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Research article

CRR-P-type and F-P-type contractions in b-metric spaces

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Abstract: This research introduced two new contraction mappings by integrating ψ-modified interpolative Ćirić-Reich-Rus-type and F-contractions with the recently established P-contraction framework. Specifically, the concepts of ψ-modified interpolative Ćirić-Reich-Rus-P-type and F-P-type contractions were formulated. Existence and uniqueness theorems for fixed points regarding these mappings were established within the context of complete b-metric spaces. To substantiate the theoretical findings, several illustrative examples were provided. Furthermore, the practical utility of the results was demonstrated by addressing the existence and uniqueness of solutions for both a specific class of nonlinear Fredholm integral equations and nonlinear fractional differential equations.

Keywords: b-metric space; fixed point; ψ-modified interpolative Ćirić-Reich-Rus-P-type contraction; F-P-type contraction; fractional derivative
Mathematics Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

The notion of distance is formalized through metric spaces, which are extensively studied in core branches of mathematics such as mathematical analysis and topology, as well as utilized in physics and engineering. However, complex structures often exhibit nonlinear features that classical metric frameworks cannot adequately accommodate. To handle such complexities, the framework of b-metric spaces (bMS) was introduced and developed by Bakhtin [3] and Czerwik [9].

The Banach contraction principle is a cornerstone of fixed-point theory. Over the years, numerous researchers have reviewed and generalized this principle by introducing various contractive conditions in metric spaces [14, 25]. Recently, interpolative contractions have emerged as a highly active topic of investigation [2, 11, 18, 22]. Within the bMS framework, fixed-point theorems concerning P-type contractions were detailed by Solak et al. [23], while modified interpolative almost ε-type contractions applied to partial b-modular spaces were explored by Kesik et al. [16].

However, building upon the standard F-contractions introduced of Wardowski (2012) and the P-type contractions of Solak (2026), a new framework is presented for analyzing more complex mappings with interpolative structures. To address these limitations, the key innovation of this paper is to bridge this gap by offering a comprehensive, hybrid contraction condition that integrates both the ψ-modified interpolative Ćirić-Reich-Rus(CRR) structure and the F-contraction conditions into P-type conditions. We formulate several fixed-point theorems for these particular mappings in the context of complete b-metric spaces (bMSs). This approach allows us to guarantee the existence of fixed points for a broader class of functions that cannot be addressed by existing independent methods. Finally, as two applications of our theoretical results, we discuss the existence of solutions to nonlinear Fredholm integral equations and nonlinear fractional differential equations.

Definition 1.1. *Let M denote a nonempty set and let the constant $s \geq 1$ be given. A mapping $u : M \times M \to [0, \infty)$ is classified as a b-metric if it satisfies the conditions of self-identity, symmetry, and the s-relaxed triangle inequality:*

$$u(\iota, \kappa) \leq s\left[u(\iota, \gamma) + u(\gamma, \kappa)\right]$$

for any elements $\iota, \gamma, \kappa \in M$. In this context, the triplet (M, u, s) constitutes a bMS [3, 10].

Definition 1.2. *[5] Suppose (M, u, s) is a bMS and $\{\iota_n\}$ is a sequence in M.*

(a) If there exists $\iota \in M$ such that $u(\iota_n, \iota) \to 0$ as $n \to \infty$, the sequence is referred to as convergent.
(b) If $u(\iota_n, \iota_m) \to 0$ as $n, m \to \infty$, the sequence is referred to as Cauchy.

A metric space (M, u, s) is classified as complete provided that every Cauchy sequence within the space converges to an element in M.

Proposition 1.3. *[5] Let (M, u, s) be a bMS. The properties listed below are satisfied:*

(1) Uniqueness of Limits: Any convergent sequence in M has a unique limit.
(2) Cauchy Property of Convergent Sequences: Every convergent sequence in M is a Cauchy sequence.
(3) Non-continuity of the b-metric: In contrast to standard metric spaces, the mapping $u : M \times M \to [0, \infty)$ is not required to be continuous in the general case.

Definition 1.4. *[20] Suppose (M, u_1, s_1) and (N, u_2, s_2) are two bMSs and $D : M \to N$ is a mapping. If for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$u_1(\iota, \iota_0) < \delta \quad \Longrightarrow \quad u_2(D\iota, D\iota_0) < \varepsilon,$$

the map D is referred to as continuous at $\iota_0 \in M$. *If D is continuous at every point of M, then D is called continuous on M.*

Proposition 1.5. *[20] Assume that (M, u_1, s_1) and (N, u_2, s_2) are bMSs and $D : M \to N$ is a mapping. For a point $\iota \in M$, D is continuous at ι iff for every sequence $\{\iota_n\} \subset M$ converging to ι with respect to u_1, the sequence $\{D\iota_n\}$ converges to $D\iota$ with respect to u_2.*

Lemma 1.6. *[19,24] Consider a bMS (M, u, s) and a sequence $\{\iota_n\}$ within M. If there exists a $z \in [0, 1)$ satisfying the condition*

$$u(\iota_{n+1}, \iota_n) \leq zu(\iota_{n-1}, \iota_n)$$

for all $n \in \mathbb{N}$, it follows that $\{\iota_n\}$ is a Cauchy sequence.

Generally, it is important to note that for $s > 1$, a b-metric function $u(\iota, \gamma)$ is not necessarily jointly continuous in both variables. Hussain et al. [12] and Chandok et al. [6] have also discussed such a b-metric example.

In this connection, a class of comparison functions, as introduced by Berinde [4], has been widely utilized.

Lemma 1.7. *[4] If $\psi : [0, \infty) \to [0, \infty)$ is a comparison function, then:*

(1) each iterate ψ^k of ψ is also a comparison function for all $k \geq 1$;
(2) ψ is continuous at 0;
(3) $\psi(t) < t$, for any $t > 0$.

