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Research article

L^p boundedness of oscillatory singular integrals with nonstandard kernels

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Abstract: In this paper, we study a kind of oscillatory singular integral operator $T_{P,\Omega,A}$ defined by

$$T_{P,\Omega,A}f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x-y)} \frac{\Omega(x-y)\,[A(x) - A(y) - \nabla A(y)\cdot(x-y)]}{|x-y|^{n+1}} f(y)dy,$$

where $P(x)$ is a real polynomial on $\mathbb{R}^n$, and $\Omega(\lambda x) = \Omega(x)$. Under some conditions, we show that $T_{P,\Omega,A}$ is bounded on $L^p(\mathbb{R}^n)$ with uniform boundedness, which improves and extends Pan's result [11].

Keywords: oscillatory integral; rough kernel; non-standard kernel; L^p spaces; BMO space
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1. Introduction

Let $n, d \in \mathbb{N}$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Let

$$P(x) = \sum_{|\alpha|\le d} a_\alpha x^\alpha, \tag{1.1}$$

wher,e $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $a_\alpha \in \mathbb{R}$.

In the paper, we consider the following operator $T_{P,\Omega,A}$ defined by

$$T_{P,\Omega,A}f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x-y)} \frac{\Omega(x-y)\,[A(x) - A(y) - \nabla A(y)\cdot(x-y)]}{|x-y|^{n+1}} f(y)dy, \tag{1.2}$$

where $A(x)$ is a function on $\mathbb{R}^n$ with $\nabla A \in BMO(\mathbb{R}^n)(BMO(\mathbb{R}^n) = \{f : \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(x) - \langle f\rangle_Q|dx < \infty$ and $\langle f\rangle_Q = \frac{1}{|Q|}\int_Q f(x)dx\})$, and $\Omega(x)$ satisfies the following conditions:

(i) $\Omega(\lambda x) = \Omega(x)$ $(\lambda, x \neq 0)$;
(ii) $\Omega \in L^\infty(\mathbb{S}^{n-1})$;

(iii) For $\forall i \in [1, n]$, $\int_{\mathbb{S}^{n-1}} \Omega(x')x'_i \, d\sigma(x') = 0$, where $d\sigma$ is the normalized Lebesgue measure on $\mathbb{S}^{n-1}$;
(iv) $\bar{\omega}(\delta) < \delta^\theta$ $(\theta > 0)$, where $\bar{\omega}(\delta)$ is defined by

$$\bar{\omega}(\delta) = \sup_{\substack{h\in\mathbb{R}^n \\ |h|\leq\delta}} \int_{\mathbb{S}^{n-1}} |\Omega(x'+h) - \Omega(x')| d\sigma(x').$$

About this operator, we first consider the operator $T_{\Omega,A}$

$$T_{\Omega,A}f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)\left[A(x) - A(y) - \nabla A(y)\cdot(x-y)\right]}{|x-y|^{n+1}} f(y)dy,$$

originating from the Calderón operator

$$T_A f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)\left[A(x) - A(y)\right]}{|x-y|^{n+1}} f(y)dy,$$

which was first introduced by Calderón [2]. For the operator T_A, many researchers have studied it and the readers can refer to [1, 2, 5, 8], [7, chapter 8], and [10, chapter 4]. However, $T_{\Omega,A}$ is not a Calderón-Zygmund operator, even if Ω is smooth enough. Those operators were first studied by Cohen, and in [3], Cohen pointed out that $T_{\Omega,A}$ is bounded on $L^p(\mathbb{R}^n)(1 < p < +\infty)$, when $\Omega \in C^1(\mathbb{S}^{n-1})$.

On the other hand, we notice that Pan [11] studied the following operator:

$$T_{P,K} = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x-y)} K(x,y) f(y)dy,$$

where $K(x,y) = K(x-y)$ and $K(x-y)$ satisfies $\mathbf{CZ}(\delta)$. Pan [11] offered the following consequence.

Theorem 1. *[11] Let $P(x)$ be a real-valued polynomial of any positive degree. Suppose that K is $\mathbf{CZ}(\delta)$ kernel (defined in [11]) for some $\delta > 0$. Then, for $1 < p < \infty$, there exists a positive constant C_P such that*

$$\|T_{P,K}f\|_{L^p(\mathbb{R}^n)} \leq C_P \|f\|_{L^p(\mathbb{R}^n)}, \tag{1.3}$$

for all $f \in L^p(\mathbb{R}^n)$. The constant C_P may depend on n, p, δ, and $\deg(P)$, *but is otherwise independent of K and the coefficients of P.*

Remark 1. *It is well-known that T_K is bounded on $L^p(\mathbb{R}^n)(1 < p < +\infty)$, where*

$$T_K = p.v. \int_{\mathbb{R}^n} e^{iP(x-y)} K(x,y) f(y)dy,$$

$K(x,y) = K(x-y)$ and $K(x-y)$ satisfies $\mathbf{CZ}(\delta)$. For the detail, please refer to [6].

