Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 20291–20308.
DOI: 10.3934/math.2026824
Received: 08 April 2026
Revised: 09 June 2026
Accepted: 11 June 2026
Published: 09 July 2026

Research article

On a generalization of the Euler line for convex polygons via averaged triangle centers

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Abstract: The Euler line is a classical geometric object defined for non-equilateral triangles as the line passing through the centroid, circumcenter, and orthocenter. In this paper, we proposed a generalization of the Euler line to convex polygons. The approach was based on constructing families of triangles associated with a given polygon and defining averaged centers, namely, the average circumcenter, centroid, and orthocenter. We proved that, for convex quadrilaterals, these averaged points are collinear, thus naturally defining a Euler line for quadrilaterals. Several geometric properties were established, including characterizations for isosceles trapezoids, right trapezoids, and cyclic quadrilaterals. Furthermore, we extended the construction to convex polygons with an arbitrary number of sides and provide an algorithmic method to compute the generalized Euler line. These results extend classical triangle geometry to polygonal settings and provided a new framework for studying geometric structures through averaged centers.

Keywords: Euler line; convex polygons; triangle centers; analytic geometry; polygon geometry
Mathematics Subject Classification: 51M04, 51M15, 51N20

1. Introduction

The Euler line is one of the most significant results in classical geometry. For any non-equilateral triangle, the centroid, circumcenter, and orthocenter are collinear, revealing a deep geometric structure among notable triangle centers. This property has been widely studied, and results related to triangle centers and Euclidean geometry have been extensively developed in the literature [1,2]. In this context, the Euler line is recognized as a fundamental object in Euclidean geometry [3–5].

Several attempts have been made to extend the notion of the Euler line beyond triangles. In particular, Dalcín [6] studied configurations of Euler-type lines in quadrilaterals, providing early insights into how classical triangle centers may behave in more general polygonal settings. Similarly, de Villiers [7] analyzed extensions of the Euler line and the nine-point circle in quadrilateral geometry, highlighting the complexity of generalizing these concepts.

A natural question arises: Can this concept be extended to polygons with more than three vertices? While some studies have addressed special cases such as cyclic quadrilaterals, a general framework for convex polygons remains largely unexplored.

In addition, previous work by Velezmoro et al. [8] explored geometric constructions involving symmetric configurations and computational approaches, which motivate the development of algorithmic methods in polygonal geometry.

In this work, we propose a method to extend the notion of the Euler line to convex polygons. The idea consists of constructing families of triangles from the vertices of a polygon and defining averaged geometric centers. Using vector methods from analytic geometry [9, 10], we establish the existence of a generalized Euler line for convex quadrilaterals and derive several geometric properties.

Additionally, we extend the construction to polygons with an arbitrary number of sides and provide an algorithmic approach for computing the corresponding line.

2. Materials and methods

All the definitions and mathematical properties developed in this work are derived using analytic and vector methods.

2.1. Linear transformations

Consider the mapping $J : \mathbb{R}^2 \to \mathbb{R}^2$ defined as a rotation of $\pi/2$, that is,

$$J(x, y) = (-y, x).$$

This transformation plays a key role in expressing orthogonality conditions in vector form [11].

2.2. Lines in vector form

A line passing through a point $P_0 \in \mathbb{R}^2$ with direction vector $\vec{d}$ is given by

$$\ell(t) = P_0 + t\vec{d}, \quad t \in \mathbb{R}.$$

Given two distinct points P_0 and P_1, the line passing through them can be written as

$$\ell(t) = P_0 + t(P_1 - P_0), \quad t \in \mathbb{R}.$$

This representation is standard in analytic geometry [12].

2.3. Intersection of lines

Let ℓ_1 and ℓ_2 be two lines given by

$$\ell_1 : P = P_1 + t\vec{d}_1, \quad \ell_2 : Q = P_2 + s\vec{d}_2.$$

If $J(\vec{d}_1) \cdot \vec{d}_2 \neq 0$, then the lines intersect at a unique point, which can be determined by solving the corresponding linear system [13].

2.4. Distance from a point to a line

Let P_1 be a point and ℓ a line passing through P_0 with direction $\vec{d}$. The distance from P_1 to ℓ is given by

$$d(P_1, \ell) = \frac{|J(\vec{d}) \cdot (P_1 - P_0)|}{||\vec{d}||}.$$

2.5. Triangle centers

Let $P_1, P_2, P_3 \in \mathbb{R}^2$ be three non-collinear points defining a triangle.

- The **circumcenter (Cir)** is the intersection of the perpendicular bisectors of the sides.
- The **centroid (Bar)** is the intersection of the medians and is given by

$$Bar = \frac{P_1 + P_2 + P_3}{3}.$$

- The **orthocenter (Ort)** is the intersection of the altitudes.

These classical constructions are fundamental in Euclidean geometry [3, 5].

2.6. Euler line for triangles

For any non-equilateral triangle, the circumcenter, centroid, and orthocenter are collinear, defining the Euler line [3–5].

