right)^2\\
&\le \left|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right|^{2(1-\varepsilon)}\left|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}\right|^{2\varepsilon} + R\left|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}} - \langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}\right|^2\\
&= \left|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right|^{2(1-\varepsilon)}\left|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}\right|^{2\varepsilon} + R\left|\langle(\mathbf{U}-\mathbf{V})\eta,\eta\rangle_{\mathbf{A}}\right|^2
\end{aligned}$$

for $\eta\in\mathcal{K}$.

Let $\eta\in\Sigma_1^{\mathbf{A}}$. Then, we obtain

$$\left|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right|^{2(1-\varepsilon)}\left|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}\right|^{2\varepsilon} \le \omega^{2(1-\varepsilon)}_{\mathbf{A}}(\mathbf{U})\omega^{2\varepsilon}_{\mathbf{A}}(\mathbf{V})$$

and

$$|\langle(\mathbf{U}-\mathbf{V})\eta,\eta\rangle_{\mathbf{A}}|^2 \leq \omega_{\mathbf{A}}^2(\mathbf{U}-\mathbf{V}).$$

Thus,

$$(1-\varepsilon)\,|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}|^2+\varepsilon\,|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}|^2 \leq \omega_{\mathbf{A}}^{2(1-\varepsilon)}(\mathbf{U})\omega_{\mathbf{A}}^{2\varepsilon}(\mathbf{V})+R\omega_{\mathbf{A}}^2(\mathbf{U}-\mathbf{V})$$

for all $\eta\in\Sigma_1^{\mathbf{A}}$. Taking the supremum over $\eta\in\Sigma_1^{\mathbf{A}}$ yields (3.1). This completes the proof.

In the next result, we complement the previous upper bound by deriving a robust lower bound for the Euclidean arithmetic mean **A**-numerical radius.

**Theorem 3.2.** *Let* $\mathbf{U},\mathbf{V}\in\mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$. *Let* $\varepsilon\in[0,1]$. *Then,*

$$\begin{aligned}\omega_{\mathbf{A},e,\varepsilon}^2(\mathbf{U},\mathbf{V}) \geq{}& \max\left\{\omega_{\mathbf{A}}\left[(1-\varepsilon)^2\mathbf{U}^2+\varepsilon^2\mathbf{V}^2\right],(1-\varepsilon)\varepsilon\omega_{\mathbf{A}}(\mathbf{UV}+\mathbf{VU})\right\}\\ &+\max\left\{(1-\varepsilon)\omega_{\mathbf{A}}(\mathbf{U}),\varepsilon\omega_{\mathbf{A}}(\mathbf{V})\right\}\times\left|\omega_{\mathbf{A}}\left[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right]-\omega_{\mathbf{A}}\left[(1-\varepsilon)\mathbf{U}-\varepsilon\mathbf{V}\right]\right|.\end{aligned}\tag{3.3}$$

*In particular, for* $\varepsilon=\frac{1}{2}$, *we obtain*

$$\begin{aligned}\omega_{\mathbf{A},e}^2(\mathbf{U},\mathbf{V}) \geq{}&\frac{1}{2}\max\left\{\omega_{\mathbf{A}}(\mathbf{U}^2+\mathbf{V}^2),\omega_{\mathbf{A}}(\mathbf{UV}+\mathbf{VU})\right\}\\ &+\frac{1}{2}\max\left\{\omega_{\mathbf{A}}(\mathbf{U}),\omega_{\mathbf{A}}(\mathbf{V})\right\}\left|\omega_{\mathbf{A}}(\mathbf{U}+\mathbf{V})-\omega_{\mathbf{A}}(\mathbf{U}-\mathbf{V})\right|.\end{aligned}$$

*Proof.* From (2.2) with $p=2$, we have

$$\omega_{\mathbf{A},e,\varepsilon}^2(\mathbf{U},\mathbf{V}) \geq \omega_{\mathbf{A}}^2\left[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right]$$

and

$$\omega_{\mathbf{A},e,\varepsilon}^2(\mathbf{U},\mathbf{V}) \geq \omega_{\mathbf{A}}^2\left[(1-\varepsilon)\mathbf{U}-\varepsilon\mathbf{V}\right].$$

Let $\varepsilon\in[0,1]$. This implies that

$$\begin{aligned}\omega_{\mathbf{A},e,\varepsilon}^2(\mathbf{U},\mathbf{V}) &\geq \max\left\{\omega_{\mathbf{A}}^2\left[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right],\omega_{\mathbf{A}}^2\left[(1-\varepsilon)\mathbf{U}-\varepsilon\mathbf{V}\right]\right\}\\ &=\frac{1}{2}\left(\omega_{\mathbf{A}}^2\left[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right]+\omega_{\mathbf{A}}^2\left[(1-\varepsilon)\mathbf{U}-\varepsilon\mathbf{V}\right]\right)\\ &\quad+\frac{1}{2}\left|\omega_{\mathbf{A}}^2\left[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right]-\omega_{\mathbf{A}}^2\left[(1-\varepsilon)\mathbf{U}-\varepsilon\mathbf{V}\right]\right|\\ &=\frac{1}{2}\left(\omega_{\mathbf{A}}^2\left[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right]+\omega_{\mathbf{A}}^2\left[(1-\varepsilon)\mathbf{U}-\varepsilon\mathbf{V}\right]\right)\\ &\quad+\frac{1}{2}\left[\omega_{\mathbf{A}}\left[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right]+\omega_{\mathbf{A}}\left[(1-\varepsilon)\mathbf{U}-\varepsilon\mathbf{V}\right]\right]\\ &\quad\times\left|\omega_{\mathbf{A}}\left[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right]-\omega_{\mathbf{A}}\left[(1-\varepsilon)\mathbf{U}-\varepsilon\mathbf{V}\right]\right|.\end{aligned}\tag{3.4}$$

Observe that

$$\begin{aligned}&\omega_{\mathbf{A}}^2\left[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right]+\omega_{\mathbf{A}}^2\left[(1-\varepsilon)\mathbf{U}-\varepsilon\mathbf{V}\right]\\ &\geq\omega_{\mathbf{A}}\left(\left[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right]^2\right)+\omega_{\mathbf{A}}\left(\left[(1-\varepsilon)\mathbf{U}-\varepsilon\mathbf{V}\right]^2\right)\\ &\geq\max\left\{\omega_{\mathbf{A}}\left(\left[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right]^2+\left[(1-\varepsilon)\mathbf{U}-\varepsilon\mathbf{V}\right]^2\right),\ \omega_{\mathbf{A}}\left(\left[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right]^2-\left[(1-\varepsilon)\mathbf{U}-\varepsilon\mathbf{V}\right]^2\right)\right\}.\end{aligned}\tag{3.5}$$

Since

$$\begin{aligned}\omega_{\mathbf{A}}\left([(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}]^2+[(1-\varepsilon)\mathbf{U}-\varepsilon\mathbf{V}]^2\right) &= \omega_{\mathbf{A}}\Big[(1-\varepsilon)^2\mathbf{U}^2+(1-\varepsilon)\varepsilon(\mathbf{UV}+\mathbf{VU})+\varepsilon^2\mathbf{V}^2\\ &\quad +(1-\varepsilon)^2\mathbf{U}^2-(1-\varepsilon)\varepsilon(\mathbf{UV}+\mathbf{VU})+\varepsilon^2\mathbf{V}^2\Big]\\ &= 2\omega_{\mathbf{A}}\left[(1-\varepsilon)^2\mathbf{U}^2+\varepsilon^2\mathbf{V}^2\right]\end{aligned}$$

and

$$\begin{aligned}\omega_{\mathbf{A}}\left([(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}]^2-[(1-\varepsilon)\mathbf{U}-\varepsilon\mathbf{V}]^2\right) &= \omega_{\mathbf{A}}\Big[(1-\varepsilon)^2\mathbf{U}^2+(1-\varepsilon)\varepsilon(\mathbf{UV}+\mathbf{VU})+\varepsilon^2\mathbf{V}^2\\ &\quad -(1-\varepsilon)^2\mathbf{U}^2+(1-\varepsilon)\varepsilon(\mathbf{UV}+\mathbf{VU})-\varepsilon^2\mathbf{V}^2\Big]\\ &= 2(1-\varepsilon)\varepsilon\omega_{\mathbf{A}}(\mathbf{UV}+\mathbf{VU}),\end{aligned}$$

by using (3.5), we obtain for $\varepsilon\in[0,1]$,

$$\begin{aligned}&\omega_{\mathbf{A}}^2\left[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right]+\omega_{\mathbf{A}}^2\left[(1-\varepsilon)\mathbf{U}-\varepsilon\mathbf{V}\right]\\ &\geq 2\max\left\{\omega_{\mathbf{A}}\left[(1-\varepsilon)^2\mathbf{U}^2+\varepsilon^2\mathbf{V}^2\right],(1-\varepsilon)\varepsilon\omega_{\mathbf{A}}(\mathbf{UV}+\mathbf{VU})\right\}.\end{aligned}\tag{3.6}$$

Additionally,

$$\omega_{\mathbf{A}}\left[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right]+\omega_{\mathbf{A}}\left[(1-\varepsilon)\mathbf{U}-\varepsilon\mathbf{V}\right]\geq 2\max\left\{\omega_{\mathbf{A}}\left[(1-\varepsilon)\mathbf{U}\right],\omega_{\mathbf{A}}(\varepsilon\mathbf{V})\right\}\tag{3.7}$$

for $\varepsilon\in[0,1]$. Combining (3.4), (3.6), and (3.7), we obtain (3.3). This completes the proof.

As a direct consequence of Theorem 3.2, we deduce the following corollary.

