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Research article

On combinatorial properties of the degenerate Krawtchouk Appell polynomials

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Abstract: The foremost aim of this study is to introduce and study several combinatorial properties and highlight specific aspects of a new class of polynomial sequences, known as degenerate Krawtchouk Appell polynomials, associated with the degenerate Pascal measure. As an application, the connection between these brand-new polynomials, Stirling numbers, the scaling operator, the translation operator, and orthogonal Bell polynomials have been investigated.

Keywords: Bell polynomials; degenerate Pascal measure; degenerate Krawtchouk Appell polynomials; scaling operator
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1. Introduction

Over the past three decades, the study of various degenerate versions of special polynomials has grown significantly in prominence, primarily because these polynomials have demonstrated applications across a wide range of scientific and engineering domains, as documented in [1–3]. In fact, through their applications to many interesting arithmetical and combinatorial results, these polynomials offer a number of useful tools for solving integral and differential equations, quantum probability, and statistics. Furthermore, they help address various problems involving specific functions of mathematical physics, along with their extensions and generalizations. The first developments in this context were carried out by Carlitz [4], who introduced the degenerate Stirling, Bernoulli, and Euler polynomials by replacing the exponential function e^z with $(1+\lambda z)^{1/\lambda}$ in their generating functions. This approach recovers the classical polynomials in the limit as $\lambda \to 0$. Building upon these developments, Haroon and Khan [5] investigated degenerate Hermite-Bernoulli polynomials using p-adic fermionic integrals, leading to new identities that establish connections with Daehee polynomials in a unified and generalized framework. Duran and Acikgoz [6, 7] introduced the generalized degenerate Gould-

Hopper-based Stirling polynomials of the second kind and the degenerate Poisson distribution, as well as their relation to degenerate Bell polynomials. In a related direction, Further contributions include the work of Araci [8,9], who studied a new class of generating functions for type 2 Bernoulli polynomials and a new family of Daehee polynomials called degenerate q-Daehee polynomials, deriving several identities involving type 2 Euler polynomials and Stirling numbers of the second kind. Frontczak and Tomovski [10] provided probabilistic interpretations for summation formulas involving generalized Bernoulli and Euler-Genocchi polynomials of order (r, m), enriching the analytical framework of these polynomial families. Recent work by T. Kim and D. S. Kim, [11] introduced probabilistic variants of degenerate Stirling numbers of the second kind and degenerate Bell polynomials, grounded in random variables that satisfy certain moment conditions. These advancements continue to deepen the interplay between combinatorial structures, probabilistic methods, and degenerate special functions, revealing new pathways for analytical exploration and interdisciplinary applications.

In recent years, there has been a great deal of interest in the study of composite Poisson processes and their application to various problems. The Pascal measure is an important example of a compound Poisson, which is a useful and popular in probability theory and statistics. Moreover, the Krawtchouk polynomials, originally introduced by Mikhail Kravchuk (Krawtchouk), represent a foundational class of orthogonal polynomials with deep connections to combinatorics and probability theory. A comprehensive overview of Kravchuk's contributions and the historical development of these polynomials up to 2004 is available in [12]. These polynomials possess intrinsic mathematical richness and have found widespread applications across diverse scientific fields, including biology, complex networks, blockchain technology, quantum probability, coding theory, electromagnetic wave propagation, and image processing (see [13] about the notion of its Darboux transformations and [14] for applications in image analysis). Recall that the probability mass function of Pascal random variable X is given by

$$Pr([X = n]) := \mathcal{NB}_{p,r}(\{n\}) = p^r \binom{-r}{n} (-q)^n, \quad q := 1 - p, \ r > 0,$$

where for $\upsilon \in \mathbb{R}$ the generalized binomial coefficient is defined by

$$\binom{\upsilon}{n} = \frac{\upsilon(\upsilon - 1) \ldots (\upsilon - n + 1)}{n!}.$$

For $s \in \mathbb{R}$ such that $|qe^s| < 1$, one can see that the Laplace transform of $\mathcal{NB}_{p,r}$, denoted by $\mathcal{L}_{\mathcal{NB}_{p,r}}$, is given by

$$\mathcal{L}_{\mathcal{NB}_{p,r}}(s) = p^r \sum_{k=0}^{\infty} \binom{-r}{k} (-qe^s)^k = \Big(\frac{p}{1 - qe^s}\Big)^r.$$

In [15], the authors investigate the generating function of the so-called classical Krawtchouk polynomials $\mathcal{K}_{n,r}(x)$ given by

$$\Psi(t, x) = (1 + t)^x (1 + qt)^{-x-r} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{K}_{n,r}(x),$$

where $\mathcal{K}_{n,r}(x)$ is the polynomial defined by

$$\mathcal{K}_{n,r}(x) = n! \sum_{k=0}^{n} (-1)^k \binom{x}{n-k} \binom{x + r + k - 1}{k} q^k. \tag{1.1}$$

Here are the first few polynomials:

$$\begin{aligned}\mathcal{K}_{0,r}(x) &= 1,\\ \mathcal{K}_{1,r}(x) &= px - qr,\\ \mathcal{K}_{2,r}(x) &= p^2x^2 - p(1+q+2qr)x + q^2r(1+r).\end{aligned}$$

Asai et al. [16] prove that the orthogonality of polynomials generated by $\Psi(t, x)$ is guaranteed under the condition that $\mathbb{E}_{\mathcal{NB}_{p,r}}(\Psi(s, x)\Psi(t, x))$ is a function of st. This approach cannot be applicable in the degenerate version of the Pascal case. So to solve this problem, the so-called normalized-Laplace exponential is introduced to produce a new class of polynomials called degenerate Krawtchouk Appell polynomials.

The structure of the paper is as follows. In Section 2, we define the degenerate Pascal measure $\mathcal{NB}^{(\lambda)}_{p,r}$, and we prove that this measure is a combination of the classical Pascal measure and a specific probability measure $\nu_{\lambda,\beta}$ on $\mathbb{R}_+$. Next, we show that the n-th moments of $\mathcal{NB}^{(\lambda)}_{p,r}$ are given explicitly in terms of the Stirling numbers of the second kind. In Section 3, we introduce and highlight key properties of a novel class of Appell polynomials associated with the degenerate Pascal measure. As an application, we explicitly obtain the chaos expansion of the scaling operator in terms of these polynomials.

2. Degenerate Pascal measure

Our first goal is to give the degenerate version of the Pascal measure. So, to define degenerate extensions of classical functions and polynomials, we introduce the degenerate exponential function. For any $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$, the degenerate exponential function [11] is defined as

$$e^{\beta}_{\lambda}(z) = (1+\lambda z)^{\frac{\beta}{\lambda}} = \sum_{k=0}^{\infty} (\beta)_{k,\lambda} \frac{z^k}{k!}, \tag{2.1}$$

where $(\beta)_{k,\lambda}$ is the degenerate falling factorial defined by

$$(\beta)_{0,\lambda} = 1, \quad (\beta)_{k,\lambda} = \beta(\beta-\lambda)(\beta-2\lambda)\ldots(\beta-(k-1)\lambda), \quad (k \geq 1). \tag{2.2}$$

Definition 2.1. For $\lambda < 0$ and $\beta > 0$, the degenerate Pascal measure $\mathcal{NB}^{(\lambda)}_{p,r}$ is defined on $\mathcal{P}(\mathbb{N})$, the σ-algebra of all subsets of $\mathbb{N}$, by

$$\mathcal{NB}^{(\lambda)}_{p,r}(\Lambda) = \sum_{k=0}^{\infty} \frac{1}{k!} (e^{\beta}_{\lambda})^{(k)} \Big[r \log\Big(\frac{p}{1-qz}\Big)\Big]_{z=0} \delta_k(\Lambda), \quad \Lambda \in \mathcal{P}(\mathbb{N}),$$

where δ_k is the Dirac measure at $k \in \mathbb{N}$ and $(e^{\beta}_{\lambda})^{(k)}(z) := \frac{d^k}{dz^k} e^{\beta}_{\lambda}(z)$ is the k-th derivative of e^{β}_{λ}. For $z \in \mathbb{C}$ such that $|qe^z| < 1$, the Laplace transform of the measure $\mathcal{NB}^{(\lambda)}_{p,r}$ is given by

$$\mathcal{L}_{\mathcal{NB}^{(\lambda)}_{p,r}}(z) = \sum_{k=0}^{\infty} \frac{(qe^z)^k}{k!} (e^{\beta}_{\lambda})^{(k)}(r\log(p)) = e^{\beta}_{\lambda}\Big\{ r \log\big(\frac{p}{1-qe^z}\big)\Big\}. \tag{2.3}$$

Proposition 2.2. *The degenerate Pascal measure $\mathcal{NB}^{(\lambda)}_{p,r}$ is related to the Pascal measure $\mathcal{NB}_{p,r}$ under the following relation:*

$$\mathcal{NB}^{(\lambda)}_{p,r} = \int_0^\infty \mathcal{NB}_{p,rs}\nu_{\lambda,\beta}(ds),$$

where

$$\nu_{\lambda,\beta}(ds) = \frac{s^{-\frac{\beta}{\lambda}-1}e^{\frac{s}{\lambda}}}{\Gamma(-\frac{\beta}{\lambda})(-\lambda)^{-\frac{\beta}{\lambda}}}ds := G_{\lambda,\beta}(s)ds.$$

Proof. First, we need to determine a function $G_{\lambda,\beta}$ such that

$$\int_0^\infty e^{-sx}G_{\lambda,\beta}(x)dx = e^{\beta}_{\lambda}(-s) = (1-\lambda s)^{\frac{\beta}{\lambda}} = \lambda^{\frac{\beta}{\lambda}}\Big(\frac{1}{\lambda} - s\Big)^{\frac{\beta}{\lambda}}. \tag{2.4}$$

Then by using the following Laplace transform

$$(-1)^k\mathcal{L}(x^{k-1}e^{ax})(t) = \frac{\Gamma(k)}{(a-t)^k}, \quad k>0, \quad \mathcal{R}(a-t)>0,$$

and by choosing $a = \dfrac{1}{\lambda}$, $k = -\dfrac{\beta}{\lambda}$, we get

$$\lambda^{\frac{\beta}{\lambda}}\Big(\frac{1}{\lambda} - s\Big)^{\frac{\beta}{\lambda}} = \mathcal{L}\Big(\frac{x^{-\frac{\beta}{\lambda}-1}e^{\frac{x}{\lambda}}}{\Gamma(-\frac{\beta}{\lambda})(-\lambda)^{-\frac{\beta}{\lambda}}}\Big)(s).$$

Since we choose $k = -\dfrac{\beta}{\lambda} > 0$, β and λ must carry opposite signs. We therefore restrict our case to $\lambda < 0$ and $\beta > 0$. Hence the function $G_{\lambda,\beta}$ is given explicitly by

$$G_{\lambda,\beta}(x) = \frac{x^{-\frac{\beta}{\lambda}-1}e^{\frac{x}{\lambda}}}{\Gamma(-\frac{\beta}{\lambda})(-\lambda)^{-\frac{\beta}{\lambda}}}.$$

Note that $G_{\lambda,\beta}$ is the probability density function of the Gamma distribution with shape parameter $-\frac{\beta}{\lambda}$ and scale parameter $-\frac{1}{\lambda}$. Denote by

$$\mu := \int_0^\infty \mathcal{NB}_{p,ry}G_{\lambda,\beta}(y)dy.$$

Then, by using Eqs (2.3) and (2.4), along with Fubini's theorem, we get

$$\begin{aligned}
\int_0^\infty e^{zx}\mu(dx) &= \int_0^\infty e^{zx}\int_0^\infty \mathcal{NB}_{p,ry}(dx)G_{\lambda,\beta}(y)dy \\
&= \int_0^\infty \Big(\int_0^\infty e^{zx}\mathcal{NB}_{p,ry}(dx)\Big)G_{\lambda,\beta}(y)dy \\
&= \int_0^\infty \exp\Big[ry\log\big(\frac{p}{1-qe^z}\big)\Big]G_{\lambda,\beta}(y)dy \\
&= e^{\beta}_{\lambda}\Big\{r\log\Big(\frac{p}{1-qe^z}\Big)\Big\} \\
&= \mathcal{L}_{\mathcal{NB}^{(\lambda)}_{p,r}}(z).
\end{aligned}$$

This gives the statement. ◻

Proposition 2.3. *The m-th moment of $\mathcal{NB}_{p,r}^{(\lambda)}$ is given explicitly by*

$$\begin{aligned}M_\lambda(m) &:= \int_{\mathbb{R}} x^m \mathcal{NB}_{p,r}^{(\lambda)}(dx)\\ &= (1+\lambda r\log p)^{\frac{\beta}{\lambda}} \sum_{k=1}^{n} S(m,k)\Big(\frac{q}{1+\lambda r\log p}\Big)^k (\beta)_{k,\lambda} e_\lambda^{\beta-\lambda k}\Big(\frac{q}{1+\lambda r\log p}\Big),\end{aligned} \tag{2.5}$$

where $S(m,k)$ denotes the Stirling numbers of the second kind defined via its generating function by

$$(e^z-1)^k = k!\sum_{m=k}^{\infty} S(m,k)\frac{z^m}{m!}. \tag{2.6}$$

Proof. First, we need to determine the n-th derivative of the degenerate exponential function. If we put $\varphi(t) = e_\lambda^\beta(t)$ and differentiate (2.1), then we obtain

$$\begin{aligned}\varphi'(t) &= \beta(1+\lambda t)^{\frac{\beta}{\lambda}-1},\\ \varphi''(t) &= \beta(\beta-\lambda)(1+\lambda t)^{\frac{\beta}{\lambda}-2},\\ \varphi^3(t) &= \beta(\beta-\lambda)(\beta-2\lambda)(1+\lambda t)^{\frac{\beta}{\lambda}-3},\\ \varphi^{(4)}(t) &= \beta(\beta-\lambda)(\beta-2\lambda)(\beta-3\lambda)(1+\lambda t)^{\frac{\beta}{\lambda}-4}.\end{aligned}$$

Thus, continuing this process n times gives

$$\varphi^{(n)}(t) = (1+\lambda t)^{\frac{\beta}{\lambda}-n}\prod_{k=0}^{n-1}(\beta-k\lambda) = (\beta)_{n,\lambda} e_\lambda^{\beta-\lambda n}(t),$$

and we get

$$\mathcal{NB}_{p,r}^{(\lambda)}(\{n\}) = \frac{q^n}{n!}\varphi^{(n)}(r\log(p)) = \frac{q^n}{n!}(\beta)_{n,\lambda} e_\lambda^{\beta-\lambda n}(r\log p). \tag{2.7}$$

Hence, by using (2.1) and (2.7), we get

$$\begin{aligned}\int_{\mathbb{R}} x^m \mathcal{NB}_{p,r}^{(\lambda)}(dx) &= \sum_{n=0}^{\infty} n^m \mathcal{NB}_{p,r}^{(\lambda)}(\{n\})\\ &= \sum_{n=0}^{\infty} n^m \frac{q^n}{n!}(\beta)_{n,\lambda} e_\lambda^{\beta-\lambda n}(r\log p)\\ &= \sum_{n=0}^{\infty} n^m \frac{q^n}{n!}(\beta)_{n,\lambda}(1+\lambda r\log p)^{\frac{\beta}{\lambda}-n}\\ &= (1+\lambda r\log p)^{\frac{\beta}{\lambda}} \sum_{n=0}^{\infty} n^m (\beta)_{n,\lambda} \frac{(\frac{q}{1+\lambda r\log p})^n}{n!}.\end{aligned} \tag{2.8}$$

To handle the n^m term, we can use the operator $x\dfrac{d}{dx}$ which acts on x^n as

$$x\frac{d}{dx}x^n = nx^n.$$

Applying this operator m times gives

$$\Big(x\frac{d}{dx}\Big)^m x^n = n^m x^n.$$

Thus, we have

$$\sum_{n=0}^{\infty} n^m (\beta)_{n,\lambda}\frac{x^n}{n!} = \Big(x\frac{d}{dx}\Big)^m (1+\lambda x)^{\frac{\beta}{\lambda}}. \tag{2.9}$$

Moreover, the operator $\Big(x\frac{d}{dx}\Big)^m$ can be expressed in terms of $S(m,k)$ as follows:

$$\Big(x\frac{d}{dx}\Big)^m = \sum_{k=1}^{m} S(m,k) x^k \frac{d^k}{dx^k}.$$

Applying this to $(1+\lambda x)^{\frac{\beta}{\lambda}}$, we obtain

$$\Big(x\frac{d}{dx}\Big)^m (1+\lambda x)^{\frac{\beta}{\lambda}} = \sum_{k=1}^{m} S(m,k) x^k \frac{d^k}{dx^k}(1+\lambda x)^{\frac{\beta}{\lambda}} = \sum_{k=1}^{m} S(m,k) x^k (\beta)_{k,\lambda} e_{\lambda}^{\beta-\lambda k}(x). \tag{2.10}$$

Therefore, Eqs (2.8)–(2.10) yield

$$M_\lambda(m) = (1+\lambda r\log(p))^{\frac{\beta}{\lambda}} \sum_{k=1}^{m} S(m,k)\Big(\frac{q}{1+\lambda r\log p}\Big)^k (\beta)_{k,\lambda} e_{\lambda}^{\beta-\lambda k}\Big(\frac{q}{1+\lambda r\log p}\Big).$$

□

3. Degenerate Krawtchouk Appell polynomials

Our main objective is to introduce a multiplicative generating function and to study a new system of Appell polynomials associated with the degenerate Pascal measure that do not form an orthogonal family.

Definition 3.1. For $z \in \mathbb{C}$ such that $|qe^z| < 1$, the normalized Laplace-exponential with respect to $\mathcal{NB}_{p,q}^{(\lambda)}$ is defined by

$$\mathcal{N}\exp_{\lambda}^{\beta}(z,x) := \frac{e^{xz}}{\mathcal{L}_{\mathcal{NB}_{p,r}^{(\lambda)}}(z)} = \frac{e^{xz}}{e_{\lambda}^{\beta}\left[r\log\left(\frac{p}{1-qe^z}\right)\right]}. \tag{3.1}$$

Consider the following entire function ϑ defined on the open unit disk by:

$$\vartheta(z) := \log\left(\frac{1+z}{1+qz}\right), \quad |z| < 1.$$

By using the analytic property of the Laplace transform and since $\mathcal{L}_{\mathcal{NB}_{p,r}^{(\lambda)}}(0) = 1$, there exists a neighborhood $\mathcal{U}$ of $0 \in \mathbb{C}$ given by

$$\mathcal{U} = \{z \in \mathbb{C}, |qe^z| < 1\}$$

such that $\mathcal{N}\exp_{\lambda}^{\beta}(\vartheta(\cdot),x)$ is expressed in terms of the so-called degenerate Krawtchouk-Appell polynomials $\mathcal{K}_{n,\lambda}^{(\beta)}$ by

$$\mathcal{N}\exp_{\lambda}^{\beta}(\vartheta(z),x) = \frac{e^{xz}}{e_{\lambda}^{\beta}\left[r\log(1+qz)\right]} = \sum_{n=0}^{\infty}\frac{z^n}{n!}\mathcal{K}_{n,\lambda}^{(\beta)}(x), \quad \forall z \in \mathcal{U},$$

with

$$\mathcal{K}_{n,\lambda}^{(\beta)}(x) = \frac{d^n}{dz^n}\left[\mathcal{N}\exp_\lambda^\beta(\vartheta(z), x)\right]_{z=0}.$$

Now our goals are to find an explicit form of these polynomials. One can see that $\mathcal{N}\exp_\lambda^\beta(\vartheta(\cdot), x)$ can be considered as a generating function of a new polynomial $\mathcal{K}_{n,\lambda}^{(\beta)}(x)$ denoted by $\Psi_\lambda(t, x)$. That is,

$$\Psi_\lambda(t, x) = \frac{e^{x\vartheta(t)}}{\mathcal{L}_{\mathcal{NB}_{p,r}^{(\lambda)}}(\vartheta(t))} = \mathcal{N}\exp_\lambda^\beta(\vartheta(t), x) = \sum_{n=0}^{\infty}\frac{z^n}{n!}\mathcal{K}_{n,\lambda}^{(\beta)}(x). \tag{3.2}$$

Remark 3.2. The polynomials $\mathcal{K}_{n,\lambda}^{(\beta)}(x)$ do not form a system of orthogonal polynomials because $\mathbb{E}_{\mathcal{NB}_{p,r}^{(\lambda)}}(\Psi_\lambda(s, x)\Psi_\lambda(t, x))$ is not a function of st. In fact, by using (2.1) and (3.5), then for $s, t > 0$, we have

$$\begin{aligned}
\mathbb{E}_{\mathcal{NB}_{p,r}^{(\lambda)}}(\Psi_\lambda(s, x)\Psi_\lambda(t, x)) &= \sum_{k=0}^{\infty}\frac{e^{k(\vartheta(t)+\vartheta(s))}}{\mathcal{L}_{\mathcal{NB}_{p,r}^{(\lambda)}}(\vartheta(t))\mathcal{L}_{\mathcal{NB}_{p,r}^{(\lambda)}}(\vartheta(s))}\mathcal{NB}_{p,r}^{(\lambda)}(\{k\}) \\
&= \frac{\mathcal{L}_{\mathcal{NB}_{p,r}^{(\lambda)}}(\vartheta(t)+\vartheta(s))}{\mathcal{L}_{\mathcal{NB}_{p,r}^{(\alpha)}}(\vartheta(t))\mathcal{L}_{\mathcal{NB}_{p,r}^{(\lambda)}}(\vartheta(s))} \\
&= \frac{e_\lambda^\beta\left[r\log\left(\frac{p(1+qt)(1+qs)}{(1+qt)(1+qs)-q(1+t)(1+s)}\right)\right]}{e_\lambda^\beta(r\log(1+qt))e_\lambda^\beta(r\log(1+qs))} \\
&= \left(\frac{1+\lambda r[\log(1+qt)+\log(1+qs)-\log(1-qst)]}{(1+\lambda r\log(1+qt))(1+\lambda r\log(1+qs))}\right)^{\beta/\lambda} \\
&= \left(\frac{1+\lambda r[\log(1+qt)+\log(1+qs)-\log(1-qst)]}{1+\lambda r[\log(1+qt)+\log(1+qs)]+\lambda^2r^2\log(1+qt)\log(1+qs)}\right)^{\beta/\lambda}.
\end{aligned}$$

As a consequence, for $\lambda \neq 0$, the expression is generally not a function of st alone, since the terms $\log(1+qt)$ and $\log(1+qs)$ appear separately. In fact, if

$$\mathbb{E}_{\mathcal{NB}_{p,r}^{(\lambda)}}(\Psi_\lambda(s, x)\Psi_\lambda(t, x)) := F(s, t) = G(st),$$

then for fixed $u = st$, F must be constant on the hyperbola $st = u$. Taking $u = 1$, the denominator contains

$$\log(1+qt)\log(1+qs),$$

which clearly changes when moving from $(1, 1)$ to $(2, \frac{1}{2})$, while st remains equal to 1. Therefore, F cannot depend only on st. In the classical limit $\lambda \to 0$, however, $e_\lambda^\beta(z) \to e^{\beta z}$ and one obtains

$$F(s, t) = (1-qst)^{-\beta r},$$

which depends only on the product st.

Theorem 3.3. *The degenerate Krawtchouk-Appell polynomials possess, the following form:*

$$\mathcal{K}_{n,\lambda}^{(\beta)}(x) = \sum_{k=0}^{n}\frac{n!}{(n-k)!}r^{n-k}c_{n-k}(r)\varepsilon_k(x),$$

where

$$\varepsilon_k(x) = \sum_{j=0}^{k} (-1)^j \binom{x}{k-j} \binom{x+j-1}{j} q^j, \tag{3.3}$$

and the coefficient sequence $(c_n(r))_n$ *fulfills the following recurrence relation:*

$$\begin{cases} c_0(r) = 1, \\ c_n(r) = -\sum_{i=1}^{n} \binom{n}{i} r^{-i} (e_\lambda^\beta \circ \xi_q)^{(i)}(0) c_{n-i}(r) \text{ with } \xi_q(t) = r \log(1+qt). \end{cases} \tag{3.4}$$

Proof. By using Eq (3.2), we have

$$\Psi_\lambda(t,x) = \frac{\omega(t)^x}{e_\lambda^\beta(r\log(1+qt))} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{K}_{n,\lambda}^{(\beta)}(x), \tag{3.5}$$

where $\omega(t) = \frac{1+t}{1+qt}$. Note that for $x = 0$, we get

$$\Omega(t) := \Psi_\lambda(t,0) = \frac{1}{e_\lambda^\beta(r\log(1+qt))} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{K}_{n,\lambda}^{(\beta)}(0). \tag{3.6}$$

Now we put

$$\Omega(t) = \sum_{k=0}^{\infty} c_k(r) \frac{(rt)^k}{k!}. \tag{3.7}$$

The binomial series expansion and the Cauchy product rule yield

$$\omega(t)^x = \sum_{n=0}^{\infty} \Big[\sum_{j=0}^{n} (-1)^j \binom{x}{n-j} \binom{x+j-1}{j} q^j \Big] t^n = \sum_{n=0}^{\infty} \varepsilon_n(x) t^n,$$

where ε_n is given as in Eq (3.3). Thus by using Eq (3.7), we get

$$\begin{aligned} \Psi_\lambda(t,x) &= \omega(t)^x \sum_{k=0}^{\infty} c_k(r) \frac{(rt)^k}{k!} \\ &= \sum_{k=0}^{\infty} \varepsilon_k(x) t^k \sum_{k=0}^{\infty} c_k(r) \frac{(rt)^k}{k!} \\ &= \sum_{n=0}^{\infty} t^n \Big(\sum_{k=0}^{n} \frac{r^{n-k}}{(n-k)!} \varepsilon_k(x) c_{n-k}(r) \Big). \end{aligned} \tag{3.8}$$

Thus (3.5) and (3.8) yield

$$\mathcal{K}_{n,\lambda}^{(\beta)}(x) = \sum_{k=0}^{n} \frac{n!}{(n-k)!} r^{n-k} c_{n-k}(r) \varepsilon_k(x).$$

Now, our goal is to find the coefficients $c_n(r)$. So, by using (3.6) and (3.7), we have

$$e_\lambda^\beta \circ \xi_q(t) \cdot \sum_{k=0}^{n} c_k(r) \frac{(rt)^k}{k!} = 1. \tag{3.9}$$

That's to say

$$\Big(\sum_{k=0}^{\infty}\frac{(e_\lambda^\beta\circ\xi_q)^{(k)}(0)}{k!}t^k\Big)\Big(\sum_{k=0}^{n}c_k(r)\frac{(rt)^k}{k!}\Big)=1.$$

Moreover, we have

$$\Big(\sum_{k=0}^{\infty}\frac{(e_\lambda^\beta\circ\xi_q)^{(k)}(0)}{k!}t^k\Big)\Big(\sum_{k=0}^{n}c_k(r)\frac{(rt)^k}{k!}\Big)=\sum_{n=0}^{\infty}\gamma_n(r)\frac{t^n}{n!},$$

with

$$\gamma_n(r)=\sum_{i=0}^{n}\frac{(e_\lambda^\beta\circ\xi_q)^{(i)}(0)}{i!}\frac{n!c_{n-i}(r)r^{n-i}}{(n-i)!}=\sum_{i=0}^{n}\binom{n}{i}r^{n-i}(e_\lambda^\beta\circ\xi_q)^{(i)}(0)c_{n-i}(r). \tag{3.10}$$

From Eqs (3.9) and (3.10), we conclude that

$$\gamma_0(r)=1,\quad \gamma_n(r)=0\quad \forall n\geq 1.$$

Thus, we get

$$c_0(r)=\gamma_0(r)=1$$

and

$$\sum_{i=0}^{n}\binom{n}{i}r^{n-i}(e_\lambda^\beta\circ\xi_q)^{(i)}(0)c_{n-i}(r)=0\quad \forall n\geq 1,$$

which gives the following recurrence formula:

$$c_n(r)=-\sum_{i=1}^{n}\binom{n}{i}r^{-i}(e_\lambda^\beta\circ\xi_q)^{(i)}(0)c_{n-i}(r).$$

□

Example 3.4. Define $f(t)=e_\lambda^\beta(\xi_q(t))$. Since $\xi_q(0)=0,\xi_q'(0)=rq,\xi_q''(0)=-rq^2$, then we have

$$f'(0)=(e_\lambda^\beta)'(0)\,\xi_q'(0)=\beta rq$$

and

$$f''(0)=(e_\lambda^\beta)''(0)(\xi_q'(0))^2+(e_\lambda^\beta)'(0)\,\xi_q''(0)=\beta rq^2((\beta-\lambda)r-1).$$

Using the recurrence relation (3.4), we obtain

$$c_1(r)=-\binom{1}{1}r^{-1}f'(0)=-\frac{1}{r}\beta rq=-\beta q,$$

$$c_2(r)=-\Big[\binom{2}{1}r^{-1}f'(0)c_1+\binom{2}{2}r^{-2}f''(0)\Big]=\beta q^2(\beta+\lambda)+\frac{\beta q^2}{r}.$$

On the other hand, the evaluation of $\varepsilon_k(x)$ gives

$$\begin{aligned}
\varepsilon_0(x) &= 1,\\
\varepsilon_1(x) &= \binom{x}{1} - \binom{x}{0}\binom{x}{1}q = x - xq = x(1-q),\\
\varepsilon_2(x) &= \binom{x}{2} - \binom{x}{1}\binom{x}{1}q + \binom{x}{0}\binom{x+1}{2}q^2\\
&= \frac{x(x-1)}{2} - x^2 q + \frac{x(x+1)}{2}q^2\\
&= \frac{(1-q)^2}{2}x^2 - \frac{(1-q)(1+q)}{2}x.
\end{aligned}$$

Thus, we obtain

$$\mathcal{K}^{(\beta)}_{0,\lambda}(x) = c_0\,\varepsilon_0(x) = 1,$$

$$\mathcal{K}^{(\beta)}_{1,\lambda}(x) = \frac{1!}{(1-0)!}r^1 c_1\varepsilon_0 + \frac{1!}{(0)!}r^0 c_0\varepsilon_1 = px - \beta qr,$$

and

$$\begin{aligned}
\mathcal{K}^{(\beta)}_{2,\lambda}(x) &= \frac{2!}{(2)!}r^2 c_2\varepsilon_0 + \frac{2!}{(1)!}r^1 c_1\varepsilon_1 + \frac{2!}{(0)!}r^0 c_0\varepsilon_2\\
&= p^2x^2 - p(1+q+2\beta qr)x + \beta q^2 r(1+(\beta+\lambda)r).
\end{aligned}$$

For $\beta = 1$ and in the limit $\lambda \to 0$, these polynomials reduce to the classical Krawtchouk polynomials defined in (1.1).

Now, consider the Taylor expansion of a normalized Laplace-exponential function at z, given in terms of the Appell polynomials $P^{(\beta)}_{n,\lambda}$ $(n \in \mathbb{N})$:

$$\mathcal{N}\mathrm{exp}^{\beta}_{\lambda}(z,x) = \frac{e^{xz}}{\mathcal{L}_{\mathcal{NB}^{(\lambda)}_{p,r}}(z)} = \sum_{n=0}^{\infty}\frac{z^n}{n!}P^{(\beta)}_{n,\lambda}(x), \quad z \in \mathcal{U}. \tag{3.11}$$

Theorem 3.5. *The polynomials $\mathcal{K}^{(\beta)}_{n,\lambda}$ satisfy the following useful combinatorial properties:*

(P1) $\mathcal{K}^{(\beta)}_{n,\lambda}(x) = \sum_{m=0}^{n}\frac{\varpi(m,n)}{m!}P^{(\beta)}_{m,\lambda}(x)$, *where* $\varpi(0,0) := 1$ *and*

$$\varpi(m,n) := \sum_{i_1+\ldots+i_m=n}(-1)^{n+m}n!\prod_{j=1}^{m}\Big(\frac{1-q^{i_j}}{i_j}\Big). \tag{3.12}$$

(P2) $x^n = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\frac{\varrho(m,k)}{m!}\mathcal{K}^{(\beta)}_{m,\lambda}(x)M_\lambda(n-k)$, *where $M_\lambda(k)$ is the k-th moment given by Eq* (2.5) *and*

$$\begin{cases}\varrho(0,0) := 1\\ \varrho(m,k) := \displaystyle\sum_{i_1+\cdots+i_m=k}\frac{k!}{i_1!\ldots i_m!}\prod_{j=1}^{m}\frac{d^{i_j}}{dz^{i_j}}\Big[\frac{e^z-1}{1-qe^z}\Big]_{z=0}.\end{cases}$$

(P3) $\mathcal{K}_{n,\lambda}^{(\beta)}(x+y) = \sum_{k+l+m=n} \frac{n!}{k!l!m!}\mathcal{K}_{k,\lambda}^{(\beta)}(x)\mathcal{K}_{l,\lambda}^{(\beta)}(x)(e_\lambda^\beta \circ \xi_q)^{(m)}(0).$

(P4) $\mathcal{K}_{n,\lambda}^{(\beta)}(x+y) = \sum_{k=0}^{n} \binom{n}{k}\mathcal{K}_{k,\lambda}^{(\beta)}(x)[y]_{n-k},$

where

$$[y]_n := \sum_{k=0}^{n} \binom{n}{k} q^k (y)_{1,n-k}(-y)_{1,k}.$$

Proof. ***(P1)*** First, by using equation (3.11), we get

$$\Psi_\lambda(x,z) = \frac{e^{x\vartheta(z)}}{\mathcal{L}_{\mathcal{NB}_{p,r}^{(\lambda)}}(\vartheta(z))} = \sum_{m=0}^{\infty} \frac{\vartheta(z)^m}{m!} P_{m,\lambda}^{(\beta)}(x)$$

for any $z \in \mathcal{U}$. It is obvious that

$$\vartheta(z) = \log\Big(\frac{1+z}{1+qz}\Big) = \sum_{k=0}^{\infty} \frac{z^k}{k!}\eta_k,$$

with

$$\eta_k = (-1)^{k+1}(k-1)!(1-q^k) \quad \forall k>0, \quad \eta_0 = 0.$$

Therefore, with the help of Eq (3.12), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{z^n}{n!}\mathcal{K}_{n,\lambda}^{(\beta)}(x) &= P_{0,\lambda}^{(\beta)} + \sum_{m=1}^{\infty} \frac{1}{m!}\Big(\sum_{k=0}^{\infty} \frac{z^k}{k!}\eta_k\Big)^m P_{m,\lambda}^{(\beta)}(x) \\
&= P_{0,\lambda}^{(\beta)} + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{n=m=0}^{\infty} \frac{z^n}{n!} \sum_{i_1+\cdots+i_m=n} \frac{n!}{i_1!\cdots i_m!} \prod_{j=1}^{m} \eta_{i_j} P_{m,\lambda}^{(\beta)}(x) \\
&= P_{0,\lambda}^{(\beta)} + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{n=m}^{\infty} \frac{z^n}{n!} \sum_{i_1+\cdots+i_m=n} (-1)^{n+m} n! \prod_{j=1}^{m} \Big(\frac{1-q^{i_j}}{i_j}\Big) P_{m,\lambda}^{(\beta)}(x) \\
&= \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{n} \frac{\varpi(m,n)}{m!} P_{m,\lambda}^{(\beta)}(x).
\end{aligned}$$

Hence we obtain

$$\mathcal{K}_{n,\lambda}^{(\beta)}(x) = \sum_{m=0}^{n} \frac{\varpi(m,n)}{m!} P_{m,\lambda}^{(\beta)}(x).$$

(P2) Let ζ be the inverse function of ϑ. Then one can see that

$$\zeta(z) = \frac{e^z - 1}{1 - qe^z} = \sum_{k=0}^{\infty} \kappa_k \frac{z^k}{k!},$$

where

$$\kappa_k = \frac{d^k}{dz^k}\Big[\frac{e^z-1}{1-qe^z}\Big]_{z=0}.$$

By using Eq (3.1), we get

$$\frac{e^{xz}}{\mathcal{L}_{\mathcal{NB}^{(\lambda)}_{p,r}}(z)} = \frac{e^{xz}}{e^{\beta}_{\lambda}\Big(r\log(\frac{p}{1-qe^{z}})\Big)} = \sum_{m=0}^{\infty}\frac{\zeta(z)^m}{m!}\mathcal{K}^{(\beta)}_{m,\lambda}(x), \quad z\in\mathcal{U}.$$

So we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{z^n}{n!}P^{(\beta)}_{n,\lambda}(x) &= \mathcal{K}^{(\beta)}_{0,\lambda}(x) + \sum_{m=1}^{\infty}\frac{1}{m!}\Big(\sum_{k=1}^{\infty}\kappa_k\frac{z^k}{k!}\Big)^m\mathcal{K}^{(\beta)}_{m,\lambda}(x)\\ &= \mathcal{K}^{(\beta)}_{0,\lambda}(x) + \sum_{m=1}^{\infty}\frac{1}{m!}\sum_{n=m}^{\infty}\frac{z^n}{n!}\sum_{i_1+\ldots+i_m=n}\frac{n!}{i_1!\ldots i_m!}\prod_{j=1}^{m}\kappa_{i_j}\mathcal{K}^{(\beta)}_{m,\lambda}(x)\\ &= \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{m=0}^{n}\frac{\varrho(m,n)}{m!}\mathcal{K}^{(\beta)}_{m,\lambda}(x).\end{aligned}$$

Thus, we have

$$P^{(\beta)}_{n,\lambda}(x) = \sum_{m=0}^{n}\frac{\varrho(m,n)}{m!}\mathcal{K}^{(\beta)}_{m,\lambda}(x). \tag{3.13}$$

On the other hand, since

$$e^{xz} = \mathcal{N}\exp^{\beta}_{\lambda}(z,x)\mathcal{L}_{\mathcal{NB}^{(\lambda)}_{p,r}}(z),$$

we get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{x^n}{n!}z^n &= \sum_{k=0}^{\infty}\frac{z^k}{k!}P^{(\beta)}_{k,\lambda}(x)\sum_{k=0}^{\infty}\frac{z^k}{k!}M_{\lambda}(k)\\ &= \sum_{n=0}^{\infty}\frac{z^n}{n!}\Big(\sum_{k=0}^{n}\binom{n}{k}P^{(\beta)}_{k,\lambda}(x)M_{\lambda}(n-k)\Big).\end{aligned}$$

Hence with the help of Eq (3.13), we obtain

$$x^n = \sum_{k=0}^{n}\binom{n}{k}P^{(\beta)}_{k,\lambda}(x)M_{\lambda}(n-k) = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\frac{\varrho(m,k)}{m!}\mathcal{K}^{(\beta)}_{m,\lambda}(x)M_{\lambda}(n-k).$$

(P3) By using the fact that

$$\mathcal{N}\exp^{\beta}_{\lambda}(\vartheta(z),x+y) = \mathcal{N}\exp^{\beta}_{\lambda}(\vartheta(z),x)\mathcal{N}\exp^{\beta}_{\lambda}(\vartheta(z),y)\mathcal{L}_{\mathcal{NB}^{(\lambda)}_{p,r}}(\vartheta(z))$$

and

$$\mathcal{L}_{\mathcal{NB}^{(\lambda)}_{p,r}}(\vartheta(z)) = e^{\beta}_{\lambda}(r\log(1+qz)) = \sum_{k=0}^{\infty}\frac{(e^{\beta}_{\lambda}\circ\xi_q)^{(k)}(0)}{k!}z^k,$$

we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{z^n}{n!}\mathcal{K}^{(\beta)}_{n,\lambda}(x+y) &= \sum_{k=0}^{\infty}\frac{z^k}{k!}\mathcal{K}^{(\beta)}_{k,\lambda}(x)\sum_{l=0}^{\infty}\frac{z^l}{l!}\mathcal{K}^{(\beta)}_{l,\lambda}(y)\sum_{m=0}^{\infty}\frac{z^m}{m!}\mathcal{K}^{(\beta)}_{m,\lambda}(x)\\ &= \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k+l+m=n}\frac{n!}{k!l!m!}\mathcal{K}^{(\beta)}_{k,\lambda}(x)\mathcal{K}^{(\beta)}_{l,\lambda}(x)(e^{\beta}_{\lambda}\circ\xi_q)^{(m)}(0).\end{aligned}$$

This gives the statement.