To verify collinearity in a general setting, one may use vector criteria: three points $A, B, C \in \mathbb{R}^n$ are collinear if and only if the vectors $B - A$ and $C - A$ are linearly dependent.

2.7. Basic definitions

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points in $\mathbb{R}^2$. The distance between them is defined by

$$d(P_1, P_2) = \sqrt{(P_1 - P_2) \cdot (P_1 - P_2)},$$

where $(\cdot)$ denotes the Euclidean scalar product [9].

2.8. Euler line for convex polygons

Definition 1. *Let*

$$P = \{P_1, P_2, \dots, P_n\},$$

be a convex polygon. Consider the consecutive triangles

$$\Delta_i = P_i P_{i+1} P_{i+2}, \qquad i = 1, \dots, (n-2),$$

$$\Delta_{n-1} = P_{n-1} P_n P_1$$

$$\Delta_n = P_n P_1 P_2.$$

Let Cir_i, Bar_i, and Ort_i denote the circumcenter, centroid, and orthocenter of Δ_i, respectively. Define

$$\mathrm{PCir} = \frac{1}{n}\sum_{i=1}^{n} Cir_i,$$

$$\mathrm{PBar} = \frac{1}{n}\sum_{i=1}^{n} Bar_i,$$

$$\mathrm{POrt} = \frac{1}{n}\sum_{i=1}^{n} Ort_i.$$

If these three points are collinear, the line containing them is called the generalized Euler line of the polygon.

Throughout this work, Mathematica [14] was employed as a computational tool to perform symbolic computations, verify analytical derivations, compute the triangle centers introduced above, and generate the graphical representations used in the subsequent sections.

Theorem 1. *Let $P_1, P_2, P_3, P_4 \in \mathbb{R}^2$ be the vertices of a convex quadrilateral given by*

$$\{P_1, P_2, P_3, P_4\} = \{(a, b), (-c, d), (-e, -f), (g, -h)\},$$

where $a, b, c, d, e, f, g, h > 0$.

Consider the four triangles

$$\Delta_1 = \triangle(P_1, P_2, P_3), \quad \Delta_2 = \triangle(P_2, P_3, P_4), \quad \Delta_3 = \triangle(P_3, P_4, P_1), \quad \Delta_4 = \triangle(P_4, P_1, P_2).$$

Let Cir_i, Bar_i, and Ort_i denote the circumcenter, centroid, and orthocenter of Δ_i, respectively. Define

$$\mathrm{PCir} = \frac{1}{4}\sum_{i=1}^{4} Cir_i, \quad \mathrm{PBar} = \frac{1}{4}\sum_{i=1}^{4} Bar_i, \quad \mathrm{POrt} = \frac{1}{4}\sum_{i=1}^{4} Ort_i.$$

Then the points PCir, PBar, and POrt are collinear and determine the generalized Euler line. Moreover,

$$\mathrm{PBar} = \frac{P_1 + P_2 + P_3 + P_4}{4}.$$

Proof. According to the definitions introduced in Subsection 2.8, we have

$$\sum_{i=1}^{4} Bar_i = \frac{1}{3}\cdot(3P_1 + 3P_2 + 3P_3 + 3P_4) = P_1 + P_2 + P_3 + P_4.$$

Therefore,

$$\mathrm{PBar} = \frac{1}{4}(P_1 + P_2 + P_3 + P_4).$$

On the other hand, for each triangle, the classical Euler line relation holds:

$$Ort_i = 3Bar_i - 2Cir_i.$$

Summing over all triangles, we obtain

$$\sum_{i=1}^{4} Ort_i = 3\sum_{i=1}^{4} Bar_i - 2\sum_{i=1}^{4} Cir_i.$$

Dividing by 4, it follows that

$$\text{POrt} = 3\,\text{PBar} - 2\,\text{PCir}\,.$$

Rewriting,

$$\text{POrt} - \text{PBar} = -2(\text{PCir} - \text{PBar})$$

which shows that the vectors POrt – PBar and PCir – PBar are linearly dependent. Therefore, the points PCir, PBar, and POrt are collinear. □

Theorem 2. *If we have an isosceles trapezoid, the generalized Euler line for quadrilaterals divides it into two congruent trapezoids. Furthermore, its parallel sides are perpendicular to the Euler line, and the trapezoid is divided into two right trapezoids of equal area.*

Proof. Let us consider the points

$$P_1 = (0, a), \quad P_2 = (-b, 0), \quad P_3 = (0, -b), \quad P_4 = (a, 0),$$

where $a, b \in \mathbb{R}^+$ and $b > a$, which define an isosceles trapezoid.

We consider the four triangles determined by consecutive triples of vertices:

$$\triangle(P_1, P_2, P_3), \quad \triangle(P_2, P_3, P_4), \quad \triangle(P_3, P_4, P_1), \quad \triangle(P_4, P_1, P_2).$$

For each triangle, we compute the circumcenter, centroid, and orthocenter. Averaging these points, we obtain the following three points:

$$\text{PCir} = \left(\frac{a-b}{2}, \frac{a-b}{2}\right), \quad \text{PBar} = \left(\frac{a-b}{4}, \frac{a-b}{4}\right), \quad \text{POrt} = \left(\frac{-a+b}{4}, \frac{-a+b}{4}\right).$$

These three points are collinear, since

$$\text{POrt} - \text{PBar} = -(\text{PCir} - \text{PBar})$$

which implies that they lie on the same line.