Regarding Pan's result, the question we raise is whether $T_{P,K}$ is still $L^p(1 < p < +\infty)$ bounded where

$$K(x,y) = \frac{\Omega(x-y)\left[A(x) - A(y) - \nabla A(y)\cdot(x-y)\right]}{|x-y|^{n+1}},$$

when $T_{\Omega,A}$ is bounded on $L^p(\mathbb{R}^n)$, and $K(x,y)$ is different from the operators in Pan's [11]. From this, we obtain the main result of this paper.

Theorem 2. *Let $P(x)$ be a real-valued polynomial of any positive degree. Suppose that Ω satisfies (i),(ii), (iii), and (iv), and a function A satisfies $\nabla A \in BMO(\mathbb{R}^n)$. Then, for $1 < p < \infty$, there exists a positive constant C_P such that*

$$\|T_{P,\Omega,A}f\|_{L^p(\mathbb{R}^n)} \leq C_P\|f\|_{L^p(\mathbb{R}^n)}, \tag{1.4}$$

for all $f \in L^p(\mathbb{R}^n)$. The constant C_P depends on $n, p, \delta, deg(P), \Omega,$ and A, but is otherwise independent of the coefficients of P.

Remark 2. *In this paper, we want to use a new method to study this operator, which is different from the general method. Now use the operator*

$$T_{P,\Omega,A,h}f(x) = p.v. \int_{|x-y|\leq h} e^{iP(x-y)} \frac{\Omega(x-y)\left[A(x) - A(y) - \nabla A(y)\cdot(x-y)\right]}{|x-y|^{n+1}} f(y)dy \tag{1.5}$$

to approach $T_{P,\Omega,A}$ with the fact that there exists a uniform C_P such that

$$\|T_{P,\Omega,A,h}f\|_{L^p(\mathbb{R}^n)} \leq C_P\|f\|_{L^p(\mathbb{R}^n)},$$

for $\forall h > 0$. In order to get this object, we consider it for two cases, which are divided by the relationship between h and 8. When $h \leq 8$, It is easy to check by Pan's method [11]. However, for $h > 8$, we cannot get the expected L^2 decay of P_j (defined below) by the Fourier decay estimate (as Pan did in [11]). In order to obtain the decay, we will reduce the problem L^2 estimate of S_j, and combine with the nature of function A. More details can be found in Section 4. It is valuable to point out that we only use simple characteristic functions to decompose the operator $T_{P,\Omega,A}$.

Remark 3. *It needs to be pointed out that our method to deal with the First oscillatory integral estimate follows the approach of Mirek, Stein, and Zorin-Kranich [9, Lemma 2.1], which is different from the classical van der Corput lemma. In the classical van der Corput lemma, for oscillatory integral $\int_{\mathbb{R}^n} e^{iP(x)}\psi(x)dx$, ψ is required to own some regularity. However, we only make ψ integrable.*

In the rest of the paper, we shall use $A \lesssim B$ to mean that $A \leq CB$ for a certain constant C which depends on some essential parameters only. Let Mf denote the Hardy-Littlewood maximal operator defined in [6], and

$$\|Mf\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} (1 < p < +\infty).$$

2. Basic definition

It might be difficult to process $P(x)$ as an oscillatory factor, so we can handle the difficulties we may encounter in the proof process using the following lemma.

Lemma 1. *[9] Let $P(x)$ be given as in (1.1). Let $R > 0$ and $\psi : \mathbb{R}^n \to \mathbb{C}$ be an integrable function supported in $B(0, R/2)$. Then,*

$$\left|\int_{\mathbb{R}^n} e^{iP(x)}\psi(x)dx\right| \lesssim_{d,n} \sup_{v\in B(0,R\Lambda^{-1/d})} \int_{\mathbb{R}^n} |\psi(x) - \psi(x-v)|dx, \tag{2.1}$$

where $\Lambda := \sum_{1\leq|\alpha|\leq d} |a_\alpha|R^{|\alpha|}$.

Lemma 2. *[4] Let A be a function on $\mathbb{R}^n$ with derivatives of order one in $L^q(\mathbb{R}^n)$ for some $q \in (n, \infty]$. Then,*

$$|A(x) - A(y)| \lesssim |x - y|(\frac{1}{I_x^y} \int_{I_x^y} |\nabla A(z)|^q dz)^{\frac{1}{q}},$$

where I_x^y is the cube which is centered at x and has side length $2|x - y|$.

