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Research article

Initial-boundary value problem for a nonlinear wave equation with strong damping and structural damping

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Abstract: In this paper, we are concerned with an initial boundary value problem of wave equations incorporating strong damping and structural damping. By virtue of semigroup theory, we establish the global well-posedness of weak solutions, where the nonlinear damping and source terms satisfy subcritical growth conditions. Furthermore, based on Nakao's inequality, we derive the exponential decay estimate of solutions. Notably, we relax the constraints on the coefficients of structural damping to a general non-negativity condition.

Keywords: wave equation; nonlinear damping; well-posedness; decay rates
Mathematics Subject Classification: 35G31, 35L05, 35L30

1. Introduction

This paper is mainly concerned with the initial boundary value problem for a nonlinear wave equation involving strong damping and structural damping:

$$u_{tt} - \Delta u - \Delta u_t + f(u) + a(x)g(u_t) = 0, (x,t) \in \Omega \times (0, +\infty), \tag{1.1}$$

$$u(x,t) = 0, x \in \partial\Omega, \tag{1.2}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, $a(x) \geq 0$ with $a(x) \in L^\infty(\Omega)$, and f and g are real-valued functions satisfying Assumptions 1.1 and 1.2 below, respectively:

Assumption 1.1. *Assume that $f \in C^2(\mathbb{R})$, and some constant $k_0 > 0$ exists such that*

$$f(0) = 0, \quad |f^{(j)}(s)| \leq k_0(1 + |s|)^{p-j}, \quad j = 1, 2, \quad \forall s \in \mathbb{R}, \tag{1.4}$$

where

$$1 \leq p \leq \frac{n+2}{n-2}, \quad \text{when } n \geq 3; \qquad p \geq 1, \quad \text{when } n = 1, 2.$$

Furthermore, let $F(\lambda) := \int_0^\lambda f(s)ds$, *and assume that* $F(\lambda) > 0$ *for all* $\lambda > 0$.

Assumption 1.2. *The feedback function* $g : \mathbb{R} \to \mathbb{R}$ *is a monotonically increasing continuous function with* $g(0) = 0$, *and satisfies*

$$\tilde{m}|s|^{r+1} \leq g(s)s \leq \tilde{M}|s|^{r+1}, \quad |s| > 1, \quad r > 0,$$

where $\tilde{m}$ *and* $\tilde{M}$ *are positive constants.*

Over the past few decades, wave equations with strong damping $-\Delta u_t$ have been extensively studied. In particular, in 1967, Greenberg et al. [17] first systematically investigated the following equation in one-dimensional space:

$$u_{tt} - \Delta u - \Delta u_t = 0. \tag{1.5}$$

They obtained the global existence, uniqueness, and decay properties of solutions to this equation. Since then, numerous studies on equations of the form (1.5) have emerged, such as [6, 12, 16, 30, 31]. Subsequently, many scholars have carried out extensive investigations into the abstract form of Eq (1.5)

$$u_{tt} + a(\|A^{\frac{1}{2}}u(t)\|^2)Au + Au_t = 0, t \in (0, \infty), \tag{1.6}$$

with the initial data being like that of (1.3), where $a \in C^1[0, \infty)$ with $a(s) \geq 0$ for all $s \geq 0$. Let H is a real, separable Hilbert space. The A is a linear operator defined in H with domain $\mathcal{D}(A)$. Moreover, $\mathcal{D}(A)$ is dense in H. The operator A satisfies the following conditions: (1) It is self-adjoint, positive definite, and has a discrete spectrum; (2) the embedding from $\mathcal{D}(A^r)$ to $\mathcal{D}(A^{r'})$ $(r \geq r' \geq 0)$ is compact. In 1984, Nishihara [23] proved that when $u_0 \in \mathcal{D}(A)$, $u_1 \in \mathcal{D}(A^{\frac{1}{2}})$, Eq (1.6) admits a unique global solution $u \in C^1([0, \infty); H)$. Furthermore, when $a(s) = s$, $u_0 \in \mathcal{D}(A^{\frac{3}{2}})$, $u_1 \in \mathcal{D}(A^{\frac{3}{2}})$, the following asymptotic estimate for the solution was obtained:

$$\left\|A^{\frac{1}{2}}u(t)\right\| \leq C_\epsilon(1+t)^{-\frac{1}{2+\epsilon}}, \quad \forall \epsilon > 0,$$

where $C_\epsilon \to \infty$ as $\epsilon \to 0$. In 1991, Matos and Pereira [19] weakened the initial condition in [23] to $u_0 \in \mathcal{D}(A^{\frac{1}{2}})$, $u_1 \in H$, and obtained the existence and uniqueness of the global solution $u \in L^\infty(0, \infty; \mathcal{D}(A^{\frac{1}{2}}))$, $u_t \in L^\infty(0, \infty; H) \cap L^2(0, T; \mathcal{D}(A^{\frac{1}{2}}))$ to (1.6). Furthermore, when $a(s) = s$, $u_0 \in \mathcal{D}(A^{\frac{1}{2}})$, $u_1 \in H$, they used the Nakao inequality [21] to derive the decay estimate of the solutions as follows:

$$\left\|A^{\frac{1}{2}}u(t)\right\| \leq C(1+t)^{-\frac{1}{2}}, \quad C > 0.$$

In 1993, Nishihara [24], building upon the work of Matos [19], strengthened the regularity conditions on the initial data u_0, u_1 to $u_0 \in \mathcal{D}(A^{\frac{3}{2}})$, $u_1 \in \mathcal{D}(A^{\frac{3}{2}})$. It was also assumed that the operator A satisfies the following conditions: (1) $\left\|A^{\frac{1}{2}}u_0\right\| + \|u_1\|$ is sufficiently small; (2) $\frac{1}{2}\left\|A^{\frac{1}{2}}u_0\right\|^2 + (u_1, u_0) > 0$. Under these assumptions, they proved that the solution $u(t)$ of the problem (1.6) satisfies

$$\left\|A^{\frac{1}{2}}u(t)\right\| \geq c(1+t)^{-\frac{1}{2}}, \quad \text{for } t > t_1.$$

for some constant $t_1 \geq 0$. Combining this result with the decay estimate obtained in [19] yields the optimal estimate for $\left\|A^{\frac{1}{2}}u(t)\right\|$. Furthermore, in 2013, Ikehata et al. [18] established the global existence and uniqueness of solutions to Problem (1.6) under different regularity assumptions on the initial data u_0, u_1. In 1988, Ang and Ngoc Dinh [1] studied the well-posedness of the initial boundary value problem for the following equation in the case $f = 0, a(x) = 1$:

$$u_{tt} - \Delta u_t - \sum_{i=1}^{n} \frac{d}{dx_i} (\sigma_i(u_{x_i})) + |u_t|^{\alpha} \operatorname{sgn} u_t = 0,(x, t) \in \Omega \times (0, T), \tag{1.7}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$. The initial data satisfy $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, and the nonlinear terms $\sigma_i \in C(\mathbb{R}, \mathbb{R})$ are nondecreasing and induce mappings of $L^2(\Omega)$ into itself, taking bounded sets into bounded sets. Under these assumptions, they obtained the local existence of weak solutions by a combination of compactness and monotonicity methods. If an additional assumption that each σ_i is locally Lipschitz continuous is imposed, then the uniqueness of the corresponding weak solution follows. Furthermore, stronger restrictions are imposed on the initial data, namely $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in L^2(\Omega)$. Under the assumptions that the nonlinear terms $\sigma_i \in C^1(\mathbb{R}, \mathbb{R})$, $\sigma_i(0) = 0$, $\sigma_i' > 0$, and σ_i' are locally Hölder continuous, the global existence of weak solutions in the one-dimensional case was obtained by using the Galerkin method. Consequently, by applying the results of analytic semigroups from [25], it was shown that the weak solution is actually a strong solution.

In the same year, Ang and Ngoc Dinh [2] investigated the well-posedness of the following equation in the case $f \neq 0, g = 0$:

$$u_{tt} - \Delta u - \lambda \Delta u_t + f(u) = 0, (x, t) \in \Omega \times (0, T), \lambda > 0, \tag{1.8}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$. Under the assumptions that $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and f satisfies a local Lipschitz condition, they used a linear iteration method to obtain a local solution $u \in C(0, T; H_0^1(\Omega))$ with $u_t \in C(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$, and the uniqueness of the solution. Furthermore, they established that the solution decays exponentially to zero as time goes to infinity. For studies on the local existence of solutions to Problem (1.8) under different conditions on the nonlinear function f and initial data, see references [3, 26]. In addition, a considerable body of numerical investigations has also been devoted to this equation, as exemplified by references [32, 33], which provide useful numerical benchmarks for the related theoretical studies.

In 2023, Cavalcanti et al. [9] studied the initial boundary value problem:

$$u_{tt} - \Delta u + f(u) + a(x) g(u_t) = 0, (x, t) \in \Omega \times (0, +\infty). \tag{1.9}$$

Here, f satisfies: (1) $f \in C^2(\mathbb{R}), f(0) = 0$; (2) $|f^{(j)}(s)| \leq k_0(1 + |s|)^{p-j}, j = 1, 2, \forall s \in \mathbb{R}$, where $1 \leq p < \frac{n+2}{n-2}$ $(n \geq 3)$, for some constant $k_0 > 0$; (3) $0 < F(s) < f(s)s, \forall s > 0$, where $F(\lambda) := \int_0^\lambda f(s) ds$. The function g satisfies: (1) $g : \mathbb{R} \to \mathbb{R}$ is a monotonically increasing continuous function with $g(0) = 0$; (2) $g(s)s > 0$ for $s \neq 0$, and $\hat{m}|s|^{r+1} \leq g(s)s \leq M|s|^{r+1}$ for $|s| > 1, r \geq 0$, where $\hat{m}$ and M are positive constants. We also assume that $a \in L^\infty(\Omega)$ is non-negative. Under these conditions, Cavalcanti et al. used semigroup theory combined with Strichartz estimates to prove that for $r = 1$ and $1 \leq p < \frac{n+2}{n-2}$, Problem (1.9) admits a unique global weak solution. By using the observability inequality, they obtained the energy decay [7, 10, 13]. In 2024, Gao et al. [15] adopted the semigroup theory proposed in [9] to establish the global existence and uniqueness of solutions for a coupled system closely related to (1.9). For such coupled systems, several numerical methods have also been developed; see, e.g., [34] and [35] for representative works.

This paper primarily utilizes semigroup theory and compactness methods to investigate the global existence and uniqueness of solutions to Problem (1.1)–(1.3). We obtain the global existence of solutions for $0 \le r \le \frac{n+2}{n-2}, 1 \le p \le \frac{n+2}{n-2}$ $(n \ge 3)$.

It is well known that $a(x) \ge a_0 > 0$ is a conventional assumption. In this case, $a(x)g(u_t)$ directly ensures the regularity of $u_t \in L^{r+1}(0, T; \Omega)$, and local well-posedness can be established via the Galerkin method. Notably, we only suppose $a(x) \ge 0$. While the Galerkin method still applies, it only relies on the regularity of $u_t \in L^2(0, T; H_0^1(\Omega))$ from strong damping, which restricts the growth order of g and fails to achieve the subcritical growth range obtained by semigroup theory. Nevertheless, concerning the uniqueness, we restrict the admissible range of r to $0 \le r \le \frac{(n+2)+\sqrt{(n+2)^2+8n}}{2n}$. For the long-time behavior of solutions, this paper does not follow the approach in [7, 9, 10, 13], thereby avoiding the condition $a(x) \ge a_0 > 0$ obtained through microlocal analysis and weakening the assumptions on $a(x)$. Moreover, compared with using observability inequalities, this paper uses the Nakao inequality to establish exponential energy decay, not only improving the precision of the estimates but also simplifying the complex derivation process.

The structure of this paper is as follows: Section 1 introduces the research background and preliminary assumptions; Section 2 presents the necessary preliminaries and main results; Section 3 applies semigroup theory to obtain the global existence of weak solutions to Problem (1.1)–(1.3); Section 4 further establishes the uniqueness of weak solutions and the energy identity for Problem (1.1)–(1.3); and Section 5 utilizes the Nakao inequality to derive the exponential decay property of the energy. The required assumptions for this paper are given below.

Remark 1. *According to Eq* (1.4) *in Assumption 1.1 and by applying the Lagrange mean value theorem, we obtain that for any $r, s \in \mathbb{R}$ and $c > 0$ exists such that*

$$|f(r) - f(s)| \le c(1 + |s|^{p-1} + |r|^{p-1})|r - s|. \tag{1.10}$$

We now introduce several notations and lemmas. Let Ω be a measurable set in $\mathbb{R}^n$. L^p and H^s denote the general Lebesgue and Sobolev space. For a monotonically increasing mapping $g : \mathbb{R} \to \mathbb{R}$, $g(0) = 0$, is of the order $r := O(g) \ge 0$ at infinity if

$$|s|^{r+1} \sim g(s)s, \text{ for } |s| \ge 1.$$

If the order r is bigger than, less than, or equal to 1, we say the map g is superlinear, sublinear, or linearly bounded at infinity, respectively.