Therefore, the Euler line is determined by the direction vector

$$\vec{d} = (1, 1)$$

and its equation can be written as

$$\ell(t) = PBar + t(1, 1).$$

Now, observe that the bases of the trapezoid are perpendicular to the Euler line, since

$$(P_4 - P_1) \cdot (1, 1) = 0$$

and

$$(P_3 - P_2) \cdot (1, 1) = 0.$$

This proves that the Euler line is perpendicular to the parallel sides.

Furthermore, the Euler line is the perpendicular bisector of the parallel sides of the isosceles trapezoid, since

$$d(P_1, \ell) = d(P_4, \ell), \quad d(P_2, \ell) = d(P_3, \ell),$$

which shows that the Euler line is the perpendicular bisector of both bases.

Consequently, the trapezoid is divided into two congruent regions. Each of these regions is a right trapezoid, and therefore both have equal area.

This completes the proof. □

Theorem 3. *If we have a right trapezoid, then the generalized Euler line for quadrilaterals is perpendicular to the parallel sides and divides the trapezoid into two quadrilaterals: one is a rectangle and the other is a right trapezoid. Moreover, both regions have equal area.*

Proof. Let us consider the points

$$S_1 = (a, 0), \quad S_2 = (c, b), \quad S_3 = (0, b), \quad S_4 = (0, 0),$$

where $a, b, c \in \mathbb{R}^+$ and $c < a$, which define a right trapezoid.

We consider the four triangles

$$\triangle(S_1, S_2, S_3), \quad \triangle(S_2, S_3, S_4), \quad \triangle(S_3, S_4, S_1), \quad \triangle(S_4, S_1, S_2).$$

For each triangle, we compute the circumcenter, centroid, and orthocenter. Averaging these points, we obtain:

$$\mathrm{PCir} = \left(\frac{a+c}{4}, \frac{-a^2 + 4b^2 + c^2}{8b}\right)$$

$$\mathrm{PBar} = \left(\frac{a+c}{4}, \frac{b}{2}\right)$$

$$\mathrm{POrt} = \left(\frac{a+c}{4}, \frac{a^2 + 2b^2 - c^2}{4b}\right).$$

These points are collinear, since

$$POrt - PBar = \lambda(PCir - PBar)$$

for some scalar λ, which shows that they lie on the same line.

Therefore, the Euler line is given by the vertical direction

$$x = \frac{a+c}{4}.$$

Now, observe that the bases of the trapezoid are horizontal, hence their direction vector is $(1, 0)$. Since the Euler line has direction $(0, 1)$, it follows that it is perpendicular to the parallel sides.

Furthermore, we verify orthogonality by computing

$$(S_3 - S_2) \cdot (PCir - PBar) = 0, \quad (S_4 - S_1) \cdot (PCir - PBar) = 0,$$

which confirms that the Euler line is perpendicular to the bases.

The area of the trapezoid is given by

$$A = \frac{1}{2} b(a + c).$$

On the other hand, the rectangle determined by the Euler line has base $\frac{a+c}{2}$ and height b, hence its area is

$$A_{rect} = \frac{1}{2} b(a + c).$$

Therefore, the Euler line divides the trapezoid into two regions of equal area: one is a rectangle and the other is a right trapezoid.

This completes the proof. □

Theorem 4. *If a quadrilateral is inscribed in a circle, then the generalized Euler line passes through the center of the circle. In the particular cases of squares and rectangles, the generalized Euler line degenerates into a single point, which coincides with the center of the circumcircle.*

Proof. Let $a > 0$, and consider the circle of radius a centered at (h, k). Let

$$P_i = (h + a\cos\theta_i,\ k + a\sin\theta_i), \qquad i = 1, 2, 3, 4,$$

be four distinct points on the circle, where

$$\theta_1, \theta_2, \theta_3, \theta_4 \in [0, 2\pi), \qquad \theta_1 < \theta_2 < \theta_3 < \theta_4.$$

Consider the triangle

$$\triangle(P_1, P_2, P_3).$$

The perpendicular bisectors of the sides P_1P_2 and P_2P_3 are given by

$$\ell_1(t) = \left(h + \frac{a}{2}\cos\theta_1 + \frac{a}{2}\cos\theta_2 + at(\sin\theta_1 - \sin\theta_2),\ k + at(\cos\theta_2 - \cos\theta_1) + \frac{a}{2}\sin\theta_1 + \frac{a}{2}\sin\theta_2\right)$$

and

$$\ell_2(t) = \left(h + \frac{a}{2}(\cos\theta_2 + \cos\theta_3) + at(\sin\theta_2 - \sin\theta_3),\ k + \frac{a}{2}(\sin\theta_2 + \sin\theta_3) + at(\cos\theta_3 - \cos\theta_2)\right).$$