Lemma 3. *[13] Let Ω be homogeneous of degree zero and $\Omega \in L\log L(S^{n-1})$, and let A be a function on $\mathbb{R}^n$ with derivatives of order one in $BMO(\mathbb{R}^n)$. Let $N_{\Omega,A}$ be the maximal operators defined by*

$$N_{\Omega,A}(f)(x) = \sup_{r>0} \frac{1}{r^{n+1}} \int_{|x-y|<r} |\Omega(x - y)|\,|A(x) - A(y) - \nabla A(y) \cdot (x - y)|\,|f(y)|\; dy.$$

Then, for any $p \in (1, \infty)$,

$$\|N_{\Omega,A}(f)\|_{L^p(\mathbb{R}^n)} \lesssim \|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)} \|\Omega\|_{L\log L(S^{n-1})} \|f\|_{L^p(\mathbb{R}^n)}, \tag{2.2}$$

where $\|\Omega\|_{L\log L(S^{n-1})} = \inf\{\lambda : \int_{S^{n-1}} \frac{|\Omega(x)|}{\lambda} \log(e + \frac{|\Omega(x)|}{\lambda})dx \le 1\}$.

Remark 4. *In this paper, we set $N_A(f) = N_{\Omega,A}(f)$ when $\Omega = 1$, and we need to point out that $L^\infty(S^{n-1}) \subsetneq L\log L(S^{n-1})$.*

Furthermore, to address issues in the part of $h > 8$, as required by our method, we need to introduce some facts about the Luxemburg norms related to the methods we will use. We list some known facts about the Luxemburg norms. Details are given in [12]. Let $\Psi : [0, \infty) \to [0, \infty)$ be a Young function, namely, Ψ is convex and continuous with $\Psi(0) = 0$, $\Psi(t) \to \infty$ as $t \to \infty$, and $\Psi(t)/t \to \infty$ as $t \to \infty$.

We also assume that Ψ satisfies a doubling condition, that is, there is $M \ge 1$ such that $\Psi(2t) \le M\Psi(t)$ for all $t \in [0, \infty)$.

Let Ψ be a Young function and $Q \subset \mathbb{R}^n$ be a cube. Define the Luxemburg norm $\|\cdot\|_{\mathcal{L}^\Psi(Q)}$ by

$$\|f\|_{\mathcal{L}^\Psi(Q)} = \inf\left\{\lambda > 0 : \frac{1}{|Q|} \int_Q \Psi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

It is well known that

$$\frac{1}{|Q|} \int_Q \Psi(|f(x)|)dx \le 1 \Leftrightarrow \|f\|_{\mathcal{L}^\Psi(Q)} \le 1,$$

and

$$\|f\|_{\mathcal{L}^\Psi(Q)} \le \inf\left\{\mu + \frac{\mu}{|Q|} \int_Q \Psi\left(\frac{|f(x)|}{\mu}\right) dx : \mu > 0\right\} \le 2\|f\|_{\mathcal{L}^\Psi(Q)},$$

see [12, p.54] and [12, p.69], respectively. For $p \in [1, \infty)$ and $\gamma \in \mathbb{R}$, set $\Psi_{p,\gamma}(t) = t^p \log^\gamma(e + t)$. We denote $\|f\|_{\mathcal{L}^{\Psi_{p,\gamma}}(Q)}$ as $\|f\|_{L^p(\log L)^\gamma, Q}$.

Let Ψ be a Young function. Ψ^*, the complementary function of Ψ, is defined on $[0, \infty)$ by

$$\Psi^*(t) = \sup\{st - \Psi(s) : s \ge 0\}.$$

The generalization of Hölder inequality

$$\frac{1}{|Q|} \int_Q |f(x)h(x)|dx \le \|f\|_{\mathcal{L}^\Psi(Q)} \|h\|_{\mathcal{L}^{\Psi^*}(Q)} \tag{2.3}$$

holds for $f \in \mathcal{L}^{\Psi}(Q)$ and $h \in \mathcal{L}^{\Psi^*}(Q)$, see [12, p.6].

For a cube $Q \subset \mathbb{R}^n$ and $\gamma > 0$, we also denote $\|f\|_{\exp L^{\gamma},Q}$ by

$$\|f\|_{\exp L^{\gamma},Q} = \inf\left\{t > 0 : \frac{1}{|Q|}\int_Q \exp\left(\frac{|f(y)|}{t}\right)^{\gamma} dy \le 2\right\}.$$

It is well known that, for $\Psi(t) = \log(e + t)$, its complementary function $\Psi^*(t) \approx e^t - 1$. Let $b \in BMO(\mathbb{R}^n)$. Then the John-Nirenberg inequality [6] tells us that for any $Q \subset \mathbb{R}^n$ and $p \in [1, \infty)$,

$$\|b - \langle b\rangle_Q\|_{\exp L^{1/p},Q} \lesssim \|b\|^p_{BMO(\mathbb{R}^n)}.$$

This, together with the inequality (2.3), shows that

$$\frac{1}{|Q|}\int_Q |b(x) - \langle b\rangle_Q|^p |h(x)|^p\, dx \lesssim \|h\|^p_{L^p(\log L),Q}\|b\|^p_{BMO(\mathbb{R}^n)}. \tag{2.4}$$

3. Main proof

Proof. Without loss of generality, we assume that $P(x)$ is non constant and $P(0) = 0$. In order to prove (1.4), we shall use induction on $deg(P)$.

