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Research article

Generating functions and formal dilation structures for a fractional Hermite family

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Abstract: We develop a generating function framework for a fractional Hermite family $\{H_n^{(\alpha)}(x)\}$ defined through an explicit gamma function representation, within the setting of formal power series. The gamma normalized generating function $\mathcal{G}_\alpha(x,t) = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x)\frac{t^n}{\Gamma(\alpha n+1)}$. encodes the entire family in a single algebraic object. Working in the formal power series ring $\mathbb{C}[x^\alpha][[t]]$, we establish several structural properties of $\mathcal{G}_\alpha(x,t)$. We derive coefficient extraction and reconstruction formulas, obtain explicit double-sum representations, and establish an even-odd decomposition of the generating series. These results show that the generating function completely determines the underlying fractional Hermite family. We further introduce the Euler operator $\mathcal{E}_t = t\partial_t$ and develop a formal functional calculus in the index variable. This leads to the dilation identity $\exp(s\mathcal{E}_t)\,\mathcal{G}_\alpha(x,t) = \mathcal{G}_\alpha(x,e^s t)$, together with the associated semigroup and infinitesimal identities. As additional algebraic consequences, we establish a homogeneous decomposition of the formal power series space, characterize the corresponding homogeneous components as eigenspaces of the Euler operator, construct the associated projection operators, derive commutation relations, and prove the invariance of the even and odd subspaces under the Euler action. All results are obtained within a purely formal coefficientwise framework and require no additional analytic assumptions. The resulting theory provides a consistent algebraic setting for studying this fractional Hermite family through generating function methods and suggests possible extensions to other classes of fractional special functions.

Keywords: fractional Hermite functions; generating function; Caputo derivative; fractional operators; formal power series
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1. Introduction

Special functions and orthogonal polynomials play a central role in mathematical analysis and its applications, including spectral theory, approximation theory, mathematical physics, and numerical computation [1, 2]. Among the most important examples are the Hermite polynomials, which form a classical orthogonal system associated with the Gaussian weight and possess a rich algebraic and analytic structure. Their theory is closely connected with differential equations, generating functions, recurrence relations, and operator methods, making them a fundamental object in both pure and applied mathematics.

Fractional calculus has emerged as an important extension of classical differential and integral calculus, providing effective tools for the description of memory effects, hereditary phenomena, and nonlocal processes [3, 4]. During the last decades, fractional differential operators have found numerous applications in mathematical modeling and have stimulated extensive research on fractional analogs of classical special functions and orthogonal polynomial systems. Such developments are motivated both by theoretical considerations and by the need for suitable basis functions in the analysis of fractional differential equations.

Recent years have witnessed increasing interest in fractional Hermite-type systems and related fractional orthogonal structures. Awadalla [5] introduced a family of fractional Hermite functions associated with Caputo fractional derivatives and derived several of their basic properties. Subsequent developments include extensions to Atangana-Baleanu fractional operators and fractional inner-product settings [6, 7]. At the same time, Hermite-type techniques have been employed in the analysis and numerical treatment of fractional differential equations [8–10]. Related developments concerning fractional orthogonal polynomials and generalized special functions may be found in [11–13].

Generating functions constitute one of the most powerful tools in the theory of special functions [12, 14]. They provide a compact representation of an entire family of functions within a single formal object and often serve as a source of structural identities, recurrence relations, and operational formulas. Although generating function methods are classical in the theory of orthogonal polynomials, a systematic generating function framework for the fractional Hermite family introduced in [5] has not previously been developed. Existing studies have mainly focused on explicit series representations, fractional differential equations, numerical methods, or particular fractional operators [8, 9, 15].

Motivated by this observation, we investigate the fractional Hermite family through a generating function approach. The starting point is the explicit representation of the functions $H_n^{(\alpha)}(x)$ introduced in [5]. Because gamma functions appear naturally in the coefficients of this representation, it is natural to consider the gamma normalized generating series

$$\mathcal{G}_\alpha(x,t) = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x)\,\frac{t^n}{\Gamma(\alpha n+1)}.$$

Rather than pursuing analytic convergence questions, we work entirely in the formal power series ring $\mathbb{C}[x^\alpha][[t]]$, which provides a natural algebraic setting for the study of the generating function and its associated structures [16–18].

Within this formal framework, we establish several structural properties of the generating function. We derive coefficient extraction and reconstruction formulas, obtain explicit double-sum representations, and establish an even odd decomposition of the generating series. We further introduce the Euler operator

$$\mathcal{E}_t = t\frac{\partial}{\partial t},$$

which acts as an index-counting operator on the generating function. This leads to a formal functional calculus in the index variable and to the dilation identity

$$\exp(s\mathcal{E}_t)\,\mathcal{G}_\alpha(x,t) = \mathcal{G}_\alpha(x, e^s t),$$

together with the associated semigroup and infinitesimal relations. In addition, we investigate several algebraic consequences of the Euler operator framework, including homogeneous decompositions, projection operators, commutation relations, and parity invariance properties.