A direct computation shows that the intersection of these two perpendicular bisectors is

$$\ell_1 \cap \ell_2 = (h, k).$$

Therefore, the circumcenter of the triangle
$\triangle(P_1, P_2, P_3)$ is

$$\mathrm{Cir(P_1, P_2, P_3)} = (h, k).$$

Since the above computation depends only on the fact that the three vertices lie on the same circle, the same argument applies to every triangle determined by three vertices of the quadrilateral. Consequently, all such triangles have the same circumcenter, namely, the center of the given circle. □

Remark 1. *The proof of Theorem 4 does not rely on any specific property of quadrilaterals other than the fact that their vertices lie on the same circle. Indeed, if*

$$P = \{P_1, P_2, \ldots, P_n\}$$

is any convex polygon inscribed in a circle with center (h, k), then every triangle determined by three vertices of the polygon has circumcenter equal to (h, k). Consequently,

$$\mathrm{PCir} = \frac{1}{n}\sum_{i=1}^{n} Cir_i = (h, k).$$

Therefore, whenever the generalized Euler line exists as a non-degenerate line, it necessarily passes through the center of the circumcircle. In particular, for highly symmetric configurations, such as regular polygons, the averaged circumcenter, centroid, and orthocenter may coincide, in which case the generalized Euler line degenerates into a single point.

Theorem 5. *Let $P_1P_2P_3P_4$ be a quadrilateral inscribed in a semicircle, where P_1P_4 is the diameter of the circle. Then the generalized Euler line passes through the center of the circle and is the perpendicular bisector of the segment P_2P_3. Moreover, the intersection point Q_1 between the Euler line and the segment P_2P_3 is a trisection point between $POrt$ and $PCir$ on the Euler line.*

Proof. Without loss of generality, points on a unit circle centered at the origin will be used. Define the points

$$P_1 = (1, 0), \quad P_2 = (\cos x_1, \sin x_1), \quad P_3 = (\cos x_2, \sin x_2), \quad P_4 = (-1, 0)$$

with $0 < x_1 < \pi$, $0 < x_2 < \pi$, and $x_2 > x_1$.

We consider the four triangles determined by consecutive triples of vertices and compute their circumcenters, centroids, and orthocenters. Averaging these points, we obtain the three points defining the Euler line:

$$\mathrm{PCir} = (0,0)$$

$$\mathrm{PBar} = \left(\frac{\cos x_1 + \cos x_2}{4}, \frac{\sin x_1 + \sin x_2}{4}\right)$$

$$\mathrm{POrt} = \left(\frac{3}{4}(\cos x_1 + \cos x_2), \frac{3}{4}(\sin x_1 + \sin x_2)\right).$$

Hence, the Euler line passes through the origin, which is the center of the circle.

Now, the midpoint of the segment P_2P_3 is

$$M = \frac{P_2 + P_3}{2} = \left(\frac{\cos x_1 + \cos x_2}{2}, \frac{\sin x_1 + \sin x_2}{2}\right).$$

On the other hand, consider the point that divides the segment from p_{Cir} to p_{Ort} in the ratio $2:1$, that is,

$$Q_1 = \frac{2}{3}\,\mathrm{POrt} = \left(\frac{\cos x_1 + \cos x_2}{2}, \frac{\sin x_1 + \sin x_2}{2}\right).$$

Thus,

$$Q_1 = M,$$

which shows that the intersection point between the Euler line and the segment P_2P_3 is the midpoint of P_2P_3.

Therefore, the Euler line is the perpendicular bisector of the segment P_2P_3.

Finally, since Q_1 divides the segment joining p_{Cir} and p_{Ort} in the ratio $2:1$, it follows that Q_1 is a trisection point of the Euler line.

This completes the proof. □

Theorem 6. *If we have a non-regular convex pentagon symmetric with respect to a line, then the line of symmetry is the generalized Euler line.*

Proof. Let us consider the points

$$P_1 = (a,0), \quad P_2 = (b,c), \quad P_3 = (0,d), \quad P_4 = (-b,c), \quad P_5 = (-a,0)$$

with $a, b, c, d > 0$ and $b > a$, $d > c$.

By construction, the points P_1 and P_5, as well as P_2 and P_4, are symmetric with respect to the y-axis. Hence, the polygon is symmetric with respect to the line $x = 0$.