When $deg(P) = 0$, it is easy to check by [4, Lemma 3.2]. For $deg(P) = 1$, we view the operator $T_{P,\Omega,A}$ as $T_{\Omega,A}$ due to the fact that we can always make the linear term of P adsorbed by f. Then, we get the L^p boundedness of $T_{P,\Omega,A}$ when $deg(P) \le 1$.

Then, for a $d \ge 2$, we assume (1.4) holds for all P with $deg(P) \le d - 1$.

Final step: We now assume that $deg(P) = d$, i.e.,

$$P(x) = \sum_{|\alpha|\le d} a_\alpha x^\alpha,$$

with $\sum_{|\alpha|=d} |a_\alpha| \ne 0$. Let $\sum_{|\alpha|=d} |a_\alpha| = t_0^d$. By rescaling,

$$T_{P,\Omega,A}f(x) = t_0^n \text{p.v.} \int_{\mathbb{R}^n} e^{iP(\frac{x-y}{t_0})}\frac{\Omega(x-y)}{|x-y|^n}\frac{\left[A(\frac{x}{t_0}) - A(\frac{y}{t_0}) - \nabla A(\frac{y}{t_0})\cdot(\frac{x-y}{t_0})\right]}{\frac{|x-y|}{t_0}} f(\frac{y}{t_0})dy, \tag{3.1}$$

and it is easy check that $\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)} = \|\nabla A(\frac{\cdot}{t_0})\|_{\mathrm{BMO}(\mathbb{R}^n)}$. For convenience, we may assume that $\sum_{|\alpha|=d} |a_\alpha| = 1$, and let

$$R(x) = P(x) - \sum_{|\alpha|=d} a_\alpha x^\alpha.$$

Then, let $deg(R) \le d - 1$, and for $0 < h \le 8$,

$$\begin{aligned}|T_{P,\Omega,A,h}f - T_{R,\Omega,A,h}f| \lesssim \int_{|x-y|\le 8} &|x-y|^d \frac{|\Omega(x-y)|}{|x-y|^n}|f(y)| \\ &\times \frac{|A(x) - A(y) - \nabla A(y)\cdot(x-y)|}{|x-y|}dy.\end{aligned}$$

According to Lemma 3,

$$\begin{aligned}
&\int_{|x-y|\leq 8} |x-y|^d \Big|\frac{\Omega(x-y)}{|x-y|^n}\Big| \frac{|A(x)-A(y)-\nabla A(y)\cdot(x-y)|}{|x-y|}|f(y)|dy \\
&\lesssim \sum_{j=-\infty}^{2} \int_{2^j<|x-y|\leq 2^{j+1}} |x-y|^d \frac{1}{|x-y|^n} \frac{|A(x)-A(y)-\nabla A(y)\cdot(x-y)|}{|x-y|}|f(y)|dy \\
&\lesssim \sum_{j=-\infty}^{2} 2^{jd} \int_{2^j<|x-y|\leq 2^{j+1}} \frac{1}{|x-y|^n} \frac{|A(x)-A(y)-\nabla A(y)\cdot(x-y)|}{|x-y|}|f(y)|dy \\
&\lesssim \sum_{j=-\infty}^{2} 2^{jd} \times \frac{1}{2^{j(n+1)}} \int_{2^j<|x-y|\leq 2^{j+1}} |A(x)-A(y)-\nabla A(y)\cdot(x-y)||f(y)|dy \\
&\lesssim \sum_{j=-\infty}^{2} 2^{jd} \times N_A(f)(x).
\end{aligned}$$

Thus, we have

$$\|T_{P,\Omega,h}f - T_{R,\Omega,h}f\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{j=-\infty}^{2} 2^{jd} \|N_A f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

By the Assumption

$$\|T_{P,\Omega,A,h}f\|_{L^p(\mathbb{R}^n)} \lesssim \|T_{R,\Omega,A,h}f\|_{L^p(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}. \tag{3.2}$$

For the rest of this proof, we assume that $h > 8$. Let $I_j = [2^j, 2^{j+1}]$, for $j \geq 3$, Using a partition of unity and considering

$$\psi_j(x) = \chi_{I_j}(|x|)\frac{\Omega(x')}{|x|^{n+1}}$$

with characteristic function χ_{I_j}. Now we turn to consider the following operator:

$$P_j f(x) = \int_{\mathbb{R}^n} e^{iP(x-y)}\psi_j(x-y)[A(x)-A(y)-\nabla A(y)\cdot(x-y)]f(y)dy.$$

However we can't find the Fourier transform of $P_j f$. By the idea from [4], we only need to use the operator

$$S_j f(x) = \int_{\mathbb{R}^n} e^{iP(x-y)}\psi_j(x-y)f(y)dy$$

to get the estimate of $P_j f$, which will be discussed in Section 4.