**Corollary 3.1.** *Let* $\mathbf{X},\mathbf{Y}\in\mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$. *For all* $\varepsilon\in[0,1]$, *we have*

$$\begin{aligned}&\omega_{\mathbf{A},e}^2(\mathbf{X},\mathbf{Y})+2\left|1-2\varepsilon\right|\omega_{\mathbf{A}}(\mathbf{X})\omega_{\mathbf{A}}(\mathbf{Y})\\ &\geq \omega_{\mathbf{A},e,\varepsilon}^2(\mathbf{X}+\mathbf{Y},\mathbf{X}-\mathbf{Y})\\ &\geq \max\left\{\omega_{\mathbf{A}}\left[\left[(1-\varepsilon)^2+\varepsilon^2\right](\mathbf{X}^2+\mathbf{Y}^2)+(1-2\varepsilon)(\mathbf{XY}+\mathbf{YX})\right],\ 2(1-\varepsilon)\varepsilon\omega_{\mathbf{A}}(\mathbf{X}^2-\mathbf{Y}^2)\right\}\\ &\quad +\max\left\{(1-\varepsilon)\omega_{\mathbf{A}}(\mathbf{X}+\mathbf{Y}),\varepsilon\omega_{\mathbf{A}}(\mathbf{X}-\mathbf{Y})\right\}\times\left|\omega_{\mathbf{A}}\left[\mathbf{X}+(1-2\varepsilon)\mathbf{Y}\right]-\omega_{\mathbf{A}}\left[(1-2\varepsilon)\mathbf{X}+\mathbf{Y}\right]\right|.\end{aligned}\tag{3.8}$$

*In particular, for* $\varepsilon=\frac{1}{2}$, *we obtain*

$$\begin{aligned}\omega_{\mathbf{A},e}^2(\mathbf{X},\mathbf{Y})\geq&\frac{1}{2}\max\left\{\omega_{\mathbf{A}}(\mathbf{X}^2+\mathbf{Y}^2),\omega_{\mathbf{A}}(\mathbf{X}^2-\mathbf{Y}^2)\right\}\\ &+\frac{1}{2}\max\left\{\omega_{\mathbf{A}}(\mathbf{X}+\mathbf{Y}),\omega_{\mathbf{A}}(\mathbf{X}-\mathbf{Y})\right\}\left|\omega_{\mathbf{A}}(\mathbf{X})-\omega_{\mathbf{A}}(\mathbf{Y})\right|.\end{aligned}$$

*Proof.* Applying (3.3) with $\mathbf{U}=\mathbf{X}+\mathbf{Y}$ and $\mathbf{V}=\mathbf{X}-\mathbf{Y}$, we obtain

$$\begin{aligned}\omega_{\mathbf{A},2,\varepsilon}^2(\mathbf{X}+\mathbf{Y},\mathbf{X}-\mathbf{Y})\geq\max&\left\{\omega_{\mathbf{A}}\left[(1-\varepsilon)^2(\mathbf{X}+\mathbf{Y})^2+\varepsilon^2(\mathbf{X}-\mathbf{Y})^2\right],\right.\\ &\left.(1-\varepsilon)\varepsilon\omega_{\mathbf{A}}\left((\mathbf{X}+\mathbf{Y})(\mathbf{X}-\mathbf{Y})+(\mathbf{X}-\mathbf{Y})(\mathbf{X}+\mathbf{Y})\right)\right\}\\ &+\max\left\{(1-\varepsilon)\omega_{\mathbf{A}}(\mathbf{X}+\mathbf{Y}),\varepsilon\omega_{\mathbf{A}}(\mathbf{X}-\mathbf{Y})\right\}\\ &\times\left|\omega_{\mathbf{A}}\left[(1-\varepsilon)(\mathbf{X}+\mathbf{Y})+\varepsilon(\mathbf{X}-\mathbf{Y})\right]-\omega_{\mathbf{A}}\left[(1-\varepsilon)(\mathbf{X}+\mathbf{Y})-\varepsilon(\mathbf{X}-\mathbf{Y})\right]\right|\end{aligned}\tag{3.9}$$

for $\varepsilon \in [0,1]$. Observe that

$$\begin{aligned}(1-\varepsilon)^2(\mathbf{X}+\mathbf{Y})^2+\varepsilon^2(\mathbf{X}-\mathbf{Y})^2 &= (1-\varepsilon)^2(\mathbf{X}^2+\mathbf{XY}+\mathbf{YX}+\mathbf{Y}^2)+\varepsilon^2(\mathbf{X}^2-\mathbf{XY}-\mathbf{YX}+\mathbf{Y}^2)\\ &= \left[(1-\varepsilon)^2+\varepsilon^2\right](\mathbf{X}^2+\mathbf{Y}^2)+(1-2\varepsilon)(\mathbf{XY}+\mathbf{YX}),\end{aligned}$$

and

$$\begin{aligned}(\mathbf{X}+\mathbf{Y})(\mathbf{X}-\mathbf{Y})+(\mathbf{X}-\mathbf{Y})(\mathbf{X}+\mathbf{Y}) &= \mathbf{X}^2+\mathbf{YX}-\mathbf{XY}-\mathbf{Y}^2+\mathbf{X}^2-\mathbf{YX}+\mathbf{XY}-\mathbf{Y}^2\\ &= 2(\mathbf{X}^2-\mathbf{Y}^2).\end{aligned}$$

Applying (3.9), we obtain the second inequality in (3.8).

Further, we have

$$\begin{aligned}\omega^2_{\mathbf{A},2,\varepsilon}(\mathbf{X}+\mathbf{Y},\mathbf{X}-\mathbf{Y}) &= \sup_{\eta\in\Sigma_1^{\mathbf{A}}}\left((1-\varepsilon)\left|\langle(\mathbf{X}+\mathbf{Y})\eta,\eta\rangle_{\mathbf{A}}\right|^2+\varepsilon\left|\langle(\mathbf{X}-\mathbf{Y})\eta,\eta\rangle_{\mathbf{A}}\right|^2\right)\\ &= \sup_{\eta\in\Sigma_1^{\mathbf{A}}}\left((1-\varepsilon)\left|\langle\mathbf{X}\eta,\eta\rangle_{\mathbf{A}}+\langle\mathbf{Y}\eta,\eta\rangle_{\mathbf{A}}\right|^2+\varepsilon\left|\langle\mathbf{X}\eta,\eta\rangle_{\mathbf{A}}-\langle\mathbf{Y}\eta,\eta\rangle_{\mathbf{A}}\right|^2\right).\end{aligned}$$

Let $\eta \in \mathcal{K}$ and $\varepsilon \in [0,1]$. We have

$$\begin{aligned}&(1-\varepsilon)\left|\langle\mathbf{X}\eta,\eta\rangle_{\mathbf{A}}+\langle\mathbf{Y}\eta,\eta\rangle_{\mathbf{A}}\right|^2+\varepsilon\left|\langle\mathbf{X}\eta,\eta\rangle_{\mathbf{A}}-\langle\mathbf{Y}\eta,\eta\rangle_{\mathbf{A}}\right|^2\\ &= (1-\varepsilon)\left(\left|\langle\mathbf{X}\eta,\eta\rangle_{\mathbf{A}}\right|^2+2\Re\left[\langle\mathbf{X}\eta,\eta\rangle_{\mathbf{A}}\overline{\langle\mathbf{Y}\eta,\eta\rangle_{\mathbf{A}}}\right]+\left|\langle\mathbf{Y}\eta,\eta\rangle_{\mathbf{A}}\right|^2\right)\\ &\quad+\varepsilon\left(\left|\langle\mathbf{X}\eta,\eta\rangle_{\mathbf{A}}\right|^2-2\Re\left[\langle\mathbf{X}\eta,\eta\rangle_{\mathbf{A}}\overline{\langle\mathbf{Y}\eta,\eta\rangle_{\mathbf{A}}}\right]+\left|\langle\mathbf{Y}\eta,\eta\rangle_{\mathbf{A}}\right|^2\right)\\ &= \left|\langle\mathbf{X}\eta,\eta\rangle_{\mathbf{A}}\right|^2+\left|\langle\mathbf{Y}\eta,\eta\rangle_{\mathbf{A}}\right|^2+2\Re\left[(1-2\varepsilon)\langle\mathbf{X}\eta,\eta\rangle_{\mathbf{A}}\overline{\langle\mathbf{Y}\eta,\eta\rangle_{\mathbf{A}}}\right].\end{aligned}$$

Let $\eta \in \Sigma_1^{\mathbf{A}}$. Then,

$$\begin{aligned}\Re\left[(1-2\varepsilon)\langle\mathbf{X}\eta,\eta\rangle_{\mathbf{A}}\overline{\langle\mathbf{Y}\eta,\eta\rangle_{\mathbf{A}}}\right] &\le \left|(1-2\varepsilon)\langle\mathbf{X}\eta,\eta\rangle_{\mathbf{A}}\overline{\langle\mathbf{Y}\eta,\eta\rangle_{\mathbf{A}}}\right|\\ &\le |1-2\varepsilon|\left|\langle\mathbf{X}\eta,\eta\rangle_{\mathbf{A}}\right|\left|\langle\mathbf{Y}\eta,\eta\rangle_{\mathbf{A}}\right|\\ &\le |1-2\varepsilon|\,\omega_{\mathbf{A}}(\mathbf{X})\omega_{\mathbf{A}}(\mathbf{Y}).\end{aligned}$$

It follows that

$$\begin{aligned}&(1-\varepsilon)\left|\langle\mathbf{X}\eta,\eta\rangle_{\mathbf{A}}+\langle\mathbf{Y}\eta,\eta\rangle_{\mathbf{A}}\right|^2+\varepsilon\left|\langle\mathbf{X}\eta,\eta\rangle_{\mathbf{A}}-\langle\mathbf{Y}\eta,\eta\rangle_{\mathbf{A}}\right|^2\\ &\le \left|\langle\mathbf{X}\eta,\eta\rangle_{\mathbf{A}}\right|^2+\left|\langle\mathbf{Y}\eta,\eta\rangle_{\mathbf{A}}\right|^2+2\,|1-2\varepsilon|\,\omega_{\mathbf{A}}(\mathbf{X})\omega_{\mathbf{A}}(\mathbf{Y})\end{aligned}$$

for $\eta \in \Sigma_1^{\mathbf{A}}$. Taking the supremum over $\eta \in \Sigma_1^{\mathbf{A}}$ yields the first inequality in (3.8). This completes the proof.

The next result establishes comparative upper bounds for the generalized $p$-arithmetic mean numerical radius using reverse Jensen inequalities.

**Theorem 3.3.** *Let* $\mathbf{U}, \mathbf{V} \in \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$ *and* $p \ge 1$. *Then,*

$$\omega^{2p}_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V}) \le \omega^{2p}_{\mathbf{A},2p,\varepsilon}(\mathbf{U},\mathbf{V}) \le \omega^{2p}_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})+\frac{1}{4}p\max\left\{\omega^{2p}_{\mathbf{A}}(\mathbf{U}),\omega^{2p}_{\mathbf{A}}(\mathbf{V})\right\} \tag{3.10}$$

*and*

$$\omega^{2p}_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V}) \le \omega^{2p}_{\mathbf{A},2p,\varepsilon}(\mathbf{U},\mathbf{V}) \le \omega^{2p}_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V}) + \frac{2^{p-1}-1}{2^{p-1}} \max\left\{\omega^{2p}_{\mathbf{A}}(\mathbf{U}), \omega^{2p}_{\mathbf{A}}(\mathbf{V})\right\}. \tag{3.11}$$

*In particular, for $p = 2$, we obtain*

$$\omega^{4}_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V}) \le \omega^{4}_{\mathbf{A},4,\varepsilon}(\mathbf{U},\mathbf{V}) \le \omega^{4}_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V}) + \frac{1}{2} \max\left\{\omega^{4}_{\mathbf{A}}(\mathbf{U}), \omega^{4}_{\mathbf{A}}(\mathbf{V})\right\}.$$

*Proof.* We use a reverse of Jensen's inequality for a convex function $f$ on $[m, M]$ [40]:

$$0 \le \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \le \frac{1}{4}(M-m)\left[f'_{-}(M) - f'_{+}(m)\right],$$

where $x_i \in [m, M]$, $p_i \ge 0$ for $i \in \{1, \dots, n\}$, and $\sum_{i=1}^{n} p_i = 1$.