We consider the family of triangles formed by consecutive triples of vertices and compute, for each triangle, its circumcenter, centroid, and orthocenter. Denote by *PCir*, *PBar*, and *POrt* these centers, respectively. Due to the symmetry of the polygon, the corresponding triangle centers also occur in symmetric pairs with respect to the y-axis. Therefore, their averages

$$\mathrm{PCir} = \left(0,\ \frac{1}{5}\left(\frac{-a^2 + b^2}{c} + c + \frac{b^2 + c^2 - d^2}{2(c-d)} + \frac{-a^2 b + bd^2 + a\left(b^2 + c^2 - d^2\right)}{a(c-d) + bd}\right)\right).$$

$$\text{PBar} = \left(0,\ \frac{2c+d}{5}\right).$$

$$\text{POrt} = \left(0,\ \frac{1}{5}\left(\frac{2(a-b)(a+b)}{c} + c + \frac{b^2}{-c+d} + \frac{2b(a^2-ab+cd)}{a(c-d)+bd}\right)\right).$$

Therefore, the generalized Euler line determined by these three points is

$$x = 0,$$

which coincides with the axis of symmetry of the pentagon. This completes the proof. □

Definition 2. *A hexaangle is a convex six-sided polygon whose interior angles are equal and whose side lengths alternate between two distinct values. That is, if*

$$P_1, P_2, P_3, P_4, P_5, P_6$$

are its vertices, then

$$|P_1P_2| = |P_3P_4| = |P_5P_6|$$

and

$$|P_2P_3| = |P_4P_5| = |P_6P_1|$$

with

$$|P_1P_2| \neq |P_2P_3|.$$

Theorem 7. *Let H be a convex hexagon. If H is a hexaangle, then*

$$\text{PCir} = \text{PBar} = \text{POrt}$$

and therefore the generalized Euler line degenerates to a single point.

On the other hand, if H is neither a regular hexagon nor a hexaangle and is symmetric with respect to a line l, then the generalized Euler line coincides with the axis of symmetry l.

Proof. (Hexaangle case).

Consider the vertices

$$H_1 = (-a,-b), \qquad H_2 = (a,-b),$$

$$H_3 = \left(\frac{a}{2} + \frac{\sqrt{3}\,b}{2}, -\frac{\sqrt{3}\,a}{2} + \frac{b}{2}\right),$$

$$H_4 = \left(-\frac{a}{2} + \frac{\sqrt{3}\,b}{2}, \frac{\sqrt{3}\,a}{2} + \frac{b}{2}\right),$$

$$H_5 = \left(\frac{a}{2} - \frac{\sqrt{3}\,b}{2}, \frac{\sqrt{3}\,a}{2} + \frac{b}{2}\right),$$

$$H_6 = \left(-\frac{a}{2} - \frac{\sqrt{3}\,b}{2}, -\frac{\sqrt{3}\,a}{2} + \frac{b}{2}\right),$$

where $a, b \in \mathbb{R}^+$ and $b \neq \sqrt{3}\,a$. It follows from the definition of the vertices that the hexagon is symmetric with respect to the line

$$x = 0,$$

which is its axis of symmetry.

According to the construction described in Subsection 2.8, we compute the averages of the circumcenters, centroids, and orthocenters associated with the six consecutive triangles Δ_i.

The circumcenters of the six triangles are

$$\mathrm{Cir} = \{(0,0), (0,0), (0,0), (0,0), (0,0), (0,0)\}.$$

Hence,

$$\mathrm{PCir} = (0,0).$$

Similarly, the centroids of the six triangles are given by

$$\mathrm{Bar} = \left\{\begin{array}{l} \left(\frac{1}{6}(a + \sqrt{3}\,b), \frac{1}{6}(-\sqrt{3}\,a - 3b)\right), \left(\frac{1}{3}(a + \sqrt{3}\,b), 0\right), \\ \left(\frac{1}{6}(a + \sqrt{3}\,b), \frac{1}{6}(\sqrt{3}\,a + 3b)\right), \left(\frac{1}{6}(-a - \sqrt{3}\,b), \frac{1}{6}(\sqrt{3}\,a + 3b)\right), \\ \left(-\frac{a}{3} - \frac{b}{\sqrt{3}}, 0\right), \left(\frac{1}{6}(-a - \sqrt{3}\,b), \frac{1}{6}(-\sqrt{3}\,a - 3b)\right) \end{array}\right\}$$

from which it follows that

$$\mathrm{PBar} = (0,0).$$

Likewise, the orthocenters of the six triangles are

$$\mathrm{Ort} = \left\{\begin{array}{l} \left(\frac{1}{2}(a + \sqrt{3}\,b), \frac{1}{2}(-\sqrt{3}\,a - 3b)\right), \left(a + \sqrt{3}\,b, 0\right), \\ \left(\frac{1}{2}(a + \sqrt{3}\,b), \frac{1}{2}(\sqrt{3}\,a + 3b)\right), \left(\frac{1}{2}(-a - \sqrt{3}\,b), \frac{1}{2}(\sqrt{3}\,a + 3b)\right), \\ \left(-a - \sqrt{3}\,b, 0\right), \left(\frac{1}{2}(-a - \sqrt{3}\,b), \frac{1}{2}(-\sqrt{3}\,a - 3b)\right) \end{array}\right\}$$

and therefore

$$\mathrm{POrt} = (0,0).$$

Consequently,

$$\mathrm{PCir} = \mathrm{PBar} = \mathrm{POrt} = (0, 0).$$

Therefore, the three averaged centers coincide, and the generalized Euler line degenerates into a single point. This completes the proof.

Proof (Symmetric hexagon case, non-hexaangle).

