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Research article

A coupled hybrid Hadamard fractional system with strongly coupled two-point boundary conditions

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Abstract: In this paper, we investigated a coupled system of hybrid Hadamard fractional differential equations subject to strongly coupled non-separated two-point boundary conditions. By employing properties of Hadamard fractional calculus, the considered problem was transformed into an equivalent system of fractional integral equations. The existence of solutions was established through a Burton-type extension of Krasnoselśkii's fixed point theorem, while uniqueness was obtained via Banach's contraction principle under suitable Lipschitz-type assumptions. Furthermore, the Ulam–Hyers stability of the proposed system was analyzed, and explicit stability estimates were derived. The obtained results extend and complement several existing contributions on hybrid fractional differential systems by treating a Hadamard framework together with strongly coupled boundary interactions. Finally, illustrative examples are presented to demonstrate the applicability of the theoretical findings.

Keywords: Hadamard fractional derivative; hybrid fractional differential equations; coupled systems; fixed point theorem; existence and uniqueness; Ulam–Hyers stability
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1. Introduction

Fractional calculus, which deals with derivatives and integrals of non-integer order, has become an effective mathematical framework for describing complex phenomena with memory and hereditary effects. In contrast to classical integer-order models, fractional-order models are capable of capturing nonlocal behavior and long-term dependence, which makes them useful in many areas such as viscoelasticity, anomalous diffusion, signal processing, control theory, biological systems, and neural dynamics [1–4]. Among the different fractional operators, the Hadamard fractional derivative is characterized by its logarithmic kernel. This feature distinguishes the Hadamard framework from other commonly used fractional operators. The basic theory of Hadamard fractional integrals and derivatives, together with their applications to fractional differential equations, has been developed in several directions [5, 6]. Recent works have also investigated generalized Hadamard-type frameworks, integral inequalities, and related analytical tools for fractional models [7–9].

Hybrid fractional differential equations form an important class of fractional models in which the unknown function appears both inside and outside the fractional derivative. Such equations provide a flexible structure for describing systems in which the fractional operator acts on a perturbed or modified state variable. This structure is useful in models involving nonlinear feedback, memory-dependent perturbations, and coupled interactions. The study of hybrid fractional systems has been developed using several analytical techniques, particularly fixed point methods and topological tools [10]. Recent contributions have considered stability and solvability issues for hybrid fractional differential systems [11], while fractional-order neural and recurrent systems have further illustrated the relevance of fractional dynamics in computational and applied sciences [12].

In addition to theoretical investigations, numerical and computational methods for fractional systems have attracted considerable attention. Liu et al. studied optimal control computation for nonlinear fractional time-delay systems with state inequality constraints and developed a convergent numerical framework for Caputo-type fractional systems [13]. Yi et al. proposed a third-order numerical method for nonlinear fractional ordinary differential equations by transforming the problem into an equivalent Volterra integral equation and applying a Taylor-based time-stepping scheme [14]. More recently, Coelho et al. introduced neural fractional differential equations as a data-driven framework for learning memory-dependent dynamics [15]. Chaudhary et al. investigated separation properties of solutions for Caputo-type fractional differential equations of order $\alpha \in (1, 2)$ and derived bounds related to the differences in the associated initial data [16]. These studies demonstrate the continuing development of fractional systems from both analytical and computational perspectives.

Existence and uniqueness results for fractional differential equations are commonly obtained through fixed point theory. Banach's contraction principle, Krasnosel'skii's theorem, Mönch's fixed point theorem, and related compactness arguments have been widely used in this context [17, 18]. For Hadamard-type systems, existence results for initial value problems in locally convex spaces have been established [19]. Sequential fractional differential equations and systems with multipoint or nonlocal boundary conditions have also been investigated using fixed point approaches [20, 21]. In addition, Mönch's fixed point theorem has been applied to nonlinear Hadamard fractional systems, showing the effectiveness of fixed point techniques in dealing with such problems [22]. Recent studies by Mesmouli et al. [23] investigated the existence, uniqueness, and finite-time stability of nonlinear systems governed by the Hadamard fractional derivative using Krasnoselskii's fixed point theorem and Banach's contraction principle.

Stability analysis is another important aspect in the qualitative study of fractional differential equations. Among different stability concepts, Ulam–Hyers stability is particularly relevant because it measures the sensitivity of exact solutions with respect to approximate ones. A general method for Ulam–Hyers stability of linear fractional equations was proposed by Wang and Li [24]. Subsequent studies extended this concept to nonlinear fractional equations with integral boundary conditions in Banach spaces [25], functional equations with integral terms in b-metric spaces [26], and nonlinear piecewise fractional dynamic systems [27]. In the Hadamard and Caputo–Hadamard settings, stability results have been established for impulsive neutral stochastic equations with infinite delay [28], nonlinear differential equations with fractional integrable impulses [29], Hadamard fractional stochastic differential equations [30], and neutral fractional functional differential equations with multiple Caputo derivatives [31]. Further related results on Hyers–Ulam and Hyers–Ulam–Rassias stability for fractional evolution equations have also been obtained [32]. Variational approaches based on Caputo derivatives provide additional analytical background for fractional problems [33]. Further recent developments include the work of Naifar et al. [34], who examined finite-time stability for Hadamard fractional-order systems using Lyapunov theory, and the study by Ma and Zhang [35], who established finite-time stability criteria for Caputo-Hadamard-type fractional differential systems without and with proportional delays, employing modified Laplace transform techniques and fractional Gronwall-type inequalities.

Coupled systems of fractional differential equations are motivated by models in which two or more interacting components evolve under memory effects. Such systems appear naturally in interconnected viscoelastic structures, interacting population models, multi-agent control systems, signal transmission networks, biological systems, and neural dynamics. The existence of solutions for coupled multifractional nonlinear hybrid differential equations with coupled boundary conditions was studied by Maheswari et al. [36]. Jamil et al. applied a tripled fixed point theorem to a nonlinear system of fractional-order hybrid sequential integro-differential equations [37]. Nonlocal boundary value problems for hybrid ϕ-Caputo fractional integro-differential equations were investigated by Ji and Ge [38]. Moreover, coupled fractional impulsive hybrid integro-differential systems have been examined by Hannabou et al. [39]. Additionally, Benkerrouche et al. [40] investigated boundary value problems of Hadamard fractional differential equations of variable order using Krasnoselskii's fixed point theorem and the Banach contraction principle, and also examined the Ulam–Hyers stability of the proposed problem. Coupled hybrid fractional systems of the type studied in this paper arise naturally in several applied contexts [23, 34, 40]. For instance, in viscoelastic materials, the stress-strain relationship is often described by fractional constitutive laws, and the interaction between different material phases can be modeled through coupled fractional equations. Similarly, in multi-agent control systems, the evolution of interacting agents under memory-dependent dynamics can be formulated as a coupled hybrid fractional system. Other potential applications include population dynamics with interacting species, signal transmission in networks with memory, and neural dynamics with fractional-order coupling. These examples illustrate the practical relevance of the theoretical framework developed in this work.

Despite these developments, the analysis of coupled hybrid Hadamard fractional systems with strongly coupled non-separated two-point boundary conditions remains considerably less studied than the corresponding Caputo and Riemann-Liouville settings. The present work adopts the Hadamard fractional derivative, whose logarithmic kernel distinguishes it from the classical Riemann–Liouville and Caputo frameworks. This choice follows the standard formulation adopted in the Hadamard fractional calculus literature and provides an alternative analytical setting for the study of coupled fractional differential equations on the interval $[1, T]$. Although Caputo and Riemann–Liouville derivatives remain the most widely used operators, recent studies demonstrate that Hadamard fractional operators have also attracted increasing attention for both theoretical analysis and qualitative properties of fractional differential equations. In the existing literature on Hadamard fractional calculus, problems are commonly formulated on intervals of the form $[1, T]$, where the logarithmic kernel is well defined. Following this standard framework, the present paper is also formulated on $[1, T]$. This convention is adopted throughout the present paper. Throughout this paper, we consider the interval $[1, T]$, where $T > 1$ is a fixed real number. Table 1 summarizes the key differences between the fractional operators.

Table 1. Comparison of some characteristic features of Riemann–Liouville/Caputo and Hadamard fractional operators.