Lemma 1.1 ([29]). *Let W be a Banach space, V be a reflexive Banach space, and H be a Hilbert space containing V. Suppose V and W are contained in the same normed linear space, with $V \cap W$ dense in both V and W; H and W are contained in a linear space; and that V is densely embedded in H. Let Z be a Banach space such that $Z \to L^1(0, T; W)$ and $Z' \to L^1(0, T; W')$ are dense embeddings. If $u \in L^\infty(0, T; V)$, $u_t \in Z \cap L^\infty(0, T; H)$, and $u_{tt} - \Delta u = F \in Z' + L^1(0, T; H)$, then for any $t \in [0, T]$, we have*

$$\|u_t(t)\|_H^2 + \langle -\Delta u(t), u(t)\rangle_V - \|u_t(0)\|_H^2 - \langle -\Delta u(0), u(0)\rangle_V = 2\Re \int_0^t \langle F, u_t\rangle_{W\cap H} \mathrm{d}\tau.$$

Lemma 1.2 (Lions' lemma [5]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $\{u_k\}$ be a sequence of real measurable functions that is bounded in $L^q(\Omega)$ $(1 < q < \infty)$ and converges almost everywhere in Ω to u. Then for any $p \in [1, q)$, u_k converges strongly to u in $L^p(\Omega)$ and weakly in $L^q(\Omega)$.*

Lemma 1.3 (Lions–Aubin–Simon lemma [28]). *Let X, B, Y be three Banach spaces with $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$.*

(1) If F is bounded in $L^p(0, T; X)$, $1 \leq p < \infty$, and $\frac{dF}{dt} =: \{\frac{df}{dt} : f \in F\}$ is bounded in $L^1(0, T; Y)$, then F is relatively compact in $L^p(0, T; B)$;

(2) If F is bounded in $L^\infty(0, T; X)$ and $\frac{dF}{dt}$ is bounded in $L^r(0, T; Y)$ $(r > 1)$, then F is relatively compact in $C(0, T; B)$.

Lemma 1.4 (Gronwall's inequality [14]). *Let $u(t)$ be a non-negative locally absolutely continuous function defined on $[t_0, \infty)$, satisfying*

$$u(t) \leq c + \int_{t_0}^{t} [a(s)u(s) + b(s)u^p(s)]\mathrm{d}s,$$

where $c \geq 0$ and $p \geq 0$ are constants, and $a(t), b(t)$ are non-negative continuous functions on $[t_0, \infty)$.

(1) When $0 \leq p < 1$, for any $t \geq t_0$, we have

$$u(t) \leq \left(c^{1-p} e^{(1-p)\int_{t_0}^{t} a(s)\mathrm{d}s} + (1-p)\int_{t_0}^{t} b(s) e^{(1-p)\int_{s}^{t} a(\tau)\mathrm{d}\tau}\mathrm{d}s\right)^{\frac{1}{1-p}};$$

(2) When $p = 1$, for any $t \geq t_0$, we have

$$u(t) \leq c e^{\int_{t_0}^{t}(a(s)+b(s))ds};$$

(3) When $p > 1$ and $h > 0$ exist such that

$$c < \left((p-1)\int_{t_0}^{t_0+h} b(s) e^{-(1-p)\int_{t_0}^{s} a(\tau)\mathrm{d}\tau}\mathrm{d}s\right)^{-\frac{1}{p-1}},$$

then for any $t \in [t_0, t_0 + h]$, we have

$$u(t) \leq \left(c^{1-p} e^{(1-p)\int_{t_0}^{t} a(s)\mathrm{d}s} - (p-1)\int_{t_0}^{t} b(s) e^{(1-p)\int_{s}^{t} a(\tau)\mathrm{d}\tau}\mathrm{d}s\right)^{-\frac{1}{p-1}}.$$

Lemma 1.5 (Nakao inequality [8,22]). *Let $\phi(t)$ be a non-negative function defined on $\mathbb{R}^+$, and suppose*

$$\sup_{s\in[t,t+1]} \phi(s)^{1+\alpha} \leq C_0(1+t)^\gamma(\phi(t) - \phi(t+1)),$$

where $C_0 > 0, \alpha \geq 0$, and $0 \leq \gamma \leq 1$ are constants. Then the following conclusions hold:

(1) When $\alpha > 0$ and $\gamma = 1$, we have $\phi(t) \leq C_1(\log(1+t))^{-\frac{1}{\alpha}}$;

(2) When $\alpha > 0$ and $0 \leq \gamma < 1$, we have $\phi(t) \leq C_2(1+t)^{-\frac{1-\gamma}{\alpha}}$;

(3) When $\alpha = 0$ and $\gamma = 1$, we have $\phi(t) \leq C_3(2+t)^{-\frac{1}{C_0+1}}$;

(4) When $\alpha = 0$ and $0 \leq \gamma < 1$, we have $\phi(t) \leq C_4 \exp\left\{-\frac{1}{(C_0+1)(1-\gamma)}(t+2)^{1-\gamma}\right\}$, where the constants $C_i (i = 1, 2, 3, 4)$ depend on $\phi(0)$ and other known constants.

2. Main results

We employ semigroup theory to study the global existence of solutions to Problem (1.1)–(1.3) under the conditions $0 \le r \le \frac{n+2}{n-2}$, $1 \le p \le \frac{n+2}{n-2}$. Furthermore, using Gronwall's inequality, we investigate the uniqueness of solutions to Problem (1.1)–(1.3) under the conditions $0 \le r \le \frac{(n+2)+\sqrt{(n+2)^2+8n}}{2n}$, $1 \le p \le \frac{n+2}{n-2}$. Then, with the aid of Nakao's inequality, we study the exponential decay of energy for solutions to Problem (1.1)–(1.3) under the conditions $1 \le r \le \frac{(n+2)+\sqrt{(n+2)^2+8n}}{2n}$, $1 \le p \le \frac{n+2}{n-2}$. Before presenting the main results of this paper, we introduce the definition of a solution to Problem (1.1)–(1.3).

Definition 2.1. *A function $u = u(x,t)$ is called a weak solution of the initial boundary value Problem* (1.1)–(1.3) *in the domain $\Omega \times (0,T)$ if $u \in L^\infty(0,T;H_0^1(\Omega))$, $u_t \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, and for any test functions $\varphi \in C_0^\infty(\Omega)$, $\theta \in C_0^\infty(0,T)$, the following holds:*

$$\begin{aligned}
&-\int_0^T \theta'(t) \int_\Omega u_t(x,t)\varphi(x)\mathrm{d}x\mathrm{d}t + \int_0^T \theta(t) \int_\Omega \nabla u(x,t) \cdot \nabla\varphi(x)\mathrm{d}x\mathrm{d}t \\
&+ \int_0^T \theta(t) \int_\Omega f(u(x,t))\varphi(x)\mathrm{d}x\mathrm{d}t + \int_0^T \theta(t) \int_\Omega \nabla u_t(x,t) \cdot \nabla\varphi(x)\mathrm{d}x\mathrm{d}t \\
&\qquad + \int_0^T \theta(t) \int_\Omega a(x)g(u_t)\varphi(x)\mathrm{d}x\mathrm{d}t = 0. \qquad (2.1)
\end{aligned}$$

The main result of the paper is shown below.

Theorem 2.2 (Global solutions)**.** *If Assumptions 1.1 and 1.2 hold, and under the condition that the order of g is r, $0 \le r \le \frac{n+2}{n-2}$, and $1 \le p \le \frac{n+2}{n-2}$ $(n \ge 3)$, for any given $\{u_0, u_1\} \in H_0^1(\Omega) \times L^2(\Omega)$, Problem* (1.1)–(1.3) *admits a global solution $u(x,t)$ satisfying $u \in C(0,T;H_0^1(\Omega))$, $u_t \in C(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, and $u_{tt} \in L^2(0,T;H^{-1}(\Omega))$.*

Next, we establish the uniqueness of solutions to Problem (1.1)–(1.3), which yields the following theorem.

Theorem 2.3 (Uniqueness)**.** *If Assumptions 1.1 and 1.2 hold, and the conditions $0 \le r \le \frac{(n+2)+\sqrt{(n+2)^2+8n}}{2n}$ and $1 \le p \le \frac{n+2}{n-2}$ $(n \ge 3)$ are satisfied, then the global solution u in Theorem 2.1 is unique, and the energy identity holds, i.e.*

$$E_u(t) + \int_0^t \int_\Omega a(x)g(u_t(x,t))u_t(x,t)\mathrm{d}x\mathrm{d}t + \int_0^t \int_\Omega |\nabla u_t|^2\mathrm{d}x\mathrm{d}t = E_u(0), \qquad (2.2)$$

where

$$E_u(t) = \frac{1}{2}\int_\Omega |u_t(x,t)|^2\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2\mathrm{d}x + \int_\Omega F(u_t(x,t))\mathrm{d}x. \qquad (2.3)$$

Remark 2. *The bound $0 < r \le \frac{(n+2)+\sqrt{(n+2)^2+8n}}{2n}$ is narrower than the parameter range derived for existence, and this discrepancy arises from a technical rather than an intrinsic obstacle. To establish the energy identity in the uniqueness argument, we require the regularity condition $a(x)g(u_t) \in L^2(0,T;H^{-1}(\Omega))$ or $a(x)g(u_t) \in L^1(0,T;L^2(\Omega))$, which forces the above tighter restriction on r.*

Furthermore, we strengthen the assumptions on f and g as follows.

Assumption 2.4. *Assume that $f(s)$ and $F(s)$ satisfy Assumption 1.1, and for all $s \in \mathbb{R}$, we have*

$$F(s) \leq f(s)s. \tag{2.4}$$

Assumption 2.5. *The feedback function $g : \mathbb{R} \to \mathbb{R}$ is a monotonically increasing continuous function, $g(0) = 0$, and satisfies the following condition*

$$\hat{m}|s|^{r+1} \leq g(s)s \leq \hat{M}|s|^{r+1}, \quad \forall s \in \mathbb{R}, \tag{2.5}$$

where $\hat{m}$ and $\hat{M}$ are positive constants.

Under the above assumptions, we obtain the energy decay rates for Problem (1.1)–(1.3).

Theorem 2.6. *If Assumptions 2.4 and 2.5 hold, and under the condition that the order of g is r, $1 \leq r \leq \frac{(n+2)+\sqrt{(n+2)^2+8n}}{2n}$ and $1 \leq p \leq \frac{n+2}{n-2}$ $(n \geq 3)$, for any given $\{u_0, u_1\} \in H_0^1(\Omega) \times L^2(\Omega)$, the energy $E_u(t)$ of Problem* (1.1)–(1.3) *satisfies*

$$E_u(t) \leq C_4 \exp\left\{-\frac{1}{C_0+1}(t+2)\right\},$$

where C_0 depends on $\|a\|_{L^\infty}$ and other known constants, while C_4 depends on $E_u(0)$ and other known constants.

3. Global existence of solutions

In this section, we study the global existence of solutions to Problem (1.1)–(1.3) by using semigroup theory under the assumption that $0 \leq r \leq \frac{n+2}{n-2}$ and $1 \leq p \leq \frac{n+2}{n-2}$. In this paper, the semigroup theory approach requires the source term $f(u)$ to satisfy a local Lipschitz condition. However, directly applying semigroup theory to Problem (1.1)–(1.3) does not yield the required conditions on $f(u)$. To overcome this difficulty, we first study the approximate problem obtained by truncating Problem (1.1)–(1.3):

$$u_{ktt} - \Delta u_k - \Delta u_{kt} + f_k(u_k) + a(x)g(u_{kt}) = 0, (x,t) \in \Omega \times (0, +\infty), \tag{3.1}$$

$$u_k(x,t) = 0, x \in \partial\Omega, \tag{3.2}$$

$$u_k(x,0) = u_{0,k}(x), u_{kt}(x,0) = u_{1,k}(x). \tag{3.3}$$

Here, $\{u_{0,k}, u_{1,k}\}$ are initial data with good regularity, strongly converging to (u_0, u_1) in $H = H_0^1(\Omega) \times L^2(\Omega)$. Moreover, for each $k \in \mathbb{N}$, $f_k : \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$f_k(s) := \begin{cases} f(s), & |s| \leq k, \\ f(k), & s > k, \\ f(-k), & s < -k. \end{cases} \tag{3.4}$$

Rewrite the problem in abstract form. We define the unbounded operator $A : \mathcal{D}(A) \subset H \to H$ in the phase space H as

$$A = \begin{pmatrix} 0 & -I \\ -\Delta & -\Delta(\cdot) + a(x)g(\cdot) \end{pmatrix},$$

with the domain

$$\mathcal{D}(A)=\left\{(u,v)\in H:\begin{array}{c}v\in H_0^1(\Omega)\\ -\Delta u-\Delta v+a(x)g(v)\in L^2(\Omega)\\ a(x)g(v)\in L^1(\Omega)\cap H^{-1}(\Omega)\end{array}\right\}. \tag{3.5}$$

Define the nonlinear operator $B_k : H \to H$ as

$$B_k(u,v)^T=(0,f_k(u))^T. \tag{3.6}$$

Let $W_k(t)=(u_k,u_{kt})$, then Problem (3.1)–(3.3) can be rewritten as a Cauchy problem on H:

$$\begin{cases}\frac{\partial W_k(t)}{\partial t}+AW_k(t)+B_k(W_k(t))=0,\\ W(0)=(u_{0,k},u_{1,k}).\end{cases} \tag{3.7}$$

Thus, we transform the study of the global existence of solutions to Problem (1.1)–(1.3) into first studying the global existence of solutions to the abstract Problem (3.7) and then obtaining the global existence of solutions to Problem (1.1)–(1.3) through a limit process. Before studying the global existence of solutions to Problem (3.7), we present the following propositions.