The novelty of the present work lies in the development of a coherent generating function framework for the fractional Hermite family. Rather than introducing a new fractional special function, we focus on the structural role played by the generating series itself and show how several identities arise naturally from the formal power series setting. The resulting framework provides a self-contained algebraic viewpoint for the study of the family $\{H_n^{(\alpha)}\}_{n\geq 0}$ and reveals a natural dilation and homogeneous decomposition structure associated with its generating function.

The paper is organized as follows. Section 2 recalls essential preliminaries on Caputo fractional derivatives, the gamma function, and the fractional Hermite family adopted in this work. Section 3 introduces the gamma normalized generating function and develops its fundamental properties, including coefficient extraction, reconstruction formulas, double-sum representations, and even-odd decompositions. Section 4 develops a formal dilation structure based on the Euler operator $\mathcal{E}_t = t\partial_t$ and establishes the associated functional calculus and dilation identities. Section 5 investigates several algebraic consequences of this framework, including homogeneous decompositions, projection operators, commutation relations, and parity invariance properties. Finally, Section 6 contains concluding remarks and directions for future research.

2. Preliminaries

In this section, we collect the basic notions and previously established facts that will be used throughout the paper. We begin with the Caputo fractional derivative and the gamma function, and then we recall the fractional Hermite functions introduced in [5]. For standard background on fractional calculus, we refer to the monographs [3, 4].

Let $AC^n([0,\infty),\mathbb{R})$ denote the space of real-valued functions whose nth derivative is absolutely continuous on compact subintervals of $[0,\infty)$.

Definition 2.1 (Caputo fractional derivative [3])**.** *Let $\alpha \in (0,1]$. For a function $f \in AC^1([0,\infty),\mathbb{R})$, the Caputo fractional derivative of order α is defined by*

$$^{C}D^{\alpha}_{0+} f(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x (x-t)^{-\alpha} f'(t)\,dt, \qquad x > 0.$$

For completeness, we recall that for general $n \in \mathbb{N}$ and $\mu \in (n-1,n)$, if $f \in AC^n([0,\infty),\mathbb{R})$, then

$$^{C}D^{\mu}_{0+} f(x) = \frac{1}{\Gamma(n-\mu)}\int_0^x (x-t)^{n-\mu-1} f^{(n)}(t)\,dt, \qquad x > 0.$$

Throughout this paper, the notation ${}^{C}D_{0+}^{2\alpha}$ denotes the Caputo derivative of order 2α, where $2\alpha \in (0, 2]$ and **not** the sequential composition

$$ {}^{C}D_{0+}^{\alpha}({}^{C}D_{0+}^{\alpha}f). $$

When $2\alpha > 1$, the definition is understood with $n = 2$ in the general Caputo formula above. This distinction is important because the fractional Hermite-type equation considered below involves a single Caputo derivative of order 2α, not two successive applications of a Caputo derivative of order α.

The following power rule will be used extensively.

Lemma 2.2 (Fractional power rule [3,4])**.** *Let $\alpha \in (0, 1]$ and $\beta > -1$. Then,*

$$ {}^{C}D_{0+}^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \qquad \beta > \alpha - 1. $$

Consequently, for $k \in \mathbb{N}$ with $k \geq 1$,

$$ {}^{C}D_{0+}^{\alpha}x^{k\alpha} = \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)}x^{(k-1)\alpha}, $$

and for $k \geq 2$,

$$ {}^{C}D_{0+}^{2\alpha}x^{k\alpha} = \frac{\Gamma(k\alpha+1)}{\Gamma((k-2)\alpha+1)}x^{(k-2)\alpha}. $$

Definition 2.3 (Gamma function [1])**.** *The gamma function is defined for $z > 0$ by*

$$ \Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\,dt. $$

It satisfies $\Gamma(z + 1) = z\Gamma(z)$ and $\Gamma(n + 1) = n!$ for $n \in \mathbb{N}_0$.