Feature	Caputo / Riemann-Liouville	Hadamard
Kernel	$\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$	$\dfrac{(\ln(t/s))^{\alpha-1}}{\Gamma(\alpha)}$
Natural interval	$[0, T]$	$[1, T]$
Temporal scaling	Additive	Multiplicative
Memory structure	Polynomial-type memory	Logarithmic-type memory

In Table 1, the kernels are written in their normalized fractional integral form, including the factor $1/\Gamma(\alpha)$. For convenience, we also define the constant

$$K_\alpha := \frac{(\ln T)^\alpha}{\Gamma(\alpha+1)},$$

which appears in the integral estimates throughout the paper.

Motivated by the above developments, the present work investigates the following coupled hybrid Hadamard fractional boundary value problem

$$\begin{cases} {}^H D^\alpha(u(t) - p(t, u(t), v(t))) = \Phi(t, u(t), v(t)), & t \in [1, T], \\ {}^H D^\beta(v(t) - q(t, u(t), v(t))) = \Psi(t, u(t), v(t)), & t \in [1, T], \end{cases}$$

supplemented with the strongly coupled two-point boundary conditions

$$u(1) = \lambda_1 u(T) + \lambda_2 v(T), \qquad v(1) = \mu_1 v(T) + \mu_2 u(T),$$

where ${}^H D^\alpha$ and ${}^H D^\beta$ denote Hadamard-type fractional derivatives of orders $\alpha, \beta \in (0, 1]$, and p, q, Φ, Ψ are given continuous nonlinear functions. The boundary conditions are non-separated and fully coupled, since the boundary value of each unknown function depends explicitly on both terminal values $u(T)$ and $v(T)$. This strong coupling introduces additional analytical challenges: the boundary conditions must be solved simultaneously as a linear algebraic system, leading to the non-degeneracy condition

$$\Delta = (1 - \lambda_1)(1 - \mu_1) - \lambda_2\mu_2 \neq 0.$$

Motivated by the above observations and the limited literature on strongly coupled Hadamard hybrid systems, we establish several qualitative properties of the proposed problem. More precisely, the main contributions of this paper are as follows:

1) We transform the coupled hybrid problem into an equivalent system of Hadamard fractional integral equations using the properties of Hadamard fractional calculus.

2) We establish the existence of at least one solution using a Burton-type extension of Krasnosel'skii's fixed point theorem under suitable growth conditions on the nonlinearities.
3) We prove uniqueness via Banach's contraction principle.
4) We investigate the Ulam-Hyers stability of the proposed system and derive explicit stability constants.
5) We provide illustrative examples to demonstrate the applicability of the theoretical results.

The remainder of the paper is organized as follows: Section 2 recalls the necessary preliminaries from Hadamard fractional calculus and fixed point theory. Section 3 presents the equivalent integral formulation and establishes the existence and uniqueness results. Section 4 is devoted to Ulam-Hyers stability. Section 5 provides illustrative examples. Finally, Section 6 contains the conclusion and possible directions for future research.

2. Preliminaries

In this section, we recall some necessary definitions and results from Hadamard fractional calculus and fixed point theory. For comprehensive treatments of fractional calculus, we refer the reader to the standard monographs by Kilbas, Srivastava, and Trujillo [3], Podlubny [1], and Zhou [2].

Let $C([1,T],\mathbb{R})$ denote the Banach space of all continuous real-valued functions defined on the interval $[1,T]$ equipped with the norm

$$\|u\|_\infty = \max_{t\in[1,T]} |u(t)|.$$

For the product space $\mathcal{A} = C([1,T],\mathbb{R}) \times C([1,T],\mathbb{R})$, we define the norm

$$\|(u,v)\|_{\mathcal{A}} = \max\{\|u\|_\infty, \|v\|_\infty\},$$

which makes $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ a Banach space.

Definition 2.1 (Hadamard fractional integral). *[3] Let $\alpha > 0$ and let $g \in C([1,T],\mathbb{R})$. The Hadamard fractional integral of order α is defined by*

$$I^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} g(s)\frac{ds}{s}, \quad t \in [1,T],$$

where $\Gamma(\cdot)$ denotes the Euler gamma function.

Definition 2.2 (Hadamard fractional derivative). *[3] Let $\alpha \in (0,1]$ and let g be a Hadamard absolutely continuous function on $[1,T]$. The Hadamard fractional derivative of order α is defined by*

$${}^H D^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \left(t\frac{d}{dt}\right) \int_1^t \left(\ln \frac{t}{s}\right)^{-\alpha} g(s)\frac{ds}{s}, \quad t \in [1,T].$$

Definition 2.3 (Hadamard absolutely continuous function). *[3] A function $g : [1,T] \to \mathbb{R}$ is said to be Hadamard absolutely continuous if it is absolutely continuous with respect to the measure $\frac{dt}{t}$, i.e., there exists a function $h \in L^1\left([1,T], \frac{dt}{t}\right)$ such that*

$$g(t) = g(1) + \int_1^t h(s)\frac{ds}{s}, \quad t \in [1,T].$$

Lemma 2.4 (Composition rule for Hadamard operators)**.** *[3] For $\alpha \in (0, 1]$, if g is Hadamard absolutely continuous on $[1, T]$, then*

$$I^\alpha \left({}^H D^\alpha g \right)(t) = g(t) - g(1), \quad t \in [1, T].$$

Lemma 2.5 (Hadamard integral of a power function)**.** *[3] For $\alpha > 0$, $\beta > 0$, and $t \in [1, T]$, the following identity holds:*

$$I^\alpha \left((\ln s)^{\beta-1} \right)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)} (\ln t)^{\beta+\alpha-1}.$$

Lemma 2.6 (Integral estimate)**.** *[3] For $\alpha > 0$, define*

$$K_\alpha = \frac{(\ln T)^\alpha}{\Gamma(\alpha + 1)}.$$

Then, for any continuous function $h : [1, T] \to \mathbb{R}$ satisfying $|h(t)| \le M$ for all $t \in [1, T]$, we have

$$\left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha-1} h(s) \frac{ds}{s} \right| \le K_\alpha M, \quad t \in [1, T].$$

Proof. The result follows from the change of variables $u = \ln(t/s)$ and the definition of the Gamma function. □

Theorem 2.7 (Burton-type extension of Krasnosel'skii's fixed point theorem)**.** *[10] Let D be a nonempty closed bounded convex subset of a Banach space X. Let $S : X \to X$ be a contraction and $T : D \to X$ be a continuous and compact operator. If whenever $x = Sx + Ty$ for some $y \in D$ it follows that $x \in D$, then the equation $x = Sx + Tx$ has at least one solution in D.*

Theorem 2.8 (Banach contraction principle)**.** *[10] Let (X, d) be a complete metric space and let $F : X \to X$ be a contraction, i.e., there exists $k \in [0, 1)$ such that $d(Fx, Fy) \le k\, d(x, y)$ for all $x, y \in X$. Then F has a unique fixed point in X.*

3. Main results

Let $\mathcal{X} = C([1, T], \mathbb{R})$ be the Banach space of all continuous real-valued functions on $[1, T]$, equipped with the norm

$$\|u\|_{\mathcal{X}} = \max_{t \in [1,T]} |u(t)|.$$

We work in the product space

$$\mathcal{A} = \mathcal{X} \times \mathcal{X},$$

endowed with the norm

$$\|(u, v)\|_{\mathcal{A}} = \max\{\|u\|_{\mathcal{X}}, \|v\|_{\mathcal{X}}\}.$$

Then $\mathcal{A}$ is a Banach space.

We consider the coupled hybrid Hadamard fractional boundary value problem

$$\begin{cases} {}^H D^{\alpha}(u(t) - p(t, u(t), v(t))) = \Phi(t, u(t), v(t)), & t \in [1, T], \\ {}^H D^{\beta}(v(t) - q(t, u(t), v(t))) = \Psi(t, u(t), v(t)), & t \in [1, T], \\ u(1) = \lambda_1 u(T) + \lambda_2 v(T), \\ v(1) = \mu_1 v(T) + \mu_2 u(T), \end{cases} \tag{3.1}$$

where $0 < \alpha, \beta \leq 1$, the functions $p, q, \Phi, \Psi : [1, T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, and

$$\Delta := (1 - \lambda_1)(1 - \mu_1) - \lambda_2\mu_2 \neq 0. \quad (3.2) \tag{3.2}$$

The condition $\Delta \neq 0$ guarantees that the coupled boundary conditions generate a uniquely solvable linear algebraic system for the constants arising in the equivalent integral representation. This condition plays a crucial role throughout the subsequent fixed point analysis.

The following lemma provides the basic integral representation that will be used to transform the coupled hybrid Hadamard system into an equivalent fixed-point problem.