Definition 1.8. *[4] Let $s \geq 1$ be a real number. A mapping $\psi : [0, \infty) \to [0, \infty)$ is called a b-comparison function if the following conditions are fulfilled:*

(1) ψ is monotone increasing;
(2) there exist $k_0 \in \mathbb{N}$, $\alpha \in (0, 1)$, and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$s^{k+1}\psi^{k+1}(t) \leq \alpha s^k \psi^k(t) + v_k,$$

for $k \geq k_0$ and any $t \in [0, \infty)$.

Example 1.9. *[7] Let $s \geq 1$ be a real number and $g : [0, \infty) \to [0, \infty)$ be a function such that $g(t) = ct$ for all $t \in [0, \infty)$, where $c \in \left(0, \frac{1}{s}\right)$. Then, g forms a comparison function. The following lemma is very important in the proof of our results.*

Lemma 1.10. *[4] If $\psi : [0, \infty) \to [0, \infty)$ is a b-comparison function, then we have the following conclusions:*

(1) the series $\sum_{k=0}^{\infty} s^k \psi^k(t)$ converges for any $t \in [0, \infty)$;
(2) the function $S_b : [0, \infty) \to [0, \infty)$ defined by $S_b(t) = \sum_{k=0}^{\infty} s^k \psi^k(t)$, $t \in [0, \infty)$ is increasing and continuous at 0.

Definition 1.11. *[25] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:*
(F1) F is strictly increasing, i.e., for all $a, b \in \mathbb{R}^+$ such that $a < b$, $F(a) < F(b)$;
(F2) For each sequence $\{\iota_n\}_{n \in \mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \iota_n = 0$, if and only if, $\lim_{n\to\infty} F(\iota_n) = -\infty$;
(F3) There exists $k \in (0, 1)$ such that $\lim_{\iota \to 0^+} \iota^k F(\iota) = 0$. A mapping $D : M \to M$ is said to be an F-contraction if there exists $\tau > 0$ such that

$$(u(D\iota, D\gamma) > 0 \Rightarrow \tau + F(u(D\iota, D\gamma)) \leq F(u(\iota, \gamma))) \tag{1.1}$$

for all $\iota, \gamma \in M$.The class of functions $F : (0, \infty) \to \mathbb{R}$ satisfying only condition (F1) is denoted by W'.

2. Main results

Definition 2.1. *Let (M, u, s) be a bMS. If there exist $\psi \in \Psi$ and constants $\alpha \geq 0$, $\beta > \frac{1}{2}$ satisfying $\alpha + \beta < 1$ such that*

$$u(D\iota, D\gamma) \leq \psi\left(\beta u(\iota, \gamma) + |\alpha u(\iota, D\iota) - (1 - \alpha - \beta)u(\gamma, D\gamma)|\right) \tag{2.1}$$

for all $\iota, \gamma \in (M \backslash Fix(D))$, then a self-map $D : M \to M$ is called a ψ-modified-interpolative-CRR-P-type contraction. Moreover, if

$$u(D\iota, D\gamma) \leq \psi\left(u(\iota, \gamma) + |u(\iota, D\iota) - u(\gamma, D\gamma)|\right)$$

$\iota, \gamma \in (M \setminus Fix(D))$*, then D is called ψ-*modified-interpolative-CRR-P-type contractive.

Theorem 2.2. *Let (M, u, s) be a complete bMS and $D : M \to M$ be a continuous ψ-modified-interpolative-CRR-P-type contraction. Consequently, the mapping D has a unique fixed point.*

Proof. Consider an arbitrary element $\iota_0 \in M$ and define the Picard iteration $\{\iota_n\}$ by $\iota_n = D\iota_{n-1}$ for $n \in \mathbb{N}$. If $u(\iota_{n_0+1}, \iota_{n_0}) = 0$ holds for some $n_0 \in \mathbb{N}$, then ι_{n_0} is clearly a fixed point of D. Hence, we proceed under the assumption that $u(\iota_{n+1}, \iota_n) > 0$ for all $n \in \mathbb{N}$, which implies that $\iota_{n+1} \neq \iota_n$ for all n.

Setting $\iota = \iota_n$ and $\gamma = \iota_{n-1}$ in (2.1), the ψ-modified-interpolative-CRR-P-type contraction condition yields:

$$\begin{aligned} u(\iota_{n+1}, \iota_n) &= u(D\iota_n, D\iota_{n-1}) \\ &\leq \psi\left(\beta u(\iota_{n-1}, \iota_n) + |\alpha u(\iota_n, D\iota_n) - (1 - \alpha - \beta)u(\iota_{n-1}, D\iota_{n-1})|\right) \\ &= \psi\left(\beta u(\iota_{n-1}, \iota_n) + |\alpha u(\iota_{n+1}, \iota_n) - (1 - \alpha - \beta)u(\iota_{n-1}, \iota_n)|\right) \end{aligned} \tag{2.2}$$

where the last strict inequality follows from the contractive property $\psi(t) < t$ for all $t > 0$. We analyze this inequality under two mutually exclusive cases depending on the sign of the argument within the absolute value:

Case 1: Assume that there exists $k \in \mathbb{N}$ such that $\alpha u(\iota_{k+1}, \iota_k) - (1-\alpha-\beta)u(\iota_{k-1}, \iota_k) \geq 0$. Then, from (2.2), we have

$$\begin{aligned} u(\iota_{k+1}, \iota_k) &\leq \psi\left(\beta u(\iota_{k-1}, \iota_k) + \alpha u(\iota_{k+1}, \iota_k) - (1 - \alpha - \beta)u(\iota_{k-1}, \iota_k)\right) \\ &= \psi\left(\alpha u(\iota_{k+1}, \iota_k) + (2\beta + \alpha - 1)u(\iota_{k-1}, \iota_k)\right) \\ &< \alpha u(\iota_{k+1}, \iota_k) + (2\beta + \alpha - 1)u(\iota_{k-1}, \iota_k). \end{aligned} \tag{2.3}$$