Then,

$$\widehat{S_j f}(\xi) = m_j(\xi)\hat{f}(\xi),$$

where

$$m_j(\xi) = \int_{\mathbb{R}^n} e^{i(P(x)-2\pi\xi\cdot x)}\psi_j(x)dx.$$

For each j, we set $R = 2^{j+2}$, and

$$P(x) - 2\pi\xi\cdot x = \sum_{k=1}^{n}\Big(\frac{\partial P}{\partial x_k}(0) - 2\pi\xi_k\Big)x_k + \sum_{2\leq|\alpha|\leq d} a_\alpha x^\alpha.$$

It is easy to check that

$$\Lambda := \sum_{k=1}^{n} |\left(\frac{\partial P}{\partial x_k}(0) - 2\pi\xi_k\right) R| + \sum_{2\le|\alpha|\le d} |a_\alpha| R^{|\alpha|} \ge \left(\sum_{|\alpha|=d} |a_\alpha|\right) R^d = R^d,$$

and hence $R\Lambda^{-1/d} \le 1$. So, we apply Lemma 1 and get the following estimate that

$$|m_j(\xi)| \lesssim \sup_{v\in B(0,1)} \int_{\mathbb{R}^n} |\psi_j(x) - \psi_j(x-v)| dx$$

for any $\xi \in \mathbb{R}^n$.

Due to the estimate

$$\begin{aligned}
|\psi_j(x) - \psi_j(x-v)| =& \left|\chi_{I_j}(x)\frac{\Omega(x)}{|x|^{n+1}} - \chi_{I_j}(x-v)\frac{\Omega(x-v)}{|x-v|^{n+1}}\right| \\
\le& \frac{|\Omega(x) - \Omega(x-v)|}{|x|^{n+1}}\chi_{I_j}(|x|) + |\Omega(x-v)| \left|\frac{1}{|x|^{n+1}} - \frac{1}{|x-v|^{n+1}}\right| \chi_{I_j}(|x|) \\
&+ \frac{|\Omega(x-v)|\chi_{I_j}(x-v)}{|x-v|^{n+1}} \left|\chi_{I_j}(|x|) - \chi_{I_j}(|x-v|)\right|,
\end{aligned}$$

we get

$$\begin{aligned}
|m_j(\xi)| \lesssim& sup_{v\in B(0,1)} \int_{\mathbb{R}^n} |\psi_j(x) - \psi_j(x-v)| dx \\
\lesssim& I_1 + I_2 + I_3.
\end{aligned}$$

For the term I_1, due to the fact that $\Omega \in L^1(S^{n-1})$ is a homogeneous function of degree zero on $\mathbb{R}^n$ and $\theta-$Lipschitz condition,

$$\begin{aligned}
\int_{\mathbb{R}^n} \chi_{I_j}(|x|)\frac{|\Omega(x) - \Omega(x-v)|}{|x|^{n+1}} dx =& 2^{-j} \int_{2^j<|x|\le 2^{j+1}} \frac{|\Omega(x) - \Omega(x-v)|}{|x|^n} dx \\
=& 2^{-j} \int_{2^j}^{2^{j+1}} \int_{S^{n-1}} \frac{|\Omega(rx') - \Omega(rx'-v)|}{r} d\sigma(x') dr.
\end{aligned}$$

Now, let $\delta = \frac{1}{r}$, and due to the fact that $\Omega(rx) = \Omega(x)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \chi_{I_j}(|x|)\frac{|\Omega(rx') - \Omega(rx - r\delta v)|}{|x|^{n+1}} dx \le& 2^{-j} \int_{\frac{1}{2^{j+1}}}^{\frac{1}{2^j}} \frac{\int_{S^{n-1}} |\Omega(x') - \Omega(x' - v\delta)| d\sigma(x')}{\delta} d\delta \\
\le& 2^{-j} \int_{\frac{1}{2^{j+1}}}^{\frac{1}{2^j}} \frac{\bar{\omega}(\delta)}{\delta} d\delta = 2^{-j} \int_{\frac{1}{2^{j+1}}}^{\frac{1}{2^j}} \delta^{\theta-1} d\delta \lesssim \frac{1}{2^{j(1+\theta)}}.
\end{aligned}$$