For $f : [m, M] \subset [0, \infty) \to [0, \infty)$, $f(t) = t^p$, $p \ge 1$, we obtain

$$0 \le \sum_{i=1}^{n} p_i x_i^p - \left(\sum_{i=1}^{n} p_i x_i\right)^p \le \frac{1}{4} p(M-m)(M^{p-1} - m^{p-1}),$$

where $x_i \in [m, M]$, $p_i \ge 0$ for $i \in \{1, \dots, n\}$, and $\sum_{i=1}^{n} p_i = 1$. For $0 \le m \le a, b \le M$, this gives

$$((1-\varepsilon)a + \varepsilon b)^p \le (1-\varepsilon)a^p + \varepsilon b^p \le ((1-\varepsilon)a + \varepsilon b)^p + \frac{1}{4} p(M-m)(M^{p-1} - m^{p-1}) \tag{3.12}$$

for $\varepsilon \in [0, 1]$ and $p \ge 1$.

Let $\eta \in \Sigma_1^{\mathbf{A}}$ and $(\mathbf{U},\mathbf{V}) \in \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$. Set $a = |\langle \mathbf{U}\eta, \eta\rangle_{\mathbf{A}}|^2$, $b = |\langle \mathbf{V}\eta, \eta\rangle_{\mathbf{A}}|^2$, $m = 0$, and $M = \max\left\{\omega^{2}_{\mathbf{A}}(\mathbf{U}), \omega^{2}_{\mathbf{A}}(\mathbf{V})\right\}$. By using (3.12), we obtain

$$\begin{aligned}
&\left((1-\varepsilon)\,|\langle \mathbf{U}\eta, \eta\rangle_{\mathbf{A}}|^2 + \varepsilon\,|\langle \mathbf{V}\eta, \eta\rangle_{\mathbf{A}}|^2\right)^p \\
&\le (1-\varepsilon)\,|\langle \mathbf{U}\eta, \eta\rangle_{\mathbf{A}}|^{2p} + \varepsilon\,|\langle \mathbf{V}\eta, \eta\rangle_{\mathbf{A}}|^{2p} \\
&\le \left((1-\varepsilon)\,|\langle \mathbf{U}\eta, \eta\rangle_{\mathbf{A}}|^2 + \varepsilon\,|\langle \mathbf{V}\eta, \eta\rangle_{\mathbf{A}}|^2\right)^p + \frac{1}{4} p \max\left\{\omega^{2p}_{\mathbf{A}}(\mathbf{U}), \omega^{2p}_{\mathbf{A}}(\mathbf{V})\right\}.
\end{aligned}$$

Taking the supremum over $\eta \in \Sigma_1^{\mathbf{A}}$ yields (3.10).

Additionally, another reverse of Jensen's inequality for a convex function $f$ on $[m, M]$ [40] gives

$$0 \le \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \le 2\left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right)\right],$$

where $x_i \in [m, M]$, $p_i \ge 0$ for $i \in \{1, \dots, n\}$, and $\sum_{i=1}^{n} p_i = 1$. For $f(t) = t^p$, $p \ge 1$,

$$0 \le \sum_{i=1}^{n} p_i x_i^p - \left(\sum_{i=1}^{n} p_i x_i\right)^p \le 2\left[\frac{m^p + M^p}{2} - \left(\frac{m+M}{2}\right)^p\right].$$

Setting $m = 0$, we deduce that

$$0 \le \sum_{i=1}^{n} p_i x_i^p - \left(\sum_{i=1}^{n} p_i x_i\right)^p \le \frac{2^{p-1}-1}{2^{p-1}} M^p,$$

where $x_i \in [0, M]$, $p_i \geq 0$ for $i \in \{1, \ldots, n\}$, and $\sum_{i=1}^{n} p_i = 1$. Using $a = |\langle \mathbf{U}\eta, \eta\rangle_{\mathbf{A}}|^2$, $b = |\langle \mathbf{V}\eta, \eta\rangle_{\mathbf{A}}|^2$, $m = 0$, and $M = \max\left\{\omega_{\mathbf{A}}^2(\mathbf{U}), \omega_{\mathbf{A}}^2(\mathbf{V})\right\}$,

$$\begin{aligned}&\left((1-\varepsilon)\,|\langle \mathbf{U}\eta, \eta\rangle_{\mathbf{A}}|^2 + \varepsilon\,|\langle \mathbf{V}\eta, \eta\rangle_{\mathbf{A}}|^2\right)^p \\ &\leq (1-\varepsilon)\,|\langle \mathbf{U}\eta, \eta\rangle_{\mathbf{A}}|^{2p} + \varepsilon\,|\langle \mathbf{V}\eta, \eta\rangle_{\mathbf{A}}|^{2p} \\ &\leq \left((1-\varepsilon)\,|\langle \mathbf{U}\eta, \eta\rangle_{\mathbf{A}}|^2 + \varepsilon\,|\langle \mathbf{V}\eta, \eta\rangle_{\mathbf{A}}|^2\right)^p + \frac{2^{p-1}-1}{2^{p-1}} \max\left\{\omega_{\mathbf{A}}^{2p}(\mathbf{U}), \omega_{\mathbf{A}}^{2p}(\mathbf{V})\right\}.\end{aligned}$$

Taking the supremum over $\eta \in \Sigma_1^{\mathbf{A}}$ yields (3.11). This completes the proof.

**Remark 3.1.** *Consider the function*

$$\delta(p) := \frac{1}{4}p - \frac{2^{p-1}-1}{2^{p-1}},\ p \geq 1.$$

*By analyzing $\delta(p)$, we find that it is positive for $p \in [1, 2) \cup (3, \infty)$ and negative for $p \in (2, 3)$, with $\delta(2) = \delta(3) = 0$. Thus, the bound in* (3.11) *is sharper than* (3.10) *for $p \in (1, 2) \cup (3, \infty)$, while the reverse holds for $p \in (2, 3)$.*

To study the continuous behavior of the parameter $\varepsilon$, we introduce the integral version of our functional.

**Definition 3.1.** *For any pair of operators $(\mathbf{U}, \mathbf{V}) \in \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K}) \times \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$, we define the integral Euclidean numerical radius by*

$$\begin{aligned}\omega_{\mathbf{A},e,i}(\mathbf{U}, \mathbf{V}) &:= \int_0^1 \omega_{\mathbf{A},e,\varepsilon}(\mathbf{U}, \mathbf{V})\, d\varepsilon \\ &= \int_0^1 \sup_{\eta \in \Sigma_1^{\mathbf{A}}} \left((1-\varepsilon)\,|\langle \mathbf{U}\eta, \eta\rangle_{\mathbf{A}}|^2 + \varepsilon\,|\langle \mathbf{V}\eta, \eta\rangle_{\mathbf{A}}|^2\right)^{\frac{1}{2}}\ d\varepsilon.\end{aligned}$$

**Remark 3.2.** *We observe that $\omega_{\mathbf{A},e,i}(\mathbf{U}, \mathbf{V}) \geq 0$ for any $(\mathbf{U}, \mathbf{V}) \in \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K}) \times \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$. If $\omega_{\mathbf{A},e,i}(\mathbf{U}, \mathbf{V}) = 0$, then $\omega_{\mathbf{A},e,\varepsilon}(\mathbf{U}, \mathbf{V}) = 0$ for almost every $\varepsilon \in [0, 1]$. Since $\omega_{\mathbf{A},2,\varepsilon}(\mathbf{U}, \mathbf{V})$ is a seminorm, it follows that $\mathbf{AU} = \mathbf{AV} = 0$. Additionally, $\omega_{\mathbf{A},e,i}(\alpha\mathbf{U}, \alpha\mathbf{V}) = |\alpha|\, \omega_{\mathbf{A},e,i}(\mathbf{U}, \mathbf{V})$ for $\alpha \in \mathbb{C}$, and*

$$\begin{aligned}\omega_{\mathbf{A},e,i}(\mathbf{U}+\mathbf{X}, \mathbf{V}+\mathbf{Y}) &= \int_0^1 \omega_{\mathbf{A},e,\varepsilon}(\mathbf{U}+\mathbf{X}, \mathbf{V}+\mathbf{Y})\, d\varepsilon \\ &\leq \int_0^1 \left(\omega_{\mathbf{A},e,\varepsilon}(\mathbf{U}, \mathbf{V}) + \omega_{\mathbf{A},e,\varepsilon}(\mathbf{X}, \mathbf{Y})\right)\, d\varepsilon \\ &= \omega_{\mathbf{A},e,i}(\mathbf{U}, \mathbf{V}) + \omega_{\mathbf{A},e,i}(\mathbf{X}, \mathbf{Y}),\end{aligned}$$

*showing that $\omega_{\mathbf{A},e,i}(\cdot, \cdot)$ is a seminorm on $\mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K}) \times \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$.*

In the next result, we establish Hermite-Hadamard-type inequalities for the Euclidean arithmetic mean $\mathbf{A}$-numerical radius.