Inequality (2.3) directly implies $u(\iota_{k+1}, \iota_k) < \frac{2\beta+\alpha-1}{1-\alpha} u(\iota_{k-1}, \iota_k)$, which, since $\frac{2\beta+\alpha-1}{1-\alpha} \in (0, 1)$, yields

$$u(\iota_{k+1}, \iota_k) < u(\iota_{k-1}, \iota_k). \tag{2.4}$$

Thus, the sequence $\{u(\iota_{n-1}, \iota_n)\}$ is strictly decreasing in this case. Utilizing (2.4), the argument of ψ in (2.3) can be estimated as follows:

$$\begin{aligned} \alpha u(\iota_{k+1}, \iota_k) + (2\beta + \alpha - 1)u(\iota_{k-1}, \iota_k) &\leq \alpha u(\iota_{k-1}, \iota_k) + (2\beta + \alpha - 1)u(\iota_{k-1}, \iota_k) \\ &= (2\beta + 2\alpha - 1)u(\iota_{k-1}, \iota_k) \\ &\leq u(\iota_{k-1}, \iota_k). \end{aligned} \tag{2.5}$$

Applying the monotonicity of ψ to (2.5) and combining it with (2.3) yields

$$u(\iota_{k+1},\iota_k) \le \psi\left(\alpha u(\iota_{k+1},\iota_k) + (2\beta+\alpha-1)u(\iota_{k-1},\iota_k)\right) \le \psi\left(u(\iota_{k-1},\iota_k)\right). \tag{2.6}$$

Case 2: Assume that there exists $m \in \mathbb{N}$ such that $\alpha u(\iota_{m+1},\iota_m)-(1-\alpha-\beta)u(\iota_{m-1},\iota_m) < 0$. Evaluating the absolute value in (2.2), we obtain

$$\begin{aligned} u(\iota_{m+1},\iota_m) &\le \psi\Big(\beta u(\iota_{m-1},\iota_m) + (1-\alpha-\beta)u(\iota_{m-1},\iota_m) - \alpha u(\iota_{m+1},\iota_m)\Big) \\ &= \psi((1-\alpha)u(\iota_{m-1},\iota_m) - \alpha u(\iota_{m+1},\iota_m)) \\ &< (1-\alpha)u(\iota_{m-1},\iota_m) - \alpha u(\iota_{m+1},\iota_m). \end{aligned} \tag{2.7}$$

Inequality (2.7) can be rewritten as $u(\iota_{m+1},\iota_m) < \frac{1-\alpha}{1+\alpha}u(\iota_{m-1},\iota_m)$, from which, since $\frac{1-\alpha}{1+\alpha} \in (0,1)$, we obtain

$$u(\iota_{m+1},\iota_m) < u(\iota_{m-1},\iota_m). \tag{2.8}$$

Hence, the sequence $\{u(\iota_{n-1},\iota_n)\}$ is strictly decreasing in this case as well. By (2.8), we have

$$\begin{aligned} (1-\alpha)u(\iota_{m-1},\iota_m) - \alpha u(\iota_{m+1},\iota_m) &\le (1-\alpha)u(\iota_{m-1},\iota_m) \\ &\le u(\iota_{m-1},\iota_m). \end{aligned} \tag{2.9}$$

From (2.9) and nondecreasing property of ψ, we have

$$u(\iota_{m+1},\iota_m) \le \psi\left((1-\alpha)u(\iota_{m-1},\iota_m) - \alpha u(\iota_{m+1},\iota_m)\right) \le \psi\left(u(\iota_{m-1},\iota_m)\right). \tag{2.10}$$

Combining Case 1 and Case 2, the positive sequence $\{u(\iota_{n-1},\iota_n)\}$ is strictly decreasing and bounded below by 0. Thus, there exists $l \ge 0$ such that $\lim_{n\to\infty} u(\iota_{n-1},\iota_n) = l$.

Moreover, from (2.6) and (2.10), the inequality

$$u(\iota_{n+1},\iota_n) \le \psi(u(\iota_{n-1},\iota_n))$$

holds for all $n \in \mathbb{N}$. By induction and the nondecreasing property of ψ, we obtain

$$u(\iota_{n+1},\iota_n) \le \psi(u(\iota_{n-1},\iota_n)) \le \psi^2(u(\iota_{n-2},\iota_{n-1})) \le \cdots \le \psi^n(u(\iota_0,\iota_1)). \tag{2.11}$$

Let $m, p \in \mathbb{N}$ such that $p > m$. Using the triangle inequality in the b-metric space along with (2.11), we have

$$\begin{aligned} u(\iota_m,\iota_p) &\le s u(\iota_m,\iota_{m+1}) + s^2 u(\iota_{m+1},\iota_{m+2}) + \cdots + s^{p-m} u(\iota_{p-1},\iota_p) \\ &\le s\psi^m(u(\iota_0,\iota_1)) + s^2\psi^{m+1}(u(\iota_0,\iota_1)) + \cdots + s^{p-m}\psi^{p-1}(u(\iota_0,\iota_1)) \\ &= \frac{1}{s^{m-1}}\left[s^m\psi^m(u(\iota_0,\iota_1)) + s^{m+1}\psi^{m+1}(u(\iota_0,\iota_1)) + \cdots + s^{p-1}\psi^{p-1}(u(\iota_0,\iota_1))\right] \\ &= \frac{1}{s^{m-1}}\sum_{i=m}^{p-1} s^i\psi^i(u(\iota_0,\iota_1)). \end{aligned} \tag{2.12}$$

Since ψ is a b-comparison function, the series $\sum_{i=0}^{\infty} s^i\psi^i(u(\iota_0,\iota_1))$ is convergent.
Letting $S_n = \sum_{i=0}^{n} s^i\psi^i(u(\iota_0,\iota_1))$, inequality (2.12) becomes

$$u(\iota_m,\iota_p) \le \frac{1}{s^{m-1}}\left(S_{p-1} - S_{m-1}\right).$$

Taking the limit as $m, p \to \infty$, we have

$$\lim_{m,p\to\infty} u(\iota_m, \iota_p) = 0$$

which proves that $(\iota_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete b-metric space M.

From completeness, there exists $\omega \in M$ such that $\lim_{n\to\infty} \iota_n = \omega$. From the continuity of D and the uniqueness of the limit, we have

$$u(\omega, D\omega) = \lim_{n\to\infty} u(\iota_n, D\omega) = \lim_{n\to\infty} u(\iota_n, \iota_{n+1}) = 0$$

which implies $D\omega = \omega$.

