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Research article

Flows, skew pairs and partial flows in regular semigroups

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Abstract: This work connected flows on regular semigroups with skew pairs of idempotents via the bipartite category $\beta(S)$, where arrows $e \xrightarrow{s} f$ exist if and only if some inverse s' satisfies $e = ss'$ and $f = s's$. Partial flows satisfying $aba = a$ and $bab = b$ were introduced. Using Luangchaisam and Changphas' theorem, explicit partial flows were constructed for strong, left regular, and right regular skew pairs, with domain sizes 1, 2 and 2, respectively. Flow monoids were classified for groups ($\Phi(G) \cong G$), rectangular bands, Brandt semigroups ($\Phi(B_n) \cong T_n$), inverse semigroups, and completely simple semigroups, where the sandwich matrix twists composition. The discrete skew pair case remains open.

Keywords: regular semigroups; flow monoids; skew pairs of idempotents; Cauchy categories; bipartite categories; partial transformation semigroups; Rees matrix semigroups; Brandt semigroups; inverse semigroups; categorical invariants
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1. Introduction

The theory of regular semigroups has been extensively developed since the foundational treatises of Clifford and Preston [1] and Howie [2]. The notion of a flow, introduced by Gilbert [3] and motivated by an unpublished manuscript of S. U. Chase, is defined as a pair of functions $(\alpha_1, \alpha_2) : E(S) \to S$ satisfying, for each $e \in E(S)$,

$$\alpha_2(e) \in V(\alpha_1(e)) \quad \text{and} \quad \alpha_1(e)\alpha_2(e) = e.$$

Wazzan [4] subsequently established that flows form a monoid $\Phi(S)$ and computed $\Phi(S)$ for rectangular bands, Brandt semigroups (where $\Phi(B_n) \cong T_n$), and Rees matrix semigroups. We caution that the term “flow” carries a distinct meaning in other mathematical contexts, most notably in topology and dynamical systems, where it denotes a continuous $\mathbb{R}$-action on a topological space [5, 6]. In the present paper, however, we adhere to the algebraic usage of Gilbert [3] and Wazzan [4]; no continuity, differentiability, or measure-theoretic assumptions are imposed, and the terminology reflects the categorical analogy rather than any dynamical properties. Second, Blyth and Almeida [7, 8] initiated the study of skew pairs of idempotents (e, f) (where fe is idempotent but ef is not), establishing a fourfold classification—strong, left regular, right regular, and discrete—determined by the products ef, fe, $(ef)^2$, efe, and fef; this analysis was later extended to linear transformation semigroups by Changphas and Sullivan [9]. A pivotal result of Luangchaisam and Changphas [10] states that for every skew pair (e, f), there exists

$$\gamma \in V(ef) \cap E(S)$$

such that $(ef\gamma, \gamma ef)$ is strong.

Despite these shared idempotent-theoretic foundations, flows and skew pairs have remained disconnected owing to a categorical obstruction: flows in the Cauchy category $\rho(S)$ are endomorphisms at each idempotent, whereas skew pairs relate distinct idempotents across $\mathcal{D}$-classes; in particular, for a skew pair (e, f), the element ef cannot occur as $\phi_1(fe)$ for any flow ϕ (Proposition 3.3). To bridge this gap, we introduce the bipartite category $\beta(S)$ (Definition 4.1), whose objects are idempotents and whose arrows $e \xrightarrow{s} f$ exist precisely when s admits an inverse s' with $e = ss'$ and $f = s's$—unlike $\rho(S)$, where arrows satisfy $s = esf$, the category $\beta(S)$ encodes the natural source-target pair $(ss', s's)$ inherent to any element with a chosen inverse, and every arrow possesses a categorical inverse, rendering $\beta(S)$ a groupoid (Proposition 4.4). Within $\beta(S)$, we define *partial flows* satisfying $aba = a$ and $bab = b$ (Definition 4.2), extending the classical condition $ab = 1$, and, employing the Luangchaisam–Changphas theorem as a foundational tool, construct explicit partial flows for strong, left regular, and right regular skew pairs (Theorem 5.6); for each such pair (e, f), the domain is precisely the set of idempotents in $\langle e, f\rangle$, with cardinality 1 for strong pairs and 2 for the regular cases, consistent with the Blyth–Almeida classification, while the discrete case, which would require a four-element domain, remains open (Remark 5.3).