**Proposition 3.1.** *For* $(\mathbf{U},\mathbf{V}) \in \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K}) \times \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$ *and* $\varepsilon_1, \varepsilon_2 \in [0,1]$ *with* $\varepsilon_1 \neq \varepsilon_2$*, we have*

$$\omega_{\mathbf{A},e,(1-t)\varepsilon_1+t\varepsilon_2}(\mathbf{U},\mathbf{V}) \leq (1-t)^{\frac{1}{2}}\omega_{\mathbf{A},e,\varepsilon_1}(\mathbf{U},\mathbf{V}) + t^{\frac{1}{2}}\omega_{\mathbf{A},e,\varepsilon_2}(\mathbf{U},\mathbf{V}) \tag{3.13}$$

*for all* $t \in [0,1]$*. In particular, for* $t = \frac{1}{2}$*, we obtain*

$$\sqrt{2}\omega_{\mathbf{A},e,\frac{\varepsilon_1+\varepsilon_2}{2}}(\mathbf{U},\mathbf{V}) \leq \omega_{\mathbf{A},e,\varepsilon_1}(\mathbf{U},\mathbf{V}) + \omega_{\mathbf{A},e,\varepsilon_2}(\mathbf{U},\mathbf{V}). \tag{3.14}$$

*Additionally,*

$$\frac{\sqrt{2}}{2}\omega_{\mathbf{A},e,\frac{\varepsilon_1+\varepsilon_2}{2}}(\mathbf{U},\mathbf{V}) \leq \frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon \leq \frac{2}{3}\left[\omega_{\mathbf{A},e,\varepsilon_1}(\mathbf{U},\mathbf{V}) + \omega_{\mathbf{A},e,\varepsilon_2}(\mathbf{U},\mathbf{V})\right]. \tag{3.15}$$

*In particular, for* $\varepsilon_1 = 0$*,* $\varepsilon_2 = 1$*, we obtain*

$$\frac{1}{2}\omega_{\mathbf{A},e}(\mathbf{U},\mathbf{V}) \leq \omega_{\mathbf{A},e,i}(\mathbf{U},\mathbf{V}) \leq \frac{2}{3}\left[\omega_{\mathbf{A}}(\mathbf{U}) + \omega_{\mathbf{A}}(\mathbf{V})\right].$$

*Proof.* Consider the function

$$f_\eta(\varepsilon) := \left((1-\varepsilon)\,|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}|^2 + \varepsilon\,|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}|^2\right)^{\frac{1}{2}}, \quad \varepsilon \in [0,1].$$

For $\varepsilon_1, \varepsilon_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha+\beta = 1$, we have

$$\begin{aligned} f_\eta(\alpha\varepsilon_1+\beta\varepsilon_2) &= \left((1-\alpha\varepsilon_1-\beta\varepsilon_2)\,|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}|^2 + (\alpha\varepsilon_1+\beta\varepsilon_2)\,|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}|^2\right)^{\frac{1}{2}} \\ &= \Big(\alpha\left[(1-\varepsilon_1)\,|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}|^2 + \varepsilon_1\,|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}|^2\right] \\ &\quad +\beta\left[(1-\varepsilon_2)\,|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}|^2 + \varepsilon_2\,|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}|^2\right]\Big)^{\frac{1}{2}}. \end{aligned}$$

Using the inequality

$$(c+d)^{\frac{1}{2}} \leq c^{\frac{1}{2}} + d^{\frac{1}{2}}, \quad c,d \geq 0,$$

we obtain

$$\begin{aligned} &\Big(\alpha\left[(1-\varepsilon_1)\,|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}|^2 + \varepsilon_1\,|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}|^2\right] +\beta\left[(1-\varepsilon_2)\,|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}|^2 + \varepsilon_2\,|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}|^2\right]\Big)^{\frac{1}{2}} \\ &\leq \alpha^{\frac{1}{2}}\left[(1-\varepsilon_1)\,|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}|^2 + \varepsilon_1\,|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}|^2\right]^{\frac{1}{2}} + \beta^{\frac{1}{2}}\left[(1-\varepsilon_2)\,|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}|^2 + \varepsilon_2\,|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}|^2\right]^{\frac{1}{2}} \\ &= \alpha^{\frac{1}{2}}f_\eta(\varepsilon_1) + \beta^{\frac{1}{2}}f_\eta(\varepsilon_2). \end{aligned}$$

Thus,

$$f_\eta(\alpha\varepsilon_1+\beta\varepsilon_2) \leq \alpha^{\frac{1}{2}}f_\eta(\varepsilon_1) + \beta^{\frac{1}{2}}f_\eta(\varepsilon_2).$$

Taking the supremum over $\eta \in \Sigma_1^{\mathbf{A}}$, we obtain

$$\begin{aligned} \omega_{\mathbf{A},e,\alpha\varepsilon_1+\beta\varepsilon_2}(\mathbf{U},\mathbf{V}) &= \sup_{\eta\in\Sigma_1^{\mathbf{A}}} f_\eta(\alpha\varepsilon_1+\beta\varepsilon_2) \\ &\leq \sup_{\eta\in\Sigma_1^{\mathbf{A}}}\left[\alpha^{\frac{1}{2}}f_\eta(\varepsilon_1) + \beta^{\frac{1}{2}}f_\eta(\varepsilon_2)\right] \end{aligned}$$

$$\leq \alpha^{\frac{1}{2}} \sup_{\eta\in\Sigma_1^{\mathbf{A}}} f_\eta(\varepsilon_1) + \beta^{\frac{1}{2}} \sup_{\eta\in\Sigma_1^{\mathbf{A}}} f_\eta(\varepsilon_2)$$
$$= \alpha^{\frac{1}{2}} \omega_{\mathbf{A},e,\varepsilon_1}(\mathbf{U},\mathbf{V}) + \beta^{\frac{1}{2}} \omega_{\mathbf{A},e,\varepsilon_2}(\mathbf{U},\mathbf{V}),$$

proving (3.13).

Integrating (3.13) over $t \in [0,1]$, we obtain

$$\int_0^1 \omega_{\mathbf{A},e,(1-t)\varepsilon_1+t\varepsilon_2}(\mathbf{U},\mathbf{V})\,dt \leq \int_0^1 (1-t)^{\frac{1}{2}}\,dt\,\omega_{\mathbf{A},e,\varepsilon_1}(\mathbf{U},\mathbf{V}) + \int_0^1 t^{\frac{1}{2}}\,dt\,\omega_{\mathbf{A},e,\varepsilon_2}(\mathbf{U},\mathbf{V}). \tag{3.16}$$

Since

$$\int_0^1 \omega_{\mathbf{A},e,(1-t)\varepsilon_1+t\varepsilon_2}(\mathbf{U},\mathbf{V})\,dt = \frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2} \omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon$$

and

$$\int_0^1 (1-t)^{\frac{1}{2}}\,dt = \int_0^1 t^{\frac{1}{2}}\,dt = \frac{2}{3},$$

it follows from (3.16) that the second inequality in (3.15) holds.

By (3.14), we have

$$\sqrt{2}\omega_{\mathbf{A},e,\frac{\varepsilon_1+\varepsilon_2}{2}}(\mathbf{U},\mathbf{V}) \leq \omega_{\mathbf{A},e,(1-t)\varepsilon_1+t\varepsilon_2}(\mathbf{U},\mathbf{V}) + \omega_{\mathbf{A},e,t\varepsilon_1+(1-t)\varepsilon_2}(\mathbf{U},\mathbf{V})$$

for all $t \in [0,1]$. Integrating over $t \in [0,1]$, we obtain

$$\sqrt{2}\omega_{\mathbf{A},e,\frac{\varepsilon_1+\varepsilon_2}{2}}(\mathbf{U},\mathbf{V}) \leq \int_0^1 \omega_{\mathbf{A},e,(1-t)\varepsilon_1+t\varepsilon_2}(\mathbf{U},\mathbf{V})\,dt + \int_0^1 \omega_{\mathbf{A},e,t\varepsilon_1+(1-t)\varepsilon_2}(\mathbf{U},\mathbf{V})\,dt. \tag{3.17}$$

Since

$$\int_0^1 \omega_{\mathbf{A},e,(1-t)\varepsilon_1+t\varepsilon_2}(\mathbf{U},\mathbf{V})\,dt = \int_0^1 \omega_{\mathbf{A},e,t\varepsilon_1+(1-t)\varepsilon_2}(\mathbf{U},\mathbf{V})\,dt = \frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2} \omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon,$$

(3.17) implies the first inequality in (3.15). This completes the proof.

The next result provides precise estimations for the integral numerical radius by selecting specific symmetric integration bounds.

**Corollary 3.2.** *For* $(\mathbf{U},\mathbf{V}) \in \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K}) \times \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$ *and* $\varepsilon \in (0,1)$*, we have*

$$\begin{aligned}&\frac{\sqrt{2}}{2}\left[\varepsilon\omega_{\mathbf{A},e,\frac{\varepsilon}{2}}(\mathbf{U},\mathbf{V}) + (1-\varepsilon)\omega_{\mathbf{A},e,\frac{\varepsilon+1}{2}}(\mathbf{U},\mathbf{V})\right] \\ &\leq \omega_{\mathbf{A},e,i}(\mathbf{U},\mathbf{V}) \leq \frac{2}{3}\left[\varepsilon\omega_{\mathbf{A}}(\mathbf{U}) + \omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V}) + (1-\varepsilon)\omega_{\mathbf{A}}(\mathbf{V})\right].\end{aligned} \tag{3.18}$$

*In particular,*

$$\frac{\sqrt{2}}{4}\left[\omega_{\mathbf{A},e,\frac{1}{4}}(\mathbf{U},\mathbf{V}) + \omega_{\mathbf{A},e,\frac{3}{4}}(\mathbf{U},\mathbf{V})\right] \leq \omega_{\mathbf{A},e,i}(\mathbf{U},\mathbf{V}) \leq \frac{1}{3}\left[\omega_{\mathbf{A}}(\mathbf{U}) + \sqrt{2}\omega_{\mathbf{A},e}(\mathbf{U},\mathbf{V}) + \omega_{\mathbf{A}}(\mathbf{V})\right]. \tag{3.19}$$

*Proof.* Setting $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon \in (0,1)$ in (3.15), we obtain

$$\frac{\sqrt{2}}{2}\omega_{\mathbf{A},e,\frac{\varepsilon}{2}}(\mathbf{U},\mathbf{V}) \leq \frac{1}{\varepsilon}\int_0^{\varepsilon} \omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon \leq \frac{2}{3}\left[\omega_{\mathbf{A},e,0}(\mathbf{U},\mathbf{V}) + \omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\right],$$

equivalently,

$$\frac{\sqrt{2}}{2}\varepsilon\omega_{\mathbf{A},e,\frac{\varepsilon}{2}}(\mathbf{U},\mathbf{V}) \leq \int_0^{\varepsilon} \omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon \leq \frac{2}{3}\varepsilon\left[\omega_{\mathbf{A}}(\mathbf{U}) + \omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\right]. \tag{3.20}$$

Setting $\varepsilon_1 = \varepsilon \in (0,1)$ and $\varepsilon_2 = 1$ in (3.15), we get

$$\frac{\sqrt{2}}{2}\omega_{\mathbf{A},e,\frac{\varepsilon+1}{2}}(\mathbf{U},\mathbf{V}) \leq \frac{1}{1-\varepsilon}\int_{\varepsilon}^{1} \omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon \leq \frac{2}{3}\left[\omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V}) + \omega_{\mathbf{A},e,1}(\mathbf{U},\mathbf{V})\right],$$

equivalently,

$$\frac{\sqrt{2}}{2}(1-\varepsilon)\omega_{\mathbf{A},e,\frac{\varepsilon+1}{2}}(\mathbf{U},\mathbf{V}) \leq \int_{\varepsilon}^{1} \omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon \leq \frac{2}{3}(1-\varepsilon)\left[\omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V}) + \omega_{\mathbf{A}}(\mathbf{V})\right]. \tag{3.21}$$

Adding (3.20) and (3.21), we obtain

$$\begin{aligned}
&\frac{\sqrt{2}}{2}\varepsilon\omega_{\mathbf{A},e,\frac{\varepsilon}{2}}(\mathbf{U},\mathbf{V}) + \frac{\sqrt{2}}{2}(1-\varepsilon)\omega_{\mathbf{A},e,\frac{\varepsilon+1}{2}}(\mathbf{U},\mathbf{V}) \\
&\leq \int_0^1 \omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon \\
&\leq \frac{2}{3}\varepsilon\left[\omega_{\mathbf{A}}(\mathbf{U}) + \omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\right] + \frac{2}{3}(1-\varepsilon)\left[\omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V}) + \omega_{\mathbf{A}}(\mathbf{V})\right],
\end{aligned}$$

which implies (3.18). This completes the proof.