Lemma 3.1. *Let $h, k \in C([1, T], \mathbb{R})$, and assume that $w - k$ is Hadamard absolutely continuous on $[1, T]$. Then the fractional differential equation*

$${}^H D^{\alpha}(w(t) - k(t)) = h(t), \qquad t \in [1, T], \quad 0 < \alpha \leq 1,$$

has the general solution

$$w(t) = k(t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} h(s) \frac{ds}{s} + c,$$

where $c \in \mathbb{R}$ is an arbitrary constant.

Proof. Applying the Hadamard fractional integral operator I^{α} to both sides of

$${}^H D^{\alpha}(w(t) - k(t)) = h(t),$$

and using the identity

$$I^{\alpha}({}^H D^{\alpha}\phi)(t) = \phi(t) - \phi(1), \qquad 0 < \alpha \leq 1,$$

valid for Hadamard absolutely continuous functions, we obtain

$$w(t) - k(t) - (w(1) - k(1)) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} h(s) \frac{ds}{s}.$$

Hence,

$$w(t) = k(t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} h(s) \frac{ds}{s} + c,$$

where $c = w(1) - k(1)$. This completes the proof. □

Applying the above lemma to each equation of the system, we obtain

$$u(t) = p(t, u(t), v(t)) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \Phi(s, u(s), v(s)) \frac{ds}{s} + c_1,$$

$$v(t) = q(t, u(t), v(t)) + \frac{1}{\Gamma(\beta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\beta-1} \Psi(s, u(s), v(s)) \frac{ds}{s} + c_2,$$

where

$$c_1 = u(1) - p(1, u(1), v(1)), \qquad c_2 = v(1) - q(1, u(1), v(1)).$$

For convenience, define

$$I_\alpha \Phi(T) \coloneqq \frac{1}{\Gamma(\alpha)} \int_1^T \left(\ln \frac{T}{s}\right)^{\alpha-1} \Phi(s, u(s), v(s)) \frac{ds}{s},$$

$$I_\beta \Psi(T) \coloneqq \frac{1}{\Gamma(\beta)} \int_1^T \left(\ln \frac{T}{s}\right)^{\beta-1} \Psi(s, u(s), v(s)) \frac{ds}{s},$$

and

$$P_T \coloneqq p(T, u(T), v(T)), \qquad P_1 \coloneqq p(1, u(1), v(1)),$$

$$Q_T \coloneqq q(T, u(T), v(T)), \qquad Q_1 \coloneqq q(1, u(1), v(1)).$$

Evaluating the integral equations at $t = T$, and then using the boundary conditions, we get

$$P_1 + c_1 = \lambda_1(P_T + I_\alpha \Phi(T) + c_1) + \lambda_2(Q_T + I_\beta \Psi(T) + c_2),$$

$$Q_1 + c_2 = \mu_1(Q_T + I_\beta \Psi(T) + c_2) + \mu_2(P_T + I_\alpha \Phi(T) + c_1).$$

Therefore,

$$(1 - \lambda_1)c_1 - \lambda_2 c_2 = \lambda_1 P_T + \lambda_1 I_\alpha \Phi(T) + \lambda_2 Q_T + \lambda_2 I_\beta \Psi(T) - P_1,$$

$$-\mu_2 c_1 + (1 - \mu_1)c_2 = \mu_1 Q_T + \mu_1 I_\beta \Psi(T) + \mu_2 P_T + \mu_2 I_\alpha \Phi(T) - Q_1.$$

That is,

$$\begin{pmatrix} 1 - \lambda_1 & -\lambda_2 \\ -\mu_2 & 1 - \mu_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix},$$

where

$$R_1 \coloneqq \lambda_1 P_T + \lambda_1 I_\alpha \Phi(T) + \lambda_2 Q_T + \lambda_2 I_\beta \Psi(T) - P_1,$$

$$R_2 \coloneqq \mu_1 Q_T + \mu_1 I_\beta \Psi(T) + \mu_2 P_T + \mu_2 I_\alpha \Phi(T) - Q_1.$$

Since $\Delta \neq 0$, the constants c_1 and c_2 are uniquely determined by

$$c_1 = \frac{(1 - \mu_1)R_1 + \lambda_2 R_2}{\Delta}, \qquad c_2 = \frac{\mu_2 R_1 + (1 - \lambda_1)R_2}{\Delta}.$$

Accordingly, any solution of the boundary value problem is a fixed point of the operator $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ given by

$$\mathcal{F}(u, v)(t) = (\mathcal{F}_1(u, v)(t), \mathcal{F}_2(u, v)(t)),$$

where

$$\mathcal{F}_1(u,v)(t) = p(t,u(t),v(t)) + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\Phi(s,u(s),v(s))\frac{ds}{s} + c_1,$$

$$\mathcal{F}_2(u,v)(t) = q(t,u(t),v(t)) + \frac{1}{\Gamma(\beta)}\int_1^t \left(\ln\frac{t}{s}\right)^{\beta-1}\Psi(s,u(s),v(s))\frac{ds}{s} + c_2,$$

with c_1 and c_2 given above.

For the existence result, we decompose $\mathcal{F}$ as

$$\mathcal{F} = \mathcal{S} + \mathcal{T},$$

where

$$\mathcal{S}(u,v)(t) := (\mathcal{S}_1(u,v)(t), \mathcal{S}_2(u,v)(t)),$$

$$\mathcal{S}_1(u,v)(t) := p(t,u(t),v(t)), \qquad \mathcal{S}_2(u,v)(t) := q(t,u(t),v(t)),$$

and

$$\mathcal{T}(u,v)(t) := (\mathcal{T}_1(u,v)(t), \mathcal{T}_2(u,v)(t)),$$

with

$$\mathcal{T}_1(u,v)(t) := \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\Phi(s,u(s),v(s))\frac{ds}{s} + c_1,$$

$$\mathcal{T}_2(u,v)(t) := \frac{1}{\Gamma(\beta)}\int_1^t \left(\ln\frac{t}{s}\right)^{\beta-1}\Psi(s,u(s),v(s))\frac{ds}{s} + c_2.$$

Throughout the rest of this section, assume that

$$|p(t,u_1,v_1) - p(t,u_2,v_2)| \le L_p(|u_1-u_2| + |v_1-v_2|),$$

$$|q(t,u_1,v_1) - q(t,u_2,v_2)| \le L_q(|u_1-u_2| + |v_1-v_2|),$$

for all $t \in [1,T]$ and all $u_1, u_2, v_1, v_2 \in \mathbb{R}$, and

$$|\Phi(t,u,v)| \le \chi_1(t) + a_1|u| + a_2|v|,$$

$$|\Psi(t,u,v)| \le \chi_2(t) + b_1|u| + b_2|v|,$$

for all $(t,u,v) \in [1,T]\times\mathbb{R}^2$, where $\chi_1, \chi_2 \in C([1,T],\mathbb{R}^+)$ and $a_1, a_2, b_1, b_2 \ge 0$. We also set

$$P_0 := \max_{t\in[1,T]}|p(t,0,0)|, \qquad Q_0 := \max_{t\in[1,T]}|q(t,0,0)|,$$

$$K_\alpha := \frac{(\ln T)^\alpha}{\Gamma(\alpha+1)}, \qquad K_\beta := \frac{(\ln T)^\beta}{\Gamma(\beta+1)},$$

$$\Theta := \frac{|1-\mu_1| + |\lambda_2|}{|\Delta|}, \qquad \Theta' := \frac{|\mu_2| + |1-\lambda_1|}{|\Delta|}.$$

By Theorem 2.7 (Burton-type extension of Krasnosel'skii's theorem), it suffices to verify the following conditions.

Theorem 3.2. *Assume that*

$$2L_p < 1, \qquad 2L_q < 1,$$

and define

$$M := 2L_p + 2L_q + K_\alpha(a_1 + a_2) + K_\beta(b_1 + b_2).$$

If

$$\Gamma := \max\left\{\frac{P_0 + K_\alpha\|\chi_1\| + \Theta(P_0 + Q_0 + K_\alpha\|\chi_1\| + K_\beta\|\chi_2\|)}{1 - 2L_p - K_\alpha(a_1 + a_2) - \Theta M},\right.$$
$$\left.\frac{Q_0 + K_\beta\|\chi_2\| + \Theta'(P_0 + Q_0 + K_\alpha\|\chi_1\| + K_\beta\|\chi_2\|)}{1 - 2L_q - K_\beta(b_1 + b_2) - \Theta' M}\right\} < \infty,$$

then the boundary value problem admits at least one solution on $[1, T]$.