To prove the uniqueness of the fixed point, suppose that $\kappa \in M$ is another fixed point of D with $\kappa \neq \omega$. From the inequality (2.1), we have

$$u(D\omega, D\kappa) \leq \psi\left(\beta u(\omega, \kappa) + |\alpha u(\omega, D\omega) - (1 - \alpha - \beta)u(\kappa, D\kappa)|\right).$$

Since $D\omega = \omega$ and $D\kappa = \kappa$, this simplifies to

$$u(\omega, \kappa) \leq \psi(\beta u(\omega, \kappa)).$$

From the properties of ψ, we obtain

$$u(\omega, \kappa) \leq \psi(\beta u(\omega, \kappa)) < \beta u(\omega, \kappa) < u(\omega, \kappa),$$

which is a contradiction since $\beta \in (0, 1)$. Therefore, the fixed point of D is unique. □

Example 2.3. *Let $0 < c < 1$ and define $L_c[\mu, \nu]$ by*

$$L_c[\mu, \nu] := \left\{ \iota : [\mu, \nu] \to \mathbb{R} \;\middle|\; \int_\mu^\nu |\iota(t)|^c dt < \infty \right\},$$

equipped with the b-metric $u : L_c[\mu, \nu] \times L_c[\mu, \nu] \to \mathbb{R}^+$ defined by

$$u(\iota, \gamma) = \left(\int_\mu^\nu |\iota(t) - \gamma(t)|^c dt \right)^{\frac{1}{c}}$$

for each $\iota, \gamma \in L_c[\mu, \nu]$. Then, $(L_c[\mu, \nu], u)$ forms a complete bMS with coefficient $s = 2^{\frac{1}{c}-1}$.

Define the self-map $D : L_c[\mu, \nu] \to L_c[\mu, \nu]$ by $D\iota = \frac{1}{2}\iota(t) + \frac{1}{4}$ and choose the b-comparison function $\psi(t) = \frac{5}{7}t$. Let $\alpha = 0.1$, $\beta = 0.7$. Clearly D is continuous. Then, we have to verify the inequality (2.1).

$$\begin{aligned}
u(D\iota, D\gamma) &= u\left(\frac{1}{2}\iota(t) + \frac{1}{4}, \frac{1}{2}\gamma(t) + \frac{1}{4}\right) \\
&= \left(\int_\mu^\nu \left| \left(\frac{1}{2}\iota(t) + \frac{1}{4}\right) - \left(\frac{1}{2}\gamma(t) + \frac{1}{4}\right) \right|^c dt \right)^{\frac{1}{c}}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_\mu^\nu \left| \frac{1}{2}\iota(t) - \frac{1}{2}\gamma(t) \right|^c dt \right)^{\frac{1}{c}} \\
&= \left(\int_\mu^\nu \left(\frac{1}{2} \right)^c |\iota(t) - \gamma(t)|^c \, dt \right)^{\frac{1}{c}} \\
&= \left(\left(\frac{1}{2} \right)^c \int_\mu^\nu |(\iota(t) - \gamma(t)|^c dt \right)^{\frac{1}{c}} \\
&= \frac{1}{2} \left(\int_\mu^\nu |\iota(t) - \gamma(t)|^c dt \right)^{\frac{1}{c}} \\
&\leq \frac{5}{7} \Bigg(0.7 \left(\int_\mu^\nu |\iota(t) - \gamma(t)|^c \, dt \right)^{\frac{1}{c}} \\
&\quad + \left| 0.1 \left(\int_\mu^\nu \left| \iota(t) - \left(\frac{\iota(t)}{2} + \frac{1}{4} \right) \right|^c \, dt \right)^{\frac{1}{c}} - 0.2 \left(\int_\mu^\nu \left| \gamma(t) - \left(\frac{\gamma(t)}{2} + \frac{1}{4} \right) \right|^c \, dt \right)^{\frac{1}{c}} \right| \Bigg). \\
&= \psi \left(\beta u(\iota, \gamma) + |\alpha u(\iota, D\iota) - (1 - \alpha - \beta) u(\gamma, D\gamma)| \right)
\end{aligned}
$$

for all $\iota, \gamma \in L_c[\mu, \nu] \setminus \{\frac{1}{2}\}$. Since the hypotheses of Theorem 2.2 are fully satisfied, we conclude the existence of a unique fixed point $\iota^*(t) = \frac{1}{2}$ in D.

Corollary 2.4. *Let a complete bMS (M, u, s) and a continuous self-mapping $D : M \to M$. If there exist positive real constants $\alpha \geq 0$ and $\beta > \frac{1}{2}$ satisfying $\alpha + \beta < 1$ such that*

$$
u(D\iota, D\gamma) \leq \alpha u(\iota, \gamma) + |\beta u(\iota, D\iota) - (1 - \alpha - \beta) u(\gamma, D\gamma)|
$$

for all $\iota, \gamma \in (M \setminus Fix(D))$, then the mapping D has a unique fixed point in M.

Definition 2.5. *Let (M, u, s) be a bMS and $D : M \to M$ a mapping. If there exist $\tau > 0$, $F \in W'$, and that for all $\iota, \gamma \in M$ the inequality $u(D\iota, D\gamma) > 0$ implies*

$$
\tau + F\Big(2s \cdot u(D\iota, D\gamma)\Big) \leq F\Big(u(\iota, \gamma) + |u(\iota, D\iota) - u(\gamma, D\gamma)|\Big), \tag{2.13}
$$

then D is called a F-P-type contraction.