(P4) Recall that, for $n \in \mathbb{N}$ and $y \in \mathbb{C}$, the generating function of the classical falling factorials $(y)_n := (y)_{1,n}$ obtained by (2.2) with $\lambda = 1$ is

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}(y)_n = \exp[y \log(1+z)].$$

Then, one can directly see that

$$\exp(y\vartheta(z)) = \exp(y\log(1+z) - y\log(1+qz)) = \sum_{n=0}^{\infty}[y]_n \frac{z^n}{n!}, \tag{3.14}$$

where

$$[y]_n := \sum_{k=0}^{n} \binom{n}{k} q^k (y)_{n-k}(-y)_k.$$

It is clear that

$$\mathcal{N}\exp_\lambda^\beta(\vartheta(z), x+y) = \mathcal{N}\exp_\lambda^\beta(\vartheta(z), x)\exp(y\vartheta(z)).$$

Then with the help of Eq (3.14), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{z^n}{n!}\mathcal{K}_{n,\lambda}^{(\beta)}(x+y) &= \sum_{k=0}^{\infty} \frac{z^k}{k!}\mathcal{K}_{k,\lambda}^{(\beta)}(x) \sum_{k=0}^{\infty} \frac{z^k}{k!}[y]_k \\
&= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k+m=n} \frac{n!}{k!m!}\mathcal{K}_{k,\lambda}^{(\beta)}(x)[y]_m \\
&= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} \binom{n}{k}\mathcal{K}_{k,\lambda}^{(\beta)}(x)[y]_{n-k}.
\end{aligned}$$

Hence we deduce the expansion of $\mathcal{K}_{n,\lambda}^{(\beta)}(x+y)$ given by

$$\mathcal{K}_{n,\lambda}^{(\beta)}(x+y) = \sum_{k=0}^{n} \binom{n}{k}\mathcal{K}_{k,\lambda}^{(\beta)}(x)[y]_{n-k}.$$

□

Corollary 3.6. *The polynomials $\mathcal{K}_{n,\lambda}^{(\beta)}$ have the following form:*

$$\mathcal{K}_{n,\lambda}^{(\beta)}(x) = \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\sum_{m=j}^{n-(k-j)}\binom{n}{m}s(m,j)s(n-m,k-j)q^{n-m}P_{k,\lambda}^{(\beta)}(x), \tag{3.15}$$

where $s(n,k)$ denotes the Stirling numbers of the first kind, defined via their generating function as follows:

$$\log(1+z)^k = k!\sum_{n=k}^{\infty} s(n,k)\frac{z^n}{n!}. \tag{3.16}$$

Proof. Applying the binomial series yields

$$\vartheta(z)^k = (\log(1+z) - \log(1+qz))^k = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\log(1+z))^j(\log(1+qz))^{k-j}.$$

By using (3.16), we have

$$(\log(1+z))^j = j!\sum_{m=j}^{\infty} s(m,j)\,\frac{z^m}{m!},$$

$$(\log(1+qz))^{k-j} = (k-j)!\sum_{t=k-j}^{\infty} s(t,k-j)\,\frac{(qz)^t}{t!},$$

and multiplying the two series gives

$$(\log(1+z))^j(\log(1+qz))^{k-j} = j!\,(k-j)!\sum_{m=j}^{\infty}\sum_{t=k-j}^{\infty} s(m,j)\,s(t,k-j)\,q^t\,\frac{z^{m+t}}{m!\,t!}.$$

Set $n = m+t$. Since $t = n-m \geq k-j$, we have $m \leq n-(k-j)$. Moreover, $m \geq j$ and $t \geq k-j$ imply $n \geq k$. Thus, the double sum becomes a single sum over n, and we get

$$(\log(1+z))^j(\log(1+qz))^{k-j} = j!\,(k-j)!\sum_{n=k}^{\infty}\frac{z^n}{n!}\sum_{m=j}^{n-(k-j)}\binom{n}{m} s(m,j)\,s(n-m,k-j)\,q^{n-m}.$$

Inserting this expression into the binomial expansion, we obtain

$$\begin{aligned}\vartheta(z)^k &= \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\,j!\,(k-j)!\sum_{n=k}^{\infty}\frac{z^n}{n!}\sum_{m=j}^{n-(k-j)}\binom{n}{m} s(m,j)\,s(n-m,k-j)\,q^{n-m}\\ &= k!\sum_{n=k}^{\infty}\frac{z^n}{n!}\sum_{j=0}^{k}(-1)^{k-j}\sum_{m=j}^{n-(k-j)}\binom{n}{m} s(m,j)\,s(n-m,k-j)\,q^{n-m}.\end{aligned}$$

Then we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{z^n}{n!}\mathcal{K}^{(\beta)}_{n,\lambda}(x) &= P^{(\beta)}_{0,\lambda}(x) + \sum_{k=1}^{\infty}\frac{1}{k!}(\vartheta(z))^k P^{(\beta)}_{m,\lambda}(x)\\ &= P^{(\beta)}_{0,\lambda}(x) + \sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{z^n}{n!}\sum_{j=0}^{k}(-1)^{k-j}\sum_{m=j}^{n-(k-j)}\binom{n}{m} s(m,j)s(n-m,k-j)q^{n-m}P^{(\beta)}_{m,\lambda}(x)\\ &= \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\sum_{m=j}^{n-(k-j)}\binom{n}{m} s(m,j)s(n-m,k-j)q^{n-m}P^{(\beta)}_{m,\lambda}(x).\end{aligned}$$

As a consequence, we obtain the explicit expression of $\mathcal{K}^{(\beta)}_{n,\lambda}(x)$ as in Eq (3.15). □

Corollary 3.7. *The Appell polynomials $P^{(\beta)}_{n,\lambda}$ have the following form in terms of $\mathcal{K}^{(\beta)}_{n,\lambda}(x)$:*

$$P^{(\beta)}_{n,\lambda}(x) = \sum_{k=0}^{n}\Big(\sum_{j=k}^{n}\binom{j-1}{k-1}\frac{q^{j-k}}{p^j}\,j!S(n,j)\mathcal{K}^{(\beta)}_{k,\lambda}(x). \tag{3.17}$$

Proof. Let $u = e^z - 1$, then we have

$$\begin{aligned}
\zeta^k(z) &= \frac{1}{(1-q)^k} u^k \Big(1 - \frac{q}{1-q} u\Big)^{-k} \\
&= \frac{1}{(1-q)^k} (e^z - 1)^k \sum_{m=0}^{\infty} \binom{-k}{m} \Big(\frac{-q}{1-q} u\Big)^m \\
&= \frac{1}{(1-q)^k} (e^z - 1)^k \sum_{m=0}^{\infty} \binom{k+m-1}{m} (\frac{q}{p})^m u^m \\
&= \frac{1}{(1-q)^k} \sum_{m=0}^{\infty} \binom{k+m-1}{m} (\frac{q}{p})^m (e^z - 1)^{k+m}.
\end{aligned}$$

Thus by using (2.6), we can write

$$\begin{aligned}
\zeta^k(z) &= \frac{1}{(1-q)^k} \sum_{m=0}^{\infty} \binom{k+m-1}{m} (\frac{q}{p})^m \sum_{n=k+m}^{\infty} (k+m)! S(n, k+m) \frac{z^n}{n!} \\
&= \frac{1}{p^k} \sum_{n=k}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{n-k} \binom{k+m-1}{m} (\frac{q}{p})^m (k+m)! S(n, k+m) \\
&= \frac{1}{p^k} \sum_{n=k}^{\infty} \frac{z^n}{n!} \sum_{j=k}^{n} \binom{j-1}{k-1} (\frac{q}{p})^{j-k} j! S(n, j) \\
&= \sum_{n=k}^{\infty} \Big(\sum_{j=k}^{n} \binom{j-1}{k-1} \frac{q^{j-k}}{p^j} j! S(n, j) \Big) \frac{z^n}{n!}.
\end{aligned}$$