For the term I_2, we have a trivial estimation:

$$\left||x-v|^{n+1} - |x|^{n+1}\right| \lesssim |v|\,|x-tv|^n \lesssim |x|^n,$$

where $0 < t < 1$, $|x| > 8$, and $|v| \leq 1$. Then, for $j \geq 3$, we get $c|x| \leq |x - v|$, and we have $\Omega \in L^1(S^{n-1})$. So we have the following estimate:

$$\begin{aligned}
&\int_{2^j<|x|\leq 2^{j+1}} |\Omega(x-v)| \left| \frac{1}{|x|^{n+1}} - \frac{1}{|x-v|^{n+1}} \right| dx \\
\leq &\int_{2^j<|x|\leq 2^{j+1}} |\Omega(x-v)| \frac{\left| |x-v|^{n+1} - |x|^{n+1} \right|}{|x|^{n+1}|x-v|^{n+1}} dx \\
\lesssim &\int_{2^{j-1}<|x-v|\leq 2^{j+2}} |\Omega(x-v)| \frac{1}{|x||x-v|^{n+1}} dx \\
\lesssim &\frac{1}{2^j} \int_{2^{j-1}<|x|\leq 2^{j+1}} \frac{|\Omega(x)|}{|x|^{n+1}} dx = \frac{1}{2^j} \int_{2^{j-1}}^{2^{j+1}} \int_{S^{n-1}} \frac{|\Omega(x')|}{r^2} d\sigma(x') dr \\
\lesssim &\frac{1}{2^{2j}}.
\end{aligned}$$

Now, we turn to the estimate of the term I_3. For any $v \in B(0,1)$ and $j \geq 3$, we get the following relation:

$$[B(0,2^{j+1})\backslash B(0,2^j)] \triangle [B(v,2^{j+1})\backslash B(v,2^j)] \subset A \bigcup B,$$

where $A = \left\{x : 2^j - 1 \leq |x - v| \leq 2^j + 1\right\}$ and $B = \left\{x : 2^{j+1} - 1 \leq |x - v| \leq 2^{j+1} + 1\right\}$.

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|\Omega(x-v)|}{|x-v|^{n+1}} \left| \chi_{I_j}(|x|) - \chi_{I_j}(|x-v|) \right| dx \leq &\frac{1}{2^j} \int_{\mathbb{R}^n} \frac{|\Omega(x-v)|}{|x-v|^{n+1}} (\chi_A(x) + \chi_B(x)) dx \\
\leq &\frac{1}{2^j}(I_{31} + I_{32}).
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|\Omega(x-v)|}{|x-v|^{n+1}} \chi_A(x) dx = &\int_{2^j-1}^{2^j+1} \int_{S^{n-1}} \frac{|\Omega(x')|}{r} d\sigma(x') dr \\
\lesssim &\frac{1}{2^j} \int_{S^{n-1}} |\Omega(x')| d\sigma(x'),
\end{aligned}$$

and we also have $I_{32} \leq \frac{1}{2^j} \int_{S^{n-1}} |\Omega(x')| d\sigma(x')$.
Above all, by the Plancherel theorem, we have

$$\|S_j f\|_{L^2(\mathbb{R}^n)} \lesssim (I_1 + I_2 + I_3) \|f\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-j(1+\theta_0)} \|\Omega\|_{L^1(\mathbb{S}^{n-1})} \|f\|_{L^2(\mathbb{R}^n)}. \tag{3.3}$$

Furthermore, we can check that there exists a small enough ϵ such that

$$\|P_j f\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-j(\theta_0 - \epsilon)} \|f\|_{L^2(\mathbb{R}^n)}. \tag{3.4}$$

On the other hand, we have

$$\|P_j f\|_{L^p(\mathbb{R}^n)} \lesssim \|N_{\Omega,A} f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}. \tag{3.5}$$

So, interpolate (3.4) with (3.5), and we obtain that for some $\zeta > 0$,

$$\|P_j f\|_{L^p(\mathbb{R}^n)} \leq C 2^{-\zeta j} \|\Omega\|_{L^1(\mathbb{S}^{n-1})} \|f\|_{L^p(\mathbb{R}^n)}, \quad for \quad 1 < p < \infty. \tag{3.6}$$

So, we have the following estimate:

$$\begin{aligned}
\|T_{P,\Omega,A,h} f\|_{L^p(\mathbb{R}^n)} &\lesssim \|T_{P,\Omega,A,8} f\|_{L^p(\mathbb{R}^n)} + \sum_{j=3}^{[\log_2 h]} \|P_j f\|_{L^p(\mathbb{R}^n)} + \|Mf\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \|T_{P,\Omega,A,8} f\|_{L^p(\mathbb{R}^n)} + \sum_{j=3}^{+\infty} \|P_j f\|_{L^p(\mathbb{R}^n)} + \|Mf\|_{L^p(\mathbb{R}^n)} \\
&\leq \|f\|_{L^p(\mathbb{R}^n)}
\end{aligned}$$

where $h > 8$ and $1 < p < \infty$.