Proof. Let $r > 0$ be chosen so large that

$$r \geq \max\left\{\frac{P_0 + K_\alpha\|\chi_1\| + \Theta(P_0 + Q_0 + K_\alpha\|\chi_1\| + K_\beta\|\chi_2\|)}{1 - 2L_p - K_\alpha(a_1 + a_2) - \Theta M},\right.$$
$$\left.\frac{Q_0 + K_\beta\|\chi_2\| + \Theta'(P_0 + Q_0 + K_\alpha\|\chi_1\| + K_\beta\|\chi_2\|)}{1 - 2L_q - K_\beta(b_1 + b_2) - \Theta' M}\right\},$$

where the denominators are assumed to be positive. Define

$$D := \{(u, v) \in \mathcal{A} : \|(u, v)\|_{\mathcal{A}} \leq r\}.$$

Then D is nonempty, closed, bounded, and convex.

First, we show that $\mathcal{S}$ is a contraction on $\mathcal{A}$. Let $(u_1, v_1), (u_2, v_2) \in \mathcal{A}$. Then

$$|\mathcal{S}_1(u_1, v_1)(t) - \mathcal{S}_1(u_2, v_2)(t)| \leq L_p(|u_1(t) - u_2(t)| + |v_1(t) - v_2(t)|) \leq 2L_p\|(u_1, v_1) - (u_2, v_2)\|_{\mathcal{A}},$$

and similarly,

$$|\mathcal{S}_2(u_1, v_1)(t) - \mathcal{S}_2(u_2, v_2)(t)| \leq 2L_q\|(u_1, v_1) - (u_2, v_2)\|_{\mathcal{A}}.$$

Hence,

$$\|\mathcal{S}(u_1, v_1) - \mathcal{S}(u_2, v_2)\|_{\mathcal{A}} \leq \max\{2L_p, 2L_q\}\|(u_1, v_1) - (u_2, v_2)\|_{\mathcal{A}}.$$

Since $\max\{2L_p, 2L_q\} < 1$, the operator $\mathcal{S}$ is a contraction.

Next, we prove that $\mathcal{T}$ is continuous on D. Let $(u_n, v_n) \to (u, v)$ in $\mathcal{A}$. Since Φ and Ψ are continuous and the convergence is uniform, the continuity of the Hadamard integral operators yields

$$\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\Phi(s, u_n(s), v_n(s))\frac{ds}{s} \to \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\Phi(s, u(s), v(s))\frac{ds}{s}$$

uniformly on $[1, T]$, and analogously for the second integral term. Since c_1 and c_2 depend continuously on (u, v) through R_1 and R_2, we conclude that $\mathcal{T}$ is continuous on D.

We now show that $\mathcal{T}(D)$ is relatively compact in $\mathcal{A}$. Let $(u, v) \in D$. Then

$$|u(t)| \leq r, \qquad |v(t)| \leq r, \qquad t \in [1, T].$$

Hence,

$$|\Phi(t,u(t),v(t))| \le \|\chi_1\| + (a_1+a_2)r, \qquad |\Psi(t,u(t),v(t))| \le \|\chi_2\| + (b_1+b_2)r.$$

Therefore,

$$\left|\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\Phi(s,u(s),v(s))\frac{ds}{s}\right| \le K_\alpha(\|\chi_1\| + (a_1+a_2)r),$$

$$\left|\frac{1}{\Gamma(\beta)}\int_1^t \left(\ln\frac{t}{s}\right)^{\beta-1}\Psi(s,u(s),v(s))\frac{ds}{s}\right| \le K_\beta(\|\chi_2\| + (b_1+b_2)r).$$

Also, from the definitions of R_1 and R_2,

$$|R_1| \le |\lambda_1||P_T| + |\lambda_2||Q_T| + |P_1| + |\lambda_1||I_\alpha\Phi(T)| + |\lambda_2||I_\beta\Psi(T)|,$$

$$|R_2| \le |\mu_1||Q_T| + |\mu_2||P_T| + |Q_1| + |\mu_1||I_\beta\Psi(T)| + |\mu_2||I_\alpha\Phi(T)|.$$

Since $(u,v)\in D$, the functions p and q are bounded on the compact set $[1,T]\times[-r,r]^2$, so there exists a constant $M_0>0$ such that

$$|p(t,u(t),v(t))| \le M_0, \qquad |q(t,u(t),v(t))| \le M_0,$$

for all $(u,v)\in D$ and all $t\in[1,T]$. Moreover, the previously obtained estimates for the Hadamard integral terms imply that $I_\alpha\Phi(T)$ and $I_\beta\Psi(T)$ are uniformly bounded on D. Consequently, R_1 and R_2 are uniformly bounded on D, and therefore, c_1 and c_2 are uniformly bounded on D. It follows that $\mathcal{T}(D)$ is uniformly bounded.

To prove equicontinuity, let $(u,v)\in D$ and $1\le t_1<t_2\le T$. Then,

$$\left|\frac{1}{\Gamma(\alpha)}\int_1^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\Phi(s,u(s),v(s))\frac{ds}{s} - \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\Phi(s,u(s),v(s))\frac{ds}{s}\right|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left|\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - \left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right||\Phi(s,u(s),v(s))|\frac{ds}{s}$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}|\Phi(s,u(s),v(s))|\frac{ds}{s}.$$

Since $|\Phi(s,u(s),v(s))|$ is uniformly bounded on D, both terms tend to zero as $t_2\to t_1$ by the continuity of the kernel and the dominated convergence theorem. The same argument applies to the second integral term involving Ψ. Since c_1 and c_2 are constants with respect to t, they do not affect equicontinuity. Hence, $\mathcal{T}(D)$ is equicontinuous. By the Arzelà–Ascoli theorem, $\mathcal{T}(D)$ is relatively compact in $\mathcal{A}$. Thus, $\mathcal{T}$ is compact on D.

It remains to verify the invariance condition required by the Burton-type theorem. Let $(u,v)\in\mathcal{A}$ and $(\tilde{u},\tilde{v})\in D$ satisfy

$$(u,v) = \mathcal{S}(u,v) + \mathcal{T}(\tilde{u},\tilde{v}).$$

Then, for every $t\in[1,T]$,

$$u(t) = p(t,u(t),v(t)) + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\Phi(s,\tilde{u}(s),\tilde{v}(s))\frac{ds}{s} + \tilde{c}_1,$$

$$v(t) = q(t, u(t), v(t)) + \frac{1}{\Gamma(\beta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\beta-1} \Psi(s, \tilde{u}(s), \tilde{v}(s)) \frac{ds}{s} + \tilde{c}_2,$$

where $\tilde{c}_1, \tilde{c}_2$ denote the constants associated with $(\tilde{u}, \tilde{v})$. Since $(\tilde{u}, \tilde{v}) \in D$, all the estimates established above for the integral terms and the constants remain valid with r in place of $\|(\tilde{u}, \tilde{v})\|_{\mathcal{A}}$. Therefore,

$$|u(t)| \leq 2L_p\|(u, v)\|_{\mathcal{A}} + P_0 + K_\alpha(\|\chi_1\| + (a_1 + a_2)r) + \Theta(Mr + P_0 + Q_0 + K_\alpha\|\chi_1\| + K_\beta\|\chi_2\|),$$

and similarly,

$$|v(t)| \leq 2L_q\|(u, v)\|_{\mathcal{A}} + Q_0 + K_\beta(\|\chi_2\| + (b_1 + b_2)r) + \Theta'(Mr + P_0 + Q_0 + K_\alpha\|\chi_1\| + K_\beta\|\chi_2\|).$$

Substituting the chosen value of r (which satisfies the inequality in the theorem statement) into the estimates for $|u(t)|$ and $|v(t)|$ and taking the maximum over $t \in [1, T]$ yields $\|(u, v)\|_{\mathcal{A}} \leq r$. Thus, $(u, v) \in D$.

All assumptions of the Burton-type Krasnosel'skii theorem are satisfied. Hence, the equation

$$(u, v) = \mathcal{S}(u, v) + \mathcal{T}(u, v)$$

has at least one solution in D. Equivalently, the original boundary value problem admits at least one solution on $[1, T]$. □

For uniqueness, we impose the additional Lipschitz conditions

$$|\Phi(t, u_1, v_1) - \Phi(t, u_2, v_2)| \leq L_\Phi(|u_1 - u_2| + |v_1 - v_2|),$$

$$|\Psi(t, u_1, v_1) - \Psi(t, u_2, v_2)| \leq L_\Psi(|u_1 - u_2| + |v_1 - v_2|),$$

for all $t \in [1, T]$ and all $u_1, u_2, v_1, v_2 \in \mathbb{R}$.