Theorem 2.6. *Let (M, u, s) be a complete b-metric space and $D : M \to M$ be a continuous F-P-contraction with $F \in W'$. Consequently, D has a unique fixed point, denoted by ι^*. Furthermore, the sequence $\{\iota_n\}$ generated by the iterative scheme $\iota_{n+1} = D\iota_n$ starting from an initial point $\iota_0 \in M$ is guaranteed to converge, satisfying*

$$
\lim_{n \to \infty} \iota_n = \iota^*.
$$

Proof. Consider $\iota_0 \in M$ to be arbitrary and $\{\iota_n\}$ is a sequence defined by $\iota_{n+1} = D\iota_n$. Denote $\xi_n = u(\iota_{n+1}, \iota_n)$ as the consecutive distances. If there exists $n_0 \in \mathbb{N}$ such that $\iota_{n_0+1} = \iota_{n_0}$, then ι_{n_0} becomes a fixed point of D. In the case when $\iota_{n+1} \neq \iota_n$ for all $n \in \mathbb{N}$, we have $\xi_n > 0$ for all $n \in \mathbb{N}$. Then, from the Eq (2.13),

$$
\tau + F(2s\xi_{n+1}) \quad \leq \quad F(\xi_n + |\xi_{n+1} - \xi_n|), \tag{2.14}
$$

regardless of the sign of the argument inside the absolute value. Provided that $\xi_{k+1} \geq \xi_k$ holds for some $k \in \mathbb{N}$, Eq (2.14) yields

$$\tau + F(2s\xi_{k+1})) \leq F(\xi_k + \xi_{k+1} - \xi_k) = F(\xi_{k+1}).$$

Since F is strictly increasing, it follows that $2s\xi_{k+1} \leq \xi_{k+1}$, which is a contradiction. Hence, we have $\xi_{n+1} < \xi_n$ for all $n \in \mathbb{N}$. Applying this strictly decreasing property to (2.14) and using the monotonicity of F again, we get

$$\begin{aligned}\tau + F(2s\xi_{n+1}) \leq F(\xi_n + \xi_n - \xi_{n+1}) = F(2\xi_n - \xi_{n+1}),\\ \implies 2s\xi_{n+1} < 2\xi_n - \xi_{n+1},\end{aligned}$$

which directly simplifies to

$$\left(s + \frac{1}{2}\right)\xi_{n+1} < \xi_n$$

for all $n \in \mathbb{N}$. Setting

$$q = \frac{1}{s + \frac{1}{2}},$$

the condition $s \geq 1$ implies that $0 < q < 1$ and

$$sq = \frac{s}{s + \frac{1}{2}} < 1.$$

Consequently, we obtain

$$\xi_{n+1} < q\xi_n$$

for all n, which inductively yields

$$\xi_{n+j} \leq q^j \xi_n$$

for every $j \geq 0$. For $m \geq 1$, using the relaxed triangle inequality in the bMS, we have

$$\begin{aligned}u(\xi_n, \xi_{n+m}) &\leq s\xi_n + s^2\xi_{n+1} + \cdots + s^m\xi_{n+m-1}\\ &\leq s\xi_n + s^2 q\xi_n + \cdots + s^m q^{m-1}\xi_n\\ &= s\xi_n \sum_{j=0}^{m-1}(sq)^j\\ &\leq \frac{s}{1 - sq}\xi_n.\end{aligned}$$

Since $\xi_n \leq q^n \xi_0 \to 0$, it follows that

$$u(\xi_n, \xi_{n+m}) \to 0$$

as $n \to \infty$, uniformly with respect to m. Thus, (ξ_n) is a Cauchy sequence. Now, we show the uniqueness of the fixed point. Indeed, if ι and γ are two distinct fixed points of D, then

$$D\iota = \iota, \quad D\gamma = \gamma, \quad u(\iota, \gamma) > 0.$$

Applying the contractive condition gives

$$\tau + F(2su(\iota, \gamma)) \leq F(u(\iota, \gamma) + |u(\iota, Dx) - u(\gamma, D\gamma)|).$$

Since ι and γ are fixed points, we have

$$u(\iota, D\iota) = u(\iota, \iota) = 0, \quad u(\gamma, D\gamma) = u(\gamma, \gamma) = 0.$$

Hence,

$$\tau + F(2su(\iota, \gamma)) \leq F(u(\iota, \gamma)).$$

Since $\tau > 0$ and F is strictly increasing, this implies

$$2su(\iota, \gamma) < u(\iota, \gamma).$$

Given $s \geq 1$ and $u(\iota, \gamma) > 0$, this results in contradiction. Hence, the fixed point is unique. $\square$

We will give an example that demonstrates F-P-type contraction.

Example 2.7. *Let $M = \{0\} \cup [1, \infty)$ such that $\iota > \gamma$ and define the mapping $u : M \times M \to [0, \infty)$ by $u(\iota, \gamma) = (\iota - \gamma)^2$. Since*

$$u(\iota, \gamma) = (\iota - \gamma)^2 \leq 2\left[(\iota - \zeta)^2 + (\zeta - \gamma)^2\right] = 2\left[u(\iota, \zeta) + u(\zeta, \gamma)\right]$$

for all $\iota, \gamma, \zeta \in M$, $(M, u, 2)$ is a complete bMS with parameter $s = 2$.

Consider the continuous self-map $D : M \to M$ defined by $D\iota = \frac{\iota}{2}$. For any $\iota, \gamma \in (M \setminus Fix(D))$ with $D\iota \neq D\gamma$, we observe:

$$\begin{aligned}
2su(D\iota, D\gamma) &= 2 \cdot 2\left(\frac{\iota}{2} - \frac{\gamma}{2}\right)^2 \\
&= (\iota - \gamma)^2 \\
&= \frac{4}{5}\left[(\iota - \gamma)^2 + \frac{(\iota - \gamma)^2}{4}\right] \\
&\leq \frac{4}{5}\left[(\iota - \gamma)^2 + \left|\frac{\iota^2}{4} - \frac{\gamma^2}{4}\right|\right] \\
&= \frac{4}{5}\left[(\iota - \gamma)^2 + \left|(\iota - \frac{\iota}{2})^2 - (\gamma - \frac{\gamma}{2})^2\right|\right] \\
&= \frac{4}{5}\left[u(\iota, \gamma) + \left|u(\iota, D\iota) - u(\gamma, D\gamma)\right|\right].
\end{aligned} \tag{2.15}$$

With the natural logarithm of both sides in (2.15)*, we have*

$$\ln\left(2su(D\iota, D\gamma)\right) \leq \ln\left(\frac{4}{5}\right) + \ln\left(u(\iota, \gamma) + |u(\iota, D\iota) - u(\gamma, D\gamma)|\right). \tag{2.16}$$

Taking the mapping $F(\iota) = \ln(\iota)$ and $\tau = -\ln\left(\frac{4}{5}\right)$, from (2.16) *we have*

$$\tau + F\left(2su(D\iota, D\gamma)\right) \leq F\left(u(\iota, \gamma) + |u(\iota, D\iota) - u(\gamma, D\gamma)|\right).$$

Thus, all the conditions of the Theorem 2.6 are satisfied. That is, D has a unique fixed point.