In the context of product spaces, Kikianty and Dragomir [41] introduced a related norm framework which inspires our subsequent results.

**Definition 3.2.** *Let $(\mathcal{X}, \|\cdot\|)$ be a normed space. The p-HH-norm on the Cartesian product $\mathcal{X}\times\mathcal{X}$ is defined for elements $x, y \in \mathcal{X}$ as follows:*

$$\|(x,y)\|_{p\text{-}HH} := \begin{cases} \left(\int_0^1 \|(1-t)x + ty\|^p\,dt\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \sup_{t\in[0,1]} \|(1-t)x + ty\| & \text{if } p = \infty. \end{cases}$$

**Remark 3.3.** *By the classical Hermite-Hadamard inequality for convex functions $f : [a,b] \to \mathbb{R}$,*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(s)\,ds \leq \frac{f(a)+f(b)}{2},$$

*we seamlessly deduce that*

$$\left\|\frac{x+y}{2}\right\| \leq \|(x,y)\|_{p\text{-}HH} \leq \left(\frac{\|x\|^p + \|y\|^p}{2}\right)^{\frac{1}{p}}$$

*for $1 \leq p < \infty$ and $(x,y) \in \mathcal{X}\times\mathcal{X}$.*

The following result derives bounds connecting the continuous integration of the **A**-seminorm directly with the integral Euclidean numerical radius.

**Proposition 3.2.** *For* $(\mathbf{U},\mathbf{V}) \in \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K}) \times \mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$ *and* $\varepsilon_1, \varepsilon_2 \in [0,1]$ *with* $\varepsilon_1 \neq \varepsilon_2$*, we have*

$$\begin{aligned}\omega_{\mathbf{A}}\left(\left(1-\frac{\varepsilon_1+\varepsilon_2}{2}\right)\mathbf{U}+\frac{\varepsilon_1+\varepsilon_2}{2}\mathbf{V}\right) &\le \frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\omega_{\mathbf{A}}\left((1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right)\,d\varepsilon \\ &\le \frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon \\ &\le \left[\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\omega^2_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon\right]^{\frac{1}{2}} \\ &\le \left[\left(1-\frac{\varepsilon_1+\varepsilon_2}{2}\right)\omega^2_{\mathbf{A}}(\mathbf{U})+\frac{\varepsilon_1+\varepsilon_2}{2}\omega^2_{\mathbf{A}}(\mathbf{V})\right]^{\frac{1}{2}}.\end{aligned} \tag{3.22}$$

*In particular, for* $\varepsilon_1 = 0$ *and* $\varepsilon_2 = 1$*, we obtain*

$$\begin{aligned}\omega_{\mathbf{A}}\left(\frac{\mathbf{U}+\mathbf{V}}{2}\right) &\le \int_0^1 \omega_{\mathbf{A}}\left((1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right)\,d\varepsilon \le \omega_{\mathbf{A},e,i}(\mathbf{U},\mathbf{V}) \\ &\le \left[\int_0^1 \omega^2_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon\right]^{\frac{1}{2}} \le \frac{\sqrt{2}}{2}\left[\omega^2_{\mathbf{A}}(\mathbf{U})+\omega^2_{\mathbf{A}}(\mathbf{V})\right]^{\frac{1}{2}}.\end{aligned} \tag{3.23}$$

*Proof.* Using a Schwarz-type inequality, let $\varepsilon \in [0,1]$ and $\eta \in \mathcal{K}$. We have

$$(1-\varepsilon)\left|\langle \mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right| + \varepsilon\left|\langle \mathbf{V}\eta,\eta\rangle_{\mathbf{A}}\right| \le \left((1-\varepsilon)\left|\langle \mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right|^2 + \varepsilon\left|\langle \mathbf{V}\eta,\eta\rangle_{\mathbf{A}}\right|^2\right)^{\frac{1}{2}}.$$

Since

$$\left|\langle [(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}]\eta,\eta\rangle_{\mathbf{A}}\right| \le (1-\varepsilon)\left|\langle \mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right| + \varepsilon\left|\langle \mathbf{V}\eta,\eta\rangle_{\mathbf{A}}\right|,$$

we obtain

$$\left|\langle [(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}]\eta,\eta\rangle_{\mathbf{A}}\right| \le \left((1-\varepsilon)\left|\langle \mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right|^2 + \varepsilon\left|\langle \mathbf{V}\eta,\eta\rangle_{\mathbf{A}}\right|^2\right)^{\frac{1}{2}}$$

for $\varepsilon \in [0,1]$ and $\eta \in \mathcal{K}$. Taking the supremum over $\eta \in \Sigma_1^{\mathbf{A}}$, we obtain

$$\omega_{\mathbf{A}}\left((1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right) \le \omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})$$

for all $\varepsilon \in [0,1]$. Integrating over $[\varepsilon_1, \varepsilon_2]$, we obtain

$$\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\omega_{\mathbf{A}}\left((1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right)\,d\varepsilon \le \frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon.$$

Since $\omega_{\mathbf{A}}$ is a seminorm, we have

$$\begin{aligned}\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\omega_{\mathbf{A}}\left((1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}\right)\,d\varepsilon &\ge \omega_{\mathbf{A}}\left(\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{V}]\,d\varepsilon\right) \\ &= \omega_{\mathbf{A}}\left(\left(1-\frac{\varepsilon_1+\varepsilon_2}{2}\right)\mathbf{U}+\frac{\varepsilon_1+\varepsilon_2}{2}\mathbf{V}\right).\end{aligned}$$

By the Schwarz integral inequality,

$$
\begin{aligned}
\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon &\le \left[\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\omega^2_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon\right]^{\frac{1}{2}}\\
&= \left[\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\sup_{\eta\in\Sigma_1^{\mathbf{A}}}\Big((1-\varepsilon)\,|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}|^2+\varepsilon\,|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}|^2\Big)\,d\varepsilon\right]^{\frac{1}{2}}\\
&\le \left[\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\left((1-\varepsilon)\sup_{\eta\in\Sigma_1^{\mathbf{A}}}|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}|^2+\varepsilon\sup_{\eta\in\Sigma_1^{\mathbf{A}}}|\langle\mathbf{V}\eta,\eta\rangle_{\mathbf{A}}|^2\right)d\varepsilon\right]^{\frac{1}{2}}\\
&= \left[\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\Big((1-\varepsilon)\omega^2_{\mathbf{A}}(\mathbf{U})+\varepsilon\omega^2_{\mathbf{A}}(\mathbf{V})\Big)\,d\varepsilon\right]^{\frac{1}{2}}\\
&= \left[\left(1-\frac{\varepsilon_1+\varepsilon_2}{2}\right)\omega^2_{\mathbf{A}}(\mathbf{U})+\frac{\varepsilon_1+\varepsilon_2}{2}\omega^2_{\mathbf{A}}(\mathbf{V})\right]^{\frac{1}{2}}.
\end{aligned}
$$

This completes the proof.

In the next result, we establish an upper bound for the quadratic integral of the Euclidean Arithmetic mean **A**-numerical radius.

**Proposition 3.3.** *Let* $(\mathbf{U},\mathbf{V})\in\mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})\times\mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$ *be such that* $\mathbf{AU}$ *and* $\mathbf{AV}$ *are nonzero operators. Then,*

$$
\int_0^1\omega^2_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon\le\frac{3}{4}\omega^2_{\mathbf{A}}(\mathbf{U}-\mathbf{V})+\begin{cases}\frac{\omega^2_{\mathbf{A}}(\mathbf{V})-\omega^2_{\mathbf{A}}(\mathbf{U})}{\ln\omega^2_{\mathbf{A}}(\mathbf{V})-\ln\omega^2_{\mathbf{A}}(\mathbf{U})} & \textit{if } \omega_{\mathbf{A}}(\mathbf{U})\neq\omega_{\mathbf{A}}(\mathbf{V}),\\[2ex] \omega^2_{\mathbf{A}}(\mathbf{U}) & \textit{if } \omega_{\mathbf{A}}(\mathbf{U})=\omega_{\mathbf{A}}(\mathbf{V}).\end{cases} \tag{3.24}
$$

*Proof.* Integrating (3.1) over $[0,1]$, we obtain

$$
\begin{aligned}
\int_0^1\omega^2_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon &\le \omega^2_{\mathbf{A}}(\mathbf{U}-\mathbf{V})\int_0^1\max\{1-\varepsilon,\varepsilon\}\,d\varepsilon+\int_0^1\omega_{\mathbf{A}}^{2(1-\varepsilon)}(\mathbf{U})\omega_{\mathbf{A}}^{2\varepsilon}(\mathbf{V})\,d\varepsilon\\
&= \omega^2_{\mathbf{A}}(\mathbf{U}-\mathbf{V})\int_0^1\left(\frac{1}{2}+\left|\varepsilon-\frac{1}{2}\right|\right)d\varepsilon+\omega^2_{\mathbf{A}}(\mathbf{U})\int_0^1\left(\frac{\omega^2_{\mathbf{A}}(\mathbf{V})}{\omega^2_{\mathbf{A}}(\mathbf{U})}\right)^{\varepsilon}d\varepsilon\\
&= \frac{3}{4}\omega^2_{\mathbf{A}}(\mathbf{U}-\mathbf{V})+\begin{cases}\frac{\omega^2_{\mathbf{A}}(\mathbf{V})-\omega^2_{\mathbf{A}}(\mathbf{U})}{\ln\omega^2_{\mathbf{A}}(\mathbf{V})-\ln\omega^2_{\mathbf{A}}(\mathbf{U})} & \text{if } \omega_{\mathbf{A}}(\mathbf{U})\neq\omega_{\mathbf{A}}(\mathbf{V}),\\[2ex] \omega^2_{\mathbf{A}}(\mathbf{U}) & \text{if } \omega_{\mathbf{A}}(\mathbf{U})=\omega_{\mathbf{A}}(\mathbf{V}).\end{cases}
\end{aligned}
$$

This implies (3.24) and completes the proof.