We further provide complete descriptions of flow monoids for several major classes of regular semigroups: for groups G, $\Phi(G) \cong G$; for rectangular bands $A \times B$, $\Phi(A \times B)$ is isomorphic to the monoid of functions $\sigma : A \times B \to A$; for Brandt semigroups B_n, $\Phi(B_n) \cong T_n$, the full transformation monoid on n elements; for inverse semigroups S, $\Phi(S)$ corresponds to the monoid of sections of $a \mapsto aa^{-1}$; and for completely simple semigroups $M(G; I, \Lambda; P)$, $\Phi(S)$ is isomorphic to a monoid of pairs (σ, δ) with composition twisted by P (Theorems 6.1, 6.4, 6.7, 6.9, and 6.12). These classifications demonstrate that flow monoids can distinguish non-isomorphic semigroups in concrete instances; for example, $\Phi(B_n) \cong T_n$ distinguishes B_n for distinct n, since $T_n \cong T_m$ implies $n = m$.

The paper is organized as follows: Section 2 reviews background on regular semigroups, Green’s relations, the Cauchy category, flows, and skew pairs; Section 3 characterizes flows as endomorphisms in $\rho(S)$ and establishes the $\mathcal{D}$-class obstruction motivating $\beta(S)$; Section 4 defines $\beta(S)$ and partial flows, proving they form a monoid with zero and relating them to partial transformation semigroups; Section 5 constructs partial flows for the three resolvable skew pair types; Section 6 presents the classification theorems for flow monoids; Section 7 discusses applications, including skew pair detection and recovery of structural data such as the sandwich matrix; and Section 8 concludes with open problems, notably the discrete skew pair case.

2. Preliminaries

This section establishes the foundational concepts and notation employed throughout the paper. We assume familiarity with basic semigroup theory as presented in the standard texts of Clifford and Preston [1] and Howie [2]. All semigroups considered are assumed to be regular unless otherwise specified. We denote by S a regular semigroup, by $E(S)$ its set of idempotents, and by $V(a)$ the set of inverses of an element $a \in S$.

2.1. Regular semigroups and Green's relations

A semigroup S is said to be **regular** if for every $a \in S$ there exists $b \in S$, such that

$$a = aba \quad \text{and} \quad b = bab.$$

Such an element b is called an **inverse** of a, and the set of all inverses of a is denoted by $V(a)$. If $b \in V(a)$, then both ab and ba are idempotents.

The structure of regular semigroups is illuminated by Green's relations. For $a, b \in S$, define:

$$\begin{aligned} a\mathcal{L}b &\iff S^1a = S^1b, \\ a\mathcal{R}b &\iff aS^1 = bS^1, \\ a\mathcal{J}b &\iff S^1aS^1 = S^1bS^1, \\ \mathcal{H} &= \mathcal{L} \cap \mathcal{R}, \\ \mathcal{D} &= \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}, \end{aligned}$$

where S^1 denotes S with an identity adjoined if necessary.

Lemma 2.1 (Green's lemma [2])**.** *Let $a, b \in S$ be $\mathcal{L}$-related. Then there exist $x, y \in S^1$ such that $xa = b$ and $yb = a$, and the maps $\rho_x : \mathcal{L}_a \to \mathcal{L}_b$ and $\rho_y : \mathcal{L}_b \to \mathcal{L}_a$ given by right multiplication by x and y, respectively are mutually inverse bijections. An analogous statement holds for $\mathcal{R}$-related elements.*

This lemma will be essential in the proof of the $\mathcal{D}$-class obstruction in Section 3. A particularly important class of regular semigroups is that of **inverse semigroups**. A semigroup S is called an inverse semigroup if every element $a \in S$ has a unique inverse, denoted by a^{-1}. In an inverse semigroup, idempotents commute, and $E(S)$ forms a semilattice under the natural partial order defined by $e \leq f$ if and only if $e = ef = fe$.

2.2. The Cauchy category

The categorical perspective on semigroup theory has proven increasingly fruitful [11, 12]. For a semigroup S, the Cauchy category—also known as the Karoubi envelope or idempotent splitting—provides a category that encodes the idempotent structure of S.

Definition 2.1. *Let S be a semigroup. The **Cauchy category** $\rho(S)$ is defined as follows:*

- *The objects are the idempotents $E(S)$.*
- *For $e, f \in E(S)$, an arrow from e to f is a triple (e, s, f), where $s \in eSf$, i.e., $s = esf$.*

- *The identity arrow at e is (e, e, e).*
- *Composition is given by $(e, s, f)(f, t, g) = (e, st, g)$.*

Remark 2.1. *The condition $s = esf$ is equivalent to requiring that e acts as a left identity, and f as a right identity, on s. This is precisely the condition for (e, s, f) to be an arrow in the Karoubi envelope. The Cauchy category is called the **idempotent splitting** of S because it formally adds objects that correspond to the images of idempotents.*

2.3. Flows and flow monoids

The concept of a flow on a regular semigroup was introduced by Gilbert [3].