Theorem 3.3. *Assume that the Lipschitz conditions for p, q, Φ, Ψ hold. Define*

$$\Omega := \max\Big\{2L_p + 2L_\Phi K_\alpha + \Theta(2L_p + 2L_q + 2L_\Phi K_\alpha + 2L_\Psi K_\beta),$$
$$2L_q + 2L_\Psi K_\beta + \Theta'(2L_p + 2L_q + 2L_\Phi K_\alpha + 2L_\Psi K_\beta)\Big\}.$$

If $\Omega < 1$, then the boundary value problem has a unique solution on $[1, T]$.

Proof. Let $U_1 = (u_1, v_1)$ and $U_2 = (u_2, v_2)$ be arbitrary elements of $\mathcal{A}$. We estimate the difference $\mathcal{F}(U_1) - \mathcal{F}(U_2)$.

For the first component, we have

$$\begin{aligned}|\mathcal{F}_1(U_1)(t) - \mathcal{F}_1(U_2)(t)| \leq\ & |p(t, u_1(t), v_1(t)) - p(t, u_2(t), v_2(t))| \\ & + \left|\frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} (\Phi(s, u_1(s), v_1(s)) - \Phi(s, u_2(s), v_2(s))) \frac{ds}{s}\right| \\ & + |c_1^{(1)} - c_1^{(2)}|.\end{aligned}$$

By the Lipschitz condition for p,

$$|p(t, u_1(t), v_1(t)) - p(t, u_2(t), v_2(t))| \leq 2L_p\|U_1 - U_2\|_{\mathcal{A}}.$$

Also,

$$\left|\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}(\Phi(s,u_1(s),v_1(s))-\Phi(s,u_2(s),v_2(s)))\frac{ds}{s}\right|$$
$$\leq\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}2L_\Phi\|U_1-U_2\|_{\mathcal{A}}\frac{ds}{s}\leq 2L_\Phi K_\alpha\|U_1-U_2\|_{\mathcal{A}}.$$

Now we estimate $|c_1^{(1)}-c_1^{(2)}|$. From the explicit formula for c_1,

$$|c_1^{(1)}-c_1^{(2)}|\leq\frac{|1-\mu_1||R_1^{(1)}-R_1^{(2)}|+|\lambda_2||R_2^{(1)}-R_2^{(2)}|}{|\Delta|}.$$

Using the definitions of R_1 and R_2 together with the Lipschitz assumptions, we get

$$|R_1^{(1)}-R_1^{(2)}|\leq 2L_p\|U_1-U_2\|_{\mathcal{A}}+2L_q\|U_1-U_2\|_{\mathcal{A}}+2L_\Phi K_\alpha\|U_1-U_2\|_{\mathcal{A}}+2L_\Psi K_\beta\|U_1-U_2\|_{\mathcal{A}},$$

and similarly,

$$|R_2^{(1)}-R_2^{(2)}|\leq 2L_p\|U_1-U_2\|_{\mathcal{A}}+2L_q\|U_1-U_2\|_{\mathcal{A}}+2L_\Phi K_\alpha\|U_1-U_2\|_{\mathcal{A}}+2L_\Psi K_\beta\|U_1-U_2\|_{\mathcal{A}}.$$

Therefore,

$$|c_1^{(1)}-c_1^{(2)}|\leq\Theta(2L_p+2L_q+2L_\Phi K_\alpha+2L_\Psi K_\beta)\|U_1-U_2\|_{\mathcal{A}}.$$

Combining the above estimates yields

$$\|\mathcal{F}_1(U_1)-\mathcal{F}_1(U_2)\|_X\leq(2L_p+2L_\Phi K_\alpha+\Theta(2L_p+2L_q+2L_\Phi K_\alpha+2L_\Psi K_\beta))\|U_1-U_2\|_{\mathcal{A}}.$$

In the same way, for the second component,

$$\|\mathcal{F}_2(U_1)-\mathcal{F}_2(U_2)\|_X\leq(2L_q+2L_\Psi K_\beta+\Theta'(2L_p+2L_q+2L_\Phi K_\alpha+2L_\Psi K_\beta))\|U_1-U_2\|_{\mathcal{A}}.$$

Taking the maximum of the two bounds, we obtain

$$\|\mathcal{F}(U_1)-\mathcal{F}(U_2)\|_{\mathcal{A}}\leq\Omega\|U_1-U_2\|_{\mathcal{A}}.$$

Since $\Omega<1$, the operator $\mathcal{F}$ is a contraction on the complete metric space $\mathcal{A}$. By Banach's contraction principle, $\mathcal{F}$ has a unique fixed point in $\mathcal{A}$. Hence, the boundary value problem has a unique solution on $[1,T]$. □

4. Ulam–Hyers stability

In this section, we investigate the stability of the coupled hybrid Hadamard fractional boundary value problem in the sense of Ulam and Hyers. We retain the notation of Section 3, in particular

$$K_\alpha:=\frac{(\ln T)^\alpha}{\Gamma(\alpha+1)},\qquad K_\beta:=\frac{(\ln T)^\beta}{\Gamma(\beta+1)},$$
$$\Theta:=\frac{|1-\mu_1|+|\lambda_2|}{|\Delta|},\qquad \Theta':=\frac{|\mu_2|+|1-\lambda_1|}{|\Delta|},$$

and we assume throughout this section that the Lipschitz conditions for p, q, Φ, Ψ hold and that the contraction constant

$$\Omega := \max\Big\{2L_p + 2L_\Phi K_\alpha + \Theta(2L_p + 2L_q + 2L_\Phi K_\alpha + 2L_\Psi K_\beta),$$

$$2L_q + 2L_\Psi K_\beta + \Theta'(2L_p + 2L_q + 2L_\Phi K_\alpha + 2L_\Psi K_\beta)\Big\}$$

satisfies $\Omega < 1$.

We first introduce the relevant notions.

Definition 4.1. *The coupled hybrid Hadamard fractional boundary value problem is said to be* Ulam–Hyers stable *if there exists a constant $C > 0$ such that, for every $\varepsilon > 0$ and every pair $(u, v) \in \mathcal{A}$ satisfying*

$$\left|{}^H D^\alpha(u(t) - p(t, u(t), v(t))) - \Phi(t, u(t), v(t))\right| \le \varepsilon, \qquad t \in [1, T],$$
$$\left|{}^H D^\beta(v(t) - q(t, u(t), v(t))) - \Psi(t, u(t), v(t))\right| \le \varepsilon, \qquad t \in [1, T],$$

together with the boundary conditions

$$u(1) = \lambda_1 u(T) + \lambda_2 v(T), \qquad v(1) = \mu_1 v(T) + \mu_2 u(T),$$

there exists an exact solution $(u_0, v_0) \in \mathcal{A}$ of the boundary value problem such that

$$\|(u, v) - (u_0, v_0)\|_{\mathcal{A}} \le C\varepsilon.$$

Definition 4.2. *The coupled hybrid Hadamard fractional boundary value problem is said to be* generalized Ulam–Hyers stable *if there exists a continuous function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that, for every $\varepsilon > 0$ and every pair $(u, v) \in \mathcal{A}$ satisfying*

$$\left|{}^H D^\alpha(u(t) - p(t, u(t), v(t))) - \Phi(t, u(t), v(t))\right| \le \varepsilon, \qquad t \in [1, T],$$

$$\left|{}^H D^\beta(v(t) - q(t, u(t), v(t))) - \Psi(t, u(t), v(t))\right| \le \varepsilon, \qquad t \in [1, T],$$

together with the boundary conditions

$$u(1) = \lambda_1 u(T) + \lambda_2 v(T), \qquad v(1) = \mu_1 v(T) + \mu_2 u(T),$$

there exists an exact solution $(u_0, v_0) \in \mathcal{A}$ of the boundary value problem such that

$$\|(u, v) - (u_0, v_0)\|_{\mathcal{A}} \le \psi(\varepsilon).$$

The following lemma provides the integral representation of an approximate solution.