Following [5], we consider the family of fractional Hermite functions defined below. The explicit representation adopted in this paper was originally introduced in connection with the fractional Hermite-type equation

$$ {}^{C}D_{0+}^{2\alpha}y(x) - 2x^{\alpha}\,{}^{C}D_{0+}^{\alpha}y(x) + 2n\,y(x) = 0. $$

A formal series solution is given by

$$ y(x) = \sum_{k=0}^{\infty} a_k x^{k\alpha}. $$

Definition 2.4 (Fractional Hermite functions [5])**.** *For $\alpha \in (0, 1]$, the fractional Hermite functions $H_n^{(\alpha)}(x)$ admit the explicit representation*

$$ H_n^{(\alpha)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k\,n!\,\Gamma((n-2k+1)\alpha+1)}{k!\,(n-2k)!\,2^{2k}\,\Gamma((n-2k)\alpha+1)}x^{(n-2k)\alpha}, \qquad n \in \mathbb{N}_0. $$

Remark 2.5. *Throughout this paper, we adopt Definition 2.4 as the starting point of the analysis. The subsequent generating function framework and all structural identities are derived directly from this explicit representation.*

Example 2.6 (The case $\alpha = 1/2$)**.** *For $\alpha = 1/2$, the explicit representation above gives the first few fractional Hermite functions:*

$$H_0^{(1/2)}(x) = 1,$$
$$H_1^{(1/2)}(x) = \frac{2}{\sqrt{\pi}}\, x^{1/2},$$
$$H_2^{(1/2)}(x) = \frac{3\sqrt{\pi}}{4}\, x - \frac{\sqrt{\pi}}{4}.$$

Substituting these functions into Definition 3.1, the corresponding generating function begins as

$$\mathcal{G}_{1/2}(x,t) = 1 + \frac{2}{\sqrt{\pi}}\, x^{1/2}\, t + \left(\frac{3\sqrt{\pi}}{4} x - \frac{\sqrt{\pi}}{4}\right) \frac{t^2}{\Gamma(2)} + \cdots,$$

where $\Gamma(2) = 1$. This example illustrates how the fractional powers of x and the gamma normalized coefficients enter the generating function framework.

Remark 2.7. *Remarks concerning the behavior of Definition 2.4 in the limit $\alpha \to 1$ are deferred to Section 5.*

Remark 2.8 (Notation)**.** *Throughout this paper,*

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \qquad \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

All series are understood as formal power series, and equalities are interpreted coefficientwise unless stated otherwise.

3. Generating function framework

In this section, we introduce a generating function for the fractional Hermite functions $H_n^{(\alpha)}(x)$ defined in Section 2. The construction is fully compatible with the fractional structure of the functions introduced in Section 2 instead of the classical normalization $t^n/n!$, and we employ the gamma normalized basis $\{t^n/\Gamma(\alpha n + 1)\}_{n\geq 0}$. This choice is natural because it reduces to the classical normalization when $\alpha = 1$ and is consistent with the gamma function structure appearing in Definition 2.4.

Throughout this section, the generating function is understood as a formal power series in the auxiliary variable t. We work in the formal series ring $\mathbb{C}[x^\alpha][[t]]$, and all operators with respect to x act coefficientwise on the t-series. This formal framework is sufficient for all algebraic and coefficient identities derived below and avoids unnecessary analytic complications.

3.1. Definition and well-definedness

Definition 3.1 (Fractional generating function)**.** *Let $\alpha \in (0, 1]$. The fractional generating function associated with the family $\{H_n^{(\alpha)}(x)\}_{n\geq 0}$ is defined by*

$$\mathcal{G}_\alpha(x,t) := \sum_{n=0}^{\infty} H_n^{(\alpha)}(x)\, \frac{t^n}{\Gamma(\alpha n + 1)}.$$

This series is regarded as an element of the formal power series ring $\mathbb{C}[x^\alpha][[t]]$.

Remark 3.2. *For $\alpha = 1$, the normalization reduces to the classical factor $1/n!$. The behavior of the resulting series and its relation to the classical Hermite framework are discussed in Section 5.*

We begin by showing that the generating function is well-defined as a formal series.

Theorem 3.3. *For every fixed $\alpha \in (0, 1]$, the series defining $\mathcal{G}_\alpha(x, t)$ is a well-defined formal power series in t with coefficients in $\mathbb{C}[x^\alpha]$. For each $n \geq 0$, the coefficient of t^n is the finite expression*

$$\frac{H_n^{(\alpha)}(x)}{\Gamma(\alpha n + 1)},$$

which is a finite linear combination of powers $x^{m\alpha}$.

Proof. By the explicit representation of $H_n^{(\alpha)}(x)$ given in Section 2, each $H_n^{(\alpha)}(x)$ is a finite sum of terms $x^{(n-2k)\alpha}$ with coefficients involving gamma functions. Hence, $H_n^{(\alpha)}(x) \in \mathbb{C}[x^\alpha]$ for every n, and the result follows. □

3.2. Coefficient extraction and reconstruction

The following result gives the basic coefficient extraction property.

