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Research article

Structural properties and estimation of bounds for Jensen gaps and Hermite–Hadamard inequalities associated with σ-convex functions

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Abstract: Generalized convexity plays a prolific role in the development and refinement of inequalities. This manuscript assesses some key regularity properties of σ convex functions and related generalized fundamental inequalities. Particularly, we proved the relation between σ-convex functions and classical convexity. Based on this result, we discussed the first-order characterization of σ-convex functions. Next, we derived some new refinements of Jensen's inequality, including an interpolating Jensen inequality and bounds of Jensen's functional. By taking the benefit of σ-convexity, we constructed general a Hermite–Hadamard inequality and its weighted analog for symmetric functions as well as and several other inequalities. Our results are novel due to their unifying characteristics because several robust estimates and inequalities can be recaptured for different choices of the transformation function σ. Finally, we addressed some useful applications to normed spaces, entropy measures, and the general mean inequality.

Keywords: Convex function; Jensen inequality; Hermite–Hadamard inequality; Quasi-mean; Hölder inequality; Shannon entropy
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1. Introduction

Analysis of functions that preserve convexity is a foundational aspect of convex analysis. They are instrumental in handling the diverse and challenging problems of analysis and geometry due to their several distinctive and featured properties. To make these concepts more accessible and applicable, several advancements have been made in the literature. The nonlinear means provide the framework to generalize and assess the classical concepts of convexity. One can interpret the role of convexity as the behavior of functions under various means. Working in this direction, several classes of convexity have been introduced and explored from diverse aspects. Classical convexity (A–A), log convexity (A–G), harmonic convexity (H–A), and geometric–arithmetic convexity (G–A) are the commonly known classes of convexity. Recently, several approaches, including bi-mappings, generalized convex structures, and nonlinear weighted means, have been made. In 2019, Wu and his coauthors [1] utilized the idea of the quasi-mean to unify several known classes, stated as follows:

Definition 1.1. *Let σ is a strictly monotonic continuous function. Then any function $\Phi : [\mathrm{w}, \tau] \to \mathbb{R}$ is said to be σ-convex, if*

$$\Phi\left(\sigma^{-1}\left((1-\beta)\sigma(\vartheta)+\beta\sigma(\omega)\right)\right) \leq (1-\beta)\Phi(\vartheta)+\beta\Phi(\omega), \quad \forall \vartheta, \omega \in [\mathrm{w}, \tau].$$

From Definition 1.1, one can conclude that every σ-convex function is a convex function, harmonically convex function, p-convex function, and geometric-arithmetic function for the transformation function $\sigma(\vartheta) = \vartheta, 1/\vartheta, \vartheta^p, \ln(\vartheta)$, respectively.

Now, we give some important classes of convexities originated from Definition 1.1.

1) For $\sigma(\vartheta) = \frac{1}{\vartheta-\lambda}$, we get the class of λ-harmonic convex functions defined in [2],

$$\Phi\left(\frac{1}{\frac{(1-\beta)}{\vartheta-\lambda}+\frac{\beta}{\omega-\lambda}}+\lambda\right) \leq (1-\beta)\Phi(\vartheta)+\beta\Phi(\omega), \quad \forall \vartheta, \omega \in [w, \tau].$$

2) For $\sigma(\vartheta) = \ln(\vartheta-\lambda)$, we get the class of λ-geometric convex functions

$$\Phi\left((\vartheta-\lambda)^{(1-\beta)}(\omega-\lambda)^{\beta}+\lambda\right) \leq (1-\beta)\Phi(\vartheta)+\beta\Phi(\omega), \quad \forall \vartheta, \omega \in [w, \tau].$$

3) For $\sigma(\vartheta) = (\vartheta-\lambda)^p$, we get the class of (λ, p)-convex functions

$$\Phi\left(((1-\beta)(\vartheta-\lambda)^p+\beta(\omega-\lambda)^p)^{\frac{1}{p}}+\lambda\right) \leq (1-\beta)\Phi(\vartheta)+\beta\Phi(\omega), \quad \forall \vartheta, \omega \in [w, \tau].$$

4) For $\sigma(\vartheta) = \vartheta^2$, we get the class of quadratically convex functions

$$\Phi\left(\sqrt{(1-\beta)\vartheta^2+\beta\omega^2}\right) \leq (1-\beta)\Phi(\vartheta)+\beta\Phi(\omega), \quad \forall \vartheta, \omega \in [w, \tau].$$

5) For $\sigma(\vartheta) = \vartheta^{-2}$, we get the class of quadratically harmonic convex functions

$$\Phi\left(\frac{1}{\sqrt{\frac{(1-\beta)}{\vartheta^2}+\frac{\beta}{\omega^2}}}\right) \leq (1-\beta)\Phi(\vartheta)+\beta\Phi(\omega), \quad \forall \vartheta, \omega \in [w, \tau].$$

In [1], they proved that the sum and product of similarly ordered σ-convex functions are also σ-convex functions. Incorporating these properties, several refinements of Hermite–Hadamard-type inequalities have been developed. Moreover, their bounded, continuity, Lipschitzian, and composition properties have been discussed in [3]. For more details, see [4, 5].

The contributions of inequalities in diverse domains of analysis showcase the necessity and utility of them because several foundational concepts are explored through inequalities. In the progression of this subject, the concept of convexity of functions have a central role. The proof of various fundamental inequalities through convexity asserts its significance and dominance as compared to other classes of functions. For more details, see [6–8].

The general Hermite–Hadamard's inequality pertaining to σ-convexity is described as follows:

Theorem 1.1 ([1]). *Let $\Phi : [\mathrm{w}, \tau] \to \mathbb{R}$ be a σ-convex function. Then the following double inequality holds:*

$$\Phi\left(\sigma^{-1}\left(\frac{\sigma(\mathrm{w})+\sigma(\tau)}{2}\right)\right) \leq \frac{1}{(\sigma(\tau)-\sigma(\mathrm{w}))}\int_{\sigma(\mathrm{w})}^{\sigma(\tau)} \Phi\circ\sigma^{-1}(\vartheta)\mathrm{d}\vartheta \leq \frac{\Phi(\mathrm{w})+\Phi(\tau)}{2}. \tag{1.1}$$

The Fejér inequality for σ-convexity is given follows:

Theorem 1.2 ([4]). *Let $\Phi : [\mathrm{w}, \tau] \to \mathbb{R}$ be a σ-convex function and g be a symmetric functions about $\frac{\sigma(\mathrm{w})+\sigma(\tau)}{2}$. Then the following weighted double inequality holds:*

$$\Phi\left(\sigma^{-1}\left(\frac{\sigma(\mathrm{w})+\sigma(\tau)}{2}\right)\right)\int_{\sigma(\mathrm{w})}^{\sigma(\tau)} g\circ\sigma^{-1}(\vartheta)\mathrm{d}\vartheta \leq \int_{\sigma(\mathrm{w})}^{\sigma(\tau)} \Phi\circ g(\sigma^{-1}(\vartheta))\mathrm{d}\vartheta \leq \frac{\Phi(\mathrm{w})+\Phi(\tau)}{2}\int_{\sigma(\mathrm{w})}^{\sigma(\tau)} g\circ\sigma^{-1}(\vartheta)\mathrm{d}\vartheta.$$

In [9], the authors examined some multiplicative formations of inequalities via σ-convexity. Moreover, some reverse inequalities associated with σ-convexity were discussed in [10].

Next, we reproduce the Jensen's inequality for σ-convexity.

Theorem 1.3 ([4]). *Let $\Phi : [\mathrm{w}, \tau] \to \mathbb{R}$ be a σ-convex function. Then*

$$\Phi\left(\sigma^{-1}\left(\sum_{i=1}^{\varrho}\beta_i\sigma(\vartheta_i)\right)\right) \leq \sum_{i=1}^{\varrho}\beta_i\Phi(\vartheta_i), \quad \forall\vartheta_i \in [\mathrm{w}, \tau] \text{ and } \sum_{i=1}^{\varrho}\beta_i = 1.$$

Various studies have been carried out to estimate Jensen's gaps; for further information, see [11–15].

The above discussion purports that σ-convexity unifies several existing classes of convexity. From the literature, we observe that various structural properties and inequality results have not been explored. The main motivation of this work is to assess some regularity properties, including relation with classical convexity and first-order differentiability characterization. Additionally, we prove several new refinements of the Jensen inequality, the converse Jensen inequality, bounds of Jensen's functionals, the Hermite–Hadamard inequality and its various variants, including Fejér inequality. Additionally, we look at some distinctive applications in analysis to highlight the significance of the result. The novelty of this work is that it offers a general framework to approximate the bounds of various classical inequalities and diverse directions of applications. It is interesting to note that we have presented some new convexity structure using appropriate transformation functions. By following a similar strategy, several new generalized convexity classes can be obtained. We have structured our study in four main parts. First, we revisit some preliminary concepts, including the need and background of the problem as well as the motivation. The next sections include a few intrinsic properties of σ-convexity and various general bounds of cumulative gaps of Jensen's inequality through diverse approaches. Additionally, we will prove a few generic versions of the Hermite–Hadamard inequality and some of its counterparts. In the last part, we will focus on analyzing their applicability. This study will play a central role in developing more robust and unified estimates of several other convex inequalities.

2. Results and discussions

In this section, we present some essential properties of σ-convexity and related inequalities.

2.1. Properties of σ-convex functions

To prove our main results, we recall an interesting result of classical convexity from [8]. Let $\Phi : I = [w, \tau] \to \mathbb{R}$ be a differentiable convex function. Then for any $\vartheta \in I$, the following inequality holds:

$$\Phi(\beta) \geq \Phi(\vartheta) + \Phi'(\vartheta)(\beta - \vartheta) \quad \forall \beta \in I.$$

Theorem 2.1. *A function Φ is σ-convex on I if and only if the function $B = \Phi \circ \sigma^{-1}$ is a convex on $\sigma(I) = [\sigma(\mathrm{w}), \sigma(\tau)]$.*

Proof. Assume that Φ is σ-convex function, and consider $B = \Phi \circ \sigma^{-1}$. We choose $u = \sigma(\vartheta)$ and $v = \sigma(\omega)$ so that

$$\begin{aligned} B\left((1-\beta)u + \beta v\right) &= \Phi\left(\sigma^{-1}(1-\beta)\sigma(\vartheta) + \beta\sigma(\omega)\right) \\ &\leq (1-\beta)\Phi(\vartheta) + \beta\,\Phi(\omega) \\ &= (1-\beta)\Phi \circ \sigma^{-1}(\sigma(\vartheta)) + \beta\,\Phi \circ \sigma^{-1}(\sigma(\omega)) \\ &= (1-\beta)B(u) + \beta B(v). \end{aligned}$$

This implies that B is a convex function on $\sigma(I)$.
Now, assume that $B = \Phi \circ \sigma^{-1}$ is a convex function on $\sigma(I)$. Then,

$$\begin{aligned} \Phi\left(\sigma^{-1}(1-\beta)\sigma(\vartheta) + \beta\sigma(\omega)\right) &= B\left((1-\beta)\sigma(\vartheta) + \beta\sigma(\omega)\right) \\ &\leq (1-\beta)B(\sigma(\vartheta)) + \beta\,B(\sigma(\omega)) \\ &= (1-\beta)\Phi(\vartheta) + \beta\,\Phi(\omega). \end{aligned}$$

This implies that Φ is the σ-convex function, which completes the proof. □

The immediate result based on the above Theorem 2.1 is stated in the following remark.

Remark 2.1. *If σ and Φ are twice differentiable functions, and $\sigma'(\vartheta) \neq 0$, then Φ is σ-convex if and only if*

$$\sigma'(\vartheta)\left[\Phi''(\vartheta)\sigma'(\vartheta) - \Phi'(\vartheta)\sigma''(\vartheta)\right] \geq 0, \qquad \vartheta \in I.$$

Theorem 2.2. *Let $\Phi : I \to \mathbb{R}$ be a differentiable function; $\sigma(\vartheta)$ be a differentiable, strictly monotonic function; and $\sigma'(\vartheta) \neq 0$. Then Φ is σ-convex if and only if for every $\omega \in I$,*

$$\Phi(\vartheta) \geq \Phi(\omega) + \frac{\Phi'(\omega)}{\sigma'(\omega)}\left(\sigma(\vartheta) - \sigma(\omega)\right), \ \forall \vartheta \in \mathbb{R}. \tag{2.1}$$

Proof. Assume that Φ is σ-convex. Then $B = \Phi \circ \sigma^{-1}$ is convex on $\sigma(I)$.
Choose $\omega \in I$ and $u = \sigma(\omega)$. By the convexity, we have

$$B(\beta) \geq B(u) + B'(u)(\beta - u), \ \forall \beta \in \sigma(I).$$

Selecting $\beta = \sigma(\vartheta)$,

$$\begin{aligned}
&B(\sigma(\vartheta)) \geq B(\sigma(\omega)) + B'(\sigma(\omega))(\sigma(\vartheta) - \sigma(\omega))\\
&\Phi \circ \sigma^{-1}(\sigma(\vartheta)) \geq \Phi \circ \sigma^{-1}(\sigma(\omega)) + (\Phi \circ \sigma^{-1})'(\sigma(\omega))(\sigma(\vartheta) - \sigma(\omega))\\
&\Phi(\vartheta) \geq \Phi(\omega) + \frac{\Phi'(\omega)}{\sigma'(\omega)}\left(\sigma(\vartheta) - \sigma(\omega)\right).
\end{aligned}$$

Now, assume that inequality (2.1) holds for all $\vartheta, \omega \in I$.

Let $u, v \in \sigma(I)$ and choose $\vartheta = \sigma^{-1}(u)$ and $\omega = \sigma^{-1}(v)$. Therefore, (2.1) becomes

$$\begin{aligned}
&\Phi(\sigma^{-1}(u)) \geq \Phi(\sigma^{-1}(v)) + \frac{\Phi'(\sigma^{-1}(v))}{\sigma'(\sigma^{-1}(v))}\,(u - v)\\
&B(u) \geq B(v) + B'(v)(u - v).
\end{aligned}$$

This implies that B is a convex function on $\sigma(I)$. Then, by Theorem 2.1, Φ is σ-convex on I, which completes the result. □

2.2. Refinements of Jensen's inequality

Theorem 2.3. *Let $\Phi : [\mathrm{w}, \tau] \to \mathbb{R}$ be a σ-convex function, $\vartheta_\lambda \in [\mathrm{w}, \tau]$, and $\beta_\lambda \geq 0$ for each $\lambda \in \{1, 2, \cdots, \varrho\}$ with $P_\varrho := \sum_{\lambda=1}^{\varrho} \beta_\lambda > 0$. Then,*

$$\begin{aligned}
\Phi\left(\sigma^{-1}\left(\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\right)\right) &\leq \frac{1}{{P_\varrho}^2}\sum_{\lambda=1}^{\varrho}\sum_{\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)\\
&\leq \frac{1}{P_\varrho^2}\left(\sum_{\lambda=1}^{\varrho}\beta_\lambda^2\Phi(\vartheta_\lambda) + 2\sum_{\lambda<\ell}\beta_\lambda\beta_\ell\frac{1}{\sigma(\vartheta_\ell)-\sigma(\vartheta_\lambda)}\int_{\sigma(\vartheta_\lambda)}^{\sigma(\vartheta_\ell)}\Phi(\sigma^{-1}(\vartheta))\,d\vartheta\right)\\
&\leq \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda). \qquad (2.2)
\end{aligned}$$

The inequality (2.2) is reversed whenever Φ is σ-concave.

Proof. Observe that the following equality holds:

$$\begin{aligned}
\frac{1}{P_\varrho^2}\sum_{\lambda=1}^{\varrho}\sum_{\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right) &= \frac{1}{P_\varrho^2}\left(\sum_{\lambda=1}^{\varrho}\beta_\lambda^2\Phi(\vartheta_\lambda) + \sum_{\lambda\neq\ell}\beta_\lambda\beta_\ell\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)\right)\\
&= \frac{1}{P_\varrho^2}\left(\sum_{\lambda=1}^{\varrho}\beta_\lambda^2\Phi(\vartheta_\lambda) + \sum_{\lambda<\ell}\beta_\lambda\beta_\ell\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)\right.\\
&\quad\left. + \sum_{\lambda>\ell}\beta_\lambda\beta_\ell\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)\right)\\
&= \frac{1}{P_\varrho^2}\left(\sum_{\lambda=1}^{\varrho}\beta_\lambda^2\Phi(\vartheta_\lambda) + 2\sum_{\lambda<\ell}\beta_\lambda\beta_\ell\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)\right). \qquad (2.3)
\end{aligned}$$

Without loss of generality, we consider that ϑ_λ are distinct points for $\lambda = 1, 2, \cdots, \varrho$. By using $\sigma(w) = \sigma(\vartheta_\lambda)$ and $\sigma(\tau) = \sigma(\vartheta_\ell)$ in (1.1), we achieve

$$\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right) \le \frac{1}{\sigma(\vartheta_\ell)-\sigma(\vartheta_\lambda)}\int_{\sigma(\vartheta_\lambda)}^{\sigma(\vartheta_\ell)} \Phi(\sigma^{-1}(\vartheta))\,d\vartheta \le \frac{\Phi(\vartheta_\lambda)+\Phi(\vartheta_\ell)}{2}. \tag{2.4}$$

Substituting (2.4) into (2.3), we obtain

$$\begin{aligned}
&\frac{1}{P_\varrho^2}\sum_{\lambda=1}^{\varrho}\sum_{\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)\\
&\le \frac{1}{P_\varrho^2}\left(\sum_{\lambda=1}^{\varrho}\beta_\lambda^2\Phi(\vartheta_\lambda) + 2\sum_{\lambda<\ell}\beta_\lambda\beta_\ell\frac{1}{\sigma(\vartheta_\ell)-\sigma(\vartheta_\lambda)}\int_{\sigma(\vartheta_\lambda)}^{\sigma(\vartheta_\ell)}\Phi(\sigma^{-1}(\vartheta))\,d\vartheta\right)\\
&\le \frac{1}{P_\varrho^2}\left(\sum_{\lambda=1}^{\varrho}\beta_\lambda^2\Phi(\vartheta_\lambda) + 2\sum_{\lambda<\ell}\beta_\lambda\beta_\ell\left(\frac{\Phi(\vartheta_\lambda)+\Phi(\vartheta_\ell)}{2}\right)\right).
\end{aligned} \tag{2.5}$$

Through the application of Jensen's inequality for σ-convexity, we get

$$\begin{aligned}
\Phi\left(\sigma^{-1}\left(\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\right)\right) &= \Phi\left(\sigma^{-1}\left(\frac{1}{P_\varrho^2}\left(\frac{2P_\varrho\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)}{2}\right)\right)\right)\\
&= \Phi\left(\sigma^{-1}\left(\frac{1}{P_\varrho^2}\left(\frac{P_\varrho\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)+P_\varrho\sum_{\ell=1}^{\varrho}\beta_\ell\sigma(\vartheta_\ell)}{2}\right)\right)\right)\\
&= \Phi\left(\sigma^{-1}\left(\frac{1}{P_\varrho^2}\left(\frac{\sum_{\lambda=1}^{\varrho}\sum_{\ell=1}^{\varrho}\beta_\lambda\beta_\ell\sigma(\vartheta_\lambda)+\sum_{\lambda=1}^{\varrho}\sum_{\ell=1}^{\varrho}\beta_\lambda\beta_\ell\sigma(\vartheta_\ell)}{2}\right)\right)\right)\\
&\le \frac{1}{P_\varrho^2}\sum_{\lambda=1}^{\varrho}\sum_{\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right).
\end{aligned} \tag{2.6}$$

The right part of (2.5) can be written as

$$\begin{aligned}
&\frac{1}{P_\varrho^2}\left(\sum_{\lambda=1}^{\varrho}\beta_\lambda^2\Phi(\sigma^{-1}(\sigma(\vartheta_\lambda)) + 2\sum_{\lambda<\ell}\beta_\lambda\beta_\ell\left(\frac{\Phi(\vartheta_\lambda)+\Phi(\vartheta_\ell)}{2}\right)\right)\\
&= \frac{1}{P_\varrho^2}\left(\sum_{\lambda=1}^{\varrho}\beta_\lambda^2\Phi(\vartheta_\lambda) + \sum_{\lambda<\ell}\beta_\lambda\beta_\ell\left(\frac{\Phi(\vartheta_\lambda)+\Phi(\vartheta_\ell)}{2}\right) + \sum_{\lambda>\ell}\beta_\lambda\beta_\ell\left(\frac{\Phi(\vartheta_\lambda)+\Phi(\vartheta_\ell)}{2}\right)\right)\\
&= \frac{1}{P_\varrho^2}\left(\sum_{\lambda=1}^{\varrho}\beta_\lambda^2\Phi(\vartheta_\lambda) + \sum_{\lambda\neq\ell}\beta_\lambda\beta_\ell\left(\frac{\Phi(\vartheta_\lambda)+\Phi(\sigma^{-1}(\sigma(\vartheta_\ell)))}{2}\right)\right)\\
&= \frac{1}{2P_\varrho^2}\sum_{\lambda=1}^{\varrho}\sum_{\ell=1}^{\varrho}\beta_\lambda\beta_\ell\left(\Phi(\vartheta_\lambda)+\Phi(\sigma^{-1}(\sigma(\vartheta_\ell)))\right)\\
&= \frac{1}{2P_\varrho^2}\left(P_\varrho\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda) + P_\varrho\sum_{\ell=1}^{\varrho}\beta_\ell\Phi(\sigma^{-1}(\sigma(\vartheta_\ell)))\right)\\
&= \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda).
\end{aligned} \tag{2.7}$$

Combining Eqs (2.5)–(2.7), we get (2.2). This finishes the proof. □

Remark 2.2. *Setting $\sigma(\vartheta) = \vartheta$ in Theorem 2.3, we get the Theorem 2.1, which is given in [15].*

$$\begin{aligned}
\Phi\left(\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda\right) &\leq \frac{1}{P_\varrho^2}\sum_{\lambda=1}^{\varrho}\sum_{\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\frac{\vartheta_\lambda+\vartheta_\ell}{2}\right)\\
&\leq \frac{1}{P_\varrho^2}\left(\sum_{\lambda=1}^{\varrho}\beta_\lambda^2\Phi(\vartheta_\lambda) + 2\sum_{\lambda<\ell}\beta_\lambda\beta_\ell\frac{1}{\vartheta_\ell-\vartheta_\lambda}\int_{\vartheta_\lambda}^{\vartheta_\ell}\Phi(\vartheta)\,d\vartheta\right)\\
&\leq \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda).
\end{aligned}$$

Setting $\sigma(\vartheta) = 1/\vartheta$ in Theorem 2.3, we get

$$\begin{aligned}
\Phi\left(\frac{1}{\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\frac{\beta_\lambda}{\vartheta_\lambda}}\right) &\leq \frac{1}{P_\varrho^2}\sum_{\lambda=1}^{\varrho}\sum_{\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\frac{2\vartheta_\lambda\vartheta_\ell}{\vartheta_\lambda+\vartheta_\ell}\right)\\
&\leq \frac{1}{P_\varrho^2}\left(\sum_{\lambda=1}^{\varrho}\beta_\lambda^2\Phi(\vartheta_\lambda) + 2\sum_{\lambda<\ell}\beta_\lambda\beta_\ell\frac{\vartheta_\lambda\vartheta_\ell}{\vartheta_\ell-\vartheta_\lambda}\int_{\vartheta_\lambda}^{\vartheta_\ell}\frac{\Phi(\vartheta)}{\vartheta^2}\,d\vartheta\right)\\
&\leq \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda).
\end{aligned}$$

Setting $\sigma(\vartheta) = \vartheta^p$ in Theorem 2.3, we get

$$\begin{aligned}
\Phi\left(\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda^p\right)^{\frac{1}{p}} &\leq \frac{1}{P_\varrho^2}\sum_{\lambda=1}^{\varrho}\sum_{\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\frac{\vartheta_\lambda^p+\vartheta_\ell^p}{2}\right)^{\frac{1}{p}}\\
&\leq \frac{1}{P_\varrho^2}\left(\sum_{\lambda=1}^{\varrho}\beta_\lambda^2\Phi(\vartheta_\lambda) + 2\sum_{\lambda<\ell}\beta_\lambda\beta_\ell\frac{1}{\vartheta_\ell^p-\vartheta_\lambda^p}\int_{\vartheta_\lambda^p}^{\vartheta_\ell^p}\Phi(\vartheta^{\frac{1}{p}})\,d\vartheta\right)\\
&\leq \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda).
\end{aligned}$$

Theorem 2.4. *Let $\Phi : I \to \mathbb{R}$ be a σ-convex function and $\sigma(\vartheta) = (\sigma(\vartheta_1), \ldots, \sigma(\vartheta_\varrho)) \in I^\varrho$. Define*

$$\Phi\left(\sigma_{\gamma,\varrho}^{-1}(\sigma(\vartheta),\beta)\right) = \frac{1}{P_\varrho\binom{\varrho-1}{\gamma-1}}\sum_{1\leq\lambda_1<\cdots<\lambda_\gamma\leq\varrho}(\beta_{\lambda_1}+\cdots+\beta_{\lambda_\gamma})\,\Phi\left(\sigma^{-1}\left(\frac{\beta_{\lambda_1}\sigma(\vartheta_{\lambda_1})+\cdots+\beta_{\lambda_\gamma}\sigma(\vartheta_{\lambda_\gamma})}{\beta_{\lambda_1}+\cdots+\beta_{\lambda_\gamma}}\right)\right),$$

where β_λ are positive numbers, and $P_\varrho = \sum_{\lambda=1}^{\varrho}\beta_\lambda$. Then,

$$\Phi\left(\sigma_{\gamma,\varrho}^{-1}(\sigma(\vartheta),\beta)\right) \geq \Phi\left(\sigma_{\gamma+1,\varrho}^{-1}(\sigma(\vartheta),\beta)\right), \qquad \gamma = 1,\ldots,\varrho-1. \tag{2.8}$$

Proof. Indeed, we have

$$(\beta_{\lambda_1}+\cdots+\beta_{\lambda_{\gamma+1}})\Phi\left(\sigma^{-1}\left(\frac{\beta_{\lambda_1}\sigma(\vartheta_{\lambda_1})+\cdots+\beta_{\lambda_{\gamma+1}}\sigma(\vartheta_{\lambda_{\gamma+1}})}{\beta_{\lambda_1}+\cdots+\beta_{\lambda_{\gamma+1}}}\right)\right)$$

$$= (\beta_{\lambda_1} + \cdots + \beta_{\lambda_{\gamma+1}})\Phi\left(\sigma^{-1}\left(\frac{\sum_{\ell=1}^{\gamma+1}(\beta_{\lambda_1} + \cdots + \beta_{\lambda_{\gamma+1}} - \beta_{\lambda_\ell})\frac{\beta_{\lambda_1}\sigma(\vartheta_{\lambda_1})+\cdots+\beta_{\lambda_{\gamma+1}}\sigma(\vartheta_{\lambda_{\gamma+1}})-\beta_{\lambda_\ell}\sigma(\vartheta_{\lambda_\ell})}{\beta_{\lambda_1}+\cdots+\beta_{\lambda_{\gamma+1}}-\beta_{\lambda_\ell}}}{\sum_{\ell=1}^{\gamma+1}(\beta_{\lambda_1} + \cdots + \beta_{\lambda_{\gamma+1}} - \beta_{\lambda_\ell})}\right)\right)$$

$$\leq (\beta_{\lambda_1} + \cdots + \beta_{\lambda_{\gamma+1}})\frac{\sum_{\ell=1}^{\gamma+1}\left(\beta_{\lambda_1} + \cdots + \beta_{\lambda_{\gamma+1}} - \beta_{\lambda_\ell}\right)\Phi\left(\sigma^{-1}\left(\frac{\beta_{\lambda_1}\sigma(\vartheta_{\lambda_1})+\cdots+\beta_{\lambda_{\gamma+1}}\sigma(\vartheta_{\lambda_{\gamma+1}})-\beta_{\lambda_\ell}\sigma(\vartheta_{\lambda_\ell})}{\beta_{\lambda_1}+\cdots+\beta_{\lambda_{\gamma+1}}-\beta_{\lambda_\ell}}\right)\right)}{\sum_{\ell=1}^{\gamma+1}\left(\beta_{\lambda_1} + \cdots + \beta_{\lambda_{\gamma+1}} - \beta_{\lambda_\ell}\right)}.$$

Because

$$\sum_{\ell=1}^{\gamma+1}(\beta_{\lambda_1} + \cdots + \beta_{\lambda_{\gamma+1}} - \beta_{\lambda_\ell}) = \gamma(\beta_{\lambda_1} + \cdots + \beta_{\lambda_{\gamma+1}});$$

then, we obtain

$$= \frac{1}{\gamma}\sum_{\ell=1}^{\gamma+1}\left(\beta_{\lambda_1} + \cdots + \beta_{\lambda_{\gamma+1}} - \beta_{\lambda_\ell}\right)\Phi\left(\sigma^{-1}\left(\frac{\beta_{\lambda_1}\sigma(\vartheta_{\lambda_1}) + \cdots + \beta_{\lambda_{\gamma+1}}\sigma(\vartheta_{\lambda_{\gamma+1}}) - \beta_{\lambda_\ell}\sigma(\vartheta_{\lambda_\ell})}{\beta_{\lambda_1} + \cdots + \beta_{\lambda_{\gamma+1}} - \beta_{\lambda_\ell}}\right)\right).$$

Therefore, we have

$$\begin{aligned}
\Phi\left(\sigma^{-1}_{\gamma+1,\varrho}\right) &= \frac{1}{P_\varrho\binom{\varrho-1}{\gamma}}\sum_{1\leq\lambda_1<\cdots<\lambda_{\gamma+1}\leq\varrho}\left(\beta_{\lambda_1} + \cdots + \beta_{\lambda_{\gamma+1}}\right)\Phi\left(\sigma^{-1}\left(\frac{\beta_{\lambda_1}\sigma(\vartheta_{\lambda_1}) + \cdots + \beta_{\lambda_{\gamma+1}}\sigma(\vartheta_{\lambda_{\gamma+1}})}{\beta_{\lambda_1} + \cdots + \beta_{\lambda_{\gamma+1}}}\right)\right)\\
&\leq \frac{1}{\gamma P_\varrho\binom{\varrho-1}{\gamma}}\sum_{1\leq\lambda_1<\cdots<\lambda_{\gamma+1}\leq\varrho}\sum_{\ell=1}^{\gamma+1}\left(\beta_{\lambda_1} + \cdots + \beta_{\lambda_{\gamma+1}} - \beta_{\lambda_\ell}\right)\\
&\qquad\times\Phi\left(\sigma^{-1}\left(\frac{\beta_{\lambda_1}\sigma(\vartheta_{\lambda_1}) + \cdots + \beta_{\lambda_{\gamma+1}}\sigma(\vartheta_{\lambda_{\gamma+1}}) - \beta_{\lambda_\ell}\sigma(\vartheta_{\lambda_\ell})}{\beta_{\lambda_1} + \cdots + \beta_{\lambda_{\gamma+1}} - \beta_{\lambda_\ell}}\right)\right)\\
&= \frac{1}{P_\varrho\binom{\varrho-1}{\gamma-1}}\sum_{1\leq\lambda_1<\cdots<\lambda_{\gamma}\leq\varrho}\left(\beta_{\lambda_1} + \cdots + \beta_{\lambda_\gamma}\right)\Phi\left(\sigma^{-1}\left(\frac{\beta_{\lambda_1}\sigma(\vartheta_{\lambda_1}) + \cdots + \beta_{\lambda_\gamma}\sigma(\vartheta_{\lambda_\gamma})}{\beta_{\lambda_1} + \cdots + \beta_{\lambda_\gamma}}\right)\right)\\
&= \Phi\left(\sigma^{-1}_{\gamma,\varrho}\right).
\end{aligned}$$

This completes the proof. □

Remark 2.3. *We note that the inequality* (2.8) *is an interpolating Jensen's inequality for σ-convex functions. Indeed, we have*

$$\Phi\left(\sigma^{-1}\left(\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\right)\right) = \Phi\left(\sigma^{-1}_{\varrho,\varrho}\right) \leq \cdots \leq \Phi\left(\sigma^{-1}_{\gamma+1,\varrho}\right) \leq \Phi\left(\sigma^{-1}_{\gamma,\varrho}\right) \leq \cdots \leq \Phi\left(\sigma^{-1}_{1,\varrho}\right)$$
$$= \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda). \tag{2.9}$$

The above results are also valid for σ-convex functions defined on an arbitrary real linear space, and if β_λ, $\lambda = 1, \ldots, \varrho$, are rational numbers, they are also valid for mid σ-convex functions defined on an arbitrary real linear space.

Theorem 2.5. *Let $\Phi : I = [\mathrm{w}, \tau] \to \mathbb{R}$ be a differentiable σ-convex function with $\sigma'(\vartheta) \neq 0$ on I, $\vartheta_\lambda \in I^o$, and $\beta_\lambda \geq 0$ with $P_\varrho > 0$. Then, we have the following inequality:*

$$\begin{aligned} 0 &\leq \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda) - \Phi\left(\sigma^{-1}\left(\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\right)\right) \\ &\leq \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\frac{\Phi'(\vartheta_\lambda)}{\sigma'(\vartheta_\lambda)} - \frac{1}{P_\varrho^2}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\sum_{\lambda=1}^{\varrho}\beta_\lambda\frac{\Phi'(\vartheta_\lambda)}{\sigma'(\vartheta_\lambda)}. \end{aligned} \tag{2.10}$$

Proof. Because Φ is σ-convex on I, it satisfies inequality (2.1). By substituting $\sigma(\vartheta) = \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)$ and $\sigma(\omega) = \sigma(\vartheta_\ell)$ in (2.1), we get

$$\Phi\left(\sigma^{-1}\left(\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\right)\right) - \Phi(\vartheta_\ell) \geq \left(\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda) - \sigma(\vartheta_\ell)\right)\frac{\Phi'(\vartheta_\ell)}{\sigma'(\vartheta_\ell)}. \tag{2.11}$$

Multiplying (2.11) by $\beta_\ell \geq 0$ and summing over $\ell = 1, 2, \cdots, n$, we determine

$$\begin{aligned} &\Phi\left(\sigma^{-1}\left(\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\right)\right) - \frac{1}{P_\varrho}\sum_{\ell=1}^{\varrho}\beta_\ell\Phi(\vartheta_\ell) \\ &\geq \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\frac{1}{P_\varrho}\sum_{\ell=1}^{\varrho}\beta_\ell\frac{\Phi'(\vartheta_\ell)}{\sigma'(\vartheta_\ell)} - \frac{1}{P_\varrho}\sum_{\ell=1}^{\varrho}\sigma(\vartheta_\ell)\frac{\Phi'(\vartheta_\ell)}{\sigma'(\vartheta_\ell)}\beta_\ell. \end{aligned}$$

After simple computations, we obtain (2.10). □

Remark 2.4. *Setting $\sigma(\vartheta) = \vartheta$ in Theorem 2.5, we get the result which is given in [16].*

$$0 \leq \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda) - \Phi\left(\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda\right) \leq \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda\Phi'(\vartheta_\lambda) - \frac{1}{P_\varrho^2}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi'(\vartheta_\lambda).$$

Setting $\sigma(\vartheta) = 1/\vartheta$, we get

$$0 \leq \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda) - \Phi\left(\frac{1}{\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\frac{1}{\vartheta_\lambda}}\right) \leq \frac{1}{P_\varrho^2}\sum_{\lambda=1}^{\varrho}\beta_\lambda\frac{1}{\vartheta_\lambda}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda^2\Phi'(\vartheta_\lambda) - \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda\Phi'(\vartheta_\lambda).$$

Setting $\sigma(\vartheta) = \vartheta^p$ in Theorem 2.5, we get

$$0 \leq \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda) - \Phi\left(\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda^p\right)^{\frac{1}{p}} \leq \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda\frac{\Phi'(\vartheta_\lambda)}{p} - \frac{1}{P_\varrho^2}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda^p\sum_{\lambda=1}^{\varrho}\beta_\lambda\frac{\Phi'(\vartheta_\lambda)}{p\vartheta_\lambda^{p-1}}.$$

Theorem 2.6. *Let $\Phi : I = [\mathrm{w}, \tau] \to \mathbb{R}$ be a differentiable σ-convex function with $\sigma'(\vartheta) \neq 0$ on I,$\vartheta_\lambda \in I$ and $\beta_\lambda \geq 0$ with $P_\varrho > 0$. Then, we have the following inequalities:*

$$\begin{aligned} 0 &\leq \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda) - \frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right) \\ &\leq \frac{1}{2}\left[\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\frac{\Phi'(\vartheta_\lambda)}{\sigma'(\vartheta_\lambda)} - \frac{1}{P_\varrho^2}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\sum_{\lambda=1}^{\varrho}\beta_\lambda\frac{\Phi'(\vartheta_\lambda)}{\sigma'(\vartheta_\lambda)}\right]. \end{aligned} \tag{2.12}$$

Proof. Because Φ is σ-convex, by using the result (2.9) for $\gamma = 1$, we have

$$\frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right) \leq \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda). \tag{2.13}$$

To prove the second part of (2.12), we substitute $\sigma(\vartheta) = \frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}$ and $\sigma(\omega) = \sigma(\vartheta_\lambda)$ in (2.1) such that

$$\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right) - \Phi(\vartheta_\lambda) \geq \frac{\Phi'(\vartheta_\lambda)}{\sigma'(\vartheta_\lambda)}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2} - \sigma(\vartheta_\lambda)\right)$$

for all $\ell = 1, ..., \varrho$. By multiplying this inequality with $\beta_\ell \geq 0$ and summing over $\ell = 1, 2, \cdots, n$, we determine that

$$\begin{aligned} &\frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right) - \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda)\right) \\ &\quad\geq \frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\left(\frac{\sigma(\vartheta_\ell)-\sigma(\vartheta_\lambda)}{2}\right)\frac{\Phi'(\vartheta_\lambda)}{\sigma'(\vartheta_\lambda)}. \end{aligned} \tag{2.14}$$

Through simple computation, we get

$$\begin{aligned} &\frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\left(\frac{\sigma(\vartheta_\ell)-\sigma(\vartheta_\lambda)}{2}\right)\frac{\Phi'(\vartheta_\lambda)}{\sigma'(\vartheta_\lambda)} \\ &\quad= \frac{1}{2P_\varrho^2}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\sum_{\lambda=1}^{\varrho}\beta_\lambda\frac{\Phi'(\vartheta_\lambda)}{\sigma'(\vartheta_\lambda)} - \frac{1}{2P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\frac{\Phi'(\vartheta_\lambda)}{\sigma'(\vartheta_\lambda)}. \end{aligned}$$

Therefore, (2.14) gains the following form:

$$\begin{aligned} &\frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right) - \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda) \\ &\quad\geq \frac{1}{2P_\varrho^2}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\sum_{\lambda=1}^{\varrho}\beta_\lambda\frac{\Phi'(\vartheta_\lambda)}{\sigma'(\vartheta_\lambda)} - \frac{1}{2P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\frac{\Phi'(\vartheta_\lambda)}{\sigma'(\vartheta_\lambda)}. \end{aligned} \tag{2.15}$$

Combining (2.13) and (2.15), we get the required inequalities (2.12). □

Remark 2.5. *Setting $\sigma(\vartheta) = \vartheta$ in Theorem 2.6 , we get the following result, which is given in [16].*

$$
\begin{aligned}
0 &\le \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda)-\frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\frac{\vartheta_\lambda+\vartheta_\ell}{2}\right)\\
&\le \frac{1}{2}\left[\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda\Phi'(\vartheta_\lambda)-\frac{1}{P_\varrho^2}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi'(\vartheta_\lambda)\right].
\end{aligned}
$$

Setting $\sigma(\vartheta)=1/\vartheta$ in Theorem 2.6, we get

$$
\begin{aligned}
0 &\le \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda)-\frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\frac{2\vartheta_\lambda\vartheta_\ell}{\vartheta_\lambda+\vartheta_\ell}\right)\\
&\le \frac{1}{2}\left[\frac{1}{P_\varrho^2}\sum_{\lambda=1}^{\varrho}\beta_\lambda\frac{1}{\vartheta_\lambda}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda^2\Phi'(\vartheta_\lambda)-\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda\Phi'(\vartheta_\lambda)\right].
\end{aligned}
$$

Setting $\sigma(\vartheta)=\vartheta^p$ in Theorem 2.6, we get

$$
\begin{aligned}
0 &\le \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\Phi(\vartheta_\lambda)-\frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\frac{\vartheta_\lambda^p+\vartheta_\ell^p}{2}\right)^{\frac{1}{p}}\\
&\le \frac{1}{2}\left[\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda\frac{\Phi'(\vartheta_\lambda)}{p}-\frac{1}{P_\varrho^2}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda^p\sum_{\lambda=1}^{\varrho}\beta_\lambda\frac{\Phi'(\vartheta_\lambda)}{p\vartheta_\lambda^{p-1}}\right].
\end{aligned}
$$

Theorem 2.7. *Let $\Phi : I=[\mathrm{w},\tau]\to\mathbb{R}$ be a differentiable σ-convex function on I, $\vartheta_\lambda\in I^o$, and $\beta_\lambda\ge 0$ with $P_\varrho>0$. Then,*

$$
\begin{aligned}
0 &\le \frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)-\Phi\left(\sigma^{-1}\left(\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\right)\right)\\
&\le \frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\frac{\Phi'\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)}{\sigma'\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)}\sigma(\vartheta_\lambda)-\frac{1}{P_\varrho^3}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\frac{\Phi'\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)}{\sigma'\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)}.
\end{aligned}
\tag{2.16}
$$

Proof. Because Φ is σ-convex, by using the result (2.9) for $\gamma=1$, we have

$$
\Phi\left(\sigma^{-1}\left(\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\right)\right)\le\frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right). \tag{2.17}
$$

To prove the second part of (2.16), we use σ-convexity of Φ,

$$
\begin{aligned}
&\Phi\left(\sigma^{-1}\left(\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\right)\right)-\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)\\
&\quad\ge\frac{\Phi'\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)}{\sigma'\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)}\left(\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)-\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right),
\end{aligned}
$$

for all $\lambda, \ell = 1, ..., \varrho$.

By multiplying this inequality with $\beta_\lambda\beta_\ell \geq 0$ and summing over $\lambda, \ell = 1, 2, \cdots, \varrho$, we determine that

$$\begin{aligned}
&\Phi\left(\sigma^{-1}\left(\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\right)\right) - \frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right) \\
&\geq \frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\sigma(\vartheta_\lambda)\frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\frac{\Phi'\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)}{\sigma'\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)} \\
&- \frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\frac{\Phi'\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)}{\sigma'\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)}. \qquad (2.18)
\end{aligned}$$

Through simple computation, we get

$$\begin{aligned}
&\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\frac{\Phi'\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)}{\sigma'\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)} \\
&= \sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\sigma(\vartheta_\lambda)\frac{\Phi'\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)}{\sigma'\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda)+\sigma(\vartheta_\ell)}{2}\right)\right)}. \qquad (2.19)
\end{aligned}$$

Put (2.19) in (2.18), and then combining (2.17) and (2.18), we get (2.16).

Remark 2.6. *Setting $\sigma(\vartheta) = \vartheta$ in Theorem 2.7, we get the following result, which is given in [16].*

$$\begin{aligned}
0 &\leq \frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\frac{\vartheta_\lambda+\vartheta_\ell}{2}\right) - \Phi\left(\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda\right) \\
&\leq \frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi'\left(\frac{\vartheta_\lambda+\vartheta_\ell}{2}\right)\vartheta_\lambda - \frac{1}{P_\varrho^3}\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi'\left(\frac{\vartheta_\lambda+\vartheta_\ell}{2}\right).
\end{aligned}$$

Setting $\sigma(\vartheta) = frac1/\vartheta$ in Theorem 2.7, we get

$$\begin{aligned}
0 &\leq \frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\Phi\left(\frac{2\vartheta_\lambda\vartheta_\ell}{\vartheta_\lambda+\vartheta_\ell}\right) - \Phi\left(\frac{1}{\frac{1}{P_\varrho}\sum_{\lambda=1}^{\varrho}\beta_\lambda\frac{1}{\vartheta_\lambda}}\right) \\
&\leq \frac{1}{P_\varrho^3}\sum_{\lambda=1}^{\varrho}\beta_\lambda\frac{1}{\vartheta_\lambda}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\left(\frac{2\vartheta_\lambda\vartheta_\ell}{\vartheta_\lambda+\vartheta_\ell}\right)^2\Phi'\left(\frac{2\vartheta_\lambda\vartheta_\ell}{\vartheta_\lambda+\vartheta_\ell}\right) - \frac{1}{P_\varrho^2}\sum_{\lambda,\ell=1}^{\varrho}\beta_\lambda\beta_\ell\left(\frac{2\vartheta_\lambda\vartheta_\ell}{\vartheta_\lambda+\vartheta_\ell}\right)^2\Phi'\left(\frac{2\vartheta_\lambda\vartheta_\ell}{\vartheta_\lambda+\vartheta_\ell}\right)\frac{1}{\vartheta_\lambda}.
\end{aligned}$$

□

2.3. Bounds for generalized Jensen-functional associated with σ-convexity

The collection of non-negative ϱ-tuples $(p_1, \ldots, p_\varrho)$ is such that $\sum_{\lambda=1}^{\varrho} p_\lambda = 1$ is specified by P_ϱ. Let us start with following normalized generalized Jensen-functional.

$$\mathfrak{J}_\varrho(\Phi, \sigma(\vartheta), p) = \sum_{\lambda=1}^{\varrho} p_\lambda\Phi(\vartheta_\lambda) - \Phi\left(\sigma^{-1}\left(\sum_{\lambda=1}^{\varrho} p_\lambda\sigma(\vartheta_\lambda)\right)\right) \geq 0,$$

where $\Phi : C \to \mathbb{R}$ is a σ-convex function on the convex set C, and

$$\sigma(\vartheta) = (\sigma(\vartheta_1), \dots, \sigma(\vartheta_\varrho)) \in C^\varrho, \quad p_\lambda \in P_\varrho.$$

Theorem 2.8. *If $p, q \in P_\varrho$, with $q_\lambda > 0$ for each $\lambda \in \{1, \dots, \varrho\}$, then*

$$\max_{1\le\lambda\le\varrho} \frac{p_\lambda}{q_\lambda} \mathfrak{J}_\varrho(\Phi, \sigma(\vartheta), q) \ge \mathfrak{J}_\varrho(\Phi, \sigma(\vartheta), p) \ge \min_{1\le\lambda\le\varrho} \frac{p_\lambda}{q_\lambda} \mathfrak{J}_\varrho(\Phi, \sigma(\vartheta), q) \quad (\ge 0). \tag{2.20}$$

Proof. We provide a direct proof using Jensen's inequality with suitable choices of the elements. Denote $m := \min_{1\le\lambda\le\varrho} \frac{p_\lambda}{q_\lambda}$, and by applying Jensen's inequality, we attain

$$\sigma(z_1) = \sum_{\lambda=1}^{\varrho} q_\lambda \sigma(\vartheta_\lambda), \quad \sigma(z_{\ell+1}) = \sigma(\vartheta_\ell) \in C, \quad \ell \in \{1, \dots, \varrho\};$$

$$r_1 = m, \quad r_{\ell+1} = \frac{p_\ell}{q_\ell} - m q_\ell \ge 0, \quad \ell \in \{1, \dots, \varrho\},$$

for which

$$\sum_{\ell=1}^{\varrho+1} r_\ell = m + \sum_{\ell=1}^{\varrho} \left(\frac{p_\ell}{q_\ell} - m\right) q_\ell = 1.$$

Then, we have

$$m\Phi\left(\sigma^{-1}\left(\sum_{\lambda=1}^{\varrho} q_\lambda\sigma(\vartheta_\lambda)\right)\right) + \sum_{\ell=1}^{\varrho}\left(\frac{p_\ell}{q_\ell} - m\right) q_\ell \Phi(\vartheta_\ell) \ge \Phi\left(\sigma^{-1}\left(m\sum_{\lambda=1}^{\varrho} q_\lambda\sigma(\vartheta_\lambda) + \sum_{\ell=1}^{\varrho}\left(\frac{p_\ell}{q_\ell} - m\right) q_\ell\sigma(\vartheta_\ell)\right)\right). \tag{2.21}$$

Because

$$m\Phi\left(\sigma^{-1}\left(\sum_{\lambda=1}^{\varrho} q_\lambda\sigma(\vartheta_\lambda)\right)\right) + \sum_{\ell=1}^{\varrho}\left(\frac{p_\ell}{q_\ell} - m\right) q_\ell \Phi(\vartheta_\ell) = \sum_{\ell=1}^{\varrho} p_\ell\Phi(\vartheta_\ell) - m\left[\sum_{\ell=1}^{\varrho} q_\ell\Phi(\vartheta_\ell) - \Phi\left(\sigma^{-1}\left(\sum_{\ell=1}^{\varrho} q_\ell\sigma(\vartheta_\ell)\right)\right)\right],$$

and

$$\Phi\left(\sigma^{-1}\left(m\sum_{\lambda=1}^{\varrho} q_\lambda\sigma(\vartheta_\lambda) + \sum_{\ell=1}^{\varrho}\left(\frac{p_\ell}{q_\ell} - m\right) q_\ell\sigma(\vartheta_\ell)\right)\right) = \Phi\left(\sigma^{-1}\left(\sum_{\ell=1}^{\varrho} p_\ell\sigma(\vartheta_\ell)\right)\right),$$

then by (2.21), we acquire the second part of the inequality (2.20).

Now, let $M := \max_{1\le\lambda\le\varrho} \frac{p_\lambda}{q_\lambda}$, and notice that $M \ge 1$. By applying Jensen's inequality for $\lambda = \varrho + 1$, with

$$\sigma(z_\ell) = \sigma(\vartheta_\ell), \quad \sigma(z_{\varrho+1}) = \sum_{\lambda=1}^{\varrho} p_\lambda\sigma(\vartheta_\lambda), \quad \ell \in \{1, \dots, \varrho\},$$

$$r_\ell = \frac{1}{M}\left(M - \frac{p_\ell}{q_\ell}\right) q_\ell, \quad r_{\varrho+1} = \frac{1}{M}, \quad \ell \in \{1, \dots, \varrho\},$$

for which

$$\sum_{\ell=1}^{\varrho+1} r_\ell = \frac{1}{M}\sum_{\ell=1}^{\varrho}\left(M-\frac{p_\ell}{q_\ell}\right)q_\ell + \frac{1}{M} = 1,$$

we have

$$\begin{aligned}&\frac{1}{M}\sum_{\ell=1}^{\varrho}\left(M-\frac{p_\ell}{q_\ell}\right)q_\ell\Phi(\vartheta_\ell) + \frac{1}{M}\Phi\left(\sigma^{-1}\left(\sum_{\lambda=1}^{\varrho}p_\lambda\sigma(\vartheta_\lambda)\right)\right)\\ &\quad\geq \Phi\left(\sigma^{-1}\left(\frac{1}{M}\sum_{\ell=1}^{\varrho}\left(M-\frac{p_\ell}{q_\ell}\right)q_\ell\sigma(\vartheta_\ell) + \frac{1}{M}\sum_{\lambda=1}^{\varrho}p_\lambda\sigma(\vartheta_\lambda)\right)\right). \qquad (2.22)\end{aligned}$$

Because

$$\begin{aligned}&\frac{1}{M}\sum_{\ell=1}^{\varrho}\left(M-\frac{p_\ell}{q_\ell}\right)q_\ell\Phi(\vartheta_\ell) + \frac{1}{M}\Phi\left(\sigma^{-1}\left(\sum_{\lambda=1}^{\varrho}p_\lambda\sigma(\vartheta_\lambda)\right)\right)\\ &= \sum_{\ell=1}^{\varrho}q_\ell\Phi(\vartheta_\ell) - \frac{1}{M}\left[\sum_{\ell=1}^{\varrho}p_\ell\Phi(\vartheta_\ell) - \Phi\left(\sigma^{-1}\left(\sum_{\lambda=1}^{\varrho}p_\lambda\sigma(\vartheta_\lambda)\right)\right)\right],\end{aligned}$$

and

$$\Phi\left(\sigma^{-1}\left(\frac{1}{M}\sum_{\ell=1}^{\varrho}\left(M-\frac{p_\ell}{q_\ell}\right)q_\ell\sigma(\vartheta_\ell) + \frac{1}{M}\sum_{\lambda=1}^{\varrho}p_\lambda\sigma(\vartheta_\lambda)\right)\right) = \Phi\left(\sigma^{-1}\left(\sum_{\ell=1}^{\varrho}p_\ell\sigma(\vartheta_\ell)\right)\right),$$

by (2.22), we deduce the first part of (2.20). □

For $u = \left(\frac{1}{\varrho},\ldots,\frac{1}{\varrho}\right)$ uniform distribution, we acquire the following functional:

$$\mathfrak{J}_\varrho(\Phi,\sigma(\vartheta)) := \mathfrak{J}_\varrho(\Phi,\sigma(\vartheta),u) = \frac{1}{\varrho}\sum_{\lambda=1}^{\varrho}\Phi(\vartheta_\lambda) - \Phi\left(\sigma^{-1}\left(\frac{1}{\varrho}\sum_{\lambda=1}^{\varrho}\sigma(\vartheta_\lambda)\right)\right).$$

Through a similar argument, we acquire the following estimate for $p \in P_\varrho$.

$$\varrho\max_{1\leq\lambda\leq\varrho}\{p_\lambda\}\,\mathfrak{J}_\varrho(\Phi,\sigma(\vartheta)) \geq \mathfrak{J}_\varrho(\Phi,\sigma(\vartheta),p) \geq \varrho\min_{1\leq\lambda\leq\varrho}\{p_\lambda\}\mathfrak{J}_\varrho(\Phi,\sigma(\vartheta)) \ \ (\geq 0). \qquad (2.23)$$

2.4. Refinements of Hadamard-type inequalities for σ-convexity

Theorem 2.9. *Let* $\Phi : [\mathrm{w},\tau] \to \mathbb{R}$ *be a* σ*-convex function; then*

$$\sum_{\lambda=1}^{\varrho}\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_{\lambda-1})+\sigma(\vartheta_\lambda)}{2}\right)\right) \leq \frac{1}{B}\int_{\sigma(w)}^{\sigma(\tau)}\left(\Phi(\sigma^{-1})\right)(\beta)\,d\beta \leq \frac{1}{2}\left[\Phi(w) + 2\sum_{\lambda=1}^{\varrho-1}\Phi(\vartheta_\lambda) + \Phi(\tau)\right], \quad (2.24)$$

holds, where $\sigma(\vartheta_\lambda) = \sigma(w) + \lambda\frac{\sigma(\tau)-\sigma(w)}{\varrho}$ for $\lambda = 0,1,2,\ldots,\varrho$; with $B = \frac{\sigma(\tau)-\sigma(w)}{\varrho}$ and $\varrho \in \mathbb{N}$. The inequality is sharp.

Proof. Because Φ is σ-convex on $[w,\tau]$, it is σ-convex on each subinterval $[\vartheta_{\lambda-1},\vartheta_\lambda]$, for $\lambda=1,\ldots,\varrho$. Then, for all $\beta\in[0,1]$, we have

$$\Phi\left(\sigma^{-1}(\beta\sigma(\vartheta_{\lambda-1})+(1-\beta)\sigma(\vartheta_\lambda))\right)\le\beta\Phi(\vartheta_{\lambda-1})+(1-\beta)\Phi(\vartheta_\lambda).$$

Integrating both sides with respect to β on $[0,1]$, we get

$$\int_0^1\Phi\left(\sigma^{-1}(\beta\sigma(\vartheta_{\lambda-1})+(1-\beta)\sigma(\vartheta_\lambda))\right)d\beta\le\frac{\Phi(\vartheta_{\lambda-1})+\Phi(\vartheta_\lambda)}{2}.$$

Substituting $u=\beta\sigma(\vartheta_{\lambda-1})+(1-\beta)\sigma(\vartheta_\lambda)$, we get

$$\frac{1}{\sigma(\vartheta_\lambda)-\sigma(\vartheta_{\lambda-1})}\int_{\sigma(\vartheta_{\lambda-1})}^{\sigma(\vartheta_\lambda)}\Phi(\sigma^{-1}(u))\,du\le\frac{\Phi(\vartheta_{\lambda-1})+\Phi(\vartheta_\lambda)}{2}.$$

Taking the sum over λ from 1 to ϱ, we get

$$\begin{aligned}\sum_{\lambda=1}^{\varrho}\int_{\sigma(\vartheta_{\lambda-1})}^{\sigma(\vartheta_\lambda)}\Phi(\sigma^{-1}(u))\,du&=\int_{\sigma(w)}^{\sigma(\tau)}\Phi(\sigma^{-1}(u))\,du\\&\le\sum_{\lambda=1}^{\varrho}\frac{\sigma(\vartheta_\lambda)-\sigma(\vartheta_{\lambda-1})}{2}(\Phi(\vartheta_{\lambda-1})+\Phi(\vartheta_\lambda))\\&=\frac{B}{2}\left[\Phi(\vartheta_0)+2\sum_{\lambda=1}^{\varrho-1}\Phi(\vartheta_\lambda)+\Phi(\vartheta_\varrho)\right]\\&=\frac{B}{2}\left[\Phi(w)+2\sum_{\lambda=1}^{\varrho-1}\Phi(\vartheta_\lambda)+\Phi(\tau)\right].\end{aligned}\tag{2.25}$$

Now, for the left-hand side inequality, we consider Φ as σ-convex on $[\vartheta_{\lambda-1},\vartheta_\lambda]$. Then, for $\beta\in[0,1]$

$$\begin{aligned}\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_{\lambda-1})+\sigma(\vartheta_\lambda)}{2}\right)\right)&=\Phi\left(\sigma^{-1}\left(\frac{\beta\sigma(\vartheta_\lambda)+(1-\beta)\sigma(\vartheta_{\lambda-1})}{2}+\frac{(1-\beta)\sigma(\vartheta_\lambda)+\beta\sigma(\vartheta_{\lambda-1})}{2}\right)\right)\\&\le\frac{1}{2}\left[\Phi\left(\sigma^{-1}(\beta\sigma(\vartheta_\lambda)+(1-\beta)\sigma(\vartheta_{\lambda-1}))\right)+\Phi\left(\sigma^{-1}((1-\beta)\sigma(\vartheta_\lambda)+\beta\sigma(\vartheta_{\lambda-1}))\right)\right].\end{aligned}$$

Integrating both sides with respect to $'\beta'$ over $[0,1]$, we obtain

$$\begin{aligned}\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_{\lambda-1})+\sigma(\vartheta_\lambda)}{2}\right)\right)&\le\frac{1}{2}\int_0^1\Phi\left(\sigma^{-1}(\beta\sigma(\vartheta_\lambda)+(1-\beta)\sigma(\vartheta_{\lambda-1}))\right)d\beta\\&\quad+\frac{1}{2}\int_0^1\Phi\left(\sigma^{-1}((1-\beta)\sigma(\vartheta_\lambda)+\beta\sigma(\vartheta_{\lambda-1}))\right)d\beta\\&=\int_0^1\Phi\left(\sigma^{-1}(\beta\sigma(\vartheta_\lambda)+(1-\beta)\sigma(\vartheta_{\lambda-1}))\right)d\beta.\end{aligned}$$

Substituting $u=\beta\sigma(\vartheta_\lambda)+(1-\beta)\sigma(\vartheta_{\lambda-1})$, and then summing over $\lambda=1$ to ϱ,

$$B\sum_{\lambda=1}^{\varrho}\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_{\lambda-1})+\sigma(\vartheta_\lambda)}{2}\right)\right)\le\int_w^\tau\Phi(\sigma^{-1}(\beta))\,d\beta.\tag{2.26}$$

Thus, from (2.25) and (2.26), we attain the inequality (2.24).

To discuss the sharpness, we consider that (2.24) holds with constants $C_1, C_2 > 0$ such that

$$C_1 B \sum_{\lambda=1}^{\varrho} \Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_{\lambda-1}) + \sigma(\vartheta_\lambda)}{2}\right)\right) \leq \int_{\sigma(w)}^{\sigma(\tau)} \Phi(\sigma^{-1}(\beta))\, d\beta \leq C_2 \frac{B}{2}\left[\Phi(w) + 2\sum_{\lambda=1}^{\varrho-1} \Phi(\vartheta_\lambda) + \Phi(\tau)\right].$$

Consider, $\Phi(\vartheta) = \sigma(\vartheta)$. Then,

$$\int_{\sigma(w)}^{\sigma(\tau)} \Phi(\sigma^{-1}(\beta))\, d\beta = \int_{\sigma(w)}^{\sigma(\tau)} \beta\, d\beta = \frac{(\sigma(\tau))^2 - (\sigma(w))^2}{2}.$$

The right-hand-side is given as:

$$\begin{aligned}
C_2 \frac{B}{2}\left[\sigma(w) + 2\sum_{\lambda=1}^{\varrho-1} \sigma(\vartheta_\lambda) + \sigma(\tau)\right] &= C_2 \frac{B}{2}\left[\sigma(w) + 2\sum_{\lambda=1}^{\varrho-1}\left(\sigma(w) + \lambda\frac{\sigma(\tau) - \sigma(w)}{\varrho}\right) + \sigma(\tau)\right] \\
&= C_2 \frac{B}{2}\left[\sigma(w) + 2(\varrho - 1)\sigma(w) + 2\frac{\sigma(\tau) - \sigma(w)}{\varrho}\frac{(\varrho - 1)\varrho}{2} + \sigma(\tau)\right] \\
&= C_2((\sigma(\tau))^2 - (\sigma(w))^2).
\end{aligned}$$

Therefore,

$$\frac{(\sigma(\tau))^2 - (\sigma(w))^2}{2} \leq C_2((\sigma(\tau))^2 - (\sigma(w))^2) \Rightarrow \frac{1}{2} \leq C_2.$$

Hence, $\frac{1}{2}$ is the least possible constant. For the left-hand side, we have

$$\begin{aligned}
C_1 B \sum_{\lambda=1}^{\varrho} \frac{\sigma(\vartheta_{\lambda-1}) + \sigma(\vartheta)_\lambda}{2} &= C_1 B \sum_{\lambda=1}^{\varrho}\left(\sigma(w) + \frac{(2\lambda - 1)(\sigma(\tau) - \sigma(w))}{2\varrho}\right) \\
&= C_1 \frac{\sigma(\tau) - \sigma(w)}{\varrho} \sum_{\lambda=1}^{\varrho}\left(\sigma(w) + \frac{(2\lambda - 1)(\sigma(\tau) - \sigma(w))}{2\varrho}\right) \\
&= C_1 \frac{(\sigma(\tau))^2 - (\sigma(w))^2}{2}.
\end{aligned}$$

Thus,

$$\frac{(\sigma(\tau))^2 - (\sigma(w))^2}{2} \geq C_1 \frac{(\sigma(\tau))^2 - (\sigma(w))^2}{2} \Rightarrow 1 \geq C_1.$$

Hence, the constant 1 is the optimal value on the left-hand side.

□

Now, we present the graphical validation of Theorem 2.9. We choose $\Phi(\vartheta) = \vartheta^m$ with $m \geq 2$ and $\sigma(\vartheta) = \vartheta, 1/\vartheta$ with $w = 1$ and $\tau = 3$.

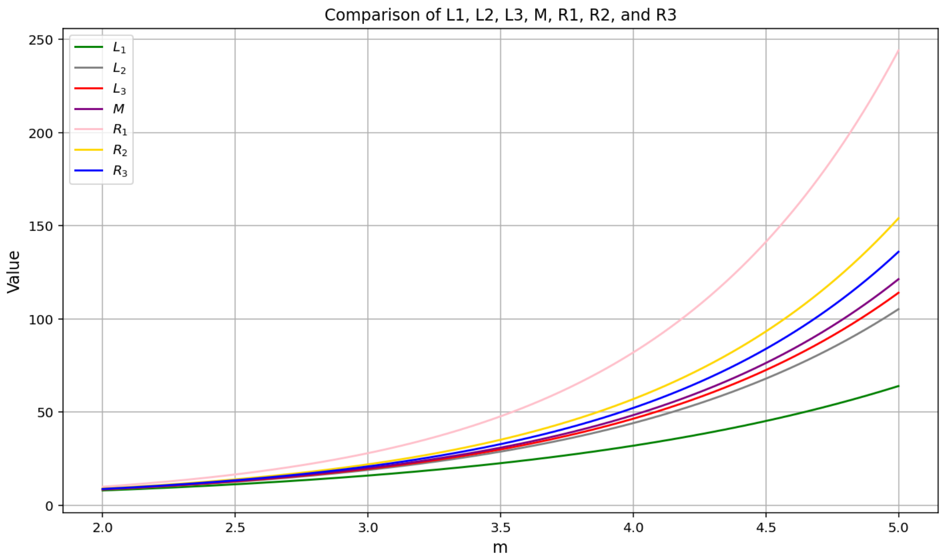

Figure 1. Graphical demonstration for $\sigma(\vartheta) = \vartheta$ and $\varrho = 1, 2, 3$.

The Figure 1 provides the different bounds of average mean integral. The L_1, L_2, L_3, M, and R_1, R_2, R_3 represent the left, middle, and right terms of Theorem 2.9 for $\sigma(\vartheta) = \vartheta$ and $\varrho = 1, 2, 3$, respectively. It is clear that for large values of ϱ, we get sharp bounds. To construct these bounds, we vary $m \in [2, 5]$.

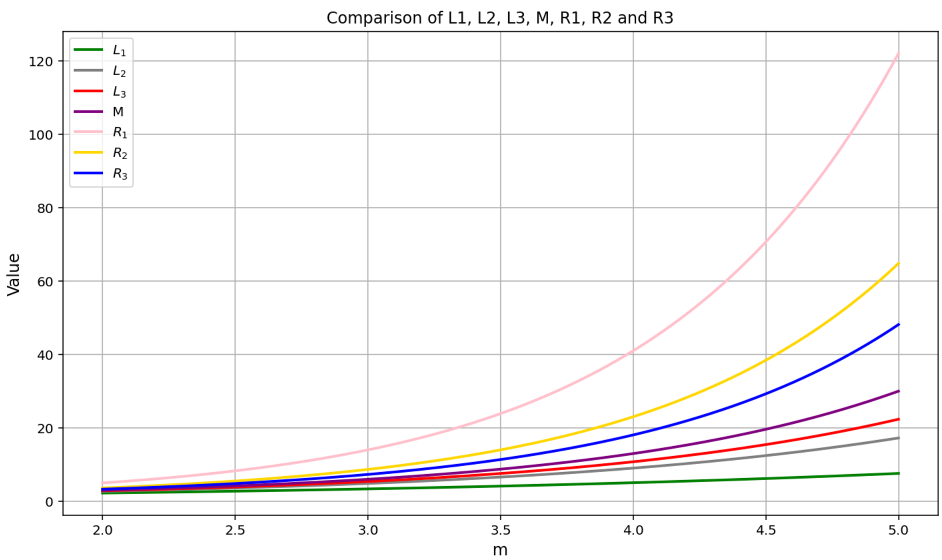

Figure 2. Graphical demonstration for $\sigma(\vartheta) = \vartheta, \frac{1}{\vartheta}$ and $\varrho = 1, 2, 3$.

Figure 2 depicts the estimates of average mean integral. The L_1, L_2, L_3, M, and R_1, R_2, R_3 represent the left, middle, and right terms of Theorem 2.9 for $\sigma(\vartheta) = \frac{1}{\vartheta}$ and $\varrho = 1, 2, 3$, respectively. It is clear that for large values of ϱ, we get sharp bounds. To construct these bounds, we vary $m \in [2, 5]$.

Remark 2.7. *In Theorem 2.9, setting $\varrho = 1$ and $\sigma(\vartheta) = \vartheta$, we obtain the classical Hadamard inequality.*

Remark 2.8. *In Theorem 2.9, setting $\sigma(\vartheta) = \vartheta$, we get the result, which is given in [17].*

$$\frac{1}{\varrho}\sum_{\lambda=1}^{\varrho}\Phi\left(\frac{\vartheta_{\lambda-1}+\vartheta_{\lambda}}{2}\right) \leq \frac{1}{\tau-w}\int_{w}^{\tau}\Phi(\beta)\,d\beta \leq \frac{1}{2\varrho}\left[\Phi(w)+2\sum_{\lambda=1}^{\varrho-1}\Phi(\vartheta_{\lambda})+\Phi(\tau)\right].$$

In Theorem 2.9, setting $\sigma(\vartheta) = 1/\vartheta$, we get

$$\frac{1}{\varrho}\sum_{\lambda=1}^{\varrho}\Phi\left(\frac{2\vartheta_{\lambda-1}\vartheta_{\lambda}}{\vartheta_{\lambda-1}+\vartheta_{\lambda}}\right) \leq \frac{w\tau}{\tau-w}\int_{w}^{\tau}\frac{1}{u^{2}}\Phi(u)\,du \leq \frac{1}{2\varrho}\left[\Phi(w)+2\sum_{\lambda=1}^{\varrho-1}\Phi(\vartheta_{\lambda})+\Phi(\tau)\right].$$

In Theorem 2.9, setting $\sigma(\vartheta) = \vartheta^{p}$, we get

$$\frac{1}{\varrho}\sum_{\lambda=1}^{\varrho}\Phi\left(\frac{\vartheta_{\lambda-1}^{p}+\vartheta_{\lambda}^{p}}{2}\right)^{\frac{1}{p}} \leq \frac{p}{\tau^{p}-w^{p}}\int_{w}^{\tau}u^{p-1}\Phi(u)\,du \leq \frac{1}{2\varrho}\left[\Phi(w)+2\sum_{\lambda=1}^{\varrho-1}\Phi(\vartheta_{\lambda})+\Phi(\tau)\right].$$

In Theorem 2.9, setting $\sigma(\vartheta) = \ln(\vartheta)$, we get

$$\ln\left(\frac{\tau}{w}\right)^{\frac{1}{\varrho}}\sum_{\lambda=1}^{\varrho}\Phi\left(\vartheta_{\lambda-1}\vartheta_{\lambda}\right)^{\frac{1}{2}} \leq \int_{w}^{\tau}\frac{1}{u}\Phi(u)\,du \leq \ln\left(\frac{\tau}{w}\right)^{\frac{1}{2\varrho}}\left[\Phi(w)+2\sum_{\lambda=1}^{\varrho-1}\Phi(\vartheta_{\lambda})+\Phi(\tau)\right].$$

Remark 2.9. *In Theorem 2.9,*

1) Setting $\varrho = 1$, we have

$$\Phi\left(\sigma^{-1}\left(\frac{\sigma(w)+\sigma(\tau)}{2}\right)\right) \leq \frac{1}{\sigma(\tau)-\sigma(w)}\int_{\sigma(w)}^{\sigma(\tau)}\Phi(\sigma^{-1}(\beta))\,d\beta \leq \left[\frac{\Phi(w)+\Phi(\tau)}{2}\right].$$

2) Setting $\varrho = 2$, we have

$$\begin{aligned}&\left(\frac{1}{2}\right)\left[\Phi\left(\sigma^{-1}\left(\frac{3\sigma(w)+\sigma(\tau)}{4}\right)\right)+\Phi\left(\sigma^{-1}\left(\frac{\sigma(w)+3\sigma(\tau)}{4}\right)\right)\right]\\ &\quad\leq \frac{1}{\sigma(\tau)-\sigma(w)}\int_{\sigma(w)}^{\sigma(\tau)}\Phi(\sigma^{-1}(\beta))\,d\beta \leq \frac{1}{4}\left[\Phi(w)+2\Phi\left(\sigma^{-1}\left(\frac{\sigma(w)+\sigma(\tau)}{2}\right)\right)+\Phi(\tau)\right].\end{aligned}$$

3) Setting $\varrho = 3$, we have

$$\begin{aligned}&\frac{1}{3}\left[\Phi\left(\sigma^{-1}\left(\frac{5\sigma(w)+\sigma(\tau)}{6}\right)\right)+\Phi\left(\sigma^{-1}\left(\frac{\sigma(w)+\sigma(\tau)}{2}\right)\right)+\Phi\left(\sigma^{-1}\left(\frac{\sigma(w)+5\sigma(\tau)}{6}\right)\right)\right]\\ &\quad\leq \frac{1}{\sigma(\tau)-\sigma(w)}\int_{\sigma(w)}^{\sigma(\tau)}\Phi(\sigma^{-1}(\beta))\,d\beta\\ &\quad\leq \frac{1}{6}\left[\Phi(w)+2\Phi\left(\sigma^{-1}\left(\frac{2\sigma(w)+\sigma(\tau)}{3}\right)\right)+2\Phi\left(\sigma^{-1}\left(\frac{\sigma(w)+2\sigma(\tau)}{3}\right)\right)+\Phi(\tau)\right].\end{aligned}$$

Theorem 2.10. *Let $\Phi : [\mathrm{w}, \tau] \to \mathbb{R}^{+}$ be a σ-convex function. Then,*

$$\frac{1}{\sigma(\tau)-\sigma(w)}\int_{\sigma(w)}^{\sigma(\tau)}\Phi(\sigma^{-1}(\vartheta))\,d\vartheta-\Phi(\omega)\leq\frac{1}{2\varrho}\left[\Phi(w)+2\sum_{\lambda=1}^{\varrho-1}\Phi(\vartheta_\lambda)+\Phi(\tau)\right] \tag{2.27}$$

holds for all $\omega\in[w,\tau]$, where $\sigma(\vartheta_\lambda)=\sigma(w)+\lambda\frac{\sigma(\tau)-\sigma(w)}{\varrho}$; and $\lambda=0,1,2,\ldots,\varrho$, with $B=\frac{\sigma(\tau)-\sigma(w)}{\varrho}$, and $\varrho\in\mathbb{N}$. The inequality is sharp because both constants can not be replaced by smaller one.

Proof. Fix $\omega\in[\vartheta_{\lambda-1},\vartheta_\lambda]$, for $\lambda=1,2,\ldots,\varrho$. Because Φ is σ-convex on $[w,\tau]$, it is also σ-convex on each subinterval $[\vartheta_{\lambda-1},\vartheta_\lambda]$. Then for all $\beta\in[0,1]$, we have

$$\Phi\left(\sigma^{-1}(\beta\sigma(\vartheta_{\lambda-1})+(1-\beta)\sigma(\omega))\right)\leq\beta\Phi(\vartheta_{\lambda-1})+(1-\beta)\Phi(\omega).$$

Integrating with respect to β on $[0,1]$ yields

$$\int_0^1\Phi\left(\sigma^{-1}(\beta\sigma(\vartheta_{\lambda-1})+(1-\beta)\sigma(\omega))\right)d\beta\leq\frac{\Phi(\vartheta_{\lambda-1})+\Phi(\omega)}{2}.$$

This implies that

$$\int_{\sigma(\vartheta_{\lambda-1})}^{\sigma(\omega)}\Phi(\sigma^{-1}(u))\,du\leq\frac{\sigma(\omega)-\sigma(\vartheta_{\lambda-1})}{2}(\Phi(\vartheta_{\lambda-1})+\Phi(\omega)). \tag{2.28}$$

Similarly, for the interval $[\omega,\vartheta_\lambda]$, we have

$$\int_{\sigma(\omega)}^{\sigma(\vartheta_\lambda)}\Phi(\sigma^{-1}(u))\,du\leq\frac{\sigma(\vartheta_\lambda)-\sigma(\omega)}{2}(\Phi(\vartheta_\lambda)+\Phi(\omega)). \tag{2.29}$$

Adding inequalities (2.28) and (2.29), we obtain

$$\begin{aligned}&\int_{\sigma(\vartheta_{\lambda-1})}^{\sigma(\vartheta_\lambda)}\Phi(\sigma^{-1}(u))\,du\\&\leq\frac{\sigma(\omega)-\sigma(\vartheta_{\lambda-1})}{2}(\Phi(\vartheta_{\lambda-1})+\Phi(\omega))+\frac{\sigma(\vartheta_\lambda)-\sigma(\omega)}{2}(\Phi(\vartheta_\lambda)+\Phi(\omega))\\&\leq\frac{\sigma(\vartheta_\lambda)-\sigma(\vartheta_{\lambda-1})}{2}\left[(\vartheta_{\lambda-1})+\Phi(\vartheta_\lambda)\right]+hf(\omega).\end{aligned} \tag{2.30}$$

Summing from $\lambda=1$ to ϱ, we obtain

$$\begin{aligned}\int_{\sigma(w)}^{\sigma(\tau)}\Phi(\sigma^{-1}(u))\,du&\leq\sum_{\lambda=1}^{\varrho}\left[\frac{\sigma(\vartheta_\lambda)-\sigma(\vartheta_{\lambda-1})}{2}(\Phi(\vartheta_{\lambda-1})+\Phi(\vartheta_\lambda))\right]+\sum_{\lambda=1}^{\varrho}hf(\omega)\\&\leq\frac{1}{2}\max_\lambda\{\sigma(\vartheta_\lambda)-\sigma(\vartheta_{\lambda-1})\}\sum_{\lambda=1}^{\varrho}\left[\Phi(\vartheta_\lambda)+\Phi(\vartheta_{\lambda-1})\right]+(\sigma(\tau)-\sigma(w))\Phi(\omega)\\&=\frac{B}{2}\left[\Phi(w)+2\sum_{\lambda=1}^{\varrho-1}\Phi(\vartheta_\lambda)+\Phi(\tau)\right]+(\sigma(\tau)-\sigma(w))\Phi(\omega).\end{aligned}$$

Through this, we get

$$\int_{\sigma(w)}^{\sigma(\tau)} \Phi(\sigma^{-1}(u))\, du - (\sigma(\tau) - \sigma(w))\Phi(\omega) \leq \frac{B}{2}\left[\Phi(w) + 2\sum_{\lambda=1}^{\varrho-1} \Phi(\vartheta_\lambda) + \Phi(\tau)\right].$$

To prove sharpness, consider that inequality (2.27) holds with another constant $C > 0$

$$\int_{\sigma(w)}^{\sigma(\tau)} \left(\Phi(\sigma^{-1})\right)(\vartheta)\, d\vartheta - (\sigma(\tau) - \sigma(w))\Phi(\omega) \leq C.B\left[\Phi(w) + 2\sum_{\lambda=1}^{\varrho-1} \Phi(\vartheta_\lambda) + \Phi(\tau)\right].$$

Consider $\Phi(\vartheta) = \sigma(\vartheta)$ defined on $[0, 1]$; then

$$\int_0^1 \vartheta\, d\vartheta - \sigma(\omega) = \frac{1}{2} - \sigma(\omega) \leq C \cdot \frac{1}{\varrho}\left[2\sum_{\lambda=1}^{\varrho-1} \sigma(\vartheta_\lambda) + 1\right].$$

Also, we get

$$\sigma(\vartheta_\lambda) = \frac{\lambda}{\varrho} \Rightarrow \sum_{\lambda=1}^{\varrho-1} \sigma(\vartheta_\lambda) = \frac{1}{\varrho} \cdot \frac{(\varrho-1)\varrho}{2\varrho} = \frac{\varrho-1}{2}.$$

Finally, we get the following inequality:

$$\frac{1}{2} - \sigma(\omega) \leq C.$$

Choosing $\sigma(\omega) = 0$ gives

$$\frac{1}{2} \leq C,$$

which proves that $\frac{1}{2}$ is the best possible constant. This ends the proof. □

Theorem 2.11. *Suppose that all the conditions of Theorem 2.10 hold. Then,*

$$\begin{aligned}
&\int_{\sigma(w)}^{\sigma(\tau)} \Phi(\sigma^{-1}(\vartheta))\, d\vartheta - (\sigma(\tau) - \sigma(w))\Phi(\omega) \\
&\leq \left(\frac{B}{2} + \max_{1\leq\lambda\leq\varrho}\left|\frac{\sigma(\omega) - \sigma(\vartheta_{\lambda-1}) + \sigma(\vartheta_\lambda)}{2}\right|\right)\left[\Phi(w) + 2\sum_{\lambda=1}^{\varrho-1} \Phi(\vartheta_\lambda) + \Phi(\tau)\right],
\end{aligned} \tag{2.31}$$

for all $\omega \in [w, \tau]$. The constant $\frac{1}{2}$ in the right-hand side is the best possible for all $\varrho \in \mathbb{N}$.

Proof. We start from Ineq (2.30):

$$\begin{aligned}\int_{\sigma(\vartheta_{\lambda-1})}^{\sigma(\vartheta_\lambda)}\Phi(\sigma^{-1}(u))\,du &\le \frac{\sigma(\omega)-\sigma(\vartheta_{\lambda-1})}{2}\Phi(\vartheta_{\lambda-1})+\frac{\sigma(\vartheta_\lambda)-\sigma(\omega)}{2}\Phi(\vartheta_\lambda)+B\Phi(\omega)\\ &\le \max\left\{\frac{\sigma(\omega)-\sigma(\vartheta_{\lambda-1})}{2},\frac{\vartheta_\lambda-\omega}{2}\right\}(\Phi(\vartheta_{\lambda-1})+\Phi(\vartheta_\lambda))+B\Phi(\omega)\\ &\le \left(\frac{\sigma(\vartheta_\lambda)-\sigma(\vartheta_{\lambda-1})}{2}+\left|\sigma(\omega)-\frac{\sigma(\vartheta_{\lambda-1})+\sigma(\vartheta_\lambda)}{2}\right|\right)(\Phi(\vartheta_{\lambda-1})+\Phi(\vartheta_\lambda))+B\Phi(\omega).\end{aligned}\tag{2.32}$$

Taking the sum over $\lambda=1$ to ϱ, we get

$$\begin{aligned}\int_{\sigma(w)}^{\sigma(\tau)}\Phi(\sigma^{-1}(u))\,du &\le \sum_{\lambda=1}^{\varrho}\left(\frac{\sigma(\vartheta_\lambda)-\sigma(\vartheta_{\lambda-1})}{2}+\left|\sigma(\omega)-\frac{\sigma(\vartheta_{\lambda-1})+\sigma(\vartheta_\lambda)}{2}\right|\right)(\Phi(\vartheta_{\lambda-1})+\Phi(\vartheta_\lambda))+\sum_{\lambda=1}^{\varrho}B\Phi(\omega)\\ &\le \max_{1\le\lambda\le\varrho}\left(\frac{\sigma(\vartheta_\lambda)-\sigma(\vartheta_{\lambda-1})}{2}+\left|\sigma(\omega)-\frac{\sigma(\vartheta_{\lambda-1})+\sigma(\vartheta_\lambda)}{2}\right|\right)\sum_{\lambda=1}^{\varrho}(\Phi(\vartheta_{\lambda-1})+\Phi(\vartheta_\lambda))\\ &\quad+(\sigma(\tau)-\sigma(w))\Phi(\omega).\end{aligned}$$

Now, simplify the summation:

$$\sum_{\lambda=1}^{\varrho}(\Phi(\vartheta_{\lambda-1})+\Phi(\vartheta_\lambda))=\Phi(\vartheta_0)+2\sum_{\lambda=1}^{\varrho-1}\Phi(\vartheta_\lambda)+\Phi(\vartheta_\varrho)=\Phi(w)+2\sum_{\lambda=1}^{\varrho-1}\Phi(\vartheta_\lambda)+\Phi(\tau).$$

This asserts that

$$\begin{aligned}&\int_{\sigma(w)}^{\sigma(\tau)}\Phi(\sigma^{-1}(u))\,du\\ &\le\left(\frac{B}{2}+\max_{1\le\lambda\le\varrho}\left|\sigma(\omega)-\frac{\sigma(\vartheta_{\lambda-1})+\sigma(\vartheta_\lambda)}{2}\right|\right)\left[\Phi(w)+2\sum_{\lambda=1}^{\varrho-1}\Phi(\vartheta_\lambda)+\Phi(\tau)\right]+(\sigma(\tau)-\sigma(w))\Phi(\omega).\end{aligned}$$

Through some simple calculations, we achieve inequality (2.31). □

Theorem 2.12. *Owing to all the conditions of Theorem 2.11, we achieve the following inequality:*

$$\frac{1}{\sigma(\tau)-\sigma(w)}\int_{\sigma(w)}^{\sigma(\tau)}\Phi(\sigma^{-1}(u))du-\frac{1}{\varrho}\sum_{\lambda=1}^{\varrho}\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_{\lambda-1})+\sigma(\vartheta_\lambda)}{2}\right)\right)-\frac{1}{\varrho}\sum_{\lambda=1}^{\varrho-1}\Phi(\vartheta_\lambda)\le\frac{\Phi(w)+\Phi(\tau)}{2\varrho},\tag{2.33}$$

for all $\lambda=1,2,\ldots,\varrho$. Moreover, the constants are sharp.

Proof. Considering (2.32) and using $\sigma(\omega)=\frac{\sigma(\vartheta_{\lambda-1})+\sigma(\vartheta_\lambda)}{2}$, we receive

$$\int_{\sigma(\vartheta_{\lambda-1})}^{\sigma(\vartheta_\lambda)}\Phi(\sigma^{-1}(u))\,du\le\frac{\sigma(\vartheta_\lambda)-\sigma(\vartheta_{\lambda-1})}{2}(\Phi(\vartheta_{\lambda-1})+\Phi(\vartheta_\lambda))+B\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_{\lambda-1})+\sigma(\vartheta_\lambda)}{2}\right)\right).$$

Taking the sum over λ from 1 to ϱ, we get

$$\int_{\sigma(w)}^{\sigma(\tau)} \Phi(\sigma^{-1}(u))\, du \leq \sum_{\lambda=1}^{\varrho} \frac{\sigma(\vartheta_\lambda) - \sigma(\vartheta_{\lambda-1})}{2} \left[\Phi(\vartheta_{\lambda-1}) + \Phi(\vartheta_\lambda)\right] + \sum_{\lambda=1}^{\varrho} B\, \Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_{\lambda-1}) + \sigma(\vartheta_\lambda)}{2}\right)\right)$$
$$\leq \frac{B}{2} \sum_{\lambda=1}^{\varrho} \left(\Phi(\vartheta_{\lambda-1}) + \Phi(\vartheta_\lambda)\right) + B \sum_{\lambda=1}^{\varrho} \Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_{\lambda-1}) + \sigma(\vartheta_\lambda)}{2}\right)\right),$$

which gives that

$$\int_{\sigma(w)}^{\sigma(\tau)} \Phi(\sigma^{-1}(u))\, du - B \sum_{\lambda=1}^{\varrho} \Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_{\lambda-1}) + \sigma(\vartheta_\lambda)}{2}\right)\right) \leq \frac{B}{2} \sum_{\lambda=1}^{\varrho} \left(\Phi(w) + 2 \sum_{\lambda=1}^{\varrho-1} \Phi(\vartheta_\lambda) + \Phi(\tau)\right),$$

which ends the proof. □

Remark 2.10. *Setting $\sigma(\vartheta) = \vartheta$ in Theorem* 2.12*, we have*

$$\frac{1}{\tau - w} \int_{w}^{\tau} \Phi(u) du - \frac{1}{\varrho} \sum_{\lambda=1}^{\varrho} \Phi\left(\frac{\vartheta_{\lambda-1} + \vartheta_\lambda}{2}\right) - \frac{1}{\varrho} \sum_{\lambda=1}^{\varrho-1} \Phi(\vartheta_\lambda) \leq \frac{\Phi(w) + \Phi(\tau)}{2\varrho}.$$

Setting $\sigma(\vartheta) = \frac{1}{\vartheta}$ in Theorem 2.12, we have

$$\frac{ab}{\tau - w} \int_{w}^{\tau} \frac{\Phi(\vartheta)}{\vartheta^2} du - \frac{1}{\varrho} \sum_{\lambda=1}^{\varrho} \Phi\left(\frac{2\vartheta_{\lambda-1}\vartheta_\lambda}{\vartheta_{\lambda-1} + \vartheta_\lambda}\right) - \frac{1}{\varrho} \sum_{\lambda=1}^{\varrho-1} \Phi(\vartheta_\lambda) \leq \frac{\Phi(w) + \Phi(\tau)}{2\varrho}.$$

2.5. Refinements of the Fejér inequality

Theorem 2.13. *Let $\Phi : [\mathrm{w}, \tau] \to \mathbb{R}$ be a σ-convex function and $g : [\mathrm{w}, \tau] \to \mathbb{R}$ be symmetric with respect to $\frac{\sigma(\mathrm{w})+\sigma(\tau)}{2}$ on $[\mathrm{w}, \tau]$. Then, we have*

$$\sum_{\lambda=1}^{\varrho} \Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_{\lambda-1}) + \sigma(\vartheta_\lambda)}{2}\right)\right) \int_{\sigma(\mathrm{w})}^{\sigma(\tau)} g(\sigma^{-1}\beta)\, d\beta \leq \int_{\sigma(\mathrm{w})}^{\sigma(\tau)} \Phi(\sigma^{-1}(\beta)) g(\sigma^{-1}\beta) d\beta$$
$$\leq \frac{1}{2}\left[\Phi(\mathrm{w}) + 2 \sum_{\lambda=1}^{\varrho-1} \Phi(\vartheta_\lambda) + \Phi(\tau)\right] \int_{\sigma(\mathrm{w})}^{\sigma(\tau)} g(\sigma^{-1}\beta) d\beta. \tag{2.34}$$

Proof. Because Φ is σ-convex on $[w, \tau]$, it is also σ-convex on each subinterval $[\vartheta_{\lambda-1}, \vartheta_\lambda]$ for $\lambda = 1, \ldots, \varrho$. Then, for all $\beta \in [0, 1]$ and using the symmetric property of g, we obtain

$$\int_0^1 (\Phi\left(\sigma^{-1}(\beta\sigma(\vartheta_{\lambda-1}) + (1-\beta)\sigma(\vartheta_\lambda))\right)\; g(\sigma^{-1}(\beta\sigma(\vartheta_{\lambda-1})) + (1-\beta)\sigma(\vartheta_\lambda))d\beta$$

$$+\int_0^1 \Phi\left(\sigma^{-1}((1-\beta)\sigma(\vartheta_{\lambda-1})+\beta\sigma(\vartheta_\lambda))\right)\, g(\sigma^{-1}((1-\beta)\sigma(\vartheta_{\lambda-1}))+\beta\sigma(\vartheta_\lambda))d\beta$$
$$\leq \int_0^1 [\Phi(\vartheta_{\lambda-1})+\Phi(\vartheta_\lambda)]\, g(\sigma^{-1}(\beta\sigma(\vartheta_{\lambda-1}))+(1-\beta)\sigma(\vartheta_\lambda))d\beta. \tag{2.35}$$

Using the substitution property in (2.35), we get

$$2\int_{\sigma(\vartheta_\lambda-1)}^{\sigma(\vartheta_\lambda)} \Phi(\sigma^{-1}(u))g(\sigma^{-1}(u))\,du \leq [\Phi(\vartheta_\lambda)+\Phi(\vartheta_{\lambda-1})]\int_{\sigma(\vartheta_\lambda-1)}^{\sigma(\vartheta_\lambda)} g(\sigma^{-1}(u))\,du.$$

Sum over λ from 1 to ϱ to get

$$\begin{aligned}
&\sum_{\lambda=1}^{\varrho}\int_{\sigma(\vartheta_\lambda-1)}^{\sigma(\vartheta_\lambda)} \Phi(\sigma^{-1}(u))g(\sigma^{-1}(u))\,du\\
&\leq \frac{1}{2}\sum_{\lambda=1}^{\varrho}(\Phi(\vartheta_\lambda)+\Phi(\vartheta_{\lambda-1}))\int_{\sigma(\vartheta_\lambda-1)}^{\sigma(\vartheta_\lambda)} g(\sigma^{1}(u))\,du\\
&= \frac{1}{2}[\Phi(w)+2\sum_{\lambda=1}^{\varrho-1}\Phi(\vartheta_{\lambda-1})+\Phi(\tau)]\int_{\sigma(w)}^{\sigma(\tau)} g(\sigma^{-1}(u))\,du.
\end{aligned}\tag{2.36}$$

Now, for the left inequality, we consider the σ-convexity of Φ:

$$\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_{\lambda-1})+\sigma(\vartheta_\lambda)}{2}\right)\right) \leq \frac{1}{2}\Phi\left(\sigma^{-1}(\beta\sigma(\vartheta_{\lambda-1})+(1-\beta)\sigma(\vartheta_\lambda))\right)+\frac{1}{2}\Phi\left(\sigma^{-1})((1-\beta)\sigma(\vartheta_{\lambda-1})+\beta\sigma(\vartheta_\lambda))\right) \tag{2.37}$$

Multiplying by (2.37) by $g(\sigma^{-1}(\beta\sigma(\vartheta_{\lambda-1})+(1-\beta)\sigma(\vartheta_\lambda)))$ and using symmetric property of g,

$$\begin{aligned}
&\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_{\lambda-1})+\sigma(\vartheta_\lambda)}{2}\right)\right)\, g(\sigma^{-1}(\beta\sigma(\vartheta_{\lambda-1})+(1-\beta)\sigma(\vartheta_\lambda)))\\
&\leq \frac{1}{2}\Phi\left(\sigma^{-1}(\beta\sigma(\vartheta_{\lambda-1}+(1-\beta)\sigma(\vartheta_\lambda))\right)\, g(\sigma^{-1}(\beta\sigma(\vartheta_{\lambda-1})+(1-\beta)\sigma(\vartheta_\lambda)))\\
&\quad+\frac{1}{2}\Phi\left(\sigma^{-1}((1-\beta)\sigma(\vartheta_{\lambda-1})+\beta\sigma(\vartheta_\lambda))\right)\, g(\sigma^{-1}((1-\beta)\sigma(\vartheta_{\lambda-1})+\beta\sigma(\vartheta_\lambda))).
\end{aligned}$$

Integrating with respect to β from 0 to 1, we have

$$\begin{aligned}
&\int_0^1 \Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_{\lambda-1})+\sigma(\vartheta_\lambda)}{2}\right)\right)\, g(\sigma^{-1}(\beta\sigma(\vartheta_{\lambda-1})+(1-\beta)\sigma(\vartheta_\lambda)))d\beta\\
&\leq \int_0^1 \Phi\left(\sigma^{-1}(\beta\sigma(\vartheta_{\lambda-1}+(1-\beta)\sigma(\vartheta_\lambda))\right)\, g(\sigma^{-1}(\beta\sigma(\vartheta_{\lambda-1})+(1-\beta)\sigma(\vartheta_\lambda)))d\beta.
\end{aligned}\tag{2.38}$$

By some simple computation and then taking the sum over λ from 1 to ϱ, we obtain

$$\sum_{\lambda=1}^{\varrho}\Phi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_{\lambda-1})+\sigma(\vartheta_\lambda)}{2}\right)\right)\int_{\sigma(w)}^{\sigma(\tau)} g(\sigma^{-1}(u))du \leq \int_{\sigma(w)}^{\sigma(\tau)}\Phi(\sigma^{-1}(u))g(\sigma^{-1}(u))du. \tag{2.39}$$

From (2.36) and (2.39), we get the desired inequality (2.34). □

3. Applications

Now, we investigate some applications of produced inequalities and bounds.