Lemma 4.3. *Let $(u, v) \in \mathcal{A}$ satisfy*

$$\left|{}^H D^\alpha(u(t) - p(t, u(t), v(t))) - \Phi(t, u(t), v(t))\right| \le \varepsilon, \qquad t \in [1, T],$$

$$\left|{}^H D^\beta(v(t) - q(t, u(t), v(t))) - \Psi(t, u(t), v(t))\right| \le \varepsilon, \qquad t \in [1, T],$$

together with

$$u(1) = \lambda_1 u(T) + \lambda_2 v(T), \qquad v(1) = \mu_1 v(T) + \mu_2 u(T).$$

Then, there exist functions $\sigma_1, \sigma_2 \in C([1,T],\mathbb{R})$ *such that*

$$|\sigma_1(t)| \leq \varepsilon, \qquad |\sigma_2(t)| \leq \varepsilon, \qquad t \in [1,T],$$

and

$$\begin{aligned} {}^H D^{\alpha}(u(t) - p(t,u(t),v(t))) &= \Phi(t,u(t),v(t)) + \sigma_1(t), \\ {}^H D^{\beta}(v(t) - q(t,u(t),v(t))) &= \Psi(t,u(t),v(t)) + \sigma_2(t), \end{aligned}$$

for all $t \in [1,T]$. *Moreover,*

$$\begin{aligned} u(t) &= p(t,u(t),v(t)) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} (\Phi(s,u(s),v(s)) + \sigma_1(s)) \frac{ds}{s} + c_1^{\varepsilon}, \\ v(t) &= q(t,u(t),v(t)) + \frac{1}{\Gamma(\beta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\beta-1} (\Psi(s,u(s),v(s)) + \sigma_2(s)) \frac{ds}{s} + c_2^{\varepsilon}, \end{aligned}$$

where

$$c_1^{\varepsilon} = u(1) - p(1,u(1),v(1)), \qquad c_2^{\varepsilon} = v(1) - q(1,u(1),v(1)).$$

Proof. Define

$$\begin{aligned} \sigma_1(t) &:= {}^H D^{\alpha}(u(t) - p(t,u(t),v(t))) - \Phi(t,u(t),v(t)), \\ \sigma_2(t) &:= {}^H D^{\beta}(v(t) - q(t,u(t),v(t))) - \Psi(t,u(t),v(t)). \end{aligned}$$

By hypothesis,

$$|\sigma_1(t)| \leq \varepsilon, \qquad |\sigma_2(t)| \leq \varepsilon, \qquad t \in [1,T].$$

Applying the integral representation established in Section 3 to the perturbed equations

$$\begin{aligned} {}^H D^{\alpha}(u(t) - p(t,u(t),v(t))) &= \Phi(t,u(t),v(t)) + \sigma_1(t), \\ {}^H D^{\beta}(v(t) - q(t,u(t),v(t))) &= \Psi(t,u(t),v(t)) + \sigma_2(t), \end{aligned}$$

we immediately obtain

$$\begin{aligned} u(t) &= p(t,u(t),v(t)) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} (\Phi(s,u(s),v(s)) + \sigma_1(s)) \frac{ds}{s} + c_1^{\varepsilon}, \\ v(t) &= q(t,u(t),v(t)) + \frac{1}{\Gamma(\beta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\beta-1} (\Psi(s,u(s),v(s)) + \sigma_2(s)) \frac{ds}{s} + c_2^{\varepsilon}, \end{aligned}$$

where

$$c_1^{\varepsilon} = u(1) - p(1,u(1),v(1)), \qquad c_2^{\varepsilon} = v(1) - q(1,u(1),v(1)).$$

This completes the proof. □

We now define the perturbed fixed point operator $\mathcal{F}_{\varepsilon} : \mathcal{A} \to \mathcal{A}$ associated with (u,v) by

$$\mathcal{F}_{\varepsilon}(u,v)(t) = (\mathcal{F}_1^{\varepsilon}(u,v)(t), \mathcal{F}_2^{\varepsilon}(u,v)(t)),$$

where

$$\mathcal{F}_1^{\varepsilon}(u,v)(t) = p(t,u(t),v(t)) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} (\Phi(s,u(s),v(s)) + \sigma_1(s)) \frac{ds}{s} + c_1^{\varepsilon},$$

$$\mathcal{F}_2^{\varepsilon}(u,v)(t) = q(t,u(t),v(t)) + \frac{1}{\Gamma(\beta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\beta-1} (\Psi(s,u(s),v(s)) + \sigma_2(s)) \frac{ds}{s} + c_2^{\varepsilon}.$$

Then, the approximate solution (u, v) satisfies

$$(u,v) = \mathcal{F}_{\varepsilon}(u,v).$$

Let $(u_0, v_0) \in \mathcal{A}$ denote the unique exact solution of the original boundary value problem. Then,

$$(u_0,v_0) = \mathcal{F}(u_0,v_0),$$

where $\mathcal{F}$ is the fixed point operator introduced in Section 3.

Theorem 4.4. *Assume that the Lipschitz conditions for p, q, Φ, Ψ hold and that $\Omega < 1$. Then, the coupled hybrid Hadamard fractional boundary value problem is Ulam–Hyers stable. In particular, it is also generalized Ulam–Hyers stable.*

Proof. Let $(u, v) \in \mathcal{A}$ be an approximate solution satisfying the inequalities of the definition with perturbation level $\varepsilon > 0$, and let $(u_0, v_0) \in \mathcal{A}$ be the unique exact solution of the boundary value problem.

We first estimate the difference between the perturbed and unperturbed boundary constants. For the exact problem, define

$$R_1^0 := \lambda_1 P_T^0 + \lambda_1 I_\alpha \Phi^0(T) + \lambda_2 Q_T^0 + \lambda_2 I_\beta \Psi^0(T) - P_1^0,$$

$$R_2^0 := \mu_1 Q_T^0 + \mu_1 I_\beta \Psi^0(T) + \mu_2 P_T^0 + \mu_2 I_\alpha \Phi^0(T) - Q_1^0,$$

where

$$P_T^0 := p(T,u(T),v(T)), \qquad P_1^0 := p(1,u(1),v(1)),$$

$$Q_T^0 := q(T,u(T),v(T)), \qquad Q_1^0 := q(1,u(1),v(1)),$$

and

$$I_\alpha \Phi^0(T) := \frac{1}{\Gamma(\alpha)} \int_1^T \left(\ln \frac{T}{s}\right)^{\alpha-1} \Phi(s,u(s),v(s)) \frac{ds}{s},$$

$$I_\beta \Psi^0(T) := \frac{1}{\Gamma(\beta)} \int_1^T \left(\ln \frac{T}{s}\right)^{\beta-1} \Psi(s,u(s),v(s)) \frac{ds}{s}.$$

For the perturbed problem, define

$$R_1^{\varepsilon} := R_1^0 + \lambda_1 I_\alpha \sigma_1(T) + \lambda_2 I_\beta \sigma_2(T),$$

$$R_2^{\varepsilon} := R_2^0 + \mu_1 I_\beta \sigma_2(T) + \mu_2 I_\alpha \sigma_1(T).$$

Accordingly,

$$c_1^0 = \frac{(1-\mu_1)R_1^0 + \lambda_2 R_2^0}{\Delta}, \qquad c_2^0 = \frac{\mu_2 R_1^0 + (1-\lambda_1)R_2^0}{\Delta},$$

and

$$c_1^{\varepsilon} = \frac{(1-\mu_1)R_1^{\varepsilon} + \lambda_2 R_2^{\varepsilon}}{\Delta}, \qquad c_2^{\varepsilon} = \frac{\mu_2 R_1^{\varepsilon} + (1-\lambda_1)R_2^{\varepsilon}}{\Delta}.$$

Since

$$|\sigma_1(t)| \leq \varepsilon, \qquad |\sigma_2(t)| \leq \varepsilon, \qquad t \in [1, T],$$

we have

$$|I_\alpha \sigma_1(T)| \leq K_\alpha \varepsilon, \qquad |I_\beta \sigma_2(T)| \leq K_\beta \varepsilon.$$

Therefore,

$$|R_1^\varepsilon - R_1^0| \leq |\lambda_1| K_\alpha \varepsilon + |\lambda_2| K_\beta \varepsilon,$$
$$|R_2^\varepsilon - R_2^0| \leq |\mu_1| K_\beta \varepsilon + |\mu_2| K_\alpha \varepsilon.$$

Hence,

$$|c_1^\varepsilon - c_1^0| \leq \frac{|1-\mu_1|\,|R_1^\varepsilon - R_1^0| + |\lambda_2|\,|R_2^\varepsilon - R_2^0|}{|\Delta|} \leq C_1 \varepsilon,$$

where

$$C_1 := \Theta(|\lambda_1| K_\alpha + |\lambda_2| K_\beta + |\mu_1| K_\beta + |\mu_2| K_\alpha),$$

and similarly,

$$|c_2^\varepsilon - c_2^0| \leq \frac{|\mu_2|\,|R_1^\varepsilon - R_1^0| + |1-\lambda_1|\,|R_2^\varepsilon - R_2^0|}{|\Delta|} \leq C_2 \varepsilon,$$

where

$$C_2 := \Theta'(|\lambda_1| K_\alpha + |\lambda_2| K_\beta + |\mu_1| K_\beta + |\mu_2| K_\alpha).$$