Proposition 3.1. *The domain $\mathcal{D}(A)$ of the operator A is a subset of H and dense in H.*

Proof. Take any $(u,v)\in\mathcal{D}(A)$. According to the definition of $\mathcal{D}(A)$, we have $(u,v)\in H$. Therefore, $\mathcal{D}(A)\subset H$.

Next, we prove that $\mathcal{D}(A)$ is dense in H. Take any $(u,v)\in C_c^\infty(\Omega)\times C_c^\infty(\Omega)$. Moreover, as $C_c^\infty(\Omega)\subset H_0^1(\Omega)$, it follows that $v\in H_0^1(\Omega)$. Since $a(x)\in L^\infty(\Omega)$ and g is a continuous function, we have $a(x)g(v)\in L^2(\Omega)$ and $a(x)g(v)\in L^1(\Omega)\cap H^{-1}(\Omega)$. Combining this with $\Delta u\in C_c^\infty(\Omega)$, $\Delta v\in C_c^\infty(\Omega)$, we obtain $-\Delta u-\Delta v+a(x)g(v)\in L^2(\Omega)$. According to the definition of $\mathcal{D}(A)$, it follows that $(u,v)\in\mathcal{D}(A)$. Thus, we have $C_c^\infty(\Omega)\times C_c^\infty(\Omega)\subset\mathcal{D}(A)\subset H$. From $C_c^\infty(\Omega)\times C_c^\infty(\Omega)$ is dense in H, $\mathcal{D}(A)$ is dense in H. □

Proposition 3.2. *For each $k\in\mathbb{N}$, let $C_k>0$ be a constant such that the function f_k satisfies*

$$|f_k(r)-f_k(s)|\le C_k|r-s|,\quad \forall r,s\in\mathbb{R}.$$

Remark 3. *The proof of this proposition only requires a simple calculation based on Eq* (1.10).

According to Proposition 3.1, $\mathcal{D}(A)$ is dense in H. Consequently, for any $(u_0,u_1)\in H$, a sequence $\{u_{0,k},u_{1,k}\}$ in $\mathcal{D}(A)$ exists such that

$$\{u_{0,k},u_{1,k}\}\to\{u_0,u_1\}\quad\text{(strong convergence)}.$$

Therefore, we may take $\{u_{0,k},u_{1,k}\}\in\mathcal{D}(A)$ in the approximate Problem (3.1), satisfying the aforementioned strong convergence property.

We now begin to study the global existence of solutions to Problem (1.1) under the conditions $0\le r\le\frac{n+2}{n-2}$ and $1\le p\le\frac{n+2}{n-2}$. To make the structure of the article clearer, we divide it into three subsections: Subsection 3.1 studies the existence of solutions to the approximate Problem (3.7); and Subsection 3.2 studies the global nature of the solutions to the approximate Problem (3.7); Subsection 3.3 studies the global existence of solutions to Problem (1.1). Since Problem (3.7) is an abstract form of Problem (3.1), Sections 1 and 2 also study the relevant properties of the solutions to Problem (3.1).

3.1. Existence of approximate solutions

This subsection primarily studies the local existence of solutions to the approximate Problem (3.1), we obtain the following conclusion.

Theorem 3.1 (Local existence of approximate solutions)**.** *If Assumptions 1.1 and 1.2 hold, and the order r of g satisfies $0 \leq r \leq \frac{n+2}{n-2}$ ($n \geq 3$), then the approximate problem* (3.1) *has a unique strong solution $(u_k, u_{k,t}) \in W^{1,\infty}(0,T;H)$, and for $t > 0$, $(u_k, u_{k,t}) \in \mathcal{D}(A)$. Here, $(0,T)$ is the maximal interval of existence of the solution.*

Lemma 3.1. *[11] Let $A : \mathcal{D}(A) \subset H \to H$ be a maximal monotone operator on a Hilbert space H with $0 \in A0$. Let $B : H \to H$ be a locally Lipschitz operator, i.e., $\|Bu - Bv\| \leq L(K)\|u - v\|$ provided $\|u\|, \|v\| \leq K$. Assume $u_0 \in \mathcal{D}(A)$, and $f \in W^{1,1}(0,t;H)$ for all $t > 0$. Then, for $T \leq \infty$, the initial value problem*

$$u_t + Au + Bu \ni f \quad \textit{and} \quad u(0) = u_0 \in H, \tag{3.8}$$

has a unique weak solution $u \in C(0,T;H)$. Furthermore, if $u_0 \in \overline{\mathcal{D}(A)}$, the Problem (3.8) *has a strong solution $u \in W^{1,\infty}(0,T;H)$ on the interval* $[0,T]$*, and for $t > 0$, $u(t) \in \mathcal{D}(A)$.*

Lemma 3.2. *[4] Let V be a reflexive Banach space, and let Y be a monotone lower semicontinuous operator from V to V^*. If W is a maximal monotone operator from $K \subset V$ to V^*, then $Y + W$ is also a maximal monotone operator from V to V^*. Moreover, if $Y + W$ is coercive, then the range satisfies $R(Y + W) = V^*$.*

We establish Theorem 3.1 on the basis of Lemma 3.1, and thus it is essential to verify all assumptions of this lemma. Specifically, we confirm that operator A is maximal monotone and each B_k is locally Lipschitz. Combined with the condition $0 \in W^{1,1}(0,t;H)$, we further derive the existence of local solutions. Prior to the main proof, we first introduce the following proposition.

Proposition 3.3. *If Assumptions 1.1 and 1.2 hold, then the operator A is maximal monotone.*

Proof. We define the operator W:

$$\begin{aligned} W : H_0^1(\Omega) &\longrightarrow H^{-1}(\Omega), \\ v &\longmapsto -\Delta v + a(x)g(v), \end{aligned}$$

and the operation

$$\langle Wv, u\rangle = \int_\Omega \nabla v(x)\nabla u(x) + a(x)g(v(x))u(x)\mathrm{d}x, \quad \forall u, v \in H_0^1(\Omega).$$

Then, the operator A can be rewritten as:

$$A = \begin{pmatrix} 0 & -I \\ -\Delta & W \end{pmatrix}.$$

The key to proving the maximality of A is to prove that W is a maximal monotone operator. For convenience, we divide the proof that W is maximal monotone into three steps: (1) W is well-defined; (2) W satisfies monotonicity; (3) W satisfies maximality.

(1) W is well-defined. For any $u, v \in H_0^1(\Omega)$, we have

$$\begin{aligned}|\langle Wv, u\rangle| &= \left|\int_\Omega \nabla v(x)\nabla u(x) + a(x)g(v(x))u(x)\mathrm{d}x\right| \\ &\le \int_\Omega |\nabla v(x)||\nabla u(x)|\mathrm{d}x + \int_\Omega |a(x)||g(v(x))||u(x)|\mathrm{d}x.\end{aligned}$$

Divide Ω into two parts Ω_0 and Ω_1, where

$$\Omega_0 = \{x \in \Omega : |v(x)| \le 1\}, \quad \Omega_1 = \{x \in \Omega : |v(x)| > 1\}.$$

Therefore, we have

$$\begin{aligned}&\int_\Omega |\nabla v(x)||\nabla u(x)|\mathrm{d}x + \int_\Omega |a(x)||g(v(x))||u(x)|\mathrm{d}x \\ \le&\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \|a(x)\|_{L^\infty(\Omega)}\left[\int_{\Omega_0} |g(v(x))||u(x)|\mathrm{d}x + \int_{\Omega_1} |g(v(x))||u(x)|\mathrm{d}x\right].\end{aligned}$$

According to Assumption 1.2, $g(s)$ has a maximum on $|s| \le 1$, denoted by M_g. Consequently, we obtain

$$\begin{aligned}|\langle Wv, u\rangle| \le& \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + M_g\|a(x)\|_{L^\infty(\Omega)}\int_{\Omega_0} |u(x)|\mathrm{d}x \\ &+ \tilde{M}\|a(x)\|_{L^\infty(\Omega)}\int_{\Omega_1} |v(x)|^r|u(x)|\mathrm{d}x \\ \le& \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + M_g(\mathrm{meas}(\Omega_0))^{\frac{1}{2}}\|a(x)\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)} \\ &+ \tilde{M}\|a(x)\|_{L^\infty(\Omega)}\|v\|^r_{L^{\frac{2n}{n-2}}(\Omega)}\|u\|_{L^{\frac{2n}{n-2}}(\Omega)}.\end{aligned}$$

Combining the Sobolev embedding theorem with the range of the exponent r, we know that $H_0^1(\Omega) \hookrightarrow L^{\frac{2nr}{n+2}}(\Omega)$. Therefore, $|\langle Wv, u\rangle|$ is bounded. This completes the proof of well-definedness.

(2) W is a monotone operator. For any $u, v \in H_0^1(\Omega)$, we have

$$\langle Wu - Wv, u - v\rangle = \int_\Omega |\nabla(u - v)|^2 + a(x)(g(u) - g(v))(u - v)\mathrm{d}x. \tag{3.9}$$

According to Assumptions 1.2, we obtain the monotonicity of W.

(3) W is maximal. We first prove that W is the Gateaux derivative of some functional J. Define the functional $J : H_0^1(\Omega) \longrightarrow \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\mathrm{d}x + \int_\Omega\int_0^{u(x)} a(x)g(s)\mathrm{d}s\mathrm{d}x, \quad \forall u \in H_0^1(\Omega).$$

For convenience, we divide the proof into three steps: (3a) Prove that W is the Gateaux derivative of the functional J at any $u \in H_0^1(\Omega)$; (3b) Prove that W equals the subdifferential of J; (3c) Prove that W is a maximal monotone operator.

(3a) W denotes the Gateaux derivative of the functional J at any $u \in H_0^1(\Omega)$. Indeed, if we take any $u \in H_0^1(\Omega)$, for any $v \in H_0^1(\Omega)$, the directional derivative of J at u in the direction v (denoted $J'(u, v)$) is:

$$J'(u, v) = \lim_{t\to 0}\frac{J(u + tv) - J(u)}{t}$$

$$
\begin{aligned}
&= \lim_{t\to 0}\frac{\frac{1}{2}\int_\Omega |\nabla u + t\nabla v|^2 - |\nabla u|^2 \mathrm{d}x}{t} + \lim_{t\to 0}\frac{\int_\Omega \int_{u(x)}^{u(x)+tv(x)} a(x)g(s)\mathrm{d}s\mathrm{d}x}{t}\\
&= \frac{1}{2}\lim_{t\to 0}\frac{\int_\Omega (2t\nabla u\cdot\nabla v + t^2|\nabla u|^2)\mathrm{d}x}{t} + \lim_{t\to 0}\frac{\int_\Omega a(x)\frac{1}{t}\int_{u(x)}^{u(x)+tv(x)} g(s)\mathrm{d}s\mathrm{d}x}{t}\\
&= \int_\Omega \nabla u\cdot\nabla v\mathrm{d}x + \lim_{t\to 0}\int_\Omega a(x)\frac{1}{t}\int_{u(x)}^{u(x)+tv(x)} g(s)\mathrm{d}s\mathrm{d}x. \qquad (3.10)
\end{aligned}
$$