Consider the following vertices:

$$P_1 = (-a, -b), \quad P_2 = (a, -b), \quad P_3 = (c, 0), \quad P_4 = (d, e), \quad P_5 = (-d, e), \quad P_6 = (-c, 0),$$

where $a, b, c, d, e \in \mathbb{R}^+$, with

$$c > d > a \quad \text{and} \quad e > b.$$

According to the construction described in Subsection 2.8, we compute the average circumcenter, average centroid, and average orthocenter, denoted by ($PCir$, $PBar$, and $POrt$), respectively.

The circumcenters of the six associated triangles are

$$\mathrm{Cir} = \left\{ \begin{array}{l} \left(0, -\dfrac{a^2 + b^2 - c^2}{2b}\right), \\ \left(\dfrac{b^2 e + (a - c)(a + c)e + b(-c^2 + d^2 + e^2)}{2(b(-c + d) + (a - c)e)}, \dfrac{-((c - d)(a^2 + b^2 + cd - a(c + d))) + (c - a)e^2}{2(b(c - d) + (c - a)e)}\right), \\ \left(0, \dfrac{-c^2 + d^2 + e^2}{2e}\right), \\ \left(0, \dfrac{-c^2 + d^2 + e^2}{2e}\right), \\ \left(\dfrac{b^2 e + (a - c)(a + c)e + b(-c^2 + d^2 + e^2)}{2(b(c - d) + (c - a)e)}, \dfrac{-((c - d)(a^2 + b^2 + cd - a(c + d))) + (c - a)e^2}{2(b(c - d) + (c - a)e)}\right), \\ \left(0, -\dfrac{a^2 + b^2 - c^2}{2b}\right) \end{array} \right\}$$

and therefore

$$\mathrm{PCir} = \left(0,\ \frac{1}{6}\left(-\frac{a^2 + b^2 - c^2}{b} + \frac{-c^2 + d^2}{e} + e + \frac{-(c - d)(a^2 + b^2 + cd - a(c + d)) + (c - a)e^2}{b(c - d) + (c - a)e}\right)\right).$$

The centroids of the six associated triangles are

$$\mathrm{Bar} =$$

$$\left\{\left(\frac{c}{3},-\frac{2b}{3}\right),\left(\frac{1}{3}(a+c+d),\frac{1}{3}(-b+e)\right),\left(\frac{c}{3},\frac{2e}{3}\right),\left(-\frac{c}{3},\frac{2e}{3}\right),\left(\frac{1}{3}(-a-c-d),\frac{1}{3}(-b+e)\right),\left(-\frac{c}{3},-\frac{2b}{3}\right)\right\},$$

which yields

$$\mathrm{PBar}=\left(0,\ \frac{e-b}{3}\right).$$

Similarly, the orthocenters of the six associated triangles are

$$\mathrm{Ort}=\left\{\begin{array}{l}\left(c,-\dfrac{-a^2+b^2+c^2}{b}\right),\\ \left(\dfrac{ab(c-d)-ade+cde+be(b+e)}{b(c-d)+(c-a)e},\dfrac{(a-d)\big(-((a-c)(c-d))-be\big)}{b(-c+d)+(a-c)e}\right),\\ \left(c,\dfrac{(c-d)(c+d)}{e}+e\right),\\ \left(-c,\dfrac{(c-d)(c+d)}{e}+e\right),\\ \left(\dfrac{ab(c-d)-ade+cde+be(b+e)}{b(-c+d)+(a-c)e},\dfrac{(a-d)\big((a-c)(c-d)+be\big)}{b(c-d)+(c-a)e}\right),\\ \left(-c,-\dfrac{-a^2+b^2+c^2}{b}\right)\end{array}\right\}$$

and consequently

$$\mathrm{POrt}=\left(0,\ \frac{b^3(d-c)e-(a-c)^2(a+c)e^2+b(a-c)e\big(2(a-d)(c-d)-e^2\big)}{3be\big(b(c-d)+(c-a)e\big)}+\frac{b^2\big((c-d)^2(c+d)+2(a-d)e^2\big)}{3be\big(b(c-d)+(c-a)e\big)}\right).$$

Hence, the generalized Euler line determined by ($PCir$, $PBar$, and $POrt$) is

$$x=0.$$

This completes the proof. □

3. Results

3.1. Application of Theorem 6

Example 1. *We illustrate Theorem 6 by considering a convex pentagon.*

Let the vertices of the polygon be given by

$$\{P_1, P_2, P_3, P_4, P_5\} = \{(1, -1), (9, -2), (7, 7), (-2, 9), (-1, 1)\}.$$

Following the construction introduced in Section 2, we compute the averaged circumcenter, centroid, and orthocenter associated with the triangles determined by consecutive triples of vertices. This yields

$$\text{PCir} = \left(\frac{141}{35}, \frac{141}{35}\right), \quad \text{PBar} = \left(\frac{14}{5}, \frac{14}{5}\right), \quad \text{POrt} = \left(\frac{12}{35}, \frac{12}{35}\right).$$

These points are collinear, thus defining the generalized Euler line of the pentagon.