We next estimate the difference between the perturbed and unperturbed operators at the approximate solution (u, v). For the first component,

$$\begin{aligned} |\mathcal{F}_1^\varepsilon(u,v)(t) - \mathcal{F}_1(u,v)(t)| &\leq \left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \sigma_1(s) \frac{ds}{s} \right| + |c_1^\varepsilon - c_1^0| \\ &\leq K_\alpha \varepsilon + C_1 \varepsilon = (K_\alpha + C_1)\varepsilon. \end{aligned}$$

Likewise,

$$|\mathcal{F}_2^\varepsilon(u,v)(t) - \mathcal{F}_2(u,v)(t)| \leq (K_\beta + C_2)\varepsilon.$$

Thus,

$$\|\mathcal{F}_\varepsilon(u,v) - \mathcal{F}(u,v)\|_{\mathcal{A}} \leq \gamma \varepsilon,$$

where

$$\gamma := \max\{K_\alpha + C_1,\ K_\beta + C_2\}.$$

Since $(u, v) = \mathcal{F}_\varepsilon(u, v)$ and $(u_0, v_0) = \mathcal{F}(u_0, v_0)$, we obtain

$$\begin{aligned} \|(u,v) - (u_0,v_0)\|_{\mathcal{A}} &= \|\mathcal{F}_\varepsilon(u,v) - \mathcal{F}(u_0,v_0)\|_{\mathcal{A}} \\ &\leq \|\mathcal{F}_\varepsilon(u,v) - \mathcal{F}(u,v)\|_{\mathcal{A}} + \|\mathcal{F}(u,v) - \mathcal{F}(u_0,v_0)\|_{\mathcal{A}}. \end{aligned}$$

By the previous estimate and the contraction property of $\mathcal{F}$,

$$\|(u,v) - (u_0,v_0)\|_{\mathcal{A}} \leq \gamma \varepsilon + \Omega \|(u,v) - (u_0,v_0)\|_{\mathcal{A}}.$$

Since $\Omega < 1$, it follows that

$$\|(u,v) - (u_0,v_0)\|_{\mathcal{A}} \leq \frac{\gamma}{1-\Omega}\, \varepsilon.$$

Therefore, the problem is Ulam–Hyers stable with stability constant

$$C = \frac{\gamma}{1-\Omega}.$$

Finally, if we define

$$\psi(\varepsilon) := \frac{\gamma}{1-\Omega}\,\varepsilon,$$

then $\psi \in C([0,\infty),[0,\infty))$ and $\psi(0) = 0$. Hence, the problem is also generalized Ulam–Hyers stable. This completes the proof. □

Remark 4.5. *The explicit Ulam–Hyers stability constant is*

$$C = \frac{1}{1-\Omega}\max\{K_\alpha + C_1,\ K_\beta + C_2\},$$

where

$$C_1 = \Theta(|\lambda_1|K_\alpha + |\lambda_2|K_\beta + |\mu_1|K_\beta + |\mu_2|K_\alpha),$$
$$C_2 = \Theta'(|\lambda_1|K_\alpha + |\lambda_2|K_\beta + |\mu_1|K_\beta + |\mu_2|K_\alpha).$$

In particular,

$$C = \frac{1}{1-\Omega}\max\Big\{K_\alpha + \Theta(|\lambda_1|K_\alpha + |\lambda_2|K_\beta + |\mu_1|K_\beta + |\mu_2|K_\alpha),$$
$$K_\beta + \Theta'(|\lambda_1|K_\alpha + |\lambda_2|K_\beta + |\mu_1|K_\beta + |\mu_2|K_\alpha)\Big\}.$$

5. Illustrative examples

In this section, we present examples to demonstrate the applicability of the theoretical results. The purpose of the following examples is to verify the assumptions of the obtained theorems and illustrate their applicability within the Hadamard framework.

Example 5.1. Existence without uniqueness. Consider the coupled hybrid Hadamard fractional boundary value problem on $[1,2]$:

$$\begin{cases} {}^{H}D^{1/2}(u(t) - p(t,u(t),v(t))) = \Phi(t,u(t),v(t)), \\ {}^{H}D^{3/4}(v(t) - q(t,u(t),v(t))) = \Psi(t,u(t),v(t)), \\ u(1) = \dfrac{1}{4}u(2) + \dfrac{1}{5}v(2), \\ v(1) = \dfrac{1}{5}v(2) + \dfrac{1}{6}u(2), \end{cases}$$

where

$$p(t,u,v) = \frac{1}{10}\sin u + \frac{1}{15}\cos v, \qquad q(t,u,v) = \frac{1}{12}\tanh u + \frac{1}{20}\frac{v}{1+|v|},$$
$$\Phi(t,u,v) = \frac{1}{20}\frac{|u|}{1+|u|} + \frac{1}{25}\arctan v + \frac{1}{1+t^2},$$
$$\Psi(t,u,v) = \frac{1}{30}\sin u + \frac{1}{35}\cos v + \frac{1}{2}e^{-t}.$$

The boundary coefficients are

$$\lambda_1 = \frac{1}{4}, \qquad \lambda_2 = \frac{1}{5}, \qquad \mu_1 = \frac{1}{5}, \qquad \mu_2 = \frac{1}{6},$$

and hence

$$\Delta = (1 - \lambda_1)(1 - \mu_1) - \lambda_2\mu_2 = \frac{17}{30} \neq 0.$$

The Lipschitz constants may be chosen as

$$L_p = \frac{1}{10}, \qquad L_q = \frac{1}{12}, \qquad L_\Phi = \frac{1}{20}, \qquad L_\Psi = \frac{1}{30}.$$

Moreover,

$$K_{1/2} = \frac{(\ln 2)^{1/2}}{\Gamma(3/2)} \approx 0.9394, \qquad K_{3/4} = \frac{(\ln 2)^{3/4}}{\Gamma(7/4)} \approx 0.8266.$$

Also,

$$\Theta = \frac{30}{17} \approx 1.7647, \qquad \Theta' = \frac{55}{34} \approx 1.6176.$$

The contraction constants are

$$\Omega_1 \approx 1.2040, \qquad \Omega_2 \approx 1.0560,$$

and therefore

$$\Omega = \max\{\Omega_1, \Omega_2\} \approx 1.2040 > 1.$$

Thus, the contraction condition is not satisfied. However, the growth assumptions of the existence theorem are fulfilled, and the corresponding denominators in the invariant-ball estimate are positive:

$$1 - 2L_p - \Theta(2L_p + 2L_q) \approx 0.1529 > 0,$$

$$1 - 2L_q - \Theta'(2L_p + 2L_q) \approx 0.2402 > 0.$$

Moreover, the continuity, growth, and boundedness assumptions required in Theorem 3.2 are readily verified for the above nonlinearities. Hence, the assumptions of the existence theorem are satisfied, and the problem admits at least one solution on $[1, 2]$. This example illustrates that existence may be obtained even when uniqueness is not guaranteed by the contraction criterion.

Example 5.2. Uniqueness and Ulam–Hyers stability. Consider the coupled hybrid Hadamard fractional boundary value problem on $[1, e]$:

$$\begin{cases} {}^{H}D^{1/2}(u(t) - p(t, u(t), v(t))) = \Phi(t, u(t), v(t)), \\ {}^{H}D^{1/2}(v(t) - q(t, u(t), v(t))) = \Psi(t, u(t), v(t)), \\ u(1) = \dfrac{1}{4}u(e) + \dfrac{1}{5}v(e), \\ v(1) = \dfrac{1}{5}v(e) + \dfrac{1}{6}u(e), \end{cases}$$

where

$$p(t,u,v) = \frac{1}{20}\frac{u}{1+|u|} + \frac{1}{25}\sin v, \qquad q(t,u,v) = \frac{1}{30}\tanh u + \frac{1}{35}\cos v,$$

$$\Phi(t,u,v) = \frac{1}{40}\arctan u + \frac{1}{45}\frac{v}{1+|v|} + \frac{1}{t^2+1},$$

$$\Psi(t,u,v) = \frac{1}{50}\sin u + \frac{1}{55}\cos v + \frac{1}{t+1}.$$

The Lipschitz constants can be taken as

$$L_p = \frac{1}{20}, \qquad L_q = \frac{1}{30}, \qquad L_\Phi = \frac{1}{40}, \qquad L_\Psi = \frac{1}{50}.$$

The boundary parameters give

$$\Delta = \frac{17}{30} \neq 0.$$

Since $T = e$ and $\alpha = \beta = 1/2$, we have

$$K_\alpha = K_\beta = \frac{1}{\Gamma(3/2)} = \frac{2}{\sqrt{\pi}} \approx 1.1284.$$

Moreover,

$$\Theta = \frac{30}{17} \approx 1.7647, \qquad \Theta' = \frac{55}{34} \approx 1.6176.$$

Direct calculation gives

$$\Omega_1 \approx 0.6298, \qquad \Omega_2 \approx 0.5457.$$

Thus,

$$\Omega = \max\{\Omega_1, \Omega_2\} \approx 0.6298 < 1.$$

Consequently, the contraction condition is satisfied. Hence, the problem has a unique solution on $[1, e]$. Moreover, by the Ulam–Hyers stability theorem, the unique solution is Ulam–Hyers stable.