3. Applications

3.1. Application to nonlinear Fredholm integral equations

In this section, we investigate the existence and uniqueness of a solution for a class of nonlinear Fredholm integral equations defined over a compact set $\Omega \subset \mathbb{R}^n$ [8, 13]. We extend the classical results by employing a generalized distance-based contractive mapping within the framework of a complete bMS (M, u, s).

Consider the integral equation defined by:

$$\iota(t) = f(t) + \lambda \int_{\Omega} K(t, \eta, \iota(\eta))d\eta \tag{3.1}$$

where λ is a constant parameter, $f \in M$, and the kernel $K : \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous. Given that $f \in C(\Omega, \mathbb{R})$ and the kernel K is continuous on the compact set $\Omega \times \Omega \times \mathbb{R}$, standard properties of integral operators ensure that the mapping D is well-defined, sending M into itself ($D : M \to M$).

Let $M = C(\Omega, \mathbb{R})$ be defined as a complete bMS equipped with a translation-invariant and q-homogeneous b-metric u [15]. This metric is defined to accommodate a Minkowski integral inequality, encompassing structures such as weighted supremum metrics or the class of L_q b-metrics for $q \geq 1$ that satisfy Minkowski's integral inequality and exhibit a q-homogeneous structure, typically derived from a norm-based metric [1, 21].

Compared to classical literature, the application presented here introduces two significant generalizations. First, instead of restricting the integral domain to a standard closed interval $[a, b]$, we extend the domain to a more general compact set $\Omega \subset \mathbb{R}^n$, which allows the equation to model multidimensional physical phenomena. Second, rather than employing standard metric spaces or specific metrics like the squared Euclidean distance $|\iota(t) - \gamma(t)|^2$, we formulate our framework within a complete b-metric space (M, u, s) equipped with translation-invariant and q-homogeneous b-metrics. This structural choice encompasses broader classes of spaces, such as L_p and L_q spaces ($q \geq 1$) that satisfy the Minkowski integral inequality. Consequently, our approach ensures that the integral operator's contraction properties remain valid across a wider variety of function spaces and under more diverse metric structures.

To establish the existence of a unique solution, we assume that the kernel K satisfies the following ψ-modified-interpolative-CRR-P-type contraction constraint for all $\iota, \gamma \in M$:

$$u(K(t, \eta, \iota), K(t, \eta, \gamma)) \leq g(t, \eta)\frac{1}{s}\psi\left(\beta u(\iota, \gamma) + |\alpha u(\iota, D\iota) - (1 - \alpha - \beta)u(\gamma, D\gamma)|\right) \tag{3.2}$$

where the following conditions hold:

(1) $\psi \in \Psi$ is a nondecreasing comparison function such that $\psi(t) < t$ for each $t > 0$.
(2) α and β are nonnegative constants satisfying $\beta > \frac{1}{2}$ and $\alpha + \beta < 1$.
(3) $g(t, \eta) \in C(\Omega \times \Omega, \mathbb{R}^+)$ is a continuous function satisfying:

$$\sup_{t \in \Omega} |\lambda|^q \int_{\Omega} g(t, \eta)d\eta \leq 1.$$

Theorem 3.1. *Under the contractive condition* (3.2)*, the Fredholm integral operator $D : M \to M$ defined by*

$$(D\iota)(t) = f(t) + \lambda \int_{\Omega} K(t, \eta, \iota(\eta))d\eta$$

has a unique fixed point in M, which is the unique solution to (3.1).

Proof. Let $\iota, \gamma \in M$. By applying the definition of the operator D, the generalized constraint (3.2), translation invariance, and the q-homogeneous b-metric, we have:

$$\begin{aligned} u(D\iota, D\gamma) &= u\left(f(t) + \lambda \int_{\Omega} K(t, \eta, \iota(\eta))d\eta, f(t) + \lambda \int_{\Omega} K(t, \eta, \gamma(\eta))d\eta\right) \\ &= u\left(\lambda \int_{\Omega} K(t, \eta, \iota(\eta))d\eta, \lambda \int_{\Omega} K(t, \eta, \gamma(\eta))d\eta\right) \quad \text{(by translation invariance)} \\ &\leq |\lambda|^q u\left(\int_{\Omega} K(t, \eta, \iota(\eta))d\eta, \int_{\Omega} K(t, \eta, \gamma(\eta))d\eta\right) \quad \text{(by q-homogeneity)} \\ &\leq |\lambda|^q \int_{\Omega} u(K(t, \eta, \iota), K(t, \eta, \gamma))d\eta \quad \text{(by integral triangle inequality)} \\ &\leq |\lambda|^q \int_{\Omega} g(t, \eta)\frac{1}{s}\psi\left(\beta u(\iota, \gamma) + |\alpha u(\iota, D\iota) - (1 - \alpha - \beta)u(\gamma, D\gamma)|\right) d\eta. \end{aligned}$$

Since the argument of ψ is independent of the integration variable η, we obtain:

$$u(D\iota, D\gamma) \leq \frac{1}{s}\left(\sup_{t\in\Omega}|\lambda|^q \int_{\Omega} g(t, \eta)d\eta\right)\psi\left(\beta u(\iota, \gamma) + |\alpha u(\iota, D\iota) - (1 - \alpha - \beta)u(\gamma, D\gamma)|\right).$$

Applying the boundary condition $\sup|\lambda|^q \int g d\eta \leq 1$ and the fact that $1/s \leq 1$, we conclude:

$$u(D\iota, D\gamma) \leq \psi\left(\beta u(\iota, \gamma) + |\alpha u(\iota, D\iota) - (1 - \alpha - \beta)u(\gamma, D\gamma)|\right).$$

This demonstrates that D is a ψ-modified-interpolative-CRR-P-type contraction in the bMS (M, u, s). Consequently, the existence and uniqueness of the solution follow from Theorem 2.2, the fixed-point theorem for ψ-modified-interpolative-CRR-P-type contractions. □

3.2. Application to nonlinear fractional differential equations

In this section, we apply the fixed-point formulation established in Theorem 2.6 to determine the existence and uniqueness of a solution for a class of nonlinear fractional differential equations. Recall the definition of Caputo fractional derivative [17]. The Caputo fractional derivative of order $\sigma > 0$ (denoted by ${}^c\mathcal{D}^\sigma$) is defined by:

$${}^c\mathcal{D}^\sigma g(t) = \frac{1}{\Gamma(m - \sigma)} \int_0^t (t - \eta)^{m-\sigma-1} g^{(m)}(\eta)d\eta,$$

where $\sigma \in [m - 1, m)$ with $m = [\sigma] + 1 \in \mathbb{N}$, $[\sigma]$ denotes the integer part of σ and $g : [0, \infty) \to \mathbb{R}$ is a continuous function.

Throughout this section, $M = C([0, 1], \mathbb{R})$ denotes the set of all continuous functions from $[0, 1]$ into $\mathbb{R}$.

Now, we deal with the existence of unique solutions to a nonlinear fractional differential equation:

$$^{c}\mathcal{D}^{\sigma}(\iota(t)) = f(t, \iota(t)),$$

with the integral boundary conditions:

$$\iota(0) = 0, \quad \iota(1) = \int_0^{\rho} \iota(t)dt, \tag{3.3}$$

where $\iota \in M$, $t, \rho \in (0, 1)$, $\sigma \in (1, 2]$, and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Note that $\iota \in M$ is a solution of (3.3) iff it is a solution of the integral equation:

$$\begin{aligned}\iota(t) =& \frac{1}{\Gamma(\sigma)} \int_0^t (t-\eta)^{\sigma-1} f(\eta, \iota(\eta))d\eta - \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^1 (1-\eta)^{\sigma-1} f(\eta, \iota(\eta))d\eta \\ &+ \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^{\rho} \left(\int_0^{\eta} (\eta - r)^{\sigma-1} f(r, \iota(r))dr \right) d\eta.\end{aligned}$$

Define the operator $D : M \to M$ by

$$\begin{aligned}D\iota(t) =& \frac{1}{\Gamma(\sigma)} \int_0^t (t-\eta)^{\sigma-1} f(\eta, \iota(\eta))d\eta - \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^1 (1-\eta)^{\sigma-1} f(\eta, \iota(\eta))d\eta \\ &+ \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^{\rho} \left(\int_0^{\eta} (\eta - r)^{\sigma-1} f(r, \iota(r))dr \right) d\eta,\end{aligned}$$

where M is a bMS with b-metric defined as

$$u(\iota, \gamma) = \sup_{t\in[0,1]} |\iota(t) - \gamma(t)|^p, \quad \text{for all } \iota, \gamma \in M,$$

with coefficient $s = 2^{p-1}$.

Theorem 3.2. *The nonlinear fractional differential equation* (3.3) *has a unique solution if the following condition holds:*

$$|f(a, \mu) - f(a, \eta)| \leq \frac{\Gamma(\sigma + 1)}{2^p} \cdot e^{-\tau} \left(\Theta(|\mu - \eta|^p)\right)^{\frac{1}{p}},$$

where $p > 1$, $\tau > 0$, and Θ is a continuous and nondecreasing function on $[0, \infty)$ *such that* $\Theta(0) = 0$ *and $\Theta(a) < a$ for all $a > 0$.*

Proof. To establish the validity of our framework, we first verify that condition 2.5 of Definition 2.6 is satisfied. Let $M_s^*(\iota, \gamma) = u(\iota, \gamma) + |u(\iota, D\iota) - u(\gamma, D\gamma)|$. For this, we have