Since $a(x)g(v) \in L^1(\Omega)$, by the fundamental theorem of calculus, we obtain

$$
\lim_{t\to 0}\int_\Omega a(x)\frac{1}{t}\int_{u(x)}^{u(x)+tv(x)} g(s)\mathrm{d}s\mathrm{d}x = \lim_{t\to 0}\int_\Omega a(x)v(x)g(u+\alpha tv)\mathrm{d}x, \qquad (3.11)
$$

where $0 \le \alpha(x) = \alpha(x, u(x), tv(x)) \le 1$. On the one hand, according to the continuity assumption on g and $a(x) \in L^\infty(\Omega)$, we obtain that $a(x)g(u+\alpha tv)$ converges for almost all $x \in \Omega$. On the other hand, we will prove that $a(x)g(u+\alpha tv)$ can be dominated by an integrable function independent of t. Indeed, according to Assumption 1.2, for any $s \in \mathbb{R}$, we have

$$
g(s) \le M_g + \tilde{M}|s|^r. \qquad (3.12)
$$

Therefore,

$$
\begin{aligned}
|a(x)v(x)g(u(x)+\alpha tv(x))| \le& |a(x)||v(x)||g(u(x)+\alpha tv(x))|\\
\le& |a(x)||v(x)|\left(M_g + \tilde{M}|u(x)+\alpha tv(x)|^r\right)\\
\le& M_g|a(x)||v(x)| + |a(x)||v(x)|\tilde{M}|u(x)+\alpha tv(x)|^r.
\end{aligned}
$$

By the basic inequality, for some constant $C > 0$, we have

$$
|u(x)+\alpha tv(x)|^r \le C(|u(x)|^r + |\alpha tv(x)|^r).
$$

Therefore, when $|t| \le 1$ and $0 \le \alpha \le 1$, we obtain

$$
|a(x)v(x)g(u(x)+\alpha tv(x))| \le M_g|a(x)||v(x)| + C\tilde{M}|a(x)||v(x)|(|u(x)|^r + |v(x)|^r). \qquad (3.13)
$$

Since $a(x) \in L^\infty(\Omega)$, and setting $0 \le r \le \frac{n+2}{n-2}$ with the Sobolev embedding theorem, we know that $H_0^1(\Omega) \hookrightarrow L^{r+1}(\Omega)$. By applying Hölder's inequality to (3.13), we conclude that $a(x)g(u+\alpha tv)$ can be dominated by an integrable function independent of t. Hence, by the Lebesgue dominated convergence theorem, we have

$$
\lim_{t\to 0}\int_\Omega a(x)v(x)g(u+\alpha tv)dx = \int_\Omega a(x)v(x)g(u)\mathrm{d}x. \qquad (3.14)
$$

Combining (3.10) and (3.14), we obtain

$$
J'(u, v) = \int_\Omega \nabla u\cdot\nabla v\mathrm{d}x + \int_\Omega a(x)v(x)g(u)\mathrm{d}x.
$$

That is, J is Gateaux differentiable at u, and the directional derivative in the direction v is $J'(u)$. Moreover, for any $u, v \in H_0^1(\Omega)$, we have

$$\langle Wu, v\rangle_{H^{-1}\times H_0^1} = \langle J'u, v\rangle_{H^{-1}\times H_0^1} = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x + \int_\Omega a(x)g(u)v(x)\mathrm{d}x.$$

Therefore, it follows that W is the Gateaux derivative of J at u, i.e., $W(u) = J'(u)$.

(3b) W equals the subdifferential of J. According to the monotonicity of W, for any $u, v \in H_0^1(\Omega)$, we have

$$\langle J'(u) - J'(v), u - v\rangle = \langle W(u) - W(v), u - v\rangle \geq 0.$$

From (3.1) and equivalent conditions for differentiability of [27], we deduce that J is a convex function and for any $u \in H_0^1(\Omega)$,

$$\partial(J(u)) = J'(u) = W(u).$$

(3c) W is a maximal monotone operator. Take any sequence $\{u_n\} \subset H_0^1(\Omega)$ and $u \in H_0^1(\Omega)$ such that u_n converges strongly to u in $H_0^1(\Omega)$. Using (3.12), we obtain

$$\begin{aligned}
|J(u_n) - J(u)| &\leq \frac{1}{2}\int_\Omega \left(|\nabla u_n|^2 - |\nabla u|^2\right)\mathrm{d}x + \left|\int_\Omega a(x)\left(\int_0^{u_n(x)} g(s)\mathrm{d}s - \int_0^{u(x)} g(s)\mathrm{d}s\right)\mathrm{d}x\right| \\
&\leq \frac{1}{2}\int_\Omega |\nabla u_n + \nabla u||\nabla u_n - \nabla u|\mathrm{d}x + M_g\|a\|_{L^\infty(\Omega)}\int_{\Omega_0}|u_n(x) - u(x)|\mathrm{d}x \\
&\quad + \frac{\tilde{M}}{r+1}\|a\|_{L^\infty(\Omega)}\int_{\Omega_1}\left||u_n(x)|^{r+1} - |u(x)|^{r+1}\right|\mathrm{d}x \\
&\leq \frac{1}{2}\|\nabla u_n + \nabla u\|_{L^2(\Omega)}\|\nabla u_n - \nabla u\|_{L^2(\Omega)} + CM_g\|a\|_{L^\infty(\Omega)}\|u_n - u\|_{L^2(\Omega)} \\
&\quad + \frac{\tilde{M}}{r+1}\|a\|_{L^\infty(\Omega)}\int_\Omega \left(|u_n(x)|^r + |u(x)|^r\right)|u_n(x) - u(x)|\mathrm{d}x \\
&\leq \frac{1}{2}\|\nabla u_n + \nabla u\|_{L^2(\Omega)}\|\nabla u_n - \nabla u\|_{L^2(\Omega)} + CM_g\|a\|_{L^\infty(\Omega)}\|u_n - u\|_{L^2(\Omega)} \\
&\quad + \frac{\tilde{M}}{r+1}\|a\|_{L^\infty(\Omega)}\left(\|u_n\|^r_{L^{\frac{2n}{n-2}}(\Omega)} + \|u\|^r_{L^{\frac{2n}{n-2}}(\Omega)}\right)\|u_n - u\|_{L^{\frac{2n}{n-2}}(\Omega)}.
\end{aligned}$$

Since u_n converges strongly to u in $H_0^1(\Omega)$, there is a constant $M > 0$ such that for all positive integers n, $\|u_n\|_{H_0^1(\Omega)} < M$. Therefore

$$\|\nabla u_n + \nabla u\|_{L^2(\Omega)} < 2M, \quad \|u_n\|_{H_0^1(\Omega)} + \|u\|_{H_0^1(\Omega)} < 2M.$$

According to the range of r and the Sobolev embedding theorem, we have that $\|u_n\|^r_{L^{\frac{2nr}{n+2}}(\Omega)} + \|u\|^r_{L^{\frac{2nr}{n+2}}(\Omega)}$ is uniformly bounded. It follows that J is continuous. We conclude that W is a maximal monotone operator defined on $H_0^1(\Omega)$. From this property, we prove that A is a maximal monotone operator. For any $u, v \in H_0^1(\Omega)$, we have

$$\begin{aligned}
&\left(A\begin{pmatrix}u_1\\v_1\end{pmatrix}-A\begin{pmatrix}u_2\\v_2\end{pmatrix},\begin{pmatrix}u_1\\v_1\end{pmatrix}-\begin{pmatrix}u_2\\v_2\end{pmatrix}\right)_H\\
=&\left(\begin{pmatrix}-(v_1-v_2)\\-\Delta(u_1-u_2)+W(v_1-v_2)\end{pmatrix},\begin{pmatrix}u_1-u_2\\v_1-v_2\end{pmatrix}\right)_H\\
=&(-(v_1-v_2),u_1-u_2)_{H_0^1(\Omega)}+(-\Delta(u_1-u_2)+W(v_1-v_2),v_1-v_2)_{L^2(\Omega)}\\
=&\int_\Omega -\nabla(v_1-v_2)\cdot\nabla(u_1-u_2)dx\\
&\quad+\int_\Omega(\nabla(u_1-u_2)\cdot\nabla(v_1-v_2)+(Wv_1-Wv_2)(v_1-v_2))\,dx\\
=&\int_\Omega(Wv_1-Wv_2)(v_1-v_2)dx.
\end{aligned}$$

From the monotonicity of W, we obtain the monotonicity of A.

Next, we prove the maximality of the operator A. For the Minty theorem [20], it suffices to verify the mapping $I+A$ is surjective from $\mathcal{D}(A)\subset H$ to H. For any given $\{u_0, v_0\}\in H$ and $\{u, v\}\in\mathcal{D}(A)\subset H$, we have

$$(I+A)(u,v)^T=(u_0,v_0)^T,$$

i.e.,

$$\begin{cases}u-v=u_0,\\-\Delta u+v+Wv=v_0.\end{cases}$$

By performing a variable substitution on the equations above, we obtain

$$-\Delta v+v+Wv=v_0+\Delta u_0. \tag{3.15}$$

Therefore, proving that the mapping $I+A$ is surjective from $D(A)\subset H$ to H and is transformed into proving that for any given $\{u_0, v_0\}\in H$ and $v\in H_0^1(\Omega)$, Eq (3.15) holds.

We now prove that the operator $-\Delta+I$ is monotone and lower semicontinuous. Obviously, the linear operator $-\Delta+I$ is continuous. Next, we prove that the operator $-\Delta+I$ is also monotone. For any $v_1, v_2\in H_0^1(\Omega)$, we have

$$\begin{aligned}
\langle(-\Delta+I)v_1-(-\Delta+I)v_2,v_1-v_2\rangle=&\langle-(\Delta v_1-\Delta v_2)+(v_1-v_2),v_1-v_2\rangle\\
=&\int_\Omega(|\nabla(v_1-v_2)|^2+|v_1-v_2|^2)\mathrm{d}x\geq 0.
\end{aligned}$$

That is, the operator $-\Delta+I$ is monotone. Combining this with the fact that W is a maximal monotone operator on $H_0^1(\Omega)$, and using Lemma 3.2, we obtain that the operator $-\Delta+I+W$ is also a maximal monotone operator on $H_0^1(\Omega)$.

Now, we prove that the operator $-\Delta+I+W$ is coercive. For any $v\in H_0^1(\Omega)$, we have

$$\begin{aligned}
\langle(-\Delta+I+W)v,v\rangle=&\langle(-\Delta+I)v,v\rangle+\langle Wv,v\rangle\\
\geq&\int_\Omega(|\nabla v|^2+|v|^2)\mathrm{d}x\geq\int_\Omega|\nabla v|^2\mathrm{d}x.
\end{aligned}$$

It follows that the operator $-\Delta+I+W$ is coercive. Therefore, by Lemma 3.2 again, we conclude that Eq (3.15) has a solution $v \in H_0^1(\Omega)$, i.e., the existence of v is proved. From $u - v = u_0$ and $u_0 \in H_0^1(\Omega)$, we obtain the existence of u and that $u \in H_0^1(\Omega)$. Noting that

$$-\Delta u + Wv = v_0 - v \in L^2(\Omega),$$

we have shown that for any given $\{u_0, v_0\} \in H$, the equation $(I + A)(u, v)^T = (u_0, v_0)^T$ has a solution $u, v \in \mathcal{D}(A)$, i.e., $R(I + A) = H$. Applying the Minty theorem [20], the A is a maximal monotone operator. □

Proposition 3.4. *If Assumption 1.1 holds, then the operator B_k is locally Lipschitz.*

Proof. Take any $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \in H, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \in H$. According to Proposition 3.2, we have

$$\begin{aligned} \|f_k(u_1) - f_k(u_2)\|^2_{L^2(\Omega)} &= \int_\Omega |f_k(u_1) - f_k(u_2)|^2 \mathrm{d}x \\ &\leq C_k \int_\Omega |u_1 - u_2|^2 \mathrm{d}x \\ &\leq C_k \|u_1 - u_2\|^2_{H_0^1(\Omega)}. \end{aligned}$$

According to the definition of B_k, we obtain

$$\left\| B_k \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - B_k \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\|_H^2 \leq C_k \left\| \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\|_H^2.$$

This completes the proof that the operator B_k is locally Lipschitz. □

We now proceed to prove Theorem 3.1.

Suppose that Assumptions 1.1 and 1.2 are satisfied. It follows from Proposition 3.3 that A is a maximal monotone operator, and Proposition 3.4 implies that each B_k is locally Lipschitz. Furthermore, in view of $0 \in W^{1,1}(0, t; H)$, we deduce from Lemma 3.1 that the approximate Problem (3.1) admits a unique solution $(u_k, u_{k,t}) \in W^{1,\infty}(0, T; H)$ satisfying $(u_k, u_{k,t}) \in \mathcal{D}(A)$ for any $t > 0$. This completes the proof of Theorem 3.1.

3.2. Global existence of approximate solutions

This subsection is devoted to investigating the global existence of solutions to the approximate Problem (3.1) established in Subsection 3.1, and we thereby state the following theorem.

Theorem 3.2. *If Assumptions 1.1 and 1.2 hold, and the order r of g satisfies $0 \leq r \leq \frac{n+2}{n-2}$ ($n \geq 3$), then the strong solution $\{u_k, u_{k,t}\}$ obtained in Theorem 3.1 is global, i.e., the existence time T can be arbitrarily large.*

Before presenting the proof, we provide a criterion for determining whether a solution is global.