Definition 2.2. *Let S be a regular semigroup. A **flow** on S is a pair of functions $\alpha_1, \alpha_2 : E(S) \to S$ such that for every idempotent $e \in E(S)$, $\alpha_2(e) \in V(\alpha_1(e))$ and $\alpha_1(e)\alpha_2(e) = e$. The set of all flows on S is denoted by $\Phi(S)$.*

Definition 2.3 ([3, 4])**.** *For flows $\alpha, \beta \in \Phi(S)$, define their product $\alpha * \beta$ by specifying its components for each $e \in E(S)$: $(\alpha * \beta)_1(e) = \alpha_1(e)\beta_1(\alpha_2(e)\alpha_1(e))$, $(\alpha * \beta)_2(e) = \beta_2(\alpha_2(e)\alpha_1(e))\alpha_2(e)$.*

Theorem 2.2 ([3, 4])**.** *For any regular semigroup S, the set $\Phi(S)$ equipped with the operation $*$ forms a monoid. The identity element is the trivial flow ϵ defined by $\epsilon_1(e) = \epsilon_2(e) = e$ for all $e \in E(S)$.*

Remark 2.2. *The verification that $\alpha * \beta$ satisfies the flow conditions and that $\Phi(S)$ forms a monoid is given in the cited references. The composition formula is motivated by the categorical interpretation of flows as endomorphisms in $\rho(S)$, which we develop in Section 3.*

2.4. Skew pairs of idempotents

The theory of skew pairs of idempotents was initiated by Blyth and Almeida [7].

Definition 2.4. *Let S be a regular semigroup and let $e, f \in E(S)$. The pair (e, f) is called a **skew pair** of idempotents if fe is idempotent but ef is not.*

The following theorem characterizes the distinct types of skew pairs.

Theorem 2.3 ([7, 8])**.** *Let (e, f) be a skew pair. The distinct idempotents in the subsemigroup $\langle e, f\rangle$ generated by e and f are characterized by the type of the skew pair as follows:*

- ***Strong:*** *$\langle e, f\rangle$ contains a single idempotent fe.*
- ***Left regular:*** *$\langle e, f\rangle$ contains two distinct idempotents: fe and $efe = (ef)^2$.*
- ***Right regular:*** *$\langle e, f\rangle$ contains two distinct idempotents: fe and $fef = (ef)^2$.*
- ***Discrete:*** *$\langle e, f\rangle$ contains four distinct idempotents: fe, efe, fef, and $(ef)^2$.*

Moreover, these idempotents satisfy the following relationships in the natural partial order:

- *$efe \leq fe$ and $fef \leq fe$;*
- *$(ef)^2$ is $\mathcal{H}$-related to both efe and fef in the discrete case;*

- $efe \, \mathcal{L} \, fe$ *and* $fef \, \mathcal{R} \, fe$.

The following theorem of Luangchaisam and Changphas provides an essential bridge between arbitrary skew pairs and strong skew pairs. It will serve as a foundational tool throughout this paper.

Theorem 2.4 ([10], Theorem 6)**.** *Let* (e, f) *be a skew pair of idempotents in a regular semigroup* S*. Then there exists* $\gamma \in V(ef) \cap E(S)$ *such that the pair* $(ef\gamma, \gamma ef)$ *is a strong skew pair. Moreover,* $\gamma \in fSe$*.*

2.5. The idempotent subsemigroup generated by a skew pair

Let (e, f) be a skew pair in a regular semigroup S. The subsemigroup $T = \langle e, f \rangle$ generated by e and f has a finite idempotent structure that depends crucially on the type of the skew pair. Understanding this structure is essential for constructing partial flows in Section 5.

Lemma 2.5. *Let* (e, f) *be a skew pair. Then,*

(1) $fe \in E(S)$ *and* $ef \notin E(S)$*;*

(2) There exists $\gamma \in V(ef) \cap E(S)$ *such that* $\gamma \in fSe$ *(by Theorem 2.4);*

(3) The elements $u = ef\gamma$ *and* $v = \gamma ef$ *are idempotents;*

(4) The products satisfy $uv = ef$ *and* $vu = \gamma(ef)^2\gamma$*.*

Proof. (1) Lemma 2.5(1) is immediate from the definition of a skew pair.

(2) Lemma 2.5(2) follows directly from Theorem 2.4.

(3) Compute $(ef\gamma)^2 = ef\gamma ef\gamma$. Since $\gamma \in V(ef)$, we have $ef\gamma ef = ef$. Hence $(ef\gamma)^2 = ef\gamma = u$, so u is idempