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Research article

A meticulous characterization of the pseudo-residual spectrum and pseudo-continuous spectrum of upper-triangular block operator matrices

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Abstract: For $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$, $C \in \mathcal{B}(Y, X)$, let $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ denote a 2×2 upper-triangular block operator matrix on $X \oplus Y$. Employing the spatial decomposition technique and stability of the closed range under small perturbations, this paper investigated the ε-injectivity and ε-density of upper-triangular block operator matrices under upper-triangular perturbations. Based on this, we studied meticulous characterization of the pseudo-residual spectrum and pseudo-continuous spectrum of upper-triangular block operator matrices under 2×2 upper-triangular perturbations.

Keywords: pseudospectra; ε-injective; pseudo-residual spectrum; pseudo-continuous spectrum; block operator matrix
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1. Introduction

Throughout this paper, let X and Y denote Hilbert spaces. $\mathcal{B}(X, Y)$ and $C(X, Y)$ represent the set of all bounded linear operators and the set of all densely defined closed linear operators from X to Y, respectively. We abbreviate $\mathcal{B}(X, X)$ and $C(X, X)$ to $\mathcal{B}(X)$ and $C(X)$, respectively. Suppose T is a linear operator in X, and the symbols $\mathcal{D}(T)$, $\mathcal{N}(T)$, $\mathcal{R}(T)$, T^* stand for the domain, the null space, the range, the adjoint of T, respectively. The symbols $\overline{\mathcal{R}(T)}$ and $\mathcal{R}(T)^\perp$ stand for the closure and the orthogonal complement of the range of T, respectively. I_X is the identity operator on X and it is abbreviated as I for simplicity. For a closed subspace M of X, P_M denotes the orthogonal projection of X onto M. In what follows, the norm $\|E\|$ of a block operator matrix $E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$ is the maximum norm of its entries E_{ij}, that is, $\|E\| = \max\limits_{i,j}\{\|E_{ij}\|\}$, $i, \ j = 1, \ 2$.

Spectral theory of linear operators is a key part of operator theory. It has played an important role as a measurement criterion in fields such as production and practice. However, after a long time, it has been discovered that some spectra of linear operators are highly sensitive to perturbations, making it impossible to accurately assess the corresponding systems solely through spectral analysis. Later, some scholars put forward the notion of the pseudospectrum. Suppose $T \in \mathcal{B}(X, Y)$ and $\varepsilon > 0$, and the pseudospectrum $\sigma_\varepsilon(T)$ of T is defined by

$$\sigma_\varepsilon(T) = \sigma(T) \cup \{\lambda \in \mathbb{C} : \ \|(T - \lambda I)^{-1}\| > \varepsilon^{-1}\},$$

where $\sigma(T)$ denotes the spectrum of T. In convention, if $\|(T - \lambda I)^{-1}\|$ is unbounded or nonexistent, i.e., $\lambda \in \sigma(T)$, we sign $\|(T - \lambda I)^{-1}\| = \infty$ (see [1]).

Also, it is very difficult for one to calculate precisely the spectra of some non-self-adjoint operators. Therefore, errors are inevitable, but it is believed that errors can be considered as the precise spectrum of perturbed operators. Under this consideration, for the pseudospectrum, Davies gives the following equivalent definition by perturbation [2]:

$$\sigma_\varepsilon(T) = \bigcup_{\|E\|<\varepsilon} \sigma(T + E) = \{\lambda \in \mathbb{C} : \lambda \in \sigma(T + E), \text{ for some } E \in \mathcal{B}(X), \|E\| < \varepsilon\}.$$

Pseudospectrum represents a generalization of the spectrum in the complex plane. It not only accurately reflects the sensitivity of spectra of the linear operators to perturbations, but also provides a new theoretical framework for analyzing the stability of non-normal systems and predicting the transient behavior of dynamical systems. Over the past 20 years, research on pseudospectra of matrices and operators has attracted significant attention. For instance, Tisseur F. discussed the structured pseudospectra of matrix polynomials and explored the relationships between structured pseudospectra, structured backward errors, and structured stability radii [3]. Rump S.M. investigated the difference between structured pseudospectrum and unstructured pseudospectrum [4]. Gerecht D. analyzed pseudospectrum of the linearized Navier-Stokes operator [5]. Jeribi A. studied essential pseudospectrum of linear operator [6–8]. Dondl W.P. studied the pseudospectrum of non-normal Schrödinger operators in the literature [9]. Guebbat H. studied the pseudospectrum of the convention-diffusion operator [10]. Roy N. depicted inner and outer approximation of pseudospectra of block triangular matrices [11]. Ammar A. proved the pseudospectrum mapping theorem of the essential pseudospectrum [12]. Smail S. analyzed the Wolf and Schechter essential pseudospectra by utilizing Fredholm perturbation [13]. For more results about pseudospectra, we refer to [14–16] and references cited therein.

Block operator matrices whose entries are composed of operators have significant applications in fields such as elasticity mechanics, fluid dynamics, and optimal control. In these applications, spectral theory of block operator matrices plays a crucial role and has garnered extensive attention from experts and scholars both domestically and internationally. Over the past two decades, substantial progress has been made in the spectral analysis and spectral complement problems of 2×2 operator matrices, as detailed in references [17–19]. Given the favorable stability properties of pseudospectra, the study of pseudospectra for block operator matrices has also attracted increasing attention. The essential pseudo-spectra of block operator matrices were studied under diagonal perturbation(see [16, 20, 21]). Tretter investigated pseudo block numerical range of operator matrix functions and spectral enclosures(see [22]). Inspired by many literatures, we first propose the definitions of the pseudo-point spectrum, the pseudo-residual spectrum, and the pseudo-continuous spectrum for densely defined closed linear operators, and characterize the pseudospectra of the upper triangular infinite-dimensional Hamiltonian operators by using the information of their diagonal entries (see [23]).

2. Preliminaries

Definition 2.1. *(See [23]) Let $T \in \mathcal{B}(X)$ and $\varepsilon > 0$. The pseudo-point spectrum $\sigma_{\varepsilon,p}(T)$, the pseudo-residual spectrum $\sigma_{\varepsilon,r}(T)$, and the pseudo-continuous spectrum $\sigma_{\varepsilon,c}(T)$ of T are, respectively, defined by*

$$\begin{aligned} \sigma_{\varepsilon,p}(T) :&= \bigcup_{\substack{E\in\mathcal{B}(X)\\ \|E\|<\varepsilon}} \sigma_p(T+E) \\ &= \{\lambda \in \mathbb{C} : \lambda \in \sigma_p(T+E),\ \text{for some } E \in \mathcal{B}(X),\ \|E\| < \varepsilon\}; \\ \sigma_{\varepsilon,r}(T) :&= \bigcup_{\substack{E\in\mathcal{B}(X)\\ \|E\|<\varepsilon}} \sigma_r(T+E) \\ &= \{\lambda \in \mathbb{C} : \lambda \in \sigma_r(T+E),\ \text{for some } E \in \mathcal{B}(X),\ \|E\| < \varepsilon\}; \\ \sigma_{\varepsilon,c}(T) :&= \bigcup_{\substack{E\in\mathcal{B}(X)\\ \|E\|<\varepsilon}} \sigma_c(T+E) \\ &= \{\lambda \in \mathbb{C} : \lambda \in \sigma_c(T+E),\ \text{for some } E \in \mathcal{B}(X),\ \|E\| < \varepsilon\}. \end{aligned}$$

It is trivial that

$$\sigma_\varepsilon(T) = \sigma_{\varepsilon,p}(T) \cup \sigma_{\varepsilon,r}(T) \cup \sigma_{\varepsilon,c}(T).$$

Remark 1. *The operator matrix $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ considered in the paper is an upper-triangular block operator matrix. When its meticulous pseudo-spectrum is studied, the perturbation operator E should be a general 2×2 block operator matrix. This paper only discusses the case where the perturbation operator E is an upper-triangular perturbation. Therefore, $\sigma^T_{\varepsilon,*}(M)(* \in \{p, r, c\})$ denotes meticulous pseudo-spectrum of M, where T denotes the perturbation operator E is an upper-triangular perturbation.*

Further, we studied the fine characterization of pseudospectra for block operator matrices under different perturbations (see [24, 25]).

Definition 2.2. *(See [24]) Let $T \in \mathcal{B}(X)$ and $\varepsilon > 0$. The ε-null space and ε-range of T are, respectively, defined by*

$$\begin{aligned} \mathcal{N}_\varepsilon(T) :&= \bigcup_{\substack{E\in\mathcal{B}(X)\\ \|E\|<\varepsilon}} \mathcal{N}(T+E) \\ &= \{x : \text{ there exist } E \in \mathcal{B}(X),\ \|E\| < \varepsilon,\ \text{ such that } (T+E)x = 0\}; \\ \mathcal{R}_\varepsilon(T) :&= \bigcup_{\substack{E\in\mathcal{B}(X)\\ \|E\|<\varepsilon}} \mathcal{R}(T+E) \end{aligned}$$

$$= \{y: \text{ there exist } E \in \mathcal{B}(X),\ \|E\| < \varepsilon,\ \text{ such that } (T+E)x = y\}.$$

Let $\alpha_\varepsilon(T) = \dim \mathcal{N}_\varepsilon(T)$, $\beta_\varepsilon(T) = \dim X \setminus \mathcal{R}_\varepsilon(T)$, $d_\varepsilon(T) = \dim \mathcal{R}_\varepsilon(T)^\perp$*. When* $\mathcal{R}_\varepsilon(T)$ *is closed,* $\beta_\varepsilon(T) = d_\varepsilon(T)$.

Remark 2. *If* $T \in \mathcal{B}(X)$*, clearly, it follows that* $\mathcal{N}(T) \subseteq \mathcal{N}_\varepsilon(T)$, $\mathcal{R}(T) \subseteq \mathcal{R}_\varepsilon(T)$.

Definition 2.3. *(See [25]) Let* $T \in \mathcal{B}(X)$ *and* $\varepsilon > 0$*. If there exist* $E \in \mathcal{B}(X)$, $\|E\| < \varepsilon$ *such that* $T + E$ *is injective, then the linear operator* T *is called* ε*-injective*; if $\overline{\mathcal{R}(T+E)} = X$, then the range of operator T is called ε-dense.

Definition 2.4. *(See [26]) Let* $T \in \mathcal{B}(X,Y)$, $S \in \mathcal{B}(Y,X)$*. If the following equalities hold:*

$$STS = S,\ TST = T,$$
$$(TS)^* = TS,\ (ST)^* = ST,$$

then S *is called the Moore-Penrose inverse of operator* T*. We write the Moore-Penrose inverse of operator* T *as* $T^\dagger$*, that is,* $T^\dagger = S$.

Definition 2.5. *Let* $T \in \mathcal{B}(X)$, $\varepsilon > 0$*. The pseudo-approximate point spectrum* $\sigma_{\varepsilon,ap}(T)$ *and the pseudo-defect spectrum* $\sigma_{\varepsilon,\delta}(T)$ *of* T *are, respectively, defined by*

$$\begin{aligned}
\sigma_{\varepsilon,ap}(T) :&= \bigcup_{\substack{E\in\mathcal{B}(X)\\ \|E\|<\varepsilon}} \sigma_{ap}(T+E)\\
&= \{\lambda \in \mathbb{C} : \lambda \in \sigma_{ap}(T+E),\ \text{for some E} \in \mathcal{B}(\text{X}),\ \|\text{E}\| < \varepsilon\};\\
\sigma_{\varepsilon,\delta}(T) :&= \bigcup_{\substack{E\in\mathcal{B}(X)\\ \|E\|<\varepsilon}} \sigma_{\delta}(T+E)\\
&= \{\lambda \in \mathbb{C} : \lambda \in \sigma_{\delta}(T+E),\ \text{for some E} \in \mathcal{B}(\text{X}),\ \|\text{E}\| < \varepsilon\}.
\end{aligned}$$

3. Main results

We begin with some basic lemmas, which are useful for the proofs of the main results of this paper.

Lemma 3.1. *(See [27]) Let* $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$, $C \in \mathcal{B}(Y,X)$*, and let* $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$*. For any* $\varepsilon > 0$, $\lambda \in \mathbb{C}$*. Then,*

$$\sigma^T_{\varepsilon,p}(M) = \sigma_{\varepsilon,p}(A) \cup \Omega_1 \cup \Omega_2,$$

where

$$\begin{aligned}
\Omega_1 &= \{\lambda \in \mathbb{C} : \mathcal{N}_\varepsilon(C) \cap \mathcal{N}_\varepsilon(B - \lambda I) \neq \{0\}\},\\
\Omega_2 &= \{\lambda \in \mathbb{C} : \mathcal{R}_\varepsilon(C\ |_{\mathcal{N}_\varepsilon(B-\lambda I)}) \cap \mathcal{R}_\varepsilon(A - \lambda I) \neq \{0\}\}.
\end{aligned}$$

Lemma 3.2. *(See [27]) Let* $T \in \mathcal{B}(X)$, $\varepsilon > 0$*. If the operator* T *has a nonzero closed range and* $0 < \varepsilon < \frac{1}{\|T^\dagger\|}$*, then for any perturbation operator* $E \in \mathcal{B}(X)$, $\|E\| < \varepsilon$*, the range* $\mathcal{R}(T+E)$ *is closed.*

In the following we study the ε-injectivity of the 2×2 bounded upper-triangular block operator matrices under upper triangular perturbations by utilizing the spatial decomposition technique.

Theorem 3.3. *Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$, $C \in \mathcal{B}(Y, X)$, and let $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. Suppose $\varepsilon > 0$ is sufficiently small, $\lambda \in \mathbb{C}$, then the following necessary and sufficient conditions hold.*

(1) $\mathcal{R}(A - \lambda I)$ is closed, and the operator $M - \lambda I$ is ε-injective if, and only if, the operators $A - \lambda I$ and $\begin{pmatrix} P_{\mathcal{R}_\varepsilon(A-\lambda I)^\perp} C \\ B - \lambda I \end{pmatrix}$ are ε-injective;

(2) $\mathcal{R}(B - \lambda I)$ is closed, and the operator $M - \lambda I$ is ε-injective if, and only if, the operators $A - \lambda I$ and $C\,|_{\mathcal{N}_\varepsilon(B-\lambda I)}$ are ε-injective, $\mathcal{R}_\varepsilon(C\,|_{\mathcal{N}_\varepsilon(B-\lambda I)}) \cap \mathcal{R}_\varepsilon(A - \lambda I) = \{0\}$.

Proof. (1) Since $\mathcal{R}(A - \lambda I)$ is closed, by virtue of Lemma 3.2, for arbitrary $E_{11} \in \mathcal{B}(X)$, $\|E_{11}\| < \varepsilon$, it concludes that $\mathcal{R}(A + E_{11} - \lambda I)$ is closed. Now we take corresponding decomposition

$$X \oplus Y = \mathcal{R}(A + E_{11} - \lambda I) \oplus \mathcal{R}(A + E_{11} - \lambda I)^\perp \oplus Y.$$

For any $E_{12} \in \mathcal{B}(Y, X)$, $\|E_{12}\| < \varepsilon$, $E_{22} \in \mathcal{B}(Y)$, $\|E_{22}\| < \varepsilon$, taking the operator

$$E = \begin{pmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{pmatrix} \in \mathcal{B}(X \oplus Y),$$

clearly, $\|E\| = \max\limits_{i,j}\{\|E_{ij}\|\} < \varepsilon$, i, $j = 1$, 2. Thus,

$$M + E - \lambda I : \ X \oplus Y \to \mathcal{R}(A + E_{11} - \lambda I) \oplus \mathcal{R}(A + E_{11} - \lambda I)^\perp \oplus Y$$

is described by

$$\begin{aligned} M + E - \lambda I &= \begin{pmatrix} A + E_{11} - \lambda I & C + E_{12} \\ 0 & B + E_{22} - \lambda I \end{pmatrix} \\ &= \begin{pmatrix} A_1 + (E_{11})_1 - \lambda I & C_1 + (E_{12})_1 \\ 0 & C_2 + (E_{12})_2 \\ 0 & B + E_{22} - \lambda I \end{pmatrix}, \end{aligned}$$

where it is clear that $A_1 + (E_{11})_1 - \lambda I : X \to \mathcal{R}(A + E_{11} - \lambda I)$ is right invertible. Suppose $(A_1 + (E_{11})_1 - \lambda I)_r^{-1}$ is a right invertible operator of $A_1 + (E_{11})_1 - \lambda I$, so there exist invertible operator

$$G = \begin{pmatrix} I & -(A_1 + (E_{11})_1 - \lambda I)_r^{-1}(C_1 + (E_{12})_1) \\ 0 & I \end{pmatrix} : X \oplus Y \to X \oplus Y$$

such that

$$\begin{pmatrix} A_1 + (E_{11})_1 - \lambda I & C_1 + (E_{12})_1 \\ 0 & C_2 + (E_{12})_2 \\ 0 & B + E_{22} - \lambda I \end{pmatrix} G = \begin{pmatrix} A_1 + (E_{11})_1 - \lambda I & 0 \\ 0 & C_2 + (E_{12})_2 \\ 0 & B + E_{22} - \lambda I \end{pmatrix}.$$

From the invertibility of the operator G, it follows that $M + E - \lambda I$ is injective if, and only if,

$$\begin{pmatrix} A_1 + (E_{11})_1 - \lambda I & 0 \\ 0 & C_2 + (E_{12})_2 \\ 0 & B + E_{22} - \lambda I \end{pmatrix}$$

is injective.

By reason of the foregoing discussion, the ε-injectivity of $M - \lambda I$ is equivalent such that $A - \lambda I$ is ε-injective and $\begin{pmatrix} C_2 \\ B - \lambda I \end{pmatrix} : Y \to \mathcal{R}_\varepsilon(A - \lambda I)^\perp \oplus Y$ is ε-injective.

(2) Suppose $M - \lambda I$ is ε-injective, and there exist

$$E = \begin{pmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{pmatrix} \in \mathcal{B}(X \oplus Y), \ \|E\| < \varepsilon,$$

such that $M+E-\lambda I$ is injective. Based on the closedness of $\mathcal{R}(B-\lambda I)$ and Lemma 3.2, for arbitrary $E_{22} \in \mathcal{B}(Y)$, $\|E_{22}\| < \varepsilon$, we conclude that $\mathcal{R}(B+E_{22}-\lambda I)$ is closed. Therefore, $M+E-\lambda I$ as an operator from the space decomposition $X \oplus \mathcal{N}(B + E_{22} - \lambda I) \oplus \mathcal{N}(B + E_{22} - \lambda I)^\perp$ into the space decomposition $X \oplus \mathcal{R}(B + E_{22} - \lambda I) \oplus \mathcal{R}(B + E_{22} - \lambda I)^\perp$ has the operator matrix

$$\begin{pmatrix} A + E_{11} - \lambda I & C_3 + (E_{12})_3 & C_4 + (E_{12})_4 \\ 0 & 0 & B_1 + (E_{22})_1 - \lambda I \end{pmatrix}.$$

It is evident that the operator $B_1 + (E_{22})_1 - \lambda I$ is invertible. So we take the invertible operator

$$T = \begin{pmatrix} I & -(C_4 + (E_{12})_4)(B_1 + (E_{22})_1 - \lambda I)^{-1} \\ 0 & I \end{pmatrix},$$

and

$$\begin{aligned} T & \begin{pmatrix} A + E_{11} - \lambda I & C_3 + (E_{12})_3 & C_4 + (E_{12})_4 \\ 0 & 0 & B_1 + (E_{22})_1 - \lambda I \end{pmatrix} \\ = & \begin{pmatrix} A + E_{11} - \lambda I & C_3 + (E_{12})_3 & 0 \\ 0 & 0 & B_1 + (E_{22})_1 - \lambda I \end{pmatrix}. \end{aligned}$$

By virtue of the invertibility of operator T, it follows that the operator $M + E - \lambda I$ is injective if, and only if,

$$\begin{pmatrix} A + E_{11} - \lambda I & C_3 + (E_{12})_3 & 0 \\ 0 & 0 & B_1 + (E_{22})_1 - \lambda I \end{pmatrix}$$

is injective. In this case,

$$\mathcal{N}(A + E_{11} - \lambda I \ \ C_3 + (E_{12})_3) = \{0\}.$$

In fact,

$$\mathcal{N}(A + E_{11} - \lambda I \ \ C_3 + (E_{12})_3) = \begin{pmatrix} \mathcal{N}(A + E_{11} - \lambda I) \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ \mathcal{N}(C_3 + (E_{12})_3) \end{pmatrix} \oplus$$

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \in \mathcal{N}(A + E_{11} - \lambda I)^\perp, \ y \in \mathcal{N}(C_3 + (E_{12})_3)^\perp, (A + E_{11} - \lambda I)x = -(C_3 + (E_{12})_3)y \right\},$$

thereby, we have

$$\mathcal{N}(A+E_{11}-\lambda I)=\{0\},\ \mathcal{N}(C_3+(E_{12})_3)=\{0\}$$

and

$$\mathcal{R}_\varepsilon(C\mid_{\mathcal{N}_\varepsilon(B-\lambda I)})\cap\mathcal{R}_\varepsilon(A-\lambda I)=\{0\}.$$

Hence, the operator $A-\lambda I$ is ε-injective, $C\mid_{\mathcal{N}_\varepsilon(B-\lambda I)}$ is ε-injective, and

$$\mathcal{R}_\varepsilon(C\mid_{\mathcal{N}_\varepsilon(B-\lambda I)})\cap\mathcal{R}_\varepsilon(A-\lambda I)=\{0\}.$$

□

Further, based on the conjugated relationship between the operator's ε-injectivity and the ε-density of its range, we obtain the following corollary.

Corollary 3.4. *Let $\varepsilon>0$ be sufficiently small, $\lambda\in\mathbb{C}$, $A\in\mathcal{B}(X)$, $B\in\mathcal{B}(Y)$, $C\in\mathcal{B}(Y,X)$, and let $M=\begin{pmatrix}A & C\\ 0 & B\end{pmatrix}$. Then, the range of the operator $M-\lambda I$ is ε-dense if, and only if, the ranges of the operator $A-\lambda I$ and $\begin{pmatrix}C_2\\ B-\lambda I\end{pmatrix}$ are ε-dense, where*

$$C_2=P_{\mathcal{R}_\varepsilon(A-\lambda I)^\perp}C:Y\to\mathcal{R}_\varepsilon(A-\lambda I)^\perp.$$

Proof. For the operator $E^*=\begin{pmatrix}E_{11}^* & 0\\ E_{12}^* & E_{22}^*\end{pmatrix}\in\mathcal{B}(X^*\oplus Y^*)$, we can decompose the space $X^*\oplus Y^*$ as follows.

$$X^*\oplus Y^*=\mathcal{N}(A^*+E_{11}^*-\overline{\lambda}I)^\perp\oplus\mathcal{N}(A^*+E_{11}^*-\overline{\lambda}I)\oplus Y^*.$$

In this manner,

$$M^*+E^*-\overline{\lambda}I:\mathcal{N}(A^*+E_{11}^*-\overline{\lambda}I)^\perp\oplus\mathcal{N}(A^*+E_{11}^*-\overline{\lambda}I)\oplus Y^*\to X^*\oplus Y^*$$

has the following matrix representation

$$\begin{aligned}M^*+E^*-\overline{\lambda}I &= \begin{pmatrix}A^*+E_{11}^*-\overline{\lambda}I & 0\\ C^*+E_{12}^* & B^*+E_{22}^*-\overline{\lambda}I\end{pmatrix}\\ &= \begin{pmatrix}A_1^*+(E_{11}^*)_1-\overline{\lambda}I & 0 & 0\\ C_1^*+(E_{12}^*)_1 & C_2^*+(E_{12}^*)_2 & B^*+E_{22}^*-\overline{\lambda}I\end{pmatrix},\end{aligned}$$

where it easily follows that $A_1^*+(E_{11}^*)_1-\overline{\lambda}I:\mathcal{N}(A^*+E_{11}^*-\overline{\lambda}I)^\perp\to X^*$ is left invertible. Suppose $(A_1^*+(E_{11}^*)_1-\overline{\lambda}I)_l^{-1}$ is a left invertible operator of $A_1^*+(E_{11}^*)_1-\overline{\lambda}I$, then there is invertible operator

$$\hat{G}=\begin{pmatrix}I & 0\\ -(C_1^*+(E_{12}^*)_1)(A_1^*+(E_{11}^*)_1-\overline{\lambda}I)_l^{-1} & I\end{pmatrix}$$

such that

$$\hat{G}(M^*+E^*-\overline{\lambda}I)=\begin{pmatrix}A_1^*+(E_{11}^*)_1-\overline{\lambda}I & 0 & 0\\ 0 & C_2^*+(E_{12}^*)_2 & B^*+E_{22}^*-\overline{\lambda}I\end{pmatrix}.$$

By invertibility of $\hat{G}$, we can claim that $M^* + E^* - \overline{\lambda} I$ is injective and it is equivalent that

$$\begin{pmatrix} A_1^* + (E_{11}^*)_1 - \overline{\lambda} I & 0 & 0 \\ 0 & C_2^* + (E_{12}^*)_2 & B^* + E_{22}^* - \overline{\lambda} I \end{pmatrix}$$

is injective. Based on the above discussion, we can get the ranges of $M - \lambda I$ as ε-dense if, and only if, the ranges of $A - \lambda I$ and $\begin{pmatrix} C_2 \\ B - \lambda I \end{pmatrix}$ are ε-dense. □

Based on the aforementioned theorems, we will next discuss the pseudo-residual spectrum and pseudo-continuous spectrum of upper-triangular block operator matrices under upper-triangular perturbations.