Lemma 3.3. *Suppose that $u_k(x, t)$ is the weak solution (strong solution) obtained from Eq (3.2), and let $[0, T)$ is the maximal interval of existence of the solution. If $T_0 \in (0, T)$, then $u_k \in C(0, T_0; H)$ ($u_k \in W^{1,\infty}(0, T_0; H)$). When $T < \infty$, then $\lim_{t \to T^-} \|u_k(t)\| = \infty$. When $T = \infty$, we call $u_k(x, t)$ a global weak solution (strong solution).*

Below, we use Lemma 3.3 to prove Theorem 3.2, i.e., we prove that $\|u_k(t)\|_H$ is uniformly bounded.

Proof. From Proposition 3.2, we have

$$\int_0^T \|f_k(u_k)\|_{L^{p+1}(\Omega)}^{p+1}\mathrm{d}t \le C_k \int_0^T \int_\Omega |u_k|^{p+1}\mathrm{d}x\mathrm{d}t.$$

By the Sobolev embedding theorem, we obtain $f_k(u_k) \in L^{p+1}(0,T;L^{p+1}(\Omega))$.

Next, we estimate the properties of $a(x)g(u_{kt})$. When $0 \le r \le 1$, using the basic inequality, we have

$$\begin{aligned}
&\|a(x)g(u_{kt})\|_{L^2(0,T;L^2(\Omega))}\\
\le& C\|a\|_{L^\infty(\Omega)}\left(\int_0^T\int_{\{x:|u_{kt}(x,t)|\le 1\}}|g(u_{kt}(x,t))|^2dx\mathrm{d}t\right)^{\frac{1}{2}}\\
&+C\|a\|_{L^\infty(\Omega)}\left(\int_0^T\int_{\{x:|u_{kt}(x,t)|>1\}}|g(u_{kt}(x,t))|^2dx\mathrm{d}t\right)^{\frac{1}{2}}\\
\le& C\|a\|_{L^\infty(\Omega)}TM_g\mathrm{meas}(\Omega_0)^{\frac{1}{2}}+C\tilde{M}\|a\|_{L^\infty(\Omega)}\int_0^T\|u_{kt}(t)\|_{L^{2r}(\Omega)}^{2r}\mathrm{d}t\\
\le& C\|a\|_{L^\infty(\Omega)}TM_g\mathrm{meas}(\Omega_0)^{\frac{1}{2}}+C\tilde{M}\|a\|_{L^\infty(\Omega)}\int_0^T\|u_{kt}(t)\|_{L^{2}(\Omega)}^{2}\mathrm{d}t.
\end{aligned}\tag{3.16}$$

From Theorem 3.1, we know that $u_{kt} \in L^\infty(0,T;H_0^1(\Omega))$. By the Sobolev embedding theorem, we have $u_{kt} \in L^2(0,T;L^2(\Omega))$. Therefore, $a(x)g(u_{kt}) \in L^2(0,T;L^2(\Omega))$. When $1 < r \le \frac{n+2}{n-2}$, we have

$$\begin{aligned}
&\|a(x)g(u_{k,t})\|_{L^{\frac{r+1}{r}}(0,T;L^{\frac{r+1}{r}}(\Omega))}^{\frac{r+1}{r}}\\
=&\int_0^T\int_\Omega|a(x)g(u_{kt})|^{\frac{r+1}{r}}\mathrm{d}x\mathrm{d}t\\
\le& C\|a\|_{L^\infty(\Omega)}^{\frac{r+1}{r}}\int_0^T\int_{\{x:|u_{kt}(x,t)|\le 1\}}|g(u_{kt}(x,t))|^{\frac{r+1}{r}}\mathrm{d}x\mathrm{d}t\\
&+C\|a\|_{L^\infty(\Omega)}^{\frac{r+1}{r}}\int_0^T\int_{\{x:|u_{k,t}(x,t)|>1\}}|g(u_{kt}(x,t))|^{\frac{r+1}{r}}\mathrm{d}x\mathrm{d}t\\
\le& C\|a\|_{L^\infty(\Omega)}^{\frac{r+1}{r}}TM_g^{\frac{r+1}{r}}\mathrm{meas}(\Omega)+C\tilde{M}^{\frac{r+1}{r}}\|a\|_{L^\infty(\Omega)}^{\frac{r+1}{r}}\int_0^T\int_\Omega|u_{kt}(x,t)|^{r+1}\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{3.17}$$

By $L^\infty(0,T;H_0^1(\Omega)) \hookrightarrow L^{r+1}(0,T;L^{r+1}(\Omega))$, we get $a(x)g(u_{kt}) \in L^{\frac{r+1}{r}}(0,T;L^{\frac{r+1}{r}}(\Omega))$. Since $L^{\frac{r+1}{r}}(0,T;L^{\frac{r+1}{r}}(\Omega)) \hookrightarrow L^{\frac{r+1}{r}}(0,T;H^{-1}(\Omega))$, it follows that $a(x)g(u_{kt}) \in L^{\frac{r+1}{r}}(0,T;H^{-1}(\Omega))$.

When $0 \le r \le \frac{n+2}{n-2}$, considering that $\Delta u_{kt} \in L^{\frac{r+1}{r}}(0,T;H^{-1}(\Omega))$, $f_k(u_k) \in L^\infty(0,T;L^{p+1}(\Omega)) \hookrightarrow L^{\frac{r+1}{r}}(0,T;H^{-1}(\Omega))$, and $0 \in L^1(0,T;L^2(\Omega))$, we obtain that $\Delta u_{kt} - f_k(u_k) - a(x)g(u_{kt}) \in L^1(0,T;L^2(\Omega)) + L^{\frac{r+1}{r}}(0,T;H^{-1}(\Omega))$. Moreover, since $u_k \in L^\infty(0,T;H_0^1(\Omega))$, $u_{kt} \in L^\infty(0,T;L^2(\Omega)) \cap L^{r+1}(0,T;H_0^1(\Omega))$ and by Lemma 1.1, for any $t > 0$, the following holds:

$$\begin{aligned}
&\|u_{kt}(t)\|_{L^2(\Omega)}^2+\|\nabla u_k(t)\|_{L^2(\Omega)}^2\\
=&\|u_{1,k}\|_{L^2(\Omega)}^2+\|\nabla u_{0,k}\|_{L^2(\Omega)}^2+2\int_0^t\langle\Delta u_{kt}-f_k(u_k)-a(x)g(u_{kt}),u_{kt}\rangle_{H^{-1}\times H_0^1}\mathrm{d}t.
\end{aligned}$$

We rewrite it as

$$
\begin{aligned}
&||u_{k,t}(t)||^2_{L^2(\Omega)} + ||\nabla u_k(t)||^2_{L^2(\Omega)} + 2\int_\Omega F_k(u_k)\mathrm{d}x + 2\int_0^t\int_\Omega |\nabla u_{kt}|^2\mathrm{d}x\mathrm{d}t \\
=&||\nabla u_{0,k}||^2_{L^2(\Omega)} + ||u_{1,k}||^2_{L^2(\Omega)} + 2\int_\Omega F_k(u_{0,k})\mathrm{d}x - 2\int_0^t\int_\Omega a(x)g(u_{kt})u_{kt}\mathrm{d}x\mathrm{d}t \\
\le&||\nabla u_{0,k}||^2_{L^2(\Omega)} + ||u_{1,k}||^2_{L^2(\Omega)} + 2\int_\Omega F_k(u_{0,k})\mathrm{d}x. \qquad (3.18)
\end{aligned}
$$

Next, we prove that the right-hand side of Inequality (3.18) is uniformly bounded. Since $\{u_{0,k}, u_{1,k}\}$ converges strongly to $\{u_0, u_1\}$ in H, for some constant $M > 0$,

$$
||u_{0,k}||_{H^1_0(\Omega)} < M, \quad ||u_{1,k}||_{L^2(\Omega)} < M.
$$

According to the definition of f_k and $F(\lambda) = \int_0^\lambda f(s)ds$, the expression for F_k is given by

$$
F_k(s) := \begin{cases} \int_0^s f(\xi)d\xi, & |s| \le k, \\ \int_0^k f(\xi)d\xi + f(k)(s-k), & s > k, \\ f(-k)(s+k) + \int_0^{-k} f(\xi)d\xi, & s < -k. \end{cases} \qquad (3.19)
$$

According Remark 1, for any $s \in \mathbb{R}$, we have

$$
|f(s)| \le c(|s| + |s|^p). \qquad (3.20)
$$

Combining (3.20) and (3.19), for any $s \in \mathbb{R}$ and $k \in \mathbb{N}$, we have

$$
|F_k(s)| \le c(|s|^2 + |s|^{p+1}),
$$

and thus

$$
\int_\Omega |F_k(u_{0,k})|dx \le c\int_\Omega (|u_{0,k}|^2 + |u_{0,k}|^{p+1})\mathrm{d}x. \qquad (3.21)
$$

Moreover, since $u_{0,k}$ converges strongly to u_0 in $H^1_0(\Omega)$, it follows that $||u_{0,k}||_{H^1_0(\Omega)}$ is uniformly bounded. According to the Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^{p+1}(\Omega)$ and $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$, we obtain that $\int_\Omega F_k(u_{0,k})\mathrm{d}x$ is uniformly bounded. This shows that the right-hand side of Inequality (3.18) is uniformly bounded. Furthermore, according to Assumption 1.1, $F_k(u_k)$ is non-negative. We conclude that $||u_{kt}(t)||^2_{L^2(\Omega)} + ||\nabla u_k(t)||^2_{L^2(\Omega)}$ is uniformly bounded. □

3.3. Global existence of weak solutions

This subsection mainly studies the global existence of weak solutions to Problem (1.1)–(1.3). From Theorem 3.2, we obtain the energy equality of the approximate solutions holds. Then, from the uniform boundedness of $E_{u_k}(0)$, we get the uniform boundedness of $E_{u_k}(t)$. On the basis of the fact that bounded sequences have convergent subsequences, we obtain the convergence of $u_k(t), u_{kt}(t)$, and $a(x)g(u_{kt})$. Then, using Lemma 1.2, we obtain the convergence of $f_k(u_k)$, verify the initial conditions, and finally prove that $a(x)g(u_{kt})$ converges to $a(x)g(u_t)$.

Lemma 3.4. *([29]) Suppose the function $u(x,t)$ satisfies the conditions of Theorem 2.2, and That the real numbers $C > 0$ and γ exist such that*

$$\langle -\Delta u, u\rangle_{H_0^1(\Omega)} + \gamma\|u\|^2_{L^2(\Omega)} \geq C\|u\|^2_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega). \tag{3.22}$$

Then the function in Theorem 2.2 satisfies

$$u \in C([0,T]; H_0^1(\Omega)), \quad u_t \in C([0,T]; L^2(\Omega)).$$

We now present the detailed proof of Theorem 2.2 concerning the global existence of solutions.