Moreover, it is observed that all three points satisfy $x = y$, which implies that the Euler line has the equation

$$y = x.$$

From the graphical representation (Figure 1), it is clear that this line acts as an axis of symmetry of the pentagon, dividing it into two congruent regions.

Therefore, the Euler line coincides with the axis of symmetry, confirming the result stated in Theorem 6.

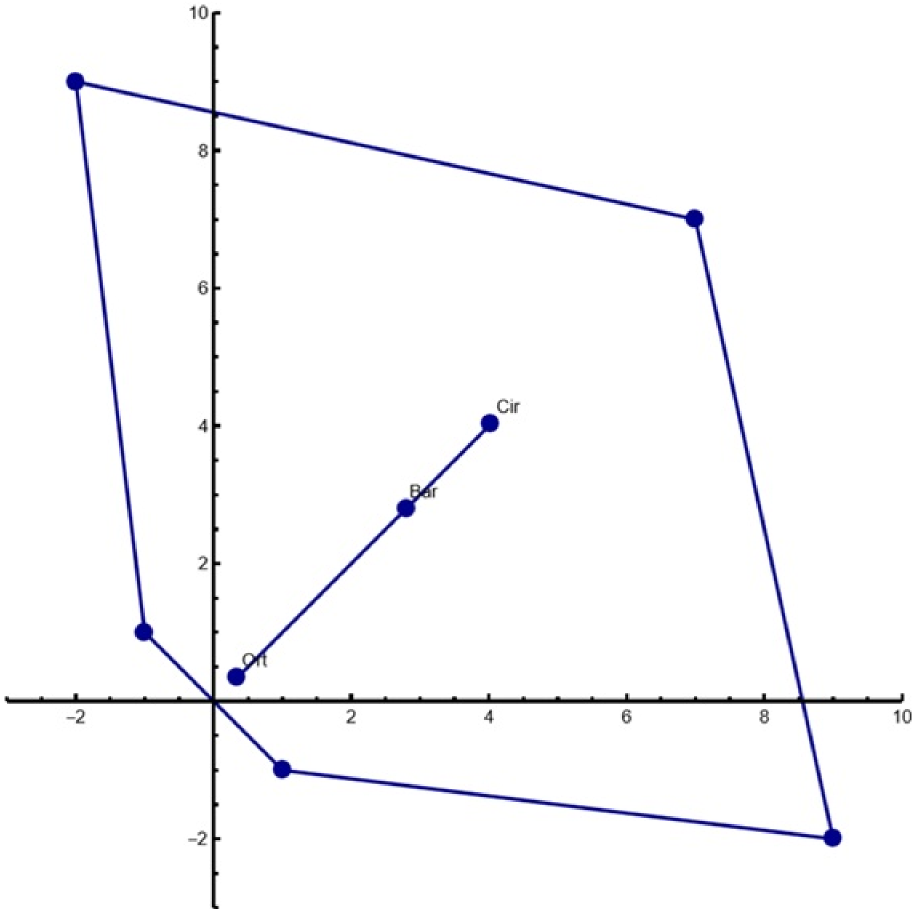

Figure 1. The generalized Euler line acting as an axis of symmetry of a convex pentagon.

3.2. Application of Theorem 7

Example 2. *We illustrate Theorem 7 by considering a convex hexagon with symmetry.*

Let the vertices be given by

$$\{P_1, P_2, P_3, P_4, P_5, P_6\} = \{(4, 0),\ (6, 2\sqrt{3}),\ (2, -4 + 6\sqrt{3}),\ (-2, -4 + 6\sqrt{3}),\ (-6, 2\sqrt{3}),\ (-4, 0)\}.$$

This configuration defines a convex hexagon symmetric with respect to the y-axis.

By direct computation, we obtain

$$\mathrm{PCir} = \left(0,\ \frac{2}{99}\left(-93 + 122\sqrt{3}\right)\right),$$

$$\mathrm{PBar} = \left(0,\ -\frac{4}{3} + \frac{8}{\sqrt{3}}\right),$$

$$\mathrm{POrt} = \left(0,\ \frac{8}{99}\left(-3 + 38\sqrt{3}\right)\right),$$

which are all aligned.

Hence, the generalized Euler line is given by the equation

$$x = 0.$$

From the graphical representation (Figure 2), it is clear that this line acts as an axis of symmetry of the hexagon.

Furthermore, this line divides the hexagon into two congruent pentagons, confirming the geometric behavior predicted by Theorem 7.