Theorem 3.4 (Coefficient extraction)**.** *For every $n \in \mathbb{N}_0$,*

$$[t^n]\mathcal{G}_\alpha(x, t) = \frac{H_n^{(\alpha)}(x)}{\Gamma(\alpha n + 1)}; \qquad \textit{equivalently,} \qquad H_n^{(\alpha)}(x) = \Gamma(\alpha n + 1)\, [t^n]\mathcal{G}_\alpha(x, t).$$

Proof. Immediate from Definition 3.1. □

The following proposition establishes the behavior of the generating function under repeated differentiation with respect to the auxiliary variable t.

Proposition 3.5 (Repeated differentiation)**.** *For every integer $r \geq 0$,*

$$\frac{\partial^r}{\partial t^r}\mathcal{G}_\alpha(x, t) = \sum_{n=0}^{\infty} \frac{(n + r)!}{n!} \frac{H_{n+r}^{(\alpha)}(x)}{\Gamma(\alpha(n + r) + 1)}\, t^n.$$

Consequently, for each $r \in \mathbb{N}_0$,

$$H_r^{(\alpha)}(x) = \frac{\Gamma(\alpha r + 1)}{r!} \left.\frac{\partial^r}{\partial t^r}\mathcal{G}_\alpha(x, t)\right|_{t=0}.$$

Proof. Differentiation is understood in the formal power series sense. Termwise differentiation of the formal power series $\mathcal{G}_\alpha(x, t) = \sum_{n=0}^{\infty} a_n t^n$ with $a_n = H_n^{(\alpha)}(x)/\Gamma(\alpha n+1)$ yields the first identity. Evaluating the differentiated series at $t = 0$ leaves only the term $n = 0$, which gives the recovery formula. □

3.3. Structural representations

Substituting Definition 2.4 into Definition 3.1 gives a double-sum formula.

Proposition 3.6. *The fractional generating function admits the representation*

$$\mathcal{G}_\alpha(x,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} \frac{(-1)^k\, n!\,\Gamma((n-2k+1)\alpha+1)}{k!\,(n-2k)!\,2^{2k}\,\Gamma((n-2k)\alpha+1)\,\Gamma(\alpha n+1)}\, x^{(n-2k)\alpha}\, t^n.$$

Equivalently, with $m = n - 2k$,

$$\mathcal{G}_\alpha(x,t) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k\,(m+2k)!\,\Gamma((m+1)\alpha+1)}{k!\,m!\,2^{2k}\,\Gamma(m\alpha+1)\,\Gamma(\alpha(m+2k)+1)}\, x^{m\alpha}\, t^{m+2k}.$$

Proof. Substituting Definition 2.4 into Definition 3.1 directly yields the first representation. Because for each fixed n, the inner sum over k is finite, the substitution is legitimate in the formal power series setting. The change of indices $m = n - 2k$ is bijective between $\{(n,k) : n \geq 0,\ 0 \leq k \leq \lfloor n/2\rfloor\}$ and $\{(m,k) : m \geq 0,\ k \geq 0\}$, giving the second form. □

The generating function separates naturally into even and odd components.

Proposition 3.7 (Even-odd decomposition)**.** *The generating function $\mathcal{G}_\alpha(x,t)$ can be written uniquely as*

$$\mathcal{G}_\alpha(x,t) = \mathcal{G}_\alpha^{\text{even}}(x,t) + \mathcal{G}_\alpha^{\text{odd}}(x,t),$$

where

$$\mathcal{G}_\alpha^{\text{even}}(x,t) = \sum_{m=0}^{\infty} H_{2m}^{(\alpha)}(x)\,\frac{t^{2m}}{\Gamma(2m\alpha+1)}, \qquad \mathcal{G}_\alpha^{\text{odd}}(x,t) = \sum_{m=0}^{\infty} H_{2m+1}^{(\alpha)}(x)\,\frac{t^{2m+1}}{\Gamma((2m+1)\alpha+1)}.$$

Moreover,

$$\mathcal{G}_\alpha^{\text{even}}(x,t) = \frac{\mathcal{G}_\alpha(x,t)+\mathcal{G}_\alpha(x,-t)}{2}, \qquad \mathcal{G}_\alpha^{\text{odd}}(x,t) = \frac{\mathcal{G}_\alpha(x,t)-\mathcal{G}_\alpha(x,-t)}{2}.$$

Proof. Splitting the sum in Definition 3.1 into even and odd indices gives the decomposition. Replacing t by $-t$ and adding/subtracting yields the expressions for $\mathcal{G}_\alpha^{\text{even}}$ and $\mathcal{G}_\alpha^{\text{odd}}$. □

Remark 3.8. *The representations obtained above are purely algebraic consequences of Definition 3.1 and do not rely on analytic convergence properties. This formal character is appropriate for the fractional setting, where the variable t is treated as a formal parameter.*

4. Formal dilation structure

In this section, we describe the dilation structure naturally associated with the generating variable t. The results below are purely formal consequences of Definition 3.1 and do not require any additional differential equation or eigenvalue assumption. All identities are understood coefficientwise in the formal power series ring $\mathbb{C}[x^\alpha][[t]]$.

4.1. *The index-counting operator*

Let

$$\mathcal{E}_t := t\frac{\partial}{\partial t}$$

denote the Euler operator acting on the formal variable t. This operator counts the index of each coefficient in the generating function.

Theorem 4.1 (Index-counting identity)**.** *For the fractional generating function*

$$\mathcal{G}_\alpha(x,t) = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x)\frac{t^n}{\Gamma(\alpha n+1)},$$

one has

$$\mathcal{E}_t\mathcal{G}_\alpha(x,t) = \sum_{n=0}^{\infty} n\, H_n^{(\alpha)}(x)\frac{t^n}{\Gamma(\alpha n+1)}.$$

More generally, for every $m \in \mathbb{N}_0$,

$$\mathcal{E}_t^m\mathcal{G}_\alpha(x,t) = \sum_{n=0}^{\infty} n^m H_n^{(\alpha)}(x)\frac{t^n}{\Gamma(\alpha n+1)}.$$

Proof. Because $\mathcal{E}_t t^n = nt^n$, applying $\mathcal{E}_t$ coefficientwise to Definition 3.1 gives the first identity. The higher-order identity follows by induction, using $\mathcal{E}_t^m t^n = n^m t^n$. □

4.2. *Functional calculus in the index variable*

The preceding identity allows any formal function of the index-counting operator to act diagonally on the generating function.

Theorem 4.2 (Formal functional calculus)**.** *Let $\Phi(z)$ be a formal power series (or a polynomial) for which the expressions below are interpreted coefficientwise. Then,*

$$\Phi(\mathcal{E}_t)\mathcal{G}_\alpha(x,t) = \sum_{n=0}^{\infty} \Phi(n)H_n^{(\alpha)}(x)\frac{t^n}{\Gamma(\alpha n+1)},$$

where $\Phi(n)$ is understood formally by substituting $z = n$ into the series expansion of Φ.

Proof. Using the previous theorem, we have

$$\Phi(\mathcal{E}_t)\mathcal{G}_\alpha(x,t) = \sum_{m=0}^{\infty} c_m\mathcal{E}_t^m\mathcal{G}_\alpha(x,t).$$

Substituting the identity for $\mathcal{E}_t^m\mathcal{G}_\alpha$ gives

$$\sum_{m=0}^{\infty} c_m \sum_{n=0}^{\infty} n^m H_n^{(\alpha)}(x)\frac{t^n}{\Gamma(\alpha n+1)}.$$

Collecting the coefficient of each t^n yields

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} c_m n^m\right)H_n^{(\alpha)}(x)\frac{t^n}{\Gamma(\alpha n+1)} = \sum_{n=0}^{\infty}\Phi(n)H_n^{(\alpha)}(x)\frac{t^n}{\Gamma(\alpha n+1)}.$$

This proves the claim. □

4.3. Operational dilation identity

Taking $\Phi(z) = e^{sz}$ in the preceding theorem gives the dilation identity in the generating variable.

Theorem 4.3 (Formal dilation identity)**.** *Let s be a formal parameter. Then,*

$$\exp(s\mathcal{E}_t)\, \mathcal{G}_\alpha(x,t) = \mathcal{G}_\alpha(x, e^s t).$$

Equivalently,

$$\exp(s\mathcal{E}_t)\, \mathcal{G}_\alpha(x,t) = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x) \frac{(e^s t)^n}{\Gamma(\alpha n + 1)}.$$

Proof. By Theorem 4.2 with $\Phi(z) = e^{sz}$, we obtain

$$\exp(s\mathcal{E}_t)\, \mathcal{G}_\alpha(x,t) = \sum_{n=0}^{\infty} e^{sn} H_n^{(\alpha)}(x) \frac{t^n}{\Gamma(\alpha n + 1)}.$$

Because $e^{sn} t^n = (e^s t)^n$, this becomes

$$\sum_{n=0}^{\infty} H_n^{(\alpha)}(x) \frac{(e^s t)^n}{\Gamma(\alpha n + 1)} = \mathcal{G}_\alpha(x, e^s t).$$

□

Corollary 4.4 (Semigroup property)**.** *For all formal parameters s_1 and s_2,*

$$\exp(s_1\mathcal{E}_t)\exp(s_2\mathcal{E}_t)\mathcal{G}_\alpha(x,t) = \exp((s_1 + s_2)\mathcal{E}_t)\mathcal{G}_\alpha(x,t).$$

Equivalently,

$$\mathcal{G}_\alpha(x, e^{s_1} e^{s_2} t) = \mathcal{G}_\alpha(x, e^{s_1+s_2} t).$$

Proof. The result follows directly from Theorem 4.3 and the identity $e^{s_1} e^{s_2} = e^{s_1+s_2}$. □

Corollary 4.5 (Infinitesimal dilation identity)**.** *The infinitesimal form of the dilation identity is*

$$\frac{\partial}{\partial s}\mathcal{G}_\alpha(x, e^s t)\bigg|_{s=0} = \mathcal{E}_t \mathcal{G}_\alpha(x,t) = t\frac{\partial}{\partial t}\mathcal{G}_\alpha(x,t).$$

Proof. The first equality follows by differentiating the formal identity

$$\mathcal{G}_\alpha(x, e^s t) = \exp(s\mathcal{E}_t)\mathcal{G}_\alpha(x,t)$$

with respect to s and setting $s = 0$. The second equality is the definition of $\mathcal{E}_t$. □

Remark 4.6. *The dilation structure established in this section is independent of any Caputo eigenvalue relation. It is a formal consequence of the gamma normalized generating function and the index structure of the series. Thus, the results remain valid for the family $H_n^{(\alpha)}$ adopted in Definition 2.4 without requiring an additional Rodrigues representation or a separate verification of a fractional differential equation.*

5. Homogeneous decomposition and algebraic consequences

In this section, we develop several algebraic consequences of the dilation framework introduced in Section 4. Throughout, all calculations are performed in the formal power series ring $\mathbb{C}[x^\alpha][[t]]$, where x-dependent expressions are treated as coefficients, and all operators act formally with respect to the variable t.

5.1. Homogeneous decomposition and the Euler operator

Recall that the Euler operator is defined by

$$\mathcal{E}_t = t\frac{\partial}{\partial t}.$$

For each $n \in \mathbb{N}_0$, define the homogeneous component

$$\mathcal{H}_n := \mathbb{C}[x^\alpha]\, t^n = \{a(x)t^n : a(x) \in \mathbb{C}[x^\alpha]\}.$$

The family $\{\mathcal{H}_n\}_{n\geq 0}$ provides a natural decomposition of the formal power series space.

Theorem 5.1 (Homogeneous decomposition)**.** *Every formal power series*

$$F(x,t) = \sum_{n=0}^{\infty} a_n(x)t^n$$

with $a_n(x) \in \mathbb{C}[x^\alpha]$ *admits the unique decomposition*

$$F = \sum_{n=0}^{\infty} F_n, \qquad F_n \in \mathcal{H}_n.$$

Moreover,

$$\mathcal{E}_t\big|_{\mathcal{H}_n} = nI,$$

where I *denotes the identity operator on* $\mathcal{H}_n$*.*

Proof. Because every formal power series has a unique coefficient expansion

$$F(x,t) = \sum_{n=0}^{\infty} a_n(x)t^n,$$

we may write

$$F_n = a_n(x)t^n \in \mathcal{H}_n.$$

The uniqueness follows from the uniqueness of the coefficient representation of formal power series.

For the second statement, let $u(x,t) = a(x)t^n \in \mathcal{H}_n$ with $a(x) \in \mathbb{C}[x^\alpha]$. Then,

$$\mathcal{E}_t u = t\frac{\partial}{\partial t}(a(x)t^n) = na(x)t^n = n\,u.$$

Hence, every element of $\mathcal{H}_n$ is an eigenvector of $\mathcal{E}_t$ associated with the eigenvalue n. □

Corollary 5.2. *The set of scalar eigenvalues of the Euler operator on* $\mathbb{C}[x^\alpha][[t]]$ *is*

$$\{0, 1, 2, \ldots\}.$$

Proof. The previous theorem shows that each nonzero element of the homogeneous component

$$\mathcal{H}_n = \mathbb{C}[x^\alpha]t^n$$

is an eigenvector corresponding to the eigenvalue n.

Conversely, let

$$F(x,t) = \sum_{k=0}^{\infty} a_k(x)t^k$$

be a nonzero eigenvector satisfying

$$\mathcal{E}_t F = \lambda F$$

for some $\lambda \in \mathbb{C}$. Then,

$$\sum_{k=0}^{\infty} k a_k(x)t^k = \sum_{k=0}^{\infty} \lambda a_k(x)t^k.$$

Comparing coefficients gives

$$(k-\lambda)a_k(x) = 0 \qquad \text{for all } k \geq 0.$$

Because $F \neq 0$, there exists at least one index k_0 such that $a_{k_0}(x) \neq 0$. Hence,

$$(k_0 - \lambda)a_{k_0}(x) = 0.$$

Because $\mathbb{C}[x^\alpha]$ has no zero divisors, and $a_{k_0}(x) \neq 0$, it follows that $\lambda = k_0$. Therefore, every scalar eigenvalue belongs to $\{0, 1, 2, \ldots\}$. □

5.2. Homogeneous projectors

The decomposition above naturally induces projection operators onto the homogeneous components.