The main constants involved in Examples 5.1 and 5.2 are summarized in Table 2.

Table 2. Summary of the main constants in Examples 5.1 and 5.2.

Quantity	**Example 5.1**	**Example 5.2**
Interval	$[1,2]$	$[1,e]$
(α,β)	$(1/2, 3/4)$	$(1/2, 1/2)$
(L_p, L_q)	$(1/10, 1/12)$	$(1/20, 1/30)$
(L_Φ, L_Ψ)	$(1/20, 1/30)$	$(1/40, 1/50)$
Δ	$17/30$	$17/30$
Ω	$\approx 1.2040 > 1$	$\approx 0.6298 < 1$
Conclusion	Existence only	Uniqueness and stability

Example 5.3. Numerical illustration of Ulam–Hyers stability. To complement the analytical results, we provide a numerical illustration based on Example 5.2. The numerical approximation is obtained

by applying a Picard-type fixed point iteration combined with a product-integration discretization in the logarithmic variable

$$\tau = \ln t, \qquad t \in [1, e].$$

This change of variables transforms the Hadamard-type integral into a weakly singular Volterra integral on $[0, 1]$, which is then approximated using product-integration weights.

Figure 1 presents representative numerical approximations of the solution components $u(t)$ and $v(t)$ associated with Example 5.2 over the interval $[1, e]$. The figure is included solely to provide a visual illustration of the computed solution profiles corresponding to the considered fractional system. No stability conclusion is inferred from the shape of the trajectories shown in the figure. The Ulam–Hyers stability of the solution is established analytically in Section 4 and is further illustrated numerically through the perturbation-response results reported in Table 3.

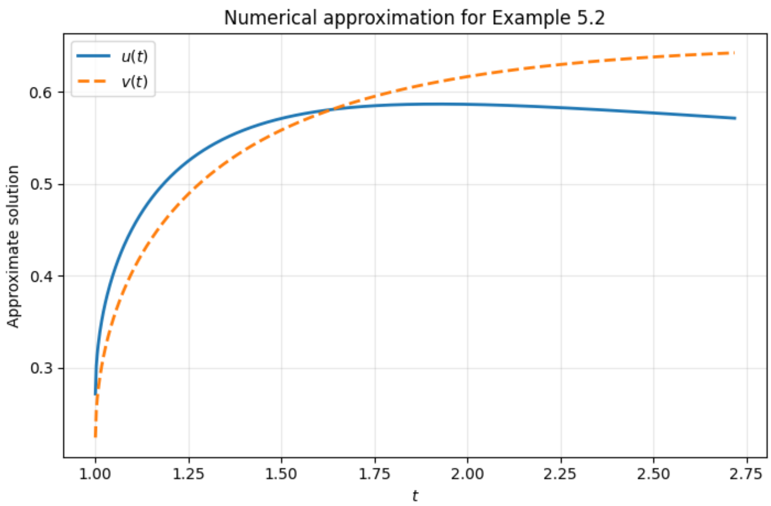

Figure 1. Representative numerical approximations of the solution components $u(t)$ and $v(t)$ for Example 5.2 on the interval $[1, e]$.

For the stability illustration, we perturb the right-hand sides by bounded functions σ_1, σ_2 satisfying

$$|\sigma_1(t)| \leq \varepsilon, \qquad |\sigma_2(t)| \leq \varepsilon, \qquad t \in [1, e].$$

For each value of ε, a numerical approximation of the perturbed solution is obtained using the same product-integration fixed-point procedure. The error is defined as

$$E(\varepsilon) = \max \{\|u_\varepsilon - u_0\|_\infty, \|v_\varepsilon - v_0\|_\infty\},$$

where (u_0, v_0) denotes the unperturbed solution, and $(u_\varepsilon, v_\varepsilon)$ denotes the perturbed approximation. The computed values are reported in Table 3.

Table 3. Computed perturbation error for Example 5.2.

ε	$E(\varepsilon)$	$E(\varepsilon)/\varepsilon$
10^{-1}	5.6842×10^{-2}	0.5684
5×10^{-2}	2.8441×10^{-2}	0.5688
10^{-2}	5.6914×10^{-3}	0.5691
5×10^{-3}	2.8459×10^{-3}	0.5692
10^{-3}	5.6922×10^{-4}	0.5692
5×10^{-4}	2.8461×10^{-4}	0.5692
10^{-4}	5.6922×10^{-5}	0.5692

The nearly constant ratio $E(\varepsilon)/\varepsilon$ provides additional numerical evidence for the linear Ulam–Hyers stability estimate.

The values reported in Table 3 are consistent with the linear dependence predicted by the Ulam–Hyers stability estimate established in Section 4. They are included only as an illustrative numerical verification of the theoretical result.

6. Conclusions

In this work, we have investigated a coupled system of hybrid Hadamard fractional differential equations with strongly coupled non-separated two-point boundary conditions. The system features distinct fractional orders $\alpha, \beta \in (0, 1]$ and a non-separated coupling expressed through the boundary conditions

$$u(1) = \lambda_1 u(T) + \lambda_2 v(T), \qquad v(1) = \mu_1 v(T) + \mu_2 u(T).$$

By converting the problem into an equivalent system of fractional integral equations via Hadamard fractional calculus, we established the existence of at least one solution using a Burton-type extension of Krasnosel'skii's fixed point theorem under suitable growth conditions on the nonlinearities. Uniqueness was proved via Banach's contraction principle under Lipschitz-type assumptions on the nonlinearities.

Furthermore, we analyzed the Ulam–Hyers stability of the proposed system and derived explicit stability constants. The theoretical findings were illustrated through concrete examples and numerical experiments, including graphical representations of representative numerical approximations and perturbation-response behavior, which support the theoretical Ulam–Hyers stability analysis.

The present work is not intended to replace the widely used Caputo or Riemann–Liouville formulations, but rather to complement the existing literature by considering a hybrid system within the Hadamard fractional framework. The logarithmic kernel of the Hadamard fractional integral naturally leads to problems posed on intervals of the form $[1, T]$ and provides an alternative mathematical setting for the analysis of coupled fractional systems. From this perspective, the results obtained here extend the current theory by combining Hadamard fractional operators with strongly coupled non-separated boundary conditions and Ulam–Hyers stability in a unified framework.

For future work, several directions can be pursued. First, the system can be extended to include nonlocal boundary conditions of integral type or multi-point conditions, which would introduce additional analytical challenges. Second, the results can be generalized to fractional differential inclusions where the right-hand sides are set-valued maps, requiring the use of appropriate fixed point theorems for multivalued operators. Third, the study of fractional hybrid systems with variable-order derivatives or with time-dependent fractional orders represents an interesting avenue for future research. Fourth, further investigation of numerical methods for coupled hybrid Hadamard systems, together with rigorous convergence and error analyses, would constitute a valuable direction for future research.

Another interesting direction is the application of the proposed analytical framework to mathematical models involving logarithmic memory effects or multiplicative temporal structures, where Hadamard-type operators arise naturally. Exploring such applications may further clarify the practical role of the Hadamard framework and broaden the range of problems that can be treated using the present approach.

We hope that the results presented in this paper will serve as a foundation for further theoretical and applied investigations in the field of hybrid fractional differential equations.

Author contributions

Dalal Alhwikem and Salma Trabelsi contributed equally to the conceptualization, methodology, formal analysis, investigation, writing of the original draft, and review and editing of the manuscript. Both authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest in this paper.

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