Proof. According to Theorem 3.2, we have

$$u_k \in L^\infty(\mathbb{R}^+; H_0^1(\Omega)), \quad u_{kt} \in L^\infty(\mathbb{R}^+; L^2(\Omega)).$$

Thus, taking the dual product of the first equation in (3.1) with u_{kt}, we conclude that for any $t \in [0,\infty)$, the following holds:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{kt}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_k(t)\|^2_{L^2(\Omega)} + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F_k(u_k(x,t))\mathrm{d}x + \int_\Omega |\nabla u_{kt}|^2\mathrm{d}x \\ &+ \int_\Omega a(x)g(u_{kt}(x,t))u_{kt}(x,t)\mathrm{d}x = 0.\end{aligned}$$

From the expression for the energy $E_u(t)$, we rewrite this equation as

$$E_{u_k}(t) + \int_0^t\int_\Omega a(x)g(u_{kt}(x,t))u_{kt}(x,t)\mathrm{d}x\mathrm{d}t + \int_0^t\int_\Omega |\nabla u_{kt}|^2\mathrm{d}x\mathrm{d}t = E_{u_k}(0). \tag{3.23}$$

Since $\frac{1}{2}\int_\Omega(|u_{1,k}|^2 + |\nabla u_{0,k}|^2)\mathrm{d}x$ is uniformly bounded and (3.21), we have the uniform boundedness of

$$E_{u_k}(0) = \frac{1}{2}\int_\Omega(|u_{1,k}|^2 + |\nabla u_{0,k}|^2)\mathrm{d}x + \int_\Omega F_k(u_{0,k})\mathrm{d}x.$$

In the other hand, according to Assumptions 1.2, for any $t > 0$, we have

$$\int_0^t\int_\Omega a(x)g(u_{k,t}(x,t))u_{k,t}(x,t)\mathrm{d}x\mathrm{d}t \geq 0.$$

Therefore, by (3.23), we have that $E_{u_k}(t)$ is uniformly bounded. As a result, there is a subsequence of $\{u_k\}$ (still denoted by $\{u_k\}$) such that

$$u_k \rightharpoonup u, \quad \text{weakly* in } L^\infty(0,\infty; H_0^1(\Omega)), \tag{3.24}$$

$$u_{k,t} \rightharpoonup u_t, \quad \text{weakly* in } L^\infty(0,\infty; L^2(\Omega)), \tag{3.25}$$

$$u_{k,t} \rightharpoonup u_t, \quad \text{weakly in } L^2(0,T; H_0^1(\Omega)), \tag{3.26}$$

$$a(x)g(u_{k,t}) \rightharpoonup \chi, \quad \text{weakly in } L^2(0,T; L^2(\Omega))/L^{\frac{r+1}{r}}(0,T; L^{\frac{r+1}{r}}(\Omega)). \tag{3.27}$$

Next, we prove the convergence of $f_k(u_k)$. From (3.20), for any $s \in \mathbb{R}$, $k \in \mathbb{N}$, a constant $c > 0$ exists such that

$$|f_k(s)| \leq c(|s| + |s|^p).$$

Therefore

$$\|f_k(u_k)\|^{\frac{p+1}{p}}_{L^{\frac{p+1}{p}}(\Omega)} = \int_\Omega |f_k(u_k)|^{\frac{p+1}{p}}\mathrm{d}x \leq c^{\frac{p+1}{p}}\left(\int_\Omega |u_k|^{\frac{p+1}{p}}\mathrm{d}x + \int_\Omega |u_k|^{p+1}\mathrm{d}x\right). \tag{3.28}$$

According to the range of p and the Sobolev embedding theorem, we obtain $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^{\frac{p+1}{p}}(\Omega)$. Then, taking the supremum over time t in (3.28) and combining it with $u_k \in L^\infty(\mathbb{R}^+; H_0^1(\Omega))$, we deduce that $f_k(u_k)$ is uniformly bounded in $L^\infty(\mathbb{R}^+; L^{\frac{p+1}{p}}(\Omega))$. By the Sobolev embedding theorem, we conclude that $f_k(u_k)$ is uniformly bounded in $L^{\frac{p+1}{p}}(0,T; L^{\frac{p+1}{p}}(\Omega))$.

We prove that $f_k(u_k)$ converges almost everywhere to $f(u)$ on $\Omega \times (0,T)$. Indeed, for any $T > 0$, according to (3.24) and (3.25), and using the Lions–Aubin–Simon lemma [28], we conclude that for any sufficiently small $\varepsilon > 0$,

$$u_k \to u, \quad \text{strongly in } L^\infty(0,T; L^{2^*-\varepsilon}(\Omega)), \tag{3.29}$$

where $2^* := \frac{2n}{n-2}$. By the property that strong convergence implies a.e. convergence, we know that $u_k(x,t)$ converges to $u(x,t)$ almost everywhere on $\Omega \times (0,T)$. Moreover, by the property that strongly convergent sequences are bounded, for some constant $L = L(x,t) > 0$ and a.e. $(x,t) \in \Omega \times (0,T)$, we have $|u_k| < L = L(x,t)$. Now taking k sufficiently large such that $k > L$, then by the definition of f_k, for almost every $(x,t) \in \Omega \times (0,T)$, we have

$$f_k(u_k(x,t)) = f(u_k(x,t)).$$

Therefore, for almost every $(x,t) \in \Omega \times (0,T)$, when $k \to \infty$, we have

$$f_k(u_k) - f(u_k) \to 0. \tag{3.30}$$

On the other hand, combining the a.e. convergence of $u_k(x,t)$ to $u(x,t)$ on $\Omega \times (0,T)$ and the continuity of f, we derive that for almost every $(x,t) \in \Omega \times (0,T)$, when $k \to \infty$, we have

$$f(u_k) - f(u) \to 0. \tag{3.31}$$

From (3.30) and (3.31), we obtain the a.e. of $f_k(u_k)$. By lemma 1.2, we deduce that

$$f_k(u_k) \rightharpoonup f(u), \quad \text{weakly in } L^{\frac{p+1}{p}}(\Omega \times (0,T)), \tag{3.32}$$

$$f_k(u_k) \to f(u), \quad \text{strongly in } L^{p_0}(\Omega \times (0,T)), \tag{3.33}$$

as $k \to \infty$, where $1 \leq p_0 < \frac{p+1}{p}$. Taking any $\varphi \in C_0^\infty(\Omega)$, $\theta \in C_0^\infty(0,T)$, obtain

$$\begin{aligned} &-\int_0^T \theta'(t)\int_\Omega u_{k,t}(x,t)\varphi(x)\mathrm{d}x\mathrm{d}t + \int_0^T \theta(t)\int_\Omega \nabla u_k(x,t)\nabla\varphi(x)\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \theta(t)\int_\Omega f_k(u_k(x,t))\varphi(x)\mathrm{d}x\mathrm{d}t + \int_0^T \theta(t)\int_\Omega \nabla u_{k,t}(x,t)\nabla\varphi(x)\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \theta(t)\int_\Omega a(x)g(u_{k,t}(x,t))\varphi(x)\mathrm{d}x\mathrm{d}t = 0. \end{aligned} \tag{3.34}$$

Take the limit in (3.34). Then by (3.24)–(3.27) and (3.32), we have

$$
\begin{aligned}
&-\int_0^T \theta'(t)\int_\Omega u_t(x,t)\varphi(x)\mathrm{d}x\mathrm{d}t+\int_0^T\theta(t)\int_\Omega \nabla u(x,t)\nabla\varphi(x)\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\theta(t)\int_\Omega f(u(x,t))\varphi(x)\mathrm{d}x\mathrm{d}t+\int_0^T\theta(t)\int_\Omega \nabla u_t(x,t)\nabla\varphi(x)\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\theta(t)\int_\Omega \chi\varphi(x)\mathrm{d}x\mathrm{d}t=0.
\end{aligned}
$$

According to the definition of weak derivatives, we have:

$$
u_{tt}-\Delta u-\Delta u_t+f(u)+\chi=0. \tag{3.35}
$$

From $\Delta u\in L^\infty(0,T;H^{-1}(\Omega))$, $\Delta u_t\in L^2(0,T;H^{-1}(\Omega))$, and $f(u)\in L^\infty(0,T;L^{\frac{p+1}{p}}(\Omega))$, we obtain $u_{tt}\in L^2(0,T;H^{-1}(\Omega))$. Next, we prove that $\chi=a(x)g(u_t)$. We now prove the strong convergence of u_k, u_{kt}. Indeed, from the strong convergence in (3.29), we have

$$
u_k\to u,\quad \text{a.e. } (x,t)\in\Omega\times(0,T).
$$

Therefore, for almost every $(x,t)\in\Omega\times(0,T)$ and some constant $L=L(x,t)>0$, we have $|u_k|<L=L(x,t)$. For sufficiently large $m,l>L$, combined with the definition of $f_k(u_k)$, we obtain that for almost every $(x,t)\in\Omega\times(0,T)$, we deduce that $f_m(u_m(x,t))=f(u_m(x,t))$ and $f_l(u_l(x,t))=f(u_l(x,t))$ hold for almost every $(x,t)\in\Omega\times(0,T)$. Let $Z_{ml}=u_m-u_l$. Then Z_{ml} satisfies (3.23). Therefore,

$$
\begin{aligned}
&\frac12\|(Z_{ml})_t\|^2_{L^2(\Omega)}+\frac12\|\nabla Z_{ml}\|^2_{L^2(\Omega)}+\int_0^t\int_\Omega|\nabla(Z_{ml})_t|^2\mathrm{d}x\mathrm{d}t\\
&+\int_0^t\int_\Omega a(x)(g(u_{mt})-g(u_{lt}))(u_{mt}-u_{lt})\mathrm{d}x\mathrm{d}t\\
=&-\int_0^t\int_\Omega(f_m(u_m)-f_l(u_l))(u_{mt}-u_{lt})\mathrm{d}x\mathrm{d}t+\frac12\|u_{1,m}-u_{1,l}\|^2+\frac12\|\nabla u_{0,m}-\nabla u_{0,l}\|^2.
\end{aligned}
$$

Next, we estimate the source term

$$
\begin{aligned}
&\left|\int_0^t\int_\Omega(f(u_m)-f(u_l))(u_{mt}-u_{lt})\mathrm{d}x\mathrm{d}t\right|\\
\le&\int_0^t\int_\Omega|f_m(u_m)-f_l(u_l)||u_{mt}-u_{lt}|\mathrm{d}x\mathrm{d}t\\
\le c&\int_0^t\int_\Omega|1+|u_m|^{p-1}+|u_l|^{p-1}||u_m-u_l||u_{mt}-u_{lt}|\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Using Hölder's inequality, we obtain

$$
\begin{aligned}
&\left|\int_0^t\int_\Omega(f(u_m)-f(u_l))(u_{mt}-u_{lt})\mathrm{d}x\mathrm{d}t\right|\\
\le c&\int_0^t(\|1\|_{L^{\frac{p+1}{p-1}}(\Omega)}+\|u_m\|^{p-1}_{L^{p+1}(\Omega)}+\|u_l\|^{p-1}_{L^{p+1}(\Omega)})\|u_m-u_l\|_{L^{p+1}(\Omega)}\|u_{mt}-u_{lt}\|_{L^{p+1}(\Omega)}\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\int_0^t (|\Omega|^{\frac{p-1}{p+1}} + \|u_m\|^{p-1}_{H_0^1(\Omega)} + \|u_l\|^{p-1}_{H_0^1(\Omega)})\|u_m - u_l\|_{H_0^1(\Omega)}\|u_{mt} - u_{lt}\|_{H_0^1(\Omega)}\mathrm{d}t\\
&\leq C\left(|\Omega|^{\frac{p-1}{p+1}} + \|u_m\|^{p-1}_{L^\infty(0,T;H_0^1(\Omega))} + \|u_l\|^{p-1}_{L^\infty(0,T;H_0^1(\Omega))}\right)\|u_m - u_l\|_{L^2(0,T;H_0^1(\Omega))}\|u_{mt} - u_{lt}\|_{L^2(0,T;H_0^1(\Omega))}.
\end{aligned}
$$

Furthermore, by the Cauchy inequality, we obtain

$$
\begin{aligned}
&\left|\int_0^t\int_\Omega (f(u_m) - f(u_l))(u_{mt} - u_{lt})\mathrm{d}x\mathrm{d}t\right|\\
\leq&\frac{1}{2}C^2\left(|\Omega|^{\frac{p-1}{p+1}} + \|u_m\|^{p-1}_{L^\infty(0,T;H_0^1(\Omega))} + \|u_l\|^{p-1}_{L^\infty(0,T;H_0^1(\Omega))}\right)^2\|u_m - u_l\|^2_{L^2(0,T;H_0^1(\Omega))}\\
&+\frac{1}{2}\|u_{mt} - u_{lt}\|^2_{L^2(0,T;H_0^1(\Omega))}.
\end{aligned}
$$

The uniform boundedness of $\|u_m\|_{L^\infty(0,T;H_0^1(\Omega))}$ and $\|u_l\|_{L^\infty(0,T;H_0^1(\Omega))}$ is readily available. Consequently, for some constant C_Ω, we have

$$
\frac{1}{2}C^2\left(|\Omega|^{\frac{p-1}{p+1}} + \|u_m\|^{p-1}_{L^\infty(0,T;H_0^1(\Omega))} + \|u_l\|^{p-1}_{L^\infty(0,T;H_0^1(\Omega))}\right)^2 \leq C_\Omega,
$$

which implies

$$
\left|\int_0^t\int_\Omega (f(u_m) - f(u_l))(u_{mt} - u_{lt})\mathrm{d}x\mathrm{d}t\right| \leq C_\Omega\|u_m - u_l\|^2_{L^2(0,T;H_0^1(\Omega))} + \frac{1}{2}\|u_{mt} - u_{lt}\|^2_{L^2(0,T;H_0^1(\Omega))}.
$$

From the Poincaré inequality, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\|(Z_{ml})_t\|^2 + \frac{1}{2}\|\nabla Z_{ml}\|^2 + \int_0^t\int_\Omega |\nabla (Z_{ml})_t|^2\mathrm{d}x\mathrm{d}t + \int_0^t\int_\Omega a(x)(g(u_{mt}) - g(u_{lt}))(u_{mt} - u_{lt})\mathrm{d}x\mathrm{d}t\\
\leq&\frac{1}{2}\|u_{1,m} - u_{1,l}\|^2 + \frac{1}{2}\|\nabla u_{0,m} - \nabla u_{0,l}\|^2 + C_\Omega\|u_m - u_l\|^2_{L^2(0,T;H_0^1(\Omega))} + \frac{1}{2}\|\nabla u_{mt} - \nabla u_{lt}\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned}
$$

Since $a(x)$ is non-negative and g is monotonically increasing, we have

$$
\begin{aligned}
&\|(Z_{ml})_t\|^2 + \|\nabla Z_{ml}\|^2 + \int_0^t\int_\Omega |\nabla (Z_{ml})_t|^2\mathrm{d}x\mathrm{d}t\\
\leq&\|u_{1,m} - u_{1,l}\|^2 + \|\nabla u_{0,m} - \nabla u_{0,l}\|^2 + C_\Omega\|Z_{ml}\|^2_{L^2(0,T;H_0^1(\Omega))}.
\end{aligned}
$$

By Lemma 1.4, we obtain

$$
\begin{aligned}
&\|(Z_{ml})_t\|^2 + \|\nabla Z_{ml}\|^2 + \int_0^t\int_\Omega |\nabla (Z_{ml})_t|^2\mathrm{d}x\mathrm{d}t\\
\leq&\left(\|u_{1,m} - u_{1,l}\|^2 + \|\nabla u_{0,m} - \nabla u_{0,l}\|^2\right)e^{C_\Omega T}.
\end{aligned} \tag{3.36}
$$

Since $\{u_{0,k}, u_{1,k}\}$ converges strongly to $\{u_0, u_1\}$ in H, for any $\varepsilon > 0$ and some $N > 0$, we get

$$
\|u_{1,m} - u_{1,l}\|_{L^2(\Omega)} < \varepsilon, \quad \|u_{0,m} - u_{0,l}\|_{H_0^1(\Omega)} < \varepsilon,
$$

when $m, l > N$. Therefore, when $m \to \infty$, $l \to \infty$, and $\varepsilon \to 0$, we obtain

$$u_m \to u, \quad \text{strongly in } L^\infty(0,T;H_0^1(\Omega)), \tag{3.37}$$
$$u_{mt} \to u_t, \quad \text{strongly in } L^\infty(0,T;L^2(\Omega)), \tag{3.38}$$
$$u_{mt} \to u_t, \quad \text{strongly in } L^2(0,T;H_0^1(\Omega)). \tag{3.39}$$

From (3.39), we know that when $k \to \infty$,

$$u_{kt} \to u_t, \quad \text{a.e. } (x,t) \in \Omega \times (0,T).$$

Since g is a continuous function and $a(x) \in L^\infty(\Omega)$, we deduce when $k \to \infty$

$$a(x)g(u_{kt}) \to a(x)g(u_t), \quad \text{a.e. } (x,t) \in \Omega \times (0,T),$$

which means $\chi = a(x)g(u_t)$.

At this point, we conclude that Problem (1.1)–(1.3) admits a global solution $u \in L^\infty(0,T;H_0^1(\Omega))$, $u_t \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, and $u_{tt} \in L^2(0,T;H^{-1}(\Omega))$. Taking $C = 1$ and $\gamma = 0$ in Lemma 3.4, (3.22) is clearly satisfied. Thus we obtain $u \in C([0,T];H_0^1(\Omega))$ and $u_t \in C([0,T];L^2(\Omega))$. This completes the proof of Theorem 2.2. □

4. Uniqueness

This section studies the uniqness of the global weak solution to Problem (1.1)–(1.3). The result has be shown as Theorem 2.3. We now prove Theorem 2.3.

Let u, v be two weak solutions of Problem (1.1)–(1.3) and set $w = u - v$. From Theorem 2.2, we have $w \in C(0,T;H_0^1(\Omega))$, and $w_t \in C(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. Similar to (3.28), we obtain $f(u) \in L^\infty(0,T;H^{-1}(\Omega))$.

When $0 \le r \le 1$, we have $a(x)g(u_t) \in L^1(0,T;L^2(\Omega))$. Similar to (3.16), we obtain

$$\begin{aligned}&\int_0^T \left(\int_\Omega |a(x)g(u_t)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}t \\ \le& C\|a\|_{L^\infty(\Omega)} T M_g \operatorname{meas}(\Omega_0)^{\frac{1}{2}} + CM\|a\|_{L^\infty(\Omega)} \int_0^T \left(\int_{\Omega_1} |u_t(x,t)|^2 \mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}t.\end{aligned}$$

Since $u_t \in C(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, by the embedding theorem, we have $u_t \in L^1(0,T;L^2(\Omega))$. Therefore, $a(x)g(u_t) \in L^1(0,T;L^2(\Omega))$. When $1 < r \le \frac{(n+2)+\sqrt{(n+2)^2+8n}}{2n}$, we have $a(x)g(u_t) \in L^2(0,T;L^{\frac{r+1}{r}}(\Omega))$. In fact,

$$\begin{aligned}&\int_0^T \left(\int_\Omega |a(x)g(u_t)|^{\frac{r+1}{r}} \mathrm{d}x\right)^{\frac{2r}{r+1}} \mathrm{d}t \\ \le& C\|a\|^2_{L^\infty(\Omega)} T M_g^2 \operatorname{meas}(\Omega_0)^{\frac{2r}{r+1}} + CM^2\|a\|^2_{L^\infty(\Omega)} \int_0^T \left(\int_{\Omega_1} |u_t(x,t)|^{r+1} \mathrm{d}x\right)^{\frac{2r}{r+1}} \mathrm{d}t.\end{aligned}$$

Let $\frac{1}{r+1} = \frac{1-\theta}{2} + \frac{(n-2)\theta}{2n}$, $0 < \theta < 1$. Using the interpolation inequality, we get

$$\begin{aligned}\int_0^T \left(\int_{\Omega_1} |u_t(x,t)|^{r+1} dx \right)^{\frac{2r}{r+1}} \mathrm{d}t \leq & \int_0^T \|u_t\|_{L^2(\Omega)}^{2r(1-\theta)} \|u_t\|_{L^{\frac{2n}{n-2}}(\Omega)}^{2r\theta} \mathrm{d}t \\ \leq & \|u_t\|_{L^\infty(0,T;L^2(\Omega))}^{2r(1-\theta)} \|u_t\|_{L^{2r\theta}(0,T;L^{\frac{2n}{n-2}}(\Omega))}^{2r\theta}.\end{aligned}$$

From $\frac{1}{r+1} = \frac{1-\theta}{2} + \frac{(n-2)\theta}{2n}$, we solve for $\theta = \frac{n(r-1)}{2(r+1)}$. Therefore,

$$2r\theta = 2r \cdot \frac{n(r-1)}{2(r+1)} \leq 2.$$

Because $u_t \in C(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, $H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ and $L^2(\Omega) \hookrightarrow L^{2r\theta}(\Omega)$, we get $a(x)g(u_t) \in L^2(0,T;L^{\frac{r+1}{r}}(\Omega))$. Thus $a(x)g(u_t) \in L^2(0,T;H^{-1}(\Omega))$.

Considering both cases, we can conclude that

$$\Delta u_t - f(u) - a(x)g(u_t) \in L^2(0,T;H^{-1}(\Omega)) + L^1(0,T;L^2(\Omega)).$$

Therefore, from Lemma 1.1, we obtain

$$\begin{aligned}&\frac{1}{2}\|w_t\|^2 + \frac{1}{2}\|\nabla w\|^2 + \int_0^t \int_\Omega |\nabla w_t|^2 \mathrm{d}x\mathrm{d}t + \int_0^t \int_\Omega a(x)(g(u_t) - g(v_t))(u_t - v_t)\mathrm{d}x\mathrm{d}t \\ =& -\int_0^t \int_\Omega (f(u) - f(v))(u_t - v_t)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.1}$$

Next, we estimate the nonlinear term $f(u)$. This process is similar to the estimation of $f_k(u_k)$ in Theorem 2.2. Therefore, we obtain

$$\begin{aligned}&\left| \int_0^t \int_\Omega (f(u) - f(v))(u_t - v_t)\mathrm{d}x\mathrm{d}t \right| \\ \leq & C_T \|u - v\|_{L^2(0,T;H_0^1(\Omega))}^2 + \frac{1}{2}\|u_t - v_t\|_{L^2(0,T;H_0^1(\Omega))}^2.\end{aligned}$$

Then, from (4.1), we obtain the inequality

$$\|w_t\|^2 + \|\nabla w\|^2 + \int_0^t \int_\Omega |\nabla w_t|^2 \mathrm{d}x\mathrm{d}t \leq C_T \int_0^t \|\nabla w\|_{L^2(\Omega)}^2 \mathrm{d}t.$$

Therefore, by Lemma 1.4, we get $w = 0$, i.e., $u = v$. From the process of getting (4.1), we also deduce the energy identity

$$E_u(t) + \int_0^t \int_\Omega a(x)g(u_t(x,t))u_t(x,t)\mathrm{d}x\mathrm{d}t + \int_0^t \int_\Omega |\nabla u_t|^2 \mathrm{d}x\mathrm{d}t = E_u(0).$$

5. Energy decay

In this section, we use the Nakao inequality–Lemma 1.5 to study the energy decay property of the solution to Problem (1.1)–(1.3). We now prove Theorem 2.6.

Differentiating the energy equality (2.3) with respect to time t, we obtain

$$E'(t) = -\int_\Omega |\nabla u_t|^2 \mathrm{d}x - \int_\Omega a(x)g(u_t)u_t \mathrm{d}x. \tag{5.1}$$

From the monotonicity of the function $g(x)$ and the non-negativity of $a(x)$, we know that for any $t > 0$, $E'(t) \le 0$. Therefore, for any $t > 0$, $E(t)$ is monotonically decreasing and always satisfies $E(t) < E(0)$. Let

$$D^2(t) = E(t) - E(t+1).$$

Then $D^2(t) < 2E(0)$, which implies that $D^2(t)$ is uniformly bounded. According to the energy equality

$$E_u(t+1) + \int_t^{t+1} \int_\Omega a(x)g(u_t(x,t))u_t(x,t)\mathrm{d}x\mathrm{d}t + \int_t^{t+1} \int_\Omega |\nabla u_t|^2 \mathrm{d}x\mathrm{d}t = E_u(t),$$

we obtain

$$\begin{aligned}\int_t^{t+1} \int_\Omega |\nabla u_t|^2 \mathrm{d}x\mathrm{d}t =& E_u(t) - E_u(t+1) - \int_t^{t+1} \int_\Omega a(x)g(u_t(x,t))u_t(x,t)\mathrm{d}x\mathrm{d}t \\ \le& E_u(t) - E_u(t+1) = D^2(t). \end{aligned}\tag{5.2}$$

Thus we have

$$\int_t^{t+\frac{1}{4}} \int_\Omega |\nabla u_t|^2 \mathrm{d}x\mathrm{d}t \le D^2(t), \quad \text{and} \quad \int_{t+\frac{3}{4}}^{t+1} \int_\Omega |\nabla u_t|^2 \mathrm{d}x\mathrm{d}t \le D^2(t).$$

By the mean value theorem of calculus, for some $t_1 \in [t, t+\frac{1}{4}]$, we deduce $\|u_t(t_1)\|_{H_0^1(\Omega)} \le 2D(t)$, and for some $t_2 \in [t+\frac{3}{4}, t+1]$, we deduce $\|u_t(t_2)\|_{H_0^1(\Omega)} \le 2D(t)$. On $[t_1, t_2]$, we have

$$\begin{aligned}&\int_{t_1}^{t_2} \langle \Delta u_t - a(x)g(u_t), u\rangle_{H_0^1(\Omega)} \mathrm{d}t \\ \le& \int_{t_1}^{t_2} \int_\Omega \nabla u \nabla u_t \mathrm{d}x\mathrm{d}t + \tilde{M}\|a\|_{L^\infty(\Omega)} \int_{t_1}^{t_2} \int_\Omega |u_t|^r |u| \mathrm{d}x\mathrm{d}t \\ \le& \int_{t_1}^{t_2} \|u\|_{H_0^1(\Omega)} \|u_t\|_{H_0^1(\Omega)} \mathrm{d}t + \tilde{M}\|a\|_{L^\infty(\Omega)} \int_{t_1}^{t_2} \|u_t\|^r_{L^{r+1}(\Omega)} \|u\|_{L^{r+1}(\Omega)} \mathrm{d}t. \end{aligned}$$

When $1 \le r \le \frac{(n+2)+\sqrt{(n+2)^2+8n}}{2n}$, by the Sobolev embedding theorem, we obtain

$$\begin{aligned}&\int_{t_1}^{t_2} \langle \Delta u_t - a(x)g(u_t), u\rangle_{H_0^1(\Omega)} \mathrm{d}t \\ \le& \int_{t_1}^{t_2} \|u\|_{H_0^1(\Omega)} \|u_t\|_{H_0^1(\Omega)} \mathrm{d}t + C\tilde{M}\|a\|_{L^\infty(\Omega)} \int_{t_1}^{t_2} \|u_t\|^r_{H_0^1(\Omega)} \|u\|_{H_0^1(\Omega)} \mathrm{d}t. \end{aligned}\tag{5.3}$$

By the Cauchy inequality, there is a sufficiently small ε_1 and corresponding C_{ε_1} such that

$$\int_{t_1}^{t_2} \|u\|_{H_0^1(\Omega)}\|u_t\|_{H_0^1(\Omega)}\mathrm{d}t \le \int_{t_1}^{t_2} \left(\varepsilon_1\|u\|^2_{H_0^1(\Omega)} + C_{\varepsilon_1}\|u_t\|^2_{H_0^1(\Omega)}\right)\mathrm{d}t. \tag{5.4}$$

Using the ε-Young inequality, for any $\varepsilon > 0$, we have

$$\begin{aligned}\int_{t_1}^{t_2} \|u_t\|^r_{H_0^1(\Omega)}\|u\|_{H_0^1(\Omega)}\mathrm{d}t \le& \int_{t_1}^{t_2} \left(\frac{r}{2}\varepsilon^{-\frac{2-r}{r}}\|u_t\|^2_{H_0^1(\Omega)} + (1-\frac{r}{2})\varepsilon\|u\|^{\frac{2}{2-r}}_{H_0^1(\Omega)}\right)\mathrm{d}t\\ \le&\varepsilon^{-\frac{2-r}{r}}\int_{t_1}^{t_2}\|u_t\|^2_{H_0^1(\Omega)}\mathrm{d}t + \varepsilon\int_{t_1}^{t_2}\|u\|^{\frac{2}{2-r}}_{H_0^1(\Omega)}\mathrm{d}t\\ \le&\varepsilon^{-\frac{2-r}{r}}D^2(t) + \varepsilon\|u\|^{2(r-1)/(2-r)}_{L^\infty(t_1,t_2;H_0^1(\Omega))}\int_{t_1}^{t_2}\|u\|^2_{H_0^1(\Omega)}\mathrm{d}t.\end{aligned}$$

Since $u \in C(\mathbb{R}^+; H_0^1(\Omega))$, for some constant $G > 0$, we have

$$\|u\|^{\frac{2(r-1)}{2-r}}_{L^\infty(t_1,t_2;H_0^1(\Omega))} < G.$$

Then, we deduce

$$\int_{t_1}^{t_2} \|u_t\|^r_{H_0^1(\Omega)}\|u\|_{H_0^1(\Omega)}\mathrm{d}t \le \varepsilon^{-\frac{2-r}{r}}D^2(t) + \varepsilon G\int_{t_1}^{t_2}\|u\|^2_{H_0^1(\Omega)}\mathrm{d}t. \tag{5.5}$$

Substituting (5.4) and (5.5) into (5.3), we obtain

$$\begin{aligned}&\int_{t_1}^{t_2}\langle \Delta u_t - a(x)g(u_t), u\rangle_{H_0^1(\Omega)}\mathrm{d}t\\ \le&(\varepsilon_1 + \varepsilon G C\tilde{M}\|a\|_{L^\infty(\Omega)})\int_{t_1}^{t_2}\|u\|^2_{H_0^1(\Omega)}\mathrm{d}t + C_{\varepsilon_1}\int_{t_1}^{t_2}\|u_t\|^2_{H_0^1(\Omega)}\mathrm{d}t + C\tilde{M}\|a\|_{L^\infty(\Omega)}\varepsilon^{-\frac{2-r}{r}}D^2(t)\\ \le&(\varepsilon_1 + \varepsilon G C\tilde{M}\|a\|_{L^\infty(\Omega)})\int_{t_1}^{t_2}\|u\|^2_{H_0^1(\Omega)}\mathrm{d}t + \left(C_{\varepsilon_1} + C\tilde{M}\|a\|_{L^\infty(\Omega)}\varepsilon^{-\frac{2-r}{r}}\right)D^2(t).\end{aligned}$$

Choose sufficiently small ε and ε_1. Set $C_{\varepsilon_1} + C\tilde{M}\|a\|_{L^\infty(\Omega)}\varepsilon^{-\frac{2-r}{r}} = C_a$. Therefore

$$\begin{aligned}\int_{t_1}^{t_2}\langle \Delta u_t - a(x)g(u_t), u\rangle_{H_0^1(\Omega)}\mathrm{d}t \le&\frac{1}{4}\int_{t_1}^{t_2}\|u\|^2_{H_0^1(\Omega)}\mathrm{d}t + C_aD^2(t)\\ \le&\frac{1}{2}\int_{t_1}^{t_2}E_u(s)\mathrm{d}s + C_aD^2(t).\end{aligned} \tag{5.6}$$

On the other hand, from (1.1), we get

$$\begin{aligned}\int_{t_1}^{t_2}\langle -\Delta u + f(u), u\rangle_{H_0^1(\Omega)}\mathrm{d}t =& \int_{t_1}^{t_2}\langle \Delta u_t - a(x)g(u_t) - u_{tt}, u\rangle_{H_0^1(\Omega)}\mathrm{d}t\\ =& \int_{t_1}^{t_2}\langle \Delta u_t - a(x)g(u_t), u\rangle_{H_0^1(\Omega)}\mathrm{d}t - \int_{t_1}^{t_2}\langle u_{tt}, u\rangle_{H_0^1(\Omega)}\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= \int_{t_1}^{t_2} \langle \Delta u_t - a(x)g(u_t), u \rangle_{H_0^1(\Omega)} \mathrm{d}t + \langle u_t(t_1), u(t_1) \rangle_{H_0^1(\Omega)} \\
&\quad - \langle u_t(t_2), u(t_2) \rangle_{H_0^1(\Omega)} + \int_{t_1}^{t_2} \|u_t\|^2 \mathrm{d}t. \qquad (5.7)
\end{aligned}
$$

From the Sobolev embedding theorem, we have

$$
\langle u_t(t_1), u(t_1) \rangle_{H_0^1(\Omega)} \leq C \|\nabla u_t(t_1)\|_{L^2(\Omega)} \sup_{s \in [t_1, t_2]} \|\nabla u(s)\|.
$$

Using the Cauchy inequality, choose ε_3 small enough such that

$$
\langle u_t(t_1), u(t_1) \rangle_{H_0^1(\Omega)} \leq \varepsilon_3 \sup_{s \in [t, t+1]} \|\nabla u(s)\|^2 + C_{\varepsilon_3} \|\nabla u_t(t_1)\|_{L^2(\Omega)}^2.
$$

Accordingly,

$$
\begin{aligned}
\langle u_t(t_1), u(t_1) \rangle_{H_0^1(\Omega)} &\leq 2\varepsilon_3 \sup_{s \in [t, t+1]} E_u(s) + C_{\varepsilon_3} \|\nabla u_t(t_1)\|_{L^2(\Omega)}^2 \\
&\leq 2\varepsilon_3 \sup_{s \in [t, t+1]} E_u(s) + 4C_{\varepsilon_3} D^2(t). \qquad (5.8)
\end{aligned}
$$

From the same way, we have

$$
-\langle u_t(t_2), u(t_2) \rangle_{H_0^1(\Omega)} \leq \left| \langle u_t(t_2), u(t_2) \rangle_{H_0^1(\Omega)} \right| \leq 2\varepsilon_3 \sup_{s \in [t, t+1]} E_u(s) + 4C_{\varepsilon_3} D^2(t). \qquad (5.9)
$$

Furthermore, by the Sobolev embedding theorem, for some constant $C > 0$, we get

$$
\int_{t_1}^{t_2} \|u_t\|^2 \mathrm{d}t \leq C \int_{t_1}^{t_2} \|u_t\|_{H_0^1(\Omega)}^2 \mathrm{d}t \leq C D^2(t). \qquad (5.10)
$$

Substituting (5.6) and (5.8)–(5.10) into (5.7), we obtain

$$
\int_{t_1}^{t_2} \langle -\Delta u + f(u), u \rangle_{H_0^1(\Omega)} \mathrm{d}t \leq \frac{1}{2} \int_{t_1}^{t_2} E_u(s) \mathrm{d}s + 4\varepsilon_3 \sup_{s \in [t, t+1]} E_u(s) + (C_a + 4C_{\varepsilon_3} + C) D^2(t). \qquad (5.11)
$$

Note that

$$
\begin{aligned}
\int_{t_1}^{t_2} E_u(s) \mathrm{d}s &= \int_{t_1}^{t_2} \left(\frac{1}{2} \|u_t(s)\|^2 + \frac{1}{2} \|\nabla u(s)\|^2 + \int_\Omega F(u(x, s)) dx \right) \mathrm{d}s \\
&\leq \frac{1}{2} \int_{t_1}^{t_2} \|u_t(s)\|^2 \mathrm{d}s + \int_{t_1}^{t_2} \int_\Omega (|\nabla u(s)|^2 + f(u)u) \mathrm{d}x \mathrm{d}s \\
&= \frac{1}{2} \int_{t_1}^{t_2} \|u_t(s)\|^2 \mathrm{d}s + \int_{t_1}^{t_2} \langle -\Delta u + f(u), u \rangle_{H_0^1(\Omega)} \mathrm{d}s. \qquad (5.12)
\end{aligned}
$$

Substituting (5.10) and (5.11) into (5.12), we obtain

$$
\int_{t_1}^{t_2} E_u(s) \mathrm{d}s \leq \frac{1}{2} \int_{t_1}^{t_2} E_u(s) \mathrm{d}s + 4\varepsilon_3 \sup_{s \in [t, t+1]} E_u(s) + (C_a + 4C_{\varepsilon_3} + 2C) D^2(t).
$$

Choose sufficiently small ε_3 such that

$$\int_{t_1}^{t_2} E_u(s)\mathrm{d}s \leq (C_a + 4C_{\varepsilon_3} + 2C)D^2(t) + \frac{1}{2}\sup_{s\in[t,t+1]} E_u(s). \tag{5.13}$$

Applying the mean value theorem to (5.13), there exists $t^* \in [t_1, t_2]$ such that

$$E_u(t^*) \leq (C_a + 4C_{\varepsilon_3} + 2C)D^2(t) + \frac{1}{2}\sup_{s\in[t,t+1]} E_u(s). \tag{5.14}$$

From the monotonic decreasing property of $E_u(t)$ and (5.14), we have

$$\begin{aligned}\sup_{s\in[t,t+1]} E_u(s) = E_u(t) &= E_u(t+1) + \int_t^{t+1}\int_\Omega \left(|\nabla u_t|^2 + a(x)g(u_t)u_t\right)\mathrm{d}x\mathrm{d}t \\ &\leq E_u(t^*) + D^2(t) \\ &\leq (C_a + 4C_{\varepsilon_3} + 2C)D^2(t) + \frac{1}{2}\sup_{s\in[t,t+1]} E_u(s) + D^2(t).\end{aligned}$$

Therefore, we obtain

$$\sup_{s\in[t,t+1]} E_u(s) = E_u(t) \leq 2(C_a + 4C_{\varepsilon_3} + 2C + 1)D^2(t). \tag{5.15}$$

Substituting $D^2(t) = E_u(t) - E_u(t+1)$ into (5.15) and letting $2(C_a + 4C_{\varepsilon_3} + 2C + 1) = C_0$, we obtain

$$\sup_{s\in[t,t+1]} E_u(s) = E_u(t) \leq C_0(E_u(t) - E_u(t+1)).$$

From the expression of $E_u(t)$, we know that $E_u(t)$ is non-negative. Hence by the fourth result of Lemma 1.5, we obtain

$$E_u(t) \leq C_4 \exp\left\{-\frac{1}{C_0+1}(t+2)\right\},$$

where C_4 depend on $E_u(0)$.

6. Conclusions

In this paper, we study the initial boundary value problem for a nonlinear wave equation with strong damping and structural damping. We establish the global existence of weak solutions for $n \geq 3$ under the subcritical growth conditions $0 \leq r \leq \frac{n+2}{n-2}$ and $1 \leq p \leq \frac{n+2}{n-2}$. By restricting r to the refined admissible range $0 \leq r \leq \frac{(n+2)+\sqrt{(n+2)^2+8n}}{2n}$, we prove the uniqueness of weak solutions. The exponential energy decay is obtained via Nakao's inequality.

We weaken the conventional positivity assumption on the variable coefficient $a(x)$ from uniform strict positivity $a(x) \geq a_0 > 0$ to mere nonnegativity $a(x) \geq 0$, significantly relaxing the restrictive prerequisites in most existing literature. Unlike the classical Galerkin method and microlocal analysis, our semigroup-theoretic compactness argument captures the full subcritical growth range.

These results open avenues for further study, including uniqueness under broader growth conditions, finite-time blow-up for large data, and extensions to coupled systems, time-dependent coefficients, and nonlinear boundary conditions.

Author contributions

Jiaxin Xu: Writing-original draft, writing - review & editing, project administration; Shubin Wang: Conceptualization, methodology; Xuelian Chen: Investigation, writing - original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest in this paper.

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