By using

$$T_{P,\Omega,A,} f = \lim_{h \to \infty} T_{P,\Omega,A,h} f$$

interpreted in the distributional sense, we obtain

$$\|T_{P,\Omega,A,h} f\|_{L^p(\mathbb{R}^n)} \leq C_P \|f\|_{L^p(\mathbb{R}^n)}, \tag{3.7}$$

for all test functions f of inequality (3.7) followed by standard arguments.

Above all, the proof of (1.4) is now completed. □

4. The proof of (3.4)

Proof. For the operator P_j, we can rewrite it as

$$P_j f(x) = \int_{\mathbb{R}^n} \Theta_j(x-y)(A(x) - A(y) - \nabla A(y) \cdot (x-y)) f(y) dy,$$

where Θ_j is the inverse Fourier transform of m_j.

Observe that $\operatorname{supp} \Theta_j \subset \{x : |x| \leq 2^{j+2}\}$. If I is a cube having side length 2^j, and $f \in L^2(\mathbb{R}^n)$ with $\operatorname{supp} f \subset I$, then $P_j f \subset C_n I$. Furthermore, for general $f \in L^2(\mathbb{R}^n)$, a function f can be writen as

$$f = \sum_k f \chi_{Q_k},$$

where $\{Q_k\}_{k \in \mathbb{Z}^n}$ is a grid of disjoint cubes of side length 2^j partitioning $\mathbb{R}^n$, and

$$\|P_j(f)\|^2_{L^2(\mathbb{R}^n)} \lesssim \sum_k \|P_j(f\chi_{Q_k})\|^2_{L^2(\mathbb{R}^n)} \lesssim \sum_k \|(f\chi_{Q_k})\|^2_{L^2(\mathbb{R}^n)} = \|f\|^2_{L^2(\mathbb{R}^n)}.$$

Therefore, to prove (3.4), we may assume that $\operatorname{supp} f \subset I$ with I being a cube having side length 2^j. Let x_0 be a point on the boundary of $2C_n I$ and $A_I^*(y) = A(y) - \sum_{m=1}^n \langle \partial_m A \rangle_{(C_n I)} \cdot y_m$, and

$$A_I(y) = (A_I^*(y) - A_I^*(x_0)) \xi_I(y),$$

where $\zeta_I \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}(\zeta_I) \subset 1.5C_nI$, and $\zeta_I(x) \equiv 1$ when $x \in 100nI$. Observe that $\|\nabla\zeta_I\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{-j}$, and, due to Lemma 2,

$$\sup_{t\in[0,1]}\left|\nabla A(x_0 + t(y - x_0)) - \langle\nabla A\rangle_{C_nI}\right| \lesssim \|\nabla A\|_{\mathrm{BMO}} \lesssim 1$$

when $t \in (0, 1)$, which implies that

$$|\nabla(A_I(y - t(y - x_0))| \le |\nabla A(x_0 + t(y - x_0)) - \langle\nabla A\rangle_{100nI}||\xi_I| + |A_I^*(y) - A_I^*(x_0)||\nabla\xi_I| \le 1.$$

So, for all $y \in 100nI$,

$$|A_I(y) - A_I(x_0)| \le |\nabla(A_I(y - t(y - x_0))||y - x_0| \lesssim 2^j.$$

Furthermore,

$$\|A_I\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^j. \tag{4.1}$$

Write

$$P_jf(x) = A_I(x)S_jf(x) - S_j(A_If)(x) - \sum_{m=1}^{n}[h_m, S_j](f\partial_mA_I)(x), \tag{4.2}$$

where $h_m(x) = x_m - c_{I,m}$ (recall that x_m denotes the mth variable of x, and $c_{I,m}$ denotes the center of I and c_I), and then it follows from (4.1) that

$$\|A_IS_jf\|_{L^2(\mathbb{R}^n)} + \|S_j(A_If)\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-j\theta_0}\|f\|_{L^2(\mathbb{R}^n)}. \tag{4.3}$$

Applying the John-Nirenberg inequality, we know that

$$\|\partial_mA_I\|^2_{\exp L^{1/2},I} \lesssim 1.$$

Recall that $\operatorname{supp}[h_m, S_j](f\partial_mA_I) \subset 100dI$. It then follows from inequality (2.4) that

$$\begin{aligned}\|[h_m, S_j](f\partial_mA_I)\|_{L^2(\mathbb{R}^n)} &= \sup_{\|g\|_{L^2(\mathbb{R}^n)}\le1}\left|\int_{\mathbb{R}^n}\partial_mA_I(x)f(x)[h_m, S_j]g(x)dx\right| \\ &\le \|f\|_{L^2(\mathbb{R}^n)}\sup_{\substack{\|g\|_{L^2(\mathbb{R}^n)}\le1\\ \operatorname{supp} g\subset100dI}}\|\partial_mA_I[h_m, S_j]g\|_{L^2(I)} \\ &\le \|f\|_{L^2(\mathbb{R}^n)}\left(|I|\sup_{\substack{\|g\|_{L^2(\mathbb{R}^n)}\le1\\ \operatorname{supp} g\subset100dI}}\|[h, S_j]g\|^2_{L^2(\log L)^2,I}\right)^{1/2}.\end{aligned}$$