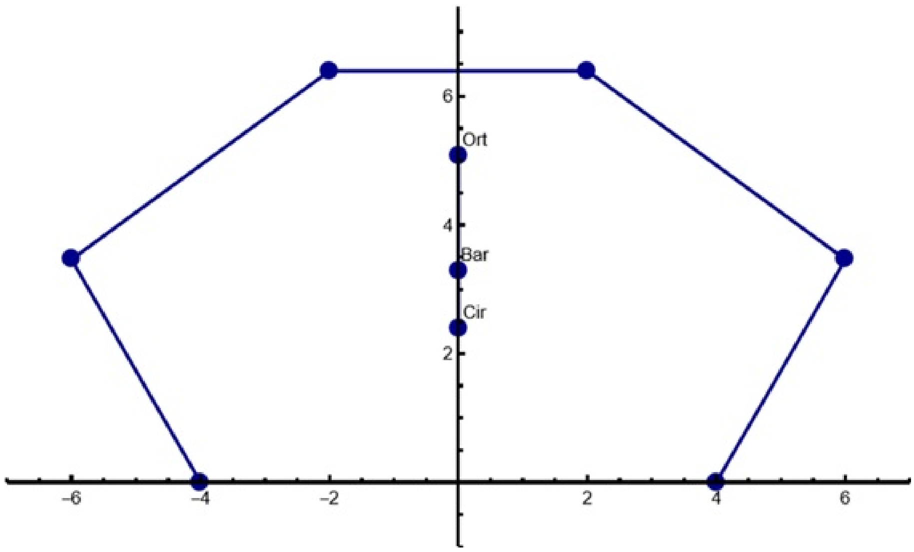

Figure 2. The generalized Euler line acting as an axis of symmetry of a convex hexagon.

4. Discussion

The results obtained in this work provide a natural extension of the classical Euler line from triangles to convex polygons. The approach based on averaging triangle centers proves to be effective in identifying geometric structures that generalize well-known properties.

In the case of quadrilaterals, the existence of a generalized Euler line is established through the collinearity of the averaged circumcenter, centroid, and orthocenter. This result differs from previous studies such as Dalcín [6] and de Villiers [7], where Euler-type configurations in quadrilaterals were analyzed under specific geometric constructions or particular cases. In contrast, the present work provides a unified and systematic framework based on averaging triangle centers, which guarantees collinearity without requiring additional geometric constraints such as cyclicity or special symmetry.

Moreover, several geometric configurations, such as isosceles trapezoids, right trapezoids, and cyclic quadrilaterals, reveal additional properties including symmetry, perpendicularity, and area decomposition. These results not only extend known properties but also provide new geometric interpretations that are not explicitly addressed in the aforementioned works.

The extension to polygons with a higher number of sides shows that symmetry plays a fundamental role. In particular, for symmetric pentagons and hexagons, the generalized Euler line coincides with the axis of symmetry. This behavior highlights a structural difference with earlier approaches, where generalizations beyond quadrilaterals are limited or not systematically developed.

Additionally, certain configurations lead to degenerate cases in which the Euler line reduces to a single point. This phenomenon does not appear in the classical triangle setting and represents a new feature arising from the polygonal generalization.

Finally, the computational approach implemented in Mathematica complements the theoretical results, allowing visualization and verification of the geometric properties in concrete examples. Unlike previous works, where results are mainly theoretical, this integration of symbolic computation and visualization provides an effective tool for exploring and validating the generalized Euler line in a broader class of polygons.

5. Conclusions

In this paper, we introduced a generalization of the Euler line to convex polygons by means of averaged triangle centers. This construction extends a classical concept from triangle geometry to more general polygonal settings.

We proved that for convex quadrilaterals, the averaged circumcenter, centroid, and orthocenter are collinear, thus defining a generalized Euler line. Several geometric properties were established, including characterizations for isosceles trapezoids, right trapezoids, cyclic quadrilaterals, and semicircular configurations.

Furthermore, the method was extended to convex polygons with an arbitrary number of vertices. It was shown that symmetry plays a key role in determining the behavior of the Euler line, and in some cases leads to degenerate configurations where the line collapses to a point.

The proposed approach provides a new framework for studying geometric structures in polygons and opens the possibility for further research in higher-dimensional settings or alternative definitions of geometric centers.

Author contributions

Ricardo Velezmoro-León contributed to the conceptualization of the study, the development of the mathematical framework, and the supervision of the research. Carlos Arellano Ramírez contributed to the theoretical analysis, validation of the mathematical results, and critical revision of the manuscript. Robert Ipanaque Chero contributed to the computational implementation, numerical verification, and visualization of the geometric constructions. Hebert Espino Aguirre contributed to the formal analysis, interpretation of the results, and review of the manuscript. Vanessa Silupu Ortega contributed to the literature review, data organization, manuscript editing, and proofreading. Eder Escobar Gómez contributed to the conceptualization of the study, mathematical analysis, preparation of the figures, writing of the original draft, revision of the manuscript, and served as the corresponding author. All authors have read and approved the final version of the manuscript.

Acknowledgments

The authors would like to thank the Vice-Rectorate for Research of the Universidad Nacional de Piura for supporting this research. The authors also thank the Department of Mathematics of the Faculty of Sciences for its academic support during the development of this work.

Use of Generative-AI tools declaration

The authors used ChatGPT (OpenAI) as a language assistance tool to improve the clarity, grammar, and academic style of the manuscript, as well as to support the translation of the text into English.

All mathematical results, definitions, proofs, and interpretations presented in this work were developed and verified by the authors. The use of AI tools was limited to linguistic refinement and did not influence the scientific content of the paper.

Conflict of interest

The authors declare no conflict of interest.

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