3.1. Applications in normed spaces

Let $(X, \|\|)$ be a real normed space and the function $\Phi_p : X \to \mathbb{R}$, $\Phi_p(\vartheta) = \|\vartheta\|^p$, $p \geq 1$ be σ-convex on X for $\sigma(\vartheta) = \vartheta^p$. From the results (2.20), then, we obtain the following inequalities:

$$\max_{1\leq\lambda\leq\varrho} \frac{p_\lambda}{q_\lambda}\left[\sum_{\ell=1}^{\varrho} q_\ell\|\vartheta_\ell\|^p - \left\|\sum_{\ell=1}^{\varrho} q_\ell\vartheta_\ell^p\right\|\right] \geq \sum_{\ell=1}^{\varrho} p_\ell\|\vartheta_\ell\|^p - \left\|\sum_{\ell=1}^{\varrho} p_\ell\vartheta_\ell^p\right\| \geq \min_{1\leq\lambda\leq\varrho} \frac{p_\lambda}{q_\lambda}\left[\sum_{\ell=1}^{\varrho} q_\ell\|\vartheta_\ell\|^p - \left\|\sum_{\ell=1}^{\varrho} q_\ell\vartheta_\ell^p\right\|\right] \quad (\geq 0). \tag{3.1}$$

$$\max_{1\leq\lambda\leq\varrho}\{p_\lambda\}\left[\sum_{\ell=1}^{\varrho} \|\vartheta_\ell\|^p - \left\|\sum_{\ell=1}^{\varrho} \vartheta_\ell^p\right\|\right] \geq \sum_{\ell=1}^{\varrho} p_\ell\|\vartheta_\ell\|^p - \left\|\sum_{\ell=1}^{\varrho} p_\ell\vartheta_\ell^p\right\| \geq \min_{1\leq\lambda\leq\varrho}\{p_\lambda\}\left[\sum_{\ell=1}^{\varrho} \|\vartheta_\ell\|^p - \left\|\sum_{\ell=1}^{\varrho} \vartheta_\ell^p\right\|\right] \quad (\geq 0). \tag{3.2}$$

If we choose $p_\ell := \frac{1}{\|\vartheta_\ell\|}$, $\vartheta_\ell \in X \setminus \{0\}$, $\ell \in \{1, \ldots, \varrho\}$ satsfying the condition $\sum_{\ell=1}^{\varrho} p_\ell = 1$ in (3.2), then, we get

$$\max_{1\leq\ell\leq\varrho}\{\|\vartheta_\ell\|\}\left[\sum_{\ell=1}^{\varrho} \|\vartheta_\ell\|^{p-1} - \left\|\sum_{\ell=1}^{\varrho} \vartheta_\ell^p\right\|\right] \geq \sum_{\ell=1}^{\varrho} \|\vartheta_\ell\|^{p-1} - \left\|\sum_{\ell=1}^{\varrho} \frac{\vartheta_\ell}{\|\vartheta_\ell^p\|}\right\| \geq \min_{1\leq\ell\leq\varrho}\{\|\vartheta_\ell\|\}\left[\sum_{\ell=1}^{\varrho} \|\vartheta_\ell\|^{p-1} - \left\|\sum_{\ell=1}^{\varrho} \vartheta_\ell^p\right\|\right] \quad (\geq 0). \tag{3.3}$$

Remark 3.1. *Setting $p = 1$, we get the results for convex functions, which are given in [11].*

3.2. Bound for Shannon entropy

The Shannon entropy of a random variable X defined over probability distribution $p_1, \ldots, p_\varrho$ (with $p_\lambda > 0$ for $\lambda = 1, \ldots, \varrho$) and $R = \{\vartheta_1, \ldots, \vartheta_\varrho\}$ is defined by [11]

$$H(X) := -\sum_{\lambda=1}^{\varrho} p_\lambda \ln p_\lambda.$$

Proposition 3.1 ([11]). *Let X be defined as above. If $p_\lambda > 0$ for each $\lambda \in \{1, \ldots, \varrho\}$, then*

$$0 \;\leq\; \varrho \min_{1\leq\lambda\leq\varrho}\{p_\lambda\} \ln\left(\frac{A_\varrho(1/p)}{G_\varrho(1/p)}\right) \;\leq\; \ln\varrho - H(X) \;\leq\; \varrho \max_{1\leq\lambda\leq\varrho}\{p_\lambda\} \ln\left(\frac{A_\varrho(1/p)}{G_\varrho(1/p)}\right).$$

Proof. It follows from the inequality (2.23) for the choice $\Phi(\beta) = -\ln\beta$, $\vartheta_\lambda = 1/p_\lambda$, $\sigma(\vartheta) = \vartheta$, and performing some essential computations. □

3.3. Applications to means

We now present some relations between means such as the quasi-mean and power mean. We start with the notion of power means:

Definition 3.1. *Assume that $p = (\xi_1, \xi_2, \cdots, \xi_\varrho)$ and $\vartheta = (\vartheta_1, \vartheta_2, \cdots, \vartheta_\varrho)$ are ordered tuples such that $\vartheta_\lambda > 0$ with $P_\varrho := \sum_{\lambda=1}^{\varrho} \xi_\lambda$. Then, it is stated as*

$$M_{\lambda_1}(p, \vartheta) = \begin{cases} \left(\frac{1}{P_\varrho} \sum_{\lambda=1}^{\varrho} \xi_\lambda \vartheta_\lambda^{\lambda_1}\right)^{\frac{1}{\lambda_1}}, & \lambda_1 \neq 0, \\ \left(\prod_{\lambda=1}^{\varrho} \xi_\lambda^{\vartheta_\lambda}\right)^{\frac{1}{P_\varrho}}, & \lambda_1 = 0. \end{cases} \tag{3.4}$$

Definition 3.2. *The nonlinear quasi-mean concerning the monotonic function σ is stated as*

$$M_\sigma(p, \vartheta) = \sigma^{-1}\left(\frac{1}{P_\varrho} \sum_{\lambda=1}^{\varrho} \xi_\lambda \sigma(\vartheta_\lambda)\right). \tag{3.5}$$

First, we develop an interesting relation using Theorem 2.3.

Proposition 3.2. *Suppose that ψ is a σ-convex function, and $\xi_\lambda, \vartheta_\lambda > 0$ for $\lambda = 1, 2, \cdots, \varrho$ with $P_\varrho := \sum_{\lambda=1}^{\varrho} \xi_\lambda$. Then*

$$\begin{aligned} \psi(M_\sigma(p, \vartheta)) &\leq \frac{1}{P_\varrho^2} \sum_{\lambda=1}^{\varrho} \sum_{\ell=1}^{\varrho} \xi_\lambda \xi_\ell \psi\left(\sigma^{-1}\left(\frac{\sigma(\vartheta_\lambda) + \sigma(\vartheta_\ell)}{2}\right)\right) \\ &\leq \frac{1}{P_\varrho^2}\left(\sum_{\lambda=1}^{\varrho} \xi_\lambda^2 \psi(\vartheta_\lambda) + 2\sum_{\lambda<\ell} \xi_\lambda \xi_\ell \frac{1}{\sigma(\vartheta_\ell) - \sigma(\vartheta_\lambda)} \int_{\vartheta_\lambda}^{\vartheta_\ell} \psi(\vartheta)\sigma'(\vartheta)\, d\vartheta\right) \\ &\leq \frac{1}{P_\varrho} \sum_{\lambda=1}^{\varrho} \xi_\lambda \psi(\vartheta_\lambda). \end{aligned} \tag{3.6}$$

Proof. This result is directly proved by Theorem 2.3, by using the definition of quasi-arithmetic means, which is defined in (3.5). □

3.4. Applications to the Hölder inequality

Theorem 3.1. *Let $\beta_\lambda, \vartheta_\lambda$ be positive real numbers for $\lambda = 1, 2, \ldots, \varrho$. If $r_1, r_2 > 1$ with $\frac{1}{r_1} + \frac{1}{r_2} = 1$; then*

$$\begin{aligned} \sum_{\lambda=1}^{\varrho} \beta_\lambda \vartheta_\lambda &\leq \frac{1}{\left(\sum_{\lambda=1}^{\varrho} \beta_\lambda^{r_1}\right)^{\frac{2}{r_2}-1}} \left[\sum_{\lambda=1}^{\varrho} \sum_{\ell=1}^{\varrho} \beta_\lambda^{r_1} \beta_\ell^{r_1} \left(\frac{\beta_\ell^{r_1-1}\vartheta_\lambda + \beta_\lambda^{r_1-1}\vartheta_\ell}{2\beta_\lambda^{r_1-1}\beta_\ell^{r_1-1}}\right)^{r_2}\right]^{\frac{1}{r_2}} \\ &\leq \frac{1}{\left(\sum_{\lambda=1}^{\varrho} \beta_\lambda^{r_1}\right)^{\frac{2}{r_2}-1}} \left[\sum_{\lambda=1}^{\varrho} \beta_\lambda^{r_1} \vartheta_\lambda^{r_2} + \frac{2}{r_2+1} \sum_{\lambda<\ell} \frac{\beta_\lambda^{r_1}\beta_\ell^{r_1}\beta_\lambda^{r_1-1}\beta_\ell^{r_1-1}}{\vartheta_\lambda \beta_\ell^{r_1-1} - \vartheta_\ell \beta_\lambda^{r_1-1}} \right. \end{aligned} \tag{3.7}$$

$$\times\left\{\left(\frac{\vartheta_\lambda}{\beta_\lambda^{r_1-1}}\right)^{r_2+1}-\left(\frac{\vartheta_\ell}{\beta_\ell^{r_1-1}}\right)^{r_2+1}\right\}\Bigg]^{\frac{1}{r_2}}$$
$$\leq\left(\sum_{\lambda=1}^{\varrho}\beta_\lambda^{r_1}\right)^{\frac{1}{r_1}}\left(\sum_{\lambda=1}^{\varrho}\vartheta_\lambda^{r_2}\right)^{\frac{1}{r_2}}. \tag{3.8}$$

Proof. Let us consider $\Phi(\vartheta)=\vartheta^{-r_2},\beta_\lambda=\beta_\lambda^{r_1},\sigma(\vartheta)=\frac{1}{\vartheta}$, and $\vartheta_\lambda=\frac{\beta_\lambda^{r_1-1}}{\vartheta_\lambda}$. Applied in (2.2), we have

$$\left(\frac{\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda}{\sum_{\lambda=1}^{\varrho}\beta_\lambda^{r_1}}\right)^{r_2} \tag{3.9}$$
$$\leq\frac{1}{\left(\sum_{\lambda=1}^{\varrho}\beta_\lambda^{r_1}\right)^2}\sum_{\lambda=1}^{\varrho}\sum_{\ell=1}^{\varrho}\beta_\lambda^{r_1}\beta_\ell^{r_1}\left(\frac{\beta_\ell^{r_1-1}\vartheta_\lambda+\beta_\lambda^{r_1-1}\vartheta_\ell}{2\beta_\lambda^{r_1-1}\beta_\ell^{r_1-1}}\right)^{r_2}$$
$$\leq\frac{1}{\left(\sum_{\lambda=1}^{\varrho}\beta_\lambda^{r_1}\right)^2}\Bigg[\sum_{\lambda=1}^{\varrho}\beta_\lambda^{2r_1}\left(\frac{\vartheta_\lambda}{\beta_\lambda^{r_1-1}}\right)^{r_2}+2\sum_{\lambda<\ell}\frac{\left(\beta_\lambda^{r_1}\beta_\ell^{r_1}\right)/\left(\beta_\lambda^{r_1-1}\beta_\ell^{r_1-1}\right)}{(1+r_2)\left(\beta_\ell^{r_1-1}\vartheta_\lambda-\beta_\lambda^{r_1-1}\vartheta_\ell\right)}$$
$$\times\left[\left(\frac{\vartheta_\lambda}{\beta_\lambda^{r_1-1}}\right)^{1+r_2}-\left(\frac{\vartheta_\ell}{\beta_\ell^{r_1-1}}\right)^{1+r_2}\right]\Bigg]$$
$$\leq\frac{\sum_{\lambda=1}^{\varrho}\vartheta_\lambda^{r_2}}{\sum_{\lambda=1}^{\varrho}\beta_\lambda^{r_1}}. \tag{3.10}$$

Now, taking the power $\frac{1}{r_2}$ on both sides, we have

$$\frac{\sum_{\lambda=1}^{\varrho}\beta_\lambda\vartheta_\lambda}{\sum_{\lambda=1}^{\varrho}\beta_\lambda^{r_1}} \tag{3.11}$$
$$\leq\frac{1}{\left(\sum_{\lambda=1}^{\varrho}\beta_\lambda^{r_1}\right)^{\frac{2}{r_2}}}\left[\sum_{\lambda=1}^{\varrho}\sum_{\ell=1}^{\varrho}\beta_\lambda^{r_1}\beta_\ell^{r_1}\left(\frac{\beta_\ell^{r_1-1}\vartheta_\lambda+\beta_\lambda^{r_1-1}\vartheta_\ell}{2\beta_\lambda^{r_1-1}\beta_\ell^{r_1-1}}\right)^{r_2}\right]^{\frac{1}{r_2}}$$

$$\leq \frac{1}{\left(\sum_{\lambda=1}^{\varrho}\beta_{\lambda}^{r_1}\right)^{\frac{2}{r_2}}}\left[\sum_{\lambda=1}^{\varrho}\beta_{\lambda}^{2r_1}\left(\frac{\vartheta_{\lambda}}{\beta_{\lambda}^{r_1-1}}\right)^{r_2}+2\sum_{\lambda<\ell}\frac{\left(\beta_{\lambda}^{r_1}\beta_{\ell}^{r_1}\right)/\left(\beta_{\lambda}^{r_1-1}\beta_{\ell}^{r_1-1}\right)}{(1+r_2)\left(\beta_{\ell}^{r_1-1}\vartheta_{\lambda}-\beta_{\lambda}^{r_1-1}\vartheta_{\ell}\right)}\right.$$

$$\left.\times\left[\left(\frac{\vartheta_{\lambda}}{\beta_{\lambda}^{r_1-1}}\right)^{1+r_2}-\left(\frac{\vartheta_{\ell}}{\beta_{\ell}^{r_1-1}}\right)^{1+r_2}\right]\right]^{\frac{1}{r_2}}$$

$$\leq \left(\frac{\sum_{\lambda=1}^{\varrho}\vartheta_{\lambda}^{r_2}}{\sum_{\lambda=1}^{\varrho}\beta_{\lambda}^{r_1}}\right)^{\frac{1}{r_2}}. \tag{3.12}$$

Multiplying by $\sum_{\lambda=1}^{\varrho}\beta_{\lambda}^{r_1}$ and using the relation $\frac{1}{r_1}+\frac{1}{r_2}=1$, we get our desired refinement of Hölder's inequality. □

For $\varrho = r_1 = r_2 = 2$, $\beta_{\lambda} = (1, 2)$, and $\vartheta_{\lambda} = (3, 4)$, inequality (3.7) reduces to $11 \leq 11.09053 \leq 11.12055 \leq 11.18033$.

4. Concluding remarks and future insights

It is a very interesting and attractive part of the analysis to conclude new improvements of the existing results of inequalities. Working in the following direction, we have focused on the characterization of σ-convexity due to its unifying behavior. First, we have established the relationship of this class of functions with classical convexity. Incorporating this property, we have presented the first-order differentiable inequality to characterize the σ-convex functions. Also, we have developed new refinements of Jensen's inequality, including interpolating Jensen's inequality. Additionally, we have derived the bounds for Jensen gaps. In the next results, some new sharp refinements of the Hermite–Hadamard inequality and its weighted form pertaining to symmetric functions have been investigated. We have discussed a few captivating special cases and applications to highlight the significance of the study. In the future, we will try to extend our results for set-valued functions and coordinated functions. Another interesting problem is to analyze the various variational problems. We hope this study will create new horizons of research, specifically to find its applications in optimization and other related sciences.

Author contributions

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Conceptualization, resources, formal analysis, validation, reviewing, editing; Hala Mostafa: Methodology, conceptualization, investigation, formal analysis, editing, software, validation, visualization. All authors have read and approved the final version of this manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used any Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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