Definition 5.3. *For each* $n \in \mathbb{N}_0$, *define*

$$P_n : \mathbb{C}[x^\alpha][[t]] \longrightarrow \mathbb{C}[x^\alpha][[t]]$$

by

$$P_n(F) := ([t^n]F)\, t^n.$$

Proposition 5.4. *For all* $m, n \in \mathbb{N}_0$,

$$P_n^2 = P_n, \qquad \textit{and} \qquad P_n P_m = 0 \quad (n \neq m).$$

Proof. Let

$$F(x,t)=\sum_{k=0}^{\infty}a_k(x)t^k.$$

Then,

$$P_n(F)=a_n(x)t^n.$$

Applying P_n again gives

$$P_n(P_n(F))=P_n(a_n(x)t^n)=a_n(x)t^n=P_n(F).$$

Hence, $P_n^2=P_n$.

Now, let $m\neq n$. Becuase

$$P_m(F)=a_m(x)t^m,$$

the coefficient of t^n in $P_m(F)$ is zero. Therefore,

$$P_n(P_m(F))=0.$$

Thus, $P_nP_m=0$. □

Proposition 5.5. *Every formal power series admits the expansion*

$$F=\sum_{n=0}^{\infty}P_nF.$$

Proof. Indeed,

$$\sum_{n=0}^{\infty}P_nF=\sum_{n=0}^{\infty}([t^n]F)t^n,$$

which is precisely the coefficient expansion of F. □

5.3. Commutation relations

The Euler operator satisfies a simple but fundamental commutation rule with multiplication by t.

Proposition 5.6 (Basic commutation relation)**.** *One has*

$$[\mathcal{E}_t,t]=t,$$

where $[A,B]=AB-BA$ *denotes the commutator.*

Proof. Let $F\in\mathbb{C}[x^\alpha][[t]]$. Then,

$$\mathcal{E}_t(tF)=t\frac{\partial}{\partial t}(tF)=tF+t^2\frac{\partial F}{\partial t},$$

and

$$t(\mathcal{E}_tF)=t^2\frac{\partial F}{\partial t}.$$

Subtracting yields

$$[\mathcal{E}_t, t]F = tF.$$

Because this holds for every F, we have

$$[\mathcal{E}_t, t] = t.$$

□

Corollary 5.7. *For every integer $m \geq 1$,*

$$[\mathcal{E}_t, t^m] = m\, t^m.$$

Proof. For $F \in \mathbb{C}[x^\alpha][[t]]$,

$$\mathcal{E}_t(t^m F) = t\frac{\partial}{\partial t}(t^m F) = mt^m F + t^{m+1}\frac{\partial F}{\partial t},$$

and

$$t^m(\mathcal{E}_t F) = t^{m+1}\frac{\partial F}{\partial t}.$$

Subtracting gives

$$[\mathcal{E}_t, t^m]F = mt^m F.$$

Hence,

$$[\mathcal{E}_t, t^m] = mt^m.$$

□

Corollary 5.8. *For every polynomial*

$$p(t) = \sum_{m=0}^{N} c_m t^m,$$

one has

$$[\mathcal{E}_t, p(t)] = t\, p'(t).$$

Proof. Using linearity of the commutator and the previous corollary,

$$[\mathcal{E}_t, p(t)] = \sum_{m=0}^{N} c_m [\mathcal{E}_t, t^m] = \sum_{m=0}^{N} c_m m t^m.$$

On the other hand,

$$t\, p'(t) = t\sum_{m=0}^{N} c_m m t^{m-1} = \sum_{m=0}^{N} c_m m t^m.$$

Therefore,

$$[\mathcal{E}_t, p(t)] = t\, p'(t).$$

□

5.4. Invariant parity subspaces

The Even-odd decomposition introduced in Section 3 is preserved by the Euler operator.
Define the even and odd subspaces

$$\mathcal{V}_{\text{even}} := \left\{ \sum_{m=0}^{\infty} a_m(x) t^{2m} : a_m(x) \in \mathbb{C}[x^\alpha] \right\},$$

and

$$\mathcal{V}_{\text{odd}} := \left\{ \sum_{m=0}^{\infty} a_m(x) t^{2m+1} : a_m(x) \in \mathbb{C}[x^\alpha] \right\}.$$