$$\begin{aligned}|D\iota(t) - D\gamma(t)| =& \left| \frac{1}{\Gamma(\sigma)} \int_0^t (t-\eta)^{\sigma-1} f(\eta, \iota(\eta))d\eta - \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^1 (1-\eta)^{\sigma-1} f(\eta, \iota(\eta))d\eta \right. \\ &+ \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^{\rho} \left(\int_0^{\eta} (\eta - r)^{\sigma-1} f(r, \iota(r))dr \right) d\eta \\ &- \frac{1}{\Gamma(\sigma)} \int_0^t (t-\eta)^{\sigma-1} f(\eta, \gamma(\eta))d\eta + \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^1 (1-\eta)^{\sigma-1} f(\eta, \gamma(\eta))d\eta \\ &\left. - \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^{\rho} \left(\int_0^{\eta} (\eta - r)^{\sigma-1} f(r, \gamma(r))dr \right) d\eta \right|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\sigma)} \int_0^t (t-\eta)^{\sigma-1} |f(\eta, \iota(\eta)) - f(\eta, \gamma(\eta))| d\eta \\
&\quad + \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^1 (1-\eta)^{\sigma-1} |f(\eta, \iota(\eta)) - f(\eta, \gamma(\eta))| d\eta \\
&\quad + \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^\rho \int_0^\eta (\eta - r)^{\sigma-1} |f(r, \iota(r)) - f(r, \gamma(r))| dr d\eta \\
&\leq \frac{1}{\Gamma(\sigma)} \int_0^t (t-\eta)^{\sigma-1} \frac{\Gamma(\sigma+1)}{2^p} \cdot e^{-\tau} \left[\Theta(|\iota(\eta) - \gamma(\eta)|^p)\right]^{\frac{1}{p}} d\eta \\
&\quad + \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^1 (1-\eta)^{\sigma-1} \frac{\Gamma(\sigma+1)}{2^p} \cdot e^{-\tau} \left[\Theta(|\iota(\eta) - \gamma(\eta)|^p)\right]^{\frac{1}{p}} d\eta \\
&\quad + \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^\rho \int_0^\eta (\eta - r)^{\sigma-1} \frac{\Gamma(\sigma+1)}{2^p} \cdot e^{-\tau} \left(\Theta(|\iota(r) - \gamma(r)|^p)\right)^{\frac{1}{p}} dr d\eta \\
&\leq \frac{1}{\Gamma(\sigma)} \int_0^t (t-\eta)^{\sigma-1} \frac{\Gamma(\sigma+1)}{2^p} \cdot e^{-\tau} \left(\Theta(u(\iota, \gamma))\right)^{\frac{1}{p}} d\eta \\
&\quad + \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^1 (1-\eta)^{\sigma-1} \frac{\Gamma(\sigma+1)}{2^p} \cdot e^{-\tau} \left(\Theta(u(\iota, \gamma))\right)^{\frac{1}{p}} d\eta \\
&\quad + \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^\rho \int_0^\eta (\eta - r)^{\sigma-1} \frac{\Gamma(\sigma+1)}{2^p} \cdot e^{-\tau} \left(\Theta(u(\iota, \gamma))\right)^{\frac{1}{p}} dr d\eta,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|D\iota(t) - D\gamma(t)| &\leq \frac{\Gamma(\sigma+1)}{2^p} \cdot e^{-\tau} \left(\Theta(M_s^*(\iota, \gamma))\right)^{\frac{1}{p}} \\
&\quad \times \left[\frac{1}{\Gamma(\sigma)} \int_0^t (t-\eta)^{\sigma-1} d\eta + \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^1 (1-\eta)^{\sigma-1} d\eta \right. \\
&\quad \left. + \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \int_0^\rho \int_0^\eta (\eta - r)^{\sigma-1} dr d\eta \right] \\
&\leq \frac{\Gamma(\sigma+1)}{2^p} \cdot e^{-\tau} \left(\Theta(M_s^*(\iota, \gamma))\right)^{\frac{1}{p}} \\
&\quad \times \left[\frac{1}{\Gamma(\sigma)} \frac{t^\sigma}{\sigma} + \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \frac{1}{\sigma} + \frac{2t}{(2-\rho^2)\Gamma(\sigma)} \frac{\rho^{\sigma+1}}{\sigma(\sigma+1)} \right] \\
&\leq \frac{\Gamma(\sigma+1)}{2^p} \cdot e^{-\tau} \left(\Theta(M_s^*(\iota, \gamma))\right)^{\frac{1}{p}} \\
&\quad \times \frac{1}{\Gamma(\sigma+1)} \sup_{t \in (0,1)} \left[t^\sigma + \frac{2t}{(2-\rho^2)} + \frac{2t}{(2-\rho^2)} \frac{\rho^{\sigma+1}}{(\sigma+1)} \right] \\
&= 2^{-p} e^{-\tau} \left(\Theta(M_s^*(\iota, \gamma))\right)^{\frac{1}{p}} \\
&< 2^{-p} e^{-\tau} \left(M_s^*(\iota, \gamma)\right)^{\frac{1}{p}}.
\end{aligned}
$$

Thus, for $p > 1$, we can write

$$
|D\iota(t) - D\gamma(t)|^p \leq 2^{-p} e^{-\tau} M_s^*(\iota, \gamma). \tag{3.4}
$$

Using inequality (3.4), we have

$$\begin{aligned} 2su(D\iota, D\gamma) &= 2^p e^{-\tau} \sup_{t\in I} |D\iota(t) - D\gamma(t)|^p \\ &\leq 2^p e^{-\tau} 2^{-p} M_s^*(\iota, \gamma) \\ &= e^{-\tau} M_s^*(\iota, \gamma). \end{aligned} \tag{3.5}$$

Taking the natural logarithm of both sides in 3.5 and $F(t) = \ln(t)$, we have

$$\tau + F\Big(2s \cdot u(D\iota, D\gamma)\Big) \leq F\Big(u(\iota, \gamma) + |u(\iota, D\iota) - u(\gamma, D\gamma)|\Big).$$

Thus all the assumptions of Theorem 2.6 are satisfied and, hence, the operator D has a fixed point in M; consequently, the nonlinear fractional differential equation (3.3) has a unique solution in M. □

4. Conclusions

In this paper, the novel concepts of ψ-modified-interpolative-CRR-P-type and F-P-type contraction mappings within the framework of bMS are successfully introduced. Corresponding fixed-point existence and uniqueness theorems for these mappings are established, supported by several concrete, illustrative examples. Furthermore, the practical utility of our theoretical results has been demonstrated by proving the existence and uniqueness of solutions for both a specific class of nonlinear Fredholm integral equations and nonlinear fractional differential equations. This work advances the current state of fixed-point theory and expands its application envelope. These theoretical advancements can be applied to establish existence and uniqueness criteria for various types of equations, including other fractional integral and differential equations, as well as in future studies focusing on loss functions. Finally, while the current work focuses exclusively on single-valued mappings, a natural and significant open direction for future research is to extend the proposed ψ-modified-interpolative-CRR–P-type and F-P-type contractions to the context of multivalued (set-valued) mappings using the Pompeiu–Hausdorff metric. Such a generalization would explicitly broaden the applicability of these results to problems arising in optimization theory, game theory, and multivalued differential inclusions. Parallel to this extension, generalizing the proposed framework to two-scale fractal spaces represents another promising avenue. Incorporating these contractions into two-scale fractal structures would allow for a more comprehensive and rigorous analysis of systems exhibiting multiscale heterogeneities, which we actively plan to investigate in future submissions.

Author contributions

Oğuz Solak: Software, Validation, Formal analysis, Investigation, Writing-original draft, Writing-review and editing; Duran Türkoğlu: Supervision, Project administration; Müzeyyen Sangurlu Sezen: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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