The next result provides a corresponding lower bound for the quadratic integral representation.

**Proposition 3.4.** *For* $(\mathbf{U},\mathbf{V})\in\mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})\times\mathcal{L}_{\sqrt{\mathbf{A}}}(\mathcal{K})$*, we have*

$$
\int_0^1\omega^2_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon\ge\frac{1}{3}\max\left\{\omega_{\mathbf{A}}(\mathbf{U}^2+\mathbf{V}^2),\omega_{\mathbf{A}}\left(\frac{\mathbf{UV}+\mathbf{VU}}{2}\right)\right\}. \tag{3.25}
$$

*Proof.* By (3.3), we have

$$\omega^2_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V}) \geq \omega_{\mathbf{A}}\left[(1-\varepsilon)^2\mathbf{U}^2+\varepsilon^2\mathbf{V}^2\right]$$

and

$$\omega^2_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V}) \geq (1-\varepsilon)\varepsilon\omega_{\mathbf{A}}(\mathbf{UV}+\mathbf{VU})$$

for all $\varepsilon \in [0,1]$. Integrating over $\varepsilon \in [0,1]$, we obtain

$$\begin{aligned}\int_0^1 \omega^2_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon &\geq \int_0^1 \omega_{\mathbf{A}}\left[(1-\varepsilon)^2\mathbf{U}^2+\varepsilon^2\mathbf{V}^2\right]d\varepsilon\\ &\geq \omega_{\mathbf{A}}\left[\int_0^1\left((1-\varepsilon)^2\mathbf{U}^2+\varepsilon^2\mathbf{V}^2\right)d\varepsilon\right]\\ &= \frac{1}{3}\omega_{\mathbf{A}}(\mathbf{U}^2+\mathbf{V}^2)\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \omega^2_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{V})\,d\varepsilon \geq&\omega_{\mathbf{A}}(\mathbf{UV}+\mathbf{VU})\int_0^1(1-\varepsilon)\varepsilon\,d\varepsilon\\ =&\frac{1}{6}\omega_{\mathbf{A}}(\mathbf{UV}+\mathbf{VU}).\end{aligned}$$

This implies (3.25) and completes the proof.

## 4. Applications

In this final section, we demonstrate the broad applicability of our established bounds. By strategically substituting specific operators into the pair $(\mathbf{U},\mathbf{V})$, we derive specialized inequalities for a single operator's numerical radius, its Cartesian decomposition, and the generalized Davis-Wielandt radius. We formally state these deductions as corollaries to emphasize their direct mathematical lineage from our main theorems.

First, we evaluate the scenario where the operator pair consists of an operator and its **A**-adjoint. Let $\mathbf{U} \in \mathcal{L}_{\mathbf{A}}(\mathcal{K})$. Let $\mathbf{V} = \mathbf{U}^{\sharp_{\mathbf{A}}}$. Then, we have

$$\begin{aligned}\omega_{\mathbf{A},e,\varepsilon}(\mathbf{U},\mathbf{U}^{\sharp_{\mathbf{A}}}) &= \sup_{\eta\in\Sigma_1^{\mathbf{A}}}\left((1-\varepsilon)\left|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right|^2+\varepsilon\left|\langle\mathbf{U}^{\sharp_{\mathbf{A}}}\eta,\eta\rangle_{\mathbf{A}}\right|^2\right)^{\frac{1}{2}}\\ &= \sup_{\eta\in\Sigma_1^{\mathbf{A}}}\left|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right| = \omega_{\mathbf{A}}(\mathbf{U})\end{aligned}$$

for all $\varepsilon \in [0,1]$, since $\left|\langle\mathbf{U}^{\sharp_{\mathbf{A}}}\eta,\eta\rangle_{\mathbf{A}}\right| = \left|\langle\mathbf{U}\eta,\eta\rangle_{\mathbf{A}}\right|$. The following result provides bounds for the classical **A**-numerical radius.

**Corollary 4.1.** *Let* $\mathbf{U} \in \mathcal{L}_{\mathbf{A}}(\mathcal{K})$*. For all* $\varepsilon \in [0,1]$*, we have*

$$\begin{aligned}\omega^2_{\mathbf{A}}(\mathbf{U}) \geq \max&\left\{\omega_{\mathbf{A}}\left[(1-\varepsilon)^2\mathbf{U}^2+\varepsilon^2(\mathbf{U}^{\sharp_{\mathbf{A}}})^2\right],(1-\varepsilon)\varepsilon\omega_{\mathbf{A}}(\mathbf{U}^{\sharp_{\mathbf{A}}}\mathbf{U}+\mathbf{U}\mathbf{U}^{\sharp_{\mathbf{A}}})\right\}\\ &+\max\{1-\varepsilon,\varepsilon\}\omega_{\mathbf{A}}(\mathbf{U})\left|\omega_{\mathbf{A}}\left[(1-\varepsilon)\mathbf{U}+\varepsilon\mathbf{U}^{\sharp_{\mathbf{A}}}\right]-\omega_{\mathbf{A}}\left[(1-\varepsilon)\mathbf{U}-\varepsilon\mathbf{U}^{\sharp_{\mathbf{A}}}\right]\right|.\end{aligned}$$

*In particular, for $\varepsilon = \frac{1}{2}$, we obtain*

$$\omega_{\mathbf{A}}^2(\mathbf{U}) \geq \frac{1}{4} \max\left\{\omega_{\mathbf{A}}(\mathbf{U}^2 + (\mathbf{U}^{\sharp_{\mathbf{A}}})^2), \omega_{\mathbf{A}}(\mathbf{U}^{\sharp_{\mathbf{A}}}\mathbf{U} + \mathbf{U}\mathbf{U}^{\sharp_{\mathbf{A}}})\right\} + \frac{1}{4}\omega_{\mathbf{A}}(\mathbf{U})\left|\omega_{\mathbf{A}}(\mathbf{U} + \mathbf{U}^{\sharp_{\mathbf{A}}}) - \omega_{\mathbf{A}}(\mathbf{U} - \mathbf{U}^{\sharp_{\mathbf{A}}})\right|.$$

*Furthermore, leveraging Corollary 3.1, we deduce the related inequality*

$$\omega_{\mathbf{A}}^2(\mathbf{U}) \geq \frac{1}{4} \max\left\{\omega_{\mathbf{A}}(\mathbf{U}^2 + (\mathbf{U}^{\sharp_{\mathbf{A}}})^2), \omega_{\mathbf{A}}(\mathbf{U}^2 - (\mathbf{U}^{\sharp_{\mathbf{A}}})^2)\right\}.$$

To further analyze an operator through its Cartesian decomposition (its **A**-real and **A**-imaginary parts), we introduce the following functional.

**Definition 4.1.** *For an operator $\mathbf{T} \in \mathcal{L}_{\mathbf{A}}(\mathcal{K})$, let the **A**-real part be $\Re_{\mathbf{A}}(\mathbf{T}) = \frac{\mathbf{T}+\mathbf{T}^{\sharp_{\mathbf{A}}}}{2}$ and the **A**-imaginary part be $\Im_{\mathbf{A}}(\mathbf{T}) = \frac{\mathbf{T}-\mathbf{T}^{\sharp_{\mathbf{A}}}}{2i}$. We define the functional $\zeta_{\mathbf{A},p,\varepsilon}(\mathbf{T})$ for $\varepsilon \in [0,1]$ and $p \geq 1$ as*

$$\begin{aligned}\zeta_{\mathbf{A},p,\varepsilon}(\mathbf{T}) &:= \omega_{\mathbf{A},p,\varepsilon}(\Re_{\mathbf{A}}(\mathbf{T}), \Im_{\mathbf{A}}(\mathbf{T})) \\ &= \sup_{\eta \in \Sigma_1^{\mathbf{A}}} \left((1-\varepsilon)\left|\langle \Re_{\mathbf{A}}(\mathbf{T})\eta, \eta\rangle_{\mathbf{A}}\right|^p + \varepsilon\left|\langle \Im_{\mathbf{A}}(\mathbf{T})\eta, \eta\rangle_{\mathbf{A}}\right|^p\right)^{\frac{1}{p}}.\end{aligned}$$

**Remark 4.1.** *We observe that $\zeta_{\mathbf{A},p,\varepsilon}(\cdot)$ is a real seminorm on $\mathcal{L}_{\mathbf{A}}(\mathcal{K})$. It satisfies $\zeta_{\mathbf{A},p,\varepsilon}(\mathbf{T}) \geq 0$ for all $\mathbf{T} \in \mathcal{L}_{\mathbf{A}}(\mathcal{K})$, with $\zeta_{\mathbf{A},p,\varepsilon}(\mathbf{T}) = 0$ implying $\mathbf{AT} = 0$. It respects the triangle inequality $\zeta_{\mathbf{A},p,\varepsilon}(\mathbf{T}+\mathbf{S}) \leq \zeta_{\mathbf{A},p,\varepsilon}(\mathbf{T}) + \zeta_{\mathbf{A},p,\varepsilon}(\mathbf{S})$, and is homogeneous over the reals: $\zeta_{\mathbf{A},p,\varepsilon}(a\mathbf{T}) = |a|\zeta_{\mathbf{A},p,\varepsilon}(\mathbf{T})$ for all $a \in \mathbb{R}$.*

*For the specific case $\varepsilon = \frac{1}{2}$, the functional simplifies to a scaled version of the unweighted sum:*

$$\zeta_{\mathbf{A},p,\frac{1}{2}}(\mathbf{T}) = \frac{1}{2^{\frac{1}{p}}}\zeta_{\mathbf{A},p}(\mathbf{T}),$$

*where*

$$\zeta_{\mathbf{A},p}(\mathbf{T}) = \sup_{\eta \in \Sigma_1^{\mathbf{A}}} \left(\left|\langle \Re_{\mathbf{A}}(\mathbf{T})\eta, \eta\rangle_{\mathbf{A}}\right|^p + \left|\langle \Im_{\mathbf{A}}(\mathbf{T})\eta, \eta\rangle_{\mathbf{A}}\right|^p\right)^{\frac{1}{p}}.$$

*Notably, when $p = 2$, the identity $|\langle \Re_{\mathbf{A}}(\mathbf{T})\eta, \eta\rangle_{\mathbf{A}}|^2 + |\langle \Im_{\mathbf{A}}(\mathbf{T})\eta, \eta\rangle_{\mathbf{A}}|^2 = |\langle \mathbf{T}\eta, \eta\rangle_{\mathbf{A}}|^2$ ensures that*

$$\zeta_{\mathbf{A},2}(\mathbf{T}) = \omega_{\mathbf{A}}(\mathbf{T}).$$

The next result establishes advanced bounds for the Cartesian decomposition functional.