Proposition 5.9 (Parity invariance)**.** *Let*

$$\mathcal{G}_\alpha^{\text{even}} = \sum_{m=0}^{\infty} H_{2m}^{(\alpha)}(x) \frac{t^{2m}}{\Gamma(2m\alpha + 1)}$$

and

$$\mathcal{G}_\alpha^{\text{odd}} = \sum_{m=0}^{\infty} H_{2m+1}^{(\alpha)}(x) \frac{t^{2m+1}}{\Gamma((2m+1)\alpha + 1)}.$$

Then,

$$\mathcal{E}_t(\mathcal{G}_\alpha^{\text{even}}) \in \mathcal{V}_{\text{even}}, \qquad \mathcal{E}_t(\mathcal{G}_\alpha^{\text{odd}}) \in \mathcal{V}_{\text{odd}}.$$

Proof. Applying $\mathcal{E}_t$ to the even component gives

$$\mathcal{E}_t(\mathcal{G}_\alpha^{\text{even}}) = \sum_{m=0}^{\infty} 2m\, H_{2m}^{(\alpha)}(x) \frac{t^{2m}}{\Gamma(2m\alpha + 1)}.$$

This is a formal series containing only even powers of t, with coefficients in $\mathbb{C}[x^\alpha]$. Hence,

$$\mathcal{E}_t(\mathcal{G}_\alpha^{\text{even}}) \in \mathcal{V}_{\text{even}}.$$

Similarly,

$$\mathcal{E}_t(\mathcal{G}_\alpha^{\text{odd}}) = \sum_{m=0}^{\infty} (2m+1) H_{2m+1}^{(\alpha)}(x) \frac{t^{2m+1}}{\Gamma((2m+1)\alpha + 1)}.$$

This is a formal series containing only odd powers of t, with coefficients in $\mathbb{C}[x^\alpha]$. Therefore,

$$\mathcal{E}_t(\mathcal{G}_\alpha^{\text{odd}}) \in \mathcal{V}_{\text{odd}}.$$

Hence, the parity decomposition is preserved. □

Remark 5.10. *The results of this section show that the Euler operator possesses a simple homogeneous structure on the formal power series space $\mathbb{C}[x^\alpha][[t]]$ and that the dilation identities of Section 4 arise naturally from this structure. In particular, the decomposition into homogeneous components, the associated projection operators, and the parity invariance properties provide an algebraic interpretation of the generating function framework developed in this paper.*

6. Conclusions

In this paper, we developed a generating function framework for the fractional Hermite family $\{H_n^{(\alpha)}(x)\}$ adopted in Definition 2.4. The construction is formulated entirely within the setting of formal power series and is based on the gamma normalized generating function

$$\mathcal{G}_\alpha(x,t) = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x)\,\frac{t^n}{\Gamma(\alpha n + 1)}.$$

Working in the formal power series ring $\mathbb{C}[x^\alpha][[t]]$, we established several structural properties of this generating function. In particular, we derived coefficientextraction and reconstruction formulas, obtained explicit double-sum representations, and established an Even-odd decomposition of the generating series. These results show that the generating function provides a natural formal representation of the family $\{H_n^{(\alpha)}\}_{n\geq 0}$.

We further introduced the Euler operator

$$\mathcal{E}_t = t\frac{\partial}{\partial t},$$

which acts naturally on the generating variable. This led to a formal functional calculus in the index variable and to the dilation identity

$$\exp(s\mathcal{E}_t)\,\mathcal{G}_\alpha(x,t) = \mathcal{G}_\alpha(x, e^s t),$$

together with the associated semigroup and infinitesimal identities.

In addition, we investigated several algebraic consequences of the Euler operator framework. We established a homogeneous decomposition of the formal power series space, characterized the corresponding homogeneous components as eigenspaces of the Euler operator, introduced projection operators onto these components, derived commutation relations involving the Euler operator, and proved the invariance of the even and odd subspaces under its action. These results provide an algebraic interpretation of the dilation structure developed in the paper.

The framework developed here is entirely formal and coefficientwise in nature. Its validity relies only on the explicit representation of the fractional Hermite family and the associated generating function construction. Consequently, the results provide a consistent and self-contained algebraic framework for studying this family through generating function methods.

Future work may include extending the present approach to other families of fractional special functions, investigating alternative normalizations and generating constructions, and studying analytic realizations of the generating function under suitable convergence assumptions.

Author contributions

Muath Awadalla and Maryam Salem Alatawi contributed equally to the conceptualization, methodology, formal analysis, investigation, writing of the original draft, and review and editing of the manuscript. Both authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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