By using Eq (3.1), we have

$$\begin{aligned}
\frac{e^{xz}}{\mathcal{L}_{\mathcal{NB}^{(\lambda)}_{p,r}}(z)} &= \sum_{n=0}^{\infty} \frac{z^n}{n!} P^{(\beta)}_{n,\lambda}(x) \\
&= \sum_{k=0}^{\infty} \frac{(\zeta(z))^k}{k!} \mathcal{K}^{(\beta)}_{k,\lambda}(x) \\
&= \sum_{k=0}^{\infty} \Big[\sum_{n=k}^{\infty} \Big(\sum_{j=k}^{n} \binom{j-1}{k-1} \frac{q^{j-k}}{p^j} j! S(n, j) \Big) \frac{z^n}{n!} \Big] \frac{\mathcal{K}^{(\beta)}_{k,\lambda}(x)}{k!} \\
&= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} \Big(\sum_{j=k}^{n} \binom{j-1}{k-1} \frac{q^{j-k}}{p^j} j! S(n, j) \Big) \frac{\mathcal{K}^{(\beta)}_{k,\lambda}(x)}{k!}.
\end{aligned}$$

Hence we obtain the explicit expression of $P^{(\beta)}_{n,\lambda}(x)$ in terms of $\mathcal{K}^{(\beta)}_{n,\lambda}(x)$ given as Eq (3.17). □

Now, our goal is to express the degenerate Krawtchouk-Appell polynomials $\mathcal{K}^{(\beta)}_{n,\lambda}$ ($n \in \mathbb{N}$) in terms of the Bell polynomials $B_{n,k}$, which are defined for any $x_1, \dots, x_{n-k+1} \in \mathbb{R}$ by

$$B_{n,k}((x_i)_{j=1}^{n-k+1}) := \sum \frac{n!}{i_1! i_2! \dots i_{n-k+1}!} \Big(\frac{x_1}{1!}\Big)^{i_1} \dots \Big(\frac{x_{n-k+1}}{(n-k+1)!}\Big)^{i_{n-k+1}},$$

where the summation is over all sequences of non negative integers $i_1, i_2, \dots, i_{n-k+1}$ satisfying the following conditions:

(i) $i_1 + i_2 + \ldots + i_{n-k+1} = k$,

(ii) $i_1 + 2i_2 + \ldots + (n-k+1)i_{n-k+1} = n$.

Recall that Faà di Bruno's formula is given as follows:

$$\frac{d^n}{dt^n}(\varphi(\psi(t))) = \sum_{k=0}^{n} \varphi^{(k)}(\psi(t))B_{n,k}(\psi'(t), \psi''(t), \ldots, \psi^{(n-k+1)}(t)). \tag{3.18}$$

For more details, see [17].

Lemma 3.8. *For every $n \in \mathbb{N}^*$, it holds that*

$$P_{n,\lambda}^{(\beta)}(0) = \sum_{k=0}^{n}(-1)^k k! B_{n,k}\Big((M_\lambda(j))_{j=1}^{n-k+1}\Big).$$

Proof. First, applying (3.11), we obtain

$$P_{n,\lambda}^{(\beta)}(0) = \frac{d^n}{dz^n}\mathcal{N}\exp_\lambda^\beta(z,x)\Big|_{z=x=0}.$$

Let φ_1 and φ_2 be the two functions defined by

$$\begin{aligned} \varphi_1(z) &:= \mathcal{N}\exp_\lambda^\beta(z,0) = \frac{1}{\mathcal{L}_{\mathcal{NB}_{p,r}^{(\lambda)}}(z)} = \frac{1}{e_\lambda^\beta[r\log(\frac{p}{1-qe^z})]}, \\ \varphi_2(z) &:= e^{xz}. \end{aligned}$$

Then, by the general Leibniz rule, we get

$$P_{n,\lambda}^{(\beta)}(0) = \frac{d^n}{dz^n}(\varphi_1(z)\varphi_2(z))\Big|_{z=x=0} = \sum_{k=0}^{n}\binom{n}{k}\varphi_1^{(n-k)}(z)\varphi_2^{(k)}(z)\Big|_{z=x=0}.$$

It is clear that $\varphi_2^{(k)}(z) = x^k\varphi_2(z)$, and by using formula (3.18) with the choice

$$\varphi_1(z) = \sigma(\mathcal{L}_{\mathcal{NB}_{p,r}^{(\lambda)}}(z)), \quad \psi(z) = \frac{1}{z},$$

we get

$$\begin{aligned} \varphi_1^{(n-k)}(z) &= \sum_{j=0}^{n-k}\psi^{(j)}(\mathcal{L}_{\mathcal{NB}_{p,r}^{(\lambda)}}(z))B_{n-k,j}(\mathcal{L}'_{\mathcal{NB}_{p,r}^{(\alpha)}}(z), \mathcal{L}''_{\mathcal{NB}_{p,r}^{(\alpha)}}(z), \ldots, \mathcal{L}^{(n-k-j+1)}_{\mathcal{NB}_{p,r}^{(\alpha)}}(z)) \\ &= \sum_{j=0}^{n-k}\frac{(-1)^j j!}{(\mathcal{L}_{\mathcal{NB}_{p,r}^{(\lambda)}}(z))^{j+1}}B_{n-k,j}((\mathcal{L}^{(i)}_{\mathcal{NB}_{p,r}^{(\lambda)}}(z))_{i=1}^{n-k-j+1}). \end{aligned}$$

Hence, we obtain

$$\frac{d^n}{dz^n}(\varphi_1(z)\varphi_2(z)) = \sum_{k=0}^{n}\binom{n}{k}\Big[\sum_{j=0}^{n-k}\frac{(-1)^j j!}{(\mathcal{L}_{\mathcal{NB}_{p,r}^{(\lambda)}}(z))^{j+1}}B_{n-k,j}((\mathcal{L}^{(i)}_{\mathcal{NB}_{p,r}^{(\lambda)}}(z))_{i=1}^{n-k-j+1})\Big]x^k\varphi_2(z),$$

and the statement follows by evaluating the previous expression at $z = x = 0$. □

Combining (3.14) and Lemma 3.8 with the uniqueness of the representation in terms of the standard powers x^k, we obtain the following technical result.

Lemma 3.9. *The Appell polynomials $P^{(\beta)}_{n,\lambda}$ have the following form:*

$$P^{(\beta)}_{n,\lambda}(x) = \sum_{k=0}^{n} \binom{n}{k} \Big[\sum_{i=0}^{n-k} (-1)^i i! B_{n-k,i}((M_\lambda(j))_{j=1}^{n-k-j+1}) \Big] x^k.$$

Finally, we obtain the following main result, which gives an explicit expression of the Krawtchouk Appell in terms of Bell polynomials.

Theorem 3.10. *The degenerate Krawtchouk Appell polynomials $\mathcal{K}^{(\lambda)}_{n,\beta}$ are given in terms of the Bell polynomials as*

$$\mathcal{K}^{(\beta)}_{n,\lambda}(x) = \sum_{k=0}^{n} \binom{n}{k} \Big[\sum_{i=0}^{n-k} (-1)^i i! B_{n-k,i}\Big((M_\alpha(j))_{j=1}^{n-k-i+1}\Big) \Big] [x]_k.$$

Proof. First, by using the general Leibniz rule, we have

$$\mathcal{K}^{(\lambda)}_{n,\beta}(0) = \frac{d^n}{dz^n}(\varphi(z)\psi(z))|_{z=x=0} = \sum_{k=0}^{n} \binom{n}{k} \varphi^{(n-k)}(z)\psi^{(k)}(z)|_{z=x=0},$$

with the choice

$$\varphi(z) := \mathcal{N}\exp^{\beta}_{\lambda}(z,0) = \frac{1}{\mathcal{L}_{\mathcal{NB}^{(\lambda)}_{p,r}}(z)} = \frac{1}{e^{\beta}_{\lambda}[r\log(\frac{p}{1-qe^z})]},$$
$$\psi(z) := e^{x\vartheta(z)} = e^{x\log(\frac{1+z}{1+qz})}, \quad |qe^z| < 1.$$

Thus, again by formula (3.18), we obtain

$$\varphi^{(n-k)}(z) = \sum_{j=0}^{n-k} \frac{(-1)^j j!}{(\mathcal{L}_{\mathcal{NB}^{(\lambda)}_{p,r}}(\vartheta(z)))^{j+1}} B_{n-k,j}((\mathcal{L}^i_{\mathcal{NB}^{(\lambda)}_{p,r}}(\vartheta(z)))_{i=1}^{n-k-j+1})$$

and

$$\psi^{(k)}(z) = \sum_{l=0}^{k} x^l e^{x\vartheta(z)} B_{k,l}((\vartheta^{(m)}(z))_{m=1}^{k-l+1}).$$