**Corollary 4.2.** *Let $\mathbf{T} \in \mathcal{L}_{\mathbf{A}}(\mathcal{K})$. Utilizing Theorem 3.1, we have for $R = \max\{1-\varepsilon, \varepsilon\}$,*

$$\zeta_{\mathbf{A},e,\varepsilon}^2(\mathbf{T}) \leq R\omega_{\mathbf{A}}^2(\Re_{\mathbf{A}}(\mathbf{T}) - \Im_{\mathbf{A}}(\mathbf{T})) + \omega_{\mathbf{A}}^{2(1-\varepsilon)}(\Re_{\mathbf{A}}(\mathbf{T}))\omega_{\mathbf{A}}^{2\varepsilon}(\Im_{\mathbf{A}}(\mathbf{T})).$$

*In particular, for $\varepsilon = \frac{1}{2}$,*

$$\omega_{\mathbf{A}}^2(\mathbf{T}) \leq \omega_{\mathbf{A}}^2(\Re_{\mathbf{A}}(\mathbf{T}) - \Im_{\mathbf{A}}(\mathbf{T})) + 2\omega_{\mathbf{A}}(\Re_{\mathbf{A}}(\mathbf{T}))\omega_{\mathbf{A}}(\Im_{\mathbf{A}}(\mathbf{T})).$$

*Furthermore, applying Theorem 3.2 and Corollary 3.1 yields the lower bounds,*

$$\zeta_{\mathbf{A},e,\varepsilon}^2(\mathbf{T}) \geq \max\left\{\omega_{\mathbf{A}}\left[(1-\varepsilon)^2(\Re_{\mathbf{A}}(\mathbf{T}))^2 + \varepsilon^2(\Im_{\mathbf{A}}(\mathbf{T}))^2\right], (1-\varepsilon)\varepsilon\omega_{\mathbf{A}}(\Re_{\mathbf{A}}(\mathbf{T})\Im_{\mathbf{A}}(\mathbf{T}) + \Im_{\mathbf{A}}(\mathbf{T})\Re_{\mathbf{A}}(\mathbf{T}))\right\}$$

$$+ \max\{(1-\varepsilon)\omega_{\mathbf{A}}(\Re_{\mathbf{A}}(\mathbf{T})), \varepsilon\omega_{\mathbf{A}}(\Im_{\mathbf{A}}(\mathbf{T}))\}$$
$$\times |\omega_{\mathbf{A}}[(1-\varepsilon)\Re_{\mathbf{A}}(\mathbf{T}) + \varepsilon\Im_{\mathbf{A}}(\mathbf{T})] - \omega_{\mathbf{A}}[(1-\varepsilon)\Re_{\mathbf{A}}(\mathbf{T}) - \varepsilon\Im_{\mathbf{A}}(\mathbf{T})]|,$$

*which reduces for $\varepsilon = \frac{1}{2}$ to*

$$\begin{aligned}\omega_{\mathbf{A}}^2(\mathbf{T}) \geq & \frac{1}{2}\max\left\{\omega_{\mathbf{A}}((\Re_{\mathbf{A}}(\mathbf{T}))^2 + (\Im_{\mathbf{A}}(\mathbf{T}))^2), \omega_{\mathbf{A}}((\Re_{\mathbf{A}}(\mathbf{T}))^2 - (\Im_{\mathbf{A}}(\mathbf{T}))^2)\right\} \\ & + \frac{1}{2}\max\{\omega_{\mathbf{A}}(\Re_{\mathbf{A}}(\mathbf{T}) + \Im_{\mathbf{A}}(\mathbf{T})), \omega_{\mathbf{A}}(\Re_{\mathbf{A}}(\mathbf{T}) - \Im_{\mathbf{A}}(\mathbf{T}))\} \times |\omega_{\mathbf{A}}(\Re_{\mathbf{A}}(\mathbf{T})) - \omega_{\mathbf{A}}(\Im_{\mathbf{A}}(\mathbf{T}))|.\end{aligned}$$

*Finally, employing Theorem 3.3, we establish the comparative upper bounds:*

$$\begin{aligned}\zeta_{\mathbf{A},e,\varepsilon}^{2p}(\mathbf{T}) &\leq \zeta_{\mathbf{A},2p,\varepsilon}^{2p}(\mathbf{T}) \\ &\leq \zeta_{\mathbf{A},e,\varepsilon}^{2p}(\mathbf{T}) + \min\left\{\frac{1}{4}p, \frac{2^{p-1}-1}{2^{p-1}}\right\}\max\left\{\omega_{\mathbf{A}}^{2p}(\Re_{\mathbf{A}}(\mathbf{T})), \omega_{\mathbf{A}}^{2p}(\Im_{\mathbf{A}}(\mathbf{T}))\right\}.\end{aligned}$$

Another significant application of our framework lies in generalizing the Davis-Wielandt radius to a $p$-arithmetic mean context.

**Definition 4.2.** *We define the p-arithmetic mean Davis-Wielandt radius of $\mathbf{T} \in \mathcal{L}_{\mathbf{A}}(\mathcal{K})$ for $p \geq 1$ and $\varepsilon \in [0,1]$ as*

$$\begin{aligned}d_{p,\varepsilon}\omega_{\mathbf{A}}(\mathbf{T}) &:= \omega_{\mathbf{A},p,\varepsilon}(\mathbf{T}, \mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T}) \\ &= \sup_{\eta\in\Sigma_1^{\mathbf{A}}}\left((1-\varepsilon)\,|\langle\mathbf{T}\eta,\eta\rangle_{\mathbf{A}}|^p + \varepsilon\,\|\mathbf{T}\eta\|_{\mathbf{A}}^{2p}\right)^{\frac{1}{p}}.\end{aligned}$$

**Remark 4.2.** *From the fundamental lower bound in* (2.2)*, this definition immediately yields the constraint*

$$d_{p,\varepsilon}\omega_{\mathbf{A}}(\mathbf{T}) \geq \omega_{\mathbf{A}}\left[(1-\varepsilon)\mathbf{T} + \varepsilon\mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T}\right]$$

*for all $p \geq 1$ and $\varepsilon \in [0,1]$. Similar to our earlier functionals, fixing $\varepsilon = \frac{1}{2}$ retrieves the standard generalized forms*

$$d_{p,\frac{1}{2}}\omega_{\mathbf{A}}(\mathbf{T}) = \frac{1}{2^{\frac{1}{p}}}d_p\omega_{\mathbf{A}}(\mathbf{T}),$$

*where*

$$d_p\omega_{\mathbf{A}}(\mathbf{T}) := \sup_{\eta\in\Sigma_1^{\mathbf{A}}}\left(|\langle\mathbf{T}\eta,\eta\rangle_{\mathbf{A}}|^p + \left|\langle\mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T}\eta,\eta\rangle_{\mathbf{A}}\right|^p\right)^{\frac{1}{p}}.$$

*When $p = 2$, this precisely recaptures the widely studied classical Davis-Wielandt radius $d\omega_{\mathbf{A}}(\mathbf{T})$, up to a scaling factor*

$$d_{e,\frac{1}{2}}\omega_{\mathbf{A}}(\mathbf{T}) = \frac{\sqrt{2}}{2}d\omega_{\mathbf{A}}(\mathbf{T}).$$

*Additionally, the linear case ($p = 1$) reduces to*

$$d_{\varepsilon}\omega_{\mathbf{A}}(\mathbf{T}) = \sup_{\eta\in\Sigma_1^{\mathbf{A}}}\left((1-\varepsilon)\,|\langle\mathbf{T}\eta,\eta\rangle_{\mathbf{A}}| + \varepsilon\,\|\mathbf{T}\eta\|_{\mathbf{A}}^2\right).$$

The following result provides bounds for the generalized Davis-Wielandt radius.

**Corollary 4.3.** *Let* $\mathbf{T} \in \mathcal{L}_{\mathbf{A}}(\mathcal{K})$*. Applying Theorem 3.1 with* $R = \max\{1-\varepsilon, \varepsilon\}$*, we have*

$$d_{e,\varepsilon}\omega_{\mathbf{A}}^2(\mathbf{T}) \le R\omega_{\mathbf{A}}^2(\mathbf{T} - \mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T}) + \omega_{\mathbf{A}}^{2(1-\varepsilon)}(\mathbf{T})\,\|\mathbf{T}\|_{\mathbf{A}}^{4\varepsilon}.$$

*In particular, for* $\varepsilon = \frac{1}{2}$*, this simplifies to*

$$d\omega_{\mathbf{A}}^2(\mathbf{T}) \le \omega_{\mathbf{A}}^2(\mathbf{T} - \mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T}) + 2\omega_{\mathbf{A}}(\mathbf{T})\,\|\mathbf{T}\|_{\mathbf{A}}^2.$$

*Conversely, applying Theorem 3.2 establishes the lower bounds:*

$$\begin{aligned} d_{e,\varepsilon}\omega_{\mathbf{A}}^2(\mathbf{T}) \ge{} & \max\left\{\omega_{\mathbf{A}}\left[(1-\varepsilon)^2\mathbf{T}^2 + \varepsilon^2(\mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T})^2\right], (1-\varepsilon)\varepsilon\omega_{\mathbf{A}}(\mathbf{T}\mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T} + \mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T}\mathbf{T})\right\} \\ & + \max\left\{(1-\varepsilon)\omega_{\mathbf{A}}(\mathbf{T}), \varepsilon\,\|\mathbf{T}\|_{\mathbf{A}}^2\right\} \times \left|\omega_{\mathbf{A}}\left[(1-\varepsilon)\mathbf{T} + \varepsilon\mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T}\right] - \omega_{\mathbf{A}}\left[(1-\varepsilon)\mathbf{T} - \varepsilon\mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T}\right]\right|, \end{aligned}$$

*which yields for the classical Davis-Wielandt radius (*$\varepsilon = \frac{1}{2}$*)*

$$\begin{aligned} d\omega_{\mathbf{A}}^2(\mathbf{T}) \ge{} & \frac{1}{2}\max\left\{\omega_{\mathbf{A}}(\mathbf{T}^2 + (\mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T})^2), \omega_{\mathbf{A}}(\mathbf{T}\mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T} + \mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T}\mathbf{T})\right\} \\ & + \frac{1}{2}\max\left\{\omega_{\mathbf{A}}(\mathbf{T}), \|\mathbf{T}\|_{\mathbf{A}}^2\right\}\left|\omega_{\mathbf{A}}(\mathbf{T} + \mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T}) - \omega_{\mathbf{A}}(\mathbf{T} - \mathbf{T}^{\sharp_{\mathbf{A}}}\mathbf{T})\right|. \end{aligned}$$

## 5. Conclusions

In this paper, we introduced two new mathematical tools for operator pairs in semi-Hilbertian spaces: the $p$-arithmetic mean **A**-numerical radius and the integral Euclidean numerical radius. By using refined reverse Young and Jensen inequalities, we found several new upper and lower bounds for these concepts. Furthermore, we showed how our results can be easily applied to other areas. For example, we derived special inequalities for the Cartesian decomposition of a single operator and generalized the well-known **A**-Davis-Wielandt radius.

The results we presented do more than just combine existing operator inequalities into a single framework. They also build a strong foundation for solving new problems. Because our approach uses a flexible parameter ($\varepsilon$) and new integral techniques, it can be adapted to many other complex settings. In particular, we believe our methods will be a great starting point for future research. Researchers can use these ideas to explore generalized Berezin numbers and to find new bounds for operator norms in reproducing kernel Hilbert spaces. Overall, this work provides simple but powerful tools to better understand operator theory in quantum mechanics and functional analysis.

## Author contributions

Salma Aljawi, Ahad Hamoud Alotaibi, Silvestru Sever Dragomir and Kais Feki: Conceptualization, Formal analysis, Investigation, Methodology, Writing–original draft, Writing–review & editing. All authors contributed equally to this work. All authors have read and approved the final version of the manuscript.

## Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

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## References

1. M. Elin, S. Reich, D. Shoikhet, Numerical range, In: *Numerical range of holomorphic mappings and applications*, Cham: Birkhäuser, 2019, 21–62. https://doi.org/10.1007/978-3-030-05020-7_2
2. H. Baklouti, S. Namouri, Spectral analysis of bounded operators on semi-Hilbertian spaces, *Banach J. Math. Anal.*, **16** (2022), 12. https://doi.org/10.1007/s43037-021-00167-1
3. M. L. Arias, G. Corach, M. C. Gonzalez, Metric properties of projections in semi-Hilbertian spaces, *Integral Equ. Oper. Theory*, **62** (2008), 11–28. https://doi.org/10.1007/s00020-008-1613-6
4. M. L. Arias, G. Corach, M. C. Gonzalez, Partial isometries in semi-Hilbertian spaces, *Linear Algebra Appl.*, **428** (2008), 1460–1475. https://doi.org/10.1016/j.laa.2007.09.031
5. H. Baklouti, K. Feki, S. A. O. A. Mahmoud, Joint normality of operators in semi-Hilbertian spaces, *Linear Multilinear Algebra*, **68** (2020), 845–866. https://doi.org/10.1080/03081087.2019.1593925
6. H. Baklouti, S. Namouri, Closed operators in semi-Hilbertian spaces, *Linear Multilinear Algebra*, **70** (2022), 5847–5858. https://doi.org/10.1080/03081087.2021.1932709
7. T. Bermúdez, A. Saddi, H. Zaway, $(A, m)$-isometries on Hilbert spaces, *Linear Algebra Appl.*, **540** (2018), 95–111. https://doi.org/10.1016/j.laa.2017.11.005
8. K. Feki, Spectral radius of semi-Hilbertian space operators and its applications, *Ann. Funct. Anal.*, **11** (2020), 929–946. https://doi.org/10.1007/s43034-020-00064-y
9. D. Krejčiřík, V. Lotoreichik, M. Znojil, The minimally anisotropic metric operator in quasi-Hermitian quantum mechanics, *Proc. A*, **474** (2018), 20180264. https://doi.org/10.1098/rspa.2018.0264
10. F. G. Scholtz, H. B. Geyer, F. J. W. Hahne, Quasi-Hermitian operators in quantum theory and the variational principle, *Ann. Phys.*, **213** (1992), 74–101. https://doi.org/10.1016/0003-4916(92)90284-S

11. A. Mostafazadeh, Pseudo-Hermitian representation of quantum mechanics, *Int. J. Geom. Methods Mod. Phys.*, **7** (2010), 1191–1306. https://doi.org/10.1142/S0219887810004816
12. T. Y. Azizov, I. S. Iokhvidov, *Linear operators in spaces with an indefinite metric*, John Wiley & Sons, 1989.
13. J. P. Antoine, C. Trapani, Partial inner product spaces, metric operators, and generalized hermiticity, *J. Phys. A Math. Theor.*, **46** (2013), 025204. https://doi.org/10.1088/1751-8113/46/2/025204
14. M. Faghih-Ahmadi, F. Gorjizadeh, $A$-numerical radius of $A$-normal operators in semi-Hilbertian spaces, *Ital. J. Pure Appl. Math.*, **36** (2016), 73–78.
15. A. Saddi, $A$-normal operators in semi-Hilbertian spaces, *Aust. J. Math. Anal. Appl.*, **9** (2012), 1–12.
16. M. S. Moslehian, Q. Xu, A. Zamani, Seminorm and numerical radius inequalities of operators in semi-Hilbertian spaces, *Linear Algebra Appl.*, **591** (2020), 299–321. https://doi.org/10.1016/j.laa.2020.01.015
17. H. W. Qiao, G. J. Hai, A. Chen, Improvements of $A$-numerical radius for semi-Hilbertian space operators, *J. Math. Inequal.*, **18** (2024), 791–810. https://doi.org/10.7153/jmi-2024-18-43
18. S. S. Dragomir, Some inequalities for the Euclidean operator radius of two operators in Hilbert spaces, *Linear Algebra Appl.*, **419** (2006), 256–264. https://doi.org/10.1016/j.laa.2006.04.017
19. A. Abu-Omar, F. Kittaneh, Numerical radius inequalities for $n\times n$ operator matrices, *Linear Algebra Appl.*, **468** (2015), 18–26. https://doi.org/10.1016/j.laa.2013.09.049
20. M. Al-Dolat, F. Kittaneh, Numerical radius inequalities for certain operator matrices, *J. Anal.*, **32** (2024), 2939–2951. https://doi.org/10.1007/s41478-024-00782-9
21. Q. X. Xu, Z. M. Ye, A. Zamani, Some upper bounds for the $A$-numerical radius of $2\times 2$ block matrices, 2020, arXiv: 2005.04590.
22. K. Feki, Some $\mathbb{A}$-numerical radius inequalities for $d\times d$ operator matrices, *Rend. Circ. Mat. Palermo Ser. 2*, **71** (2022), 85–103. https://doi.org/10.1007/s12215-021-00623-9
23. P. Bhunia, R. K. Nayak, K. Paul, Refinements of $A$-numerical radius inequalities and their applications, *Adv. Oper. Theory*, **5** (2020), 1498–1511. https://doi.org/10.1007/s43036-020-00056-8
24. F. Kittaneh, A. Zamani, Bounds for $\mathbb{A}$-numerical radius based on an extension of $A$-Buzano inequality, *J. Comput. Appl. Math.*, **426** (2023), 115070. https://doi.org/10.1016/j.cam.2023.115070
25. F. Kittaneh, A. Zamani, A refinement of $A$-Buzano inequality and applications to $A$-numerical radius inequalities, *Linear Algebra Appl.*, **697** (2024), 32–48. https://doi.org/10.1016/j.laa.2023.02.020
26. S. Sahoo, On $\mathbb{A}$-numerical radius inequalities for $2\times 2$ operator matrices-II, *Filomat*, **35** (2021), 5237–5252. https://doi.org/10.2298/FIL2115237S
27. P. Bhunia, K. Feki, K. Paul, $A$-numerical radius orthogonality and parallelism of semi-Hilbertian space operators and their applications, *Bull. Iran. Math. Soc.*, **47** (2021), 435–457. https://doi.org/10.1007/s41980-020-00392-8

28. M. Guesba, $A$-numerical radius inequalities via Dragomir inequalities and related results, *Novi Sad J. Math.*, **54** (2024), 213–229. https://doi.org/10.30755/NSJOM.15867

29. M. Guesba, P. Bhunia, K. Paul, $A$-numerical radius inequalities and $A$-translatable radii of semi-Hilbert space operators, *Filomat*, **37** (2023), 3443–3456. https://doi.org/10.2298/FIL2311443G

30. M. Guesba, Some generalizations of $A$-numerical radius inequalities for semi-Hilbert space operators, *Boll. Unione Mat. Ital.*, **14** (2021), 681–692. https://doi.org/10.1007/s40574-021-00307-3

31. A. Zamani, $A$-numerical radius inequalities for semi-Hilbertian space operators, *Linear Algebra Appl.*, **578** (2019), 159–183. https://doi.org/10.1016/j.laa.2019.05.012

32. S. Daptari, F. Kittaneh, S. Sahoo, New $\mathbb{A}$-numerical radius equalities and inequalities for certain operator matrices and applications, *Results Math.*, **80** (2025), 22. https://doi.org/10.1007/s00025-024-02325-x

33. N. Ahmad, $(\gamma, \delta)$-$B$-norm and $B$-numerical radius inequalities in semi-Hilbertian spaces, *Iran. J. Sci.*, **47** (2023), 1765–1769. https://doi.org/10.1007/s40995-023-01545-0

34. M. L. Arias, G. Corach, M. C. Gonzalez, Lifting properties in operator ranges, *Acta Sci. Math. (Szeged)*, **75** (2009), 635–653.

35. P. Bhunia, K. Paul, R. K. Nayak, On inequalities for $A$-numerical radius of operators, *Electron. J. Linear Algebra*, **36** (2020), 143–157.

36. K. Feki, Inequalities for the $A$-joint numerical radius of two operators and their applications, *Hacet. J. Math. Stat.*, **53** (2024), 22–39. https://doi.org/10.15672/hujms.1142554

37. R. G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert space, *Proc. Amer. Math. Soc.*, **17** (1966), 413–415. https://doi.org/10.1090/S0002-9939-1966-0203464-1

38. F. Kittaneh, Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.*, **361** (2010), 262–269. https://doi.org/10.1016/j.jmaa.2009.08.059

39. F. Kittaneh, Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra*, **59** (2011), 1031–1037. https://doi.org/10.1080/03081087.2010.551661

40. S. S. Dragomir, Reverses of Jensen's integral inequality and applications: a survey of recent results, In: *Applications of nonlinear analysis*, Cham: Springer, 2018, 193–264. https://doi.org/10.1007/978-3-319-89815-5_8

41. E. Kikianty, S. S. Dragomir, Hermite-Hadamard's inequality and the $p$-HH-norm on the Cartesian product of two copies of a normed space, *Math. Inequal. Appl.*, **13** (2010), 1–32.