Theorem 3.5. *Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$, $C \in \mathcal{B}(Y, X)$, and let $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. $\varepsilon > 0$ is sufficiently small, $\lambda \in \mathbb{C}$, then $\lambda \in \sigma_{\varepsilon,r}^T(M)$ if, and only if, $A - \lambda I$ is ε-injective, $\mathcal{R}(A - \lambda I)$ is closed, $\begin{pmatrix} C_2 \\ B - \lambda I \end{pmatrix}$ is ε-injective, and it satisfies one of the following conditions:*

(1) the range of $B - \lambda I$ is not ε-dense;
(2) the range of $\begin{pmatrix} A - \lambda I & C \end{pmatrix}$ is not ε-dense;
(3) the range of $\begin{pmatrix} C \\ B - \lambda I \end{pmatrix}$ is not ε-dense.

Proof. Suppose $\lambda \in \sigma_{\varepsilon,r}^T(M)$. By the definition of pseudo-residual spectrum, we can conclude that there exist

$$E = \begin{pmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{pmatrix} \in \mathcal{B}(X \oplus Y), \ \|E\| = \max_{i,j}\{\|E_{ij}\|\} < \varepsilon,$$

such that $M + E - \lambda I$ is injective and $\overline{\mathcal{R}(M + E - \lambda I)} \neq X \oplus Y$. Consequently, by means of Theorem 3.3, we obtain that $M - \lambda I$ is ε-injective if, and only if, $A - \lambda I$ is ε-injective and $\mathcal{R}(A - \lambda I)$ is closed; $\begin{pmatrix} C_2 \\ B - \lambda I \end{pmatrix}$ is ε-injective, where $C_2 = P_{\mathcal{R}_\varepsilon(A-\lambda I)^\perp} C : Y \to \mathcal{R}_\varepsilon(A - \lambda I)^\perp$.

By $\overline{\mathcal{R}(M + E - \lambda I)} \neq X \oplus Y$, we have that there exist $E^* \in \mathcal{B}(X^* \oplus Y^*)$, $\mathcal{N}(M^* + E^* - \overline{\lambda} I) \neq \{0\}$. Then, we obtain $\overline{\lambda} \in \sigma_{\varepsilon,p}^T(M^*)$. So by virtue to Lemma 3.1, it follows that

$$\overline{\lambda} \in \sigma_{\varepsilon,p}(B^*) \cup \Omega_3 \cup \Omega_4,$$

where we set

$$\begin{aligned} \Omega_3 &= \{\overline{\lambda} \in \mathbb{C} : \mathcal{N}_\varepsilon(C^*) \cap \mathcal{N}_\varepsilon(A^* - \overline{\lambda} I) \neq \{0\}\}, \\ \Omega_4 &= \{\overline{\lambda} \in \mathbb{C} : \mathcal{R}_\varepsilon(C^* \mid_{\mathcal{N}_\varepsilon(A^* - \overline{\lambda} I)}) \cap \mathcal{R}_\varepsilon(B^* - \overline{\lambda} I) \neq \{0\}\}. \end{aligned}$$

Therefore, from $\overline{\lambda} \in \sigma_{\varepsilon,p}(B^*)$, it can be inferred that the range of the operator $B - \lambda I$ is not ε-dense.

By $\overline{\lambda} \in \Omega_3$, we get that there exist $\begin{pmatrix} E_{11}^* \\ E_{12}^* \end{pmatrix} \in \mathcal{B}(X^* \oplus Y^*)$ such that $\begin{pmatrix} A^* + E_{11}^* - \overline{\lambda} I \\ C^* + E_{12}^* \end{pmatrix}$ is not injective. Thus, it concludes that the range of $\begin{pmatrix} A + E_{11} - \lambda I & C + E_{12} \end{pmatrix}$ is not dense. Therefore, the range of $\begin{pmatrix} A - \lambda I & C \end{pmatrix}$ is not ε-dense.

Through $\overline{\lambda} \in \Omega_4$, it follows that there exist $E_{22}^* \in \mathcal{B}(Y^*)$, $E_{12}^* \in \mathcal{B}(X^*, Y^*)$, and nonzero elements $(x^*, y^*)^T \in \mathcal{N}_\varepsilon(A^* - \overline{\lambda}I) \oplus Y^*$ such that

$$(C^* + E_{12}^*)x^* + (B^* + E_{22}^* - \overline{\lambda}I)y^* = 0.$$

Consequently, $\begin{pmatrix} C^* + E_{12}^* & B^* + E_{22}^* - \overline{\lambda}I \end{pmatrix}$ is not injective. Further, it is evident that the range of $\begin{pmatrix} C + E_{12} \\ B + E_{22} - \lambda I \end{pmatrix}$ is not dense. That is, the range of $\begin{pmatrix} C \\ B - \lambda I \end{pmatrix}$ is not ε-dense.

Hence, $\lambda \in \sigma_{\varepsilon,r}^T(M)$ if, and only if, $A - \lambda I$ is ε-injective, $\mathcal{R}(A - \lambda I)$ is closed, $\begin{pmatrix} C_2 \\ B - \lambda I \end{pmatrix}$ is ε-injective, and it satisfies one of the following conditions:

(1) the range of $B - \lambda I$ is not ε-dense;
(2) the range of $\begin{pmatrix} A - \lambda I & C \end{pmatrix}$ is not ε-dense;
(3) the range of $\begin{pmatrix} C \\ B - \lambda I \end{pmatrix}$ is not ε-dense.

□

Theorem 3.6. *Let $\varepsilon > 0$ be sufficiently small, $\lambda \in \mathbb{C}$, $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$, $C \in \mathcal{B}(Y, X)$, and let $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. Then, $\lambda \in \sigma_{\varepsilon,c}^T(M) \backslash \sigma_{\varepsilon,\delta}(A)$ if, and only if, $\lambda \in \sigma_{\varepsilon,\delta}(B)$, $A - \lambda I$ is ε-injective, $\mathcal{R}(A - \lambda I)$ is closed, $\begin{pmatrix} C_2 \\ B - \lambda I \end{pmatrix}$ is ε-injective, and the ranges of $A - \lambda I$ and $\begin{pmatrix} C_2 \\ B - \lambda I \end{pmatrix}$ are ε-dense.*

Proof. Suppose $\lambda \in \sigma_{\varepsilon,c}^T(M)$, so it follows that there exist

$$E = \begin{pmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{pmatrix} \in \mathcal{B}(X \oplus Y), \ \|E\| = \max_{i,j}\{\|E_{ij}\|\} < \varepsilon,$$

such that $M + E - \lambda I$ is injective, $\overline{\mathcal{R}(M + E - \lambda I)} = X \oplus Y$, and $\mathcal{R}(M + E - \lambda I) \neq X \oplus Y$. On the one hand, by Theorem 3.3, we get $M - \lambda I$ is ε-injective if, and only if, $A - \lambda I$ is ε-injective, $\mathcal{R}(A - \lambda I)$ is closed and $\begin{pmatrix} C_2 \\ B - \lambda I \end{pmatrix}$ is ε-injective.

On the other hand, by virtue of Corollary 3.4, it follows that $\overline{\mathcal{R}(M + E - \lambda I)} = X \oplus Y$ if, and only if, the ranges of $A - \lambda I$ and $\begin{pmatrix} C_2 \\ B - \lambda I \end{pmatrix}$ are ε-dense. Also, $\lambda \notin \sigma_{\varepsilon,\delta}(A)$ and $\mathcal{R}(M + E - \lambda I) \neq X \oplus Y$, therefore, $\lambda \in \sigma_{\varepsilon,\delta}(B)$.

Considered the above analysis, it follows that $\lambda \in \sigma_{\varepsilon,c}^T(M) \backslash \sigma_{\varepsilon,\delta}(A)$ if, and only if, $\lambda \in \sigma_{\varepsilon,\delta}(B)$, $A - \lambda I$ is ε-injective, $\mathcal{R}(A - \lambda I)$ is closed, $\begin{pmatrix} C_2 \\ B - \lambda I \end{pmatrix}$ is ε-injective, and the ranges of $A - \lambda I$ and $\begin{pmatrix} C_2 \\ B - \lambda I \end{pmatrix}$ are ε-dense. □

4. Conclusions

In this paper, we utilize spatial decomposition techniques and stability of the closed range under small perturbations to discuss the ε-injectivity and ε-density of upper-triangular block operator matrices under upper-triangular perturbations. Based on this, we also study meticulous characterization of the pseudo-residual spectrum and pseudo-continuous spectrum of upper-triangular block operator matrices under 2×2 upper-triangular perturbations. These characterizations explicitly describe how these parts of the ε-pseudospectrum behave under 2×2 upper-triangular perturbations .

Author contributions

Runshuan Shen: Conceptualization, Methodology, Writing original draft, Writing review and editing; GuoLin Hou: Methodology, Writing review and editing, All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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