Then we have

$$\mathcal{K}^{(\lambda)}_{n,\beta}(0) = \sum_{j=0}^{n} (-1)^j j! B_{n,j}\Big((M_\lambda(m))_{m=1}^{n-j+1}\Big). \tag{3.19}$$

Hence by using the property ***(P4)*** and (3.19), we get

$$\mathcal{K}^{(\lambda)}_{n,\beta}(x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{K}^{(\lambda)}_{n,\beta}(0)[x]_k = \sum_{k=0}^{n} \binom{n}{k} \Big[\sum_{i=0}^{n-k} (-1)^i i! B_{n-k,i}\Big((M_\lambda(j))_{j=1}^{n-k-i+1}\Big) \Big] [x]_k.$$

□

Now, let $\mathcal{P}(\mathbb{R})$ be the set of all polynomials in $\mathbb{R}$ with complex coefficients, defined by

$$\mathcal{P}(\mathbb{R}) = \Big\{\varphi : \mathbb{R} \to \mathbb{C}; \varphi(x) = \sum_{n=0}^{N_\varphi} \varphi_n \mathcal{K}_{n,\beta}^{(\lambda)}(x), \varphi_n \in \mathbb{C}, N_\varphi \in \mathbb{N}^*\Big\}.$$

Let $z \in \mathbb{C}$ be given. Define the scaling operator σ_z as a continuous operator from $\mathcal{P}(\mathbb{R})$ into itself, such that

$$(\sigma_z\varphi)(x) = \varphi(zx), \quad \varphi \in \mathcal{P}(\mathbb{R}).$$

Theorem 3.11. *Let $\varphi \in \mathcal{P}(\mathbb{R})$ be given by its chaos decomposition:*

$$\varphi(x) = \sum_{n=0}^{N_\varphi} \varphi_n \mathcal{K}_{n,\beta}^{(\lambda)}(x).$$

Then $\sigma_z\varphi$ is given explicitly as follows:

$$\sigma_z(\varphi) = \sum_{n=0}^{N_\varphi} n!\varphi_n \sum_{k=0}^{n} \varepsilon_{n-k}(zx) \sum_{m=0}^{k} \frac{\rho(m,k)(-\beta)_{m,\lambda}}{m!k!},$$

where ε_j is given by Eq (3.3)*, and*

$$\rho(m,k) = \sum_{l_1+\ldots l_m=n} (-1)^{n+m} q^m k! \prod_{i=1}^{m} \frac{1}{l_i}.$$

Proof. First, one can see that

$$\begin{aligned}
e_\lambda^{-\beta}(\xi_q(t)) &= \sum_{n=0}^{\infty} (-\beta)_{n,\lambda} \frac{\xi_q^n(t)}{n!} \\
&= 1 + \sum_{n=1}^{\infty} \frac{1}{n!}(-\beta)_{n,\lambda} \Big(\sum_{l=0}^{\infty} (-1)^{l-1} r q^l \frac{t^l}{l}\Big)^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{1}{n!}(-\beta)_{n,\lambda} \sum_{n=m=0}^{\infty} \frac{t^n}{n!} \sum_{l_1+\ldots+l_m=n} (-1)^{n+m} q^m \prod_{i=1}^{m} \frac{1}{l_i} \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{m=0}^{n} \frac{\rho(m,n)(-\beta)_{m,\lambda}}{m!}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\Psi_\lambda(zx,t) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{K}_{n,\beta}^{(\lambda)}(zx) \\
&= \frac{e^{xz\vartheta(t)}}{e_\lambda^{\beta}\{r\log(\frac{p}{1-qe^{\vartheta(t)}})\}} \\
&= e^{xz\vartheta(t)} e_\lambda^{-\beta}(r\log(1+qt))
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \varepsilon_n(zx)t^n \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\rho(m,n)(-\beta)_{m,\lambda}}{m!n!} t^n$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \varepsilon_{n-k}(zx) \sum_{m=0}^{k} \frac{\rho(m,k)(-\beta)_{m,\lambda}}{m!k!} t^n.$$

Thus, we get

$$\mathcal{K}_{n,\beta}^{(\lambda)}(zx) = n! \sum_{k=0}^{n} \varepsilon_{n-k}(zx) \sum_{m=0}^{k} \frac{\rho(m,k)(-\beta)_{m,\lambda}}{m!k!},$$

and as a consequence we obtain

$$\sigma_z(\varphi) = \sum_{n=0}^{N_\varphi} n!\varphi_n \mathcal{K}_{n,\beta}^{(\lambda)}(zx) = \sum_{n=0}^{N_\varphi} n!\varphi_n \sum_{k=0}^{n} \varepsilon_{n-k}(zx) \sum_{m=0}^{k} \frac{\rho(m,k)(-\beta)_{m,\lambda}}{m!k!}.$$

□

Example 3.12. If

$$\varphi(x) = \sum_{n=0}^{N_\varphi} \varphi_n \mathcal{K}_{n,\beta}^{(\lambda)}(x),$$

then by using ***(P4)***, the translation operator τ_y for $y \in \mathbb{R}$ can be represented as

$$\tau_y\varphi(x) = \varphi(x+y) = \sum_{n=0}^{N_\varphi} \varphi_n \mathcal{K}_{n,\beta}^{(\lambda)}(x+y) = \sum_{n=0}^{N_\varphi} \sum_{k=0}^{n} \binom{n}{k} \varphi_n \mathcal{K}_{n,\beta}^{(\lambda)}(x)[y]_{n-k}.$$

Furthermore, the action of the translation operator on the normalized-Laplace exponential function is given by

$$\tau_y \mathcal{N}\exp_\lambda^\beta(z,x) = e^{yz}\mathcal{N}\exp_\lambda^\beta(z,x).$$

Let us define an operator D_y on $\mathcal{P}(\mathbb{R})$ such that its action on $\mathcal{N}\exp_\lambda^\beta(z,x)$ is given by

$$D_y\mathcal{N}\exp_\lambda^\beta(z,x) = yz\mathcal{N}\exp_\lambda^\beta(z,x).$$

Then, we obtain

$$\langle \tau_y\mathcal{N}\exp_\lambda^\beta(z,x), \mathcal{N}\exp_\lambda^\beta(z,x')\rangle = \langle e^{yz}\mathcal{N}\exp_\lambda^\beta(z,x), \mathcal{N}\exp_\lambda^\beta(z,x')\rangle$$
$$= e^{yz}\langle \mathcal{N}\exp_\lambda^\beta(z,x), \mathcal{N}\exp_\lambda^\beta(z,x')\rangle.$$

On the other hand, we have

$$\left\langle \exp(D_y)\mathcal{N}\exp_\lambda^\beta(z,x), \mathcal{N}\exp_\lambda^\beta(z,x')\right\rangle = \left\langle \sum_{k=0}^{\infty} \frac{1}{k!} D_y^k \mathcal{N}\exp_\lambda^\beta(z,x), \mathcal{N}\exp_\lambda^\beta(z,x')\right\rangle$$
$$= \left\langle \sum_{k=0}^{\infty} \frac{1}{k!} (yz)^k \mathcal{N}\exp_\lambda^\beta(z,x), \mathcal{N}\exp_\lambda^\beta(z,x')\right\rangle$$
$$= e^{yz}\langle \mathcal{N}\exp_\lambda^\beta(z,x), \mathcal{N}\exp_\lambda^\beta(z,x')\rangle.$$

Hence we conclude that $\tau_y = \exp(D_y)$. The translation, derivative, and scaling operators obtained in the degenerate case are essential for our subsequent investigation of certain stochastic differential equations and the study of the Donsker functional of degenerate Brownian motion.

4. Conclusions

In this study, we propose a concrete method for investigating several combinatorial properties and highlight specific aspects of a new class of polynomial sequences known as degenerate Krawtchouk-Appell polynomials. Our research focuses primarily on the role of the relationship between these new polynomials, Stirling numbers, the scaling operator, translation operator, and orthogonal Bell polynomials. This combinatorial study will serve us in our future research to develop some stochastic applications, with the aim of obtaining practical applications.

Author contributions

Both authors of this article have contributed equally. Both authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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