Now let $g \in L^2(\mathbb{R}^n)$ with $\|g\|_{L^2(\mathbb{R}^n)} \le 1$ and $\operatorname{supp} g \subset 100dI$. Observe that

$$|\Theta_j| \le \sup_{2^j\le|x|\le2^{j+1}}\left|\frac{\Omega(x)}{|x|^{n+1}}\right| \lesssim \frac{1}{2^{j(n+1)}}.$$

This imply that

$$\|S_jg\|_{L^\infty(\mathbb{R}^n)} \lesssim \|\Theta_j\|_{L^\infty(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)} \lesssim \frac{1}{2^{j(n+1)}}\|g\|_{L^1(\mathbb{R}^n)}.$$

Furthermore, we have the following estimate:

$$\begin{aligned}\|\chi_I[h_m, S_j]g\|_{L^\infty(\mathbb{R}^n)} \lesssim& \|\chi_I h_m S_j g\|_{L^\infty(\mathbb{R}^n)} + \|\chi_I S_j(h_m g)\|_{L^\infty(\mathbb{R}^n)}\\ \lesssim& 2^j\|S_j g\|_{L^\infty(\mathbb{R}^n)} + \|\Theta_j\|_{L^\infty(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}\\ \lesssim& \frac{1}{2^{\frac{n}{2}j}},\end{aligned}$$

since $supp(g) \subset I$ and

$$\|g\|_{L^1(\mathbb{R}^n)} \lesssim |I|^{1/2}\|g\|_{L^2(\mathbb{R}^n)} \lesssim 2^{\frac{nj}{2}}.$$

On the other hand, it is easy to check that

$$\begin{aligned}\|[h_m, S_j]g\|_{L^2(\mathbb{R}^n)} \lesssim& \|\chi_I h_m S_j g\|_{L^2(\mathbb{R}^n)} + \|\chi_I S_j(h_m g)\|_{L^2(\mathbb{R}^n)}\\ \lesssim& 2^{-j\theta_0}\|g\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Set $\lambda_0 = [2^{-j\theta_0}]^2 2^{-jn}\log^2\left(e + 2^{j\theta_0}\right)$. A straightforward computation tells us that

$$\begin{aligned}&\int_I |[h_m, S_j]g(x)|^2 \log^2\left(e + \frac{|[h_m, S_j]g(x)|}{\sqrt{\lambda_0}}\right)dx\\ \lesssim& \log^2(e + 2^{j\theta_0})\int_I |[h_m, S_j]g(x)|^2 dx\\ \lesssim& \left[2^{-j\theta_0}\log(e + 2^{j\theta_0})\right]^2\\ \lesssim& \lambda_0 2^{jn},\end{aligned}$$

due to the face that
$$\frac{\|\chi_I[h_m, S_j]g\|_{L^\infty(\mathbb{R}^n)}}{\sqrt{\lambda_0}} \lesssim \frac{2^{-(\frac{n}{2})j}}{2^{-j\theta_0}2^{-\frac{nj}{2}}\log(e + 2^{j\theta_0})} \lesssim 2^{j\theta_0}.$$

This tells us that

$$\|[h_m, S_j]g\|_{L^2(\log L)^2, I} \lesssim \sqrt{\lambda_0},$$

and thus

$$\|[h_m, S_j](f\partial_m A_I)\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-j\theta_0}\log\left(e + 2^{j\theta_0}\right)\|f\|_{L^2(\mathbb{R}^n)}.$$

So there exist an ϵ, which is small enough, such that

$$\|[h_m, S_j](f\partial_m A_I)\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-j(\theta_0 - \epsilon)}\|f\|_{L^2(\mathbb{R}^n)}. \tag{4.4}$$

5. Conclusions

In this paper, we establish the $L^p(\mathbb{R}^n)$-boundedness for $1 < p < \infty$ of the oscillatory singular integral operator $T_{P,\Omega,A}$, under suitable assumptions on Ω and $\nabla A \in BMO$, which extends Pan's result [11] to non-standard kernels. In our proof, we use a novel oscillatory estimate to replace van der Corput lemma.

Author contributions

Jiawei Shen: Writing—original draft preparation, writing—review and editing, funding acquisition; Jie Yang: Writing—original draft preparation, writing—review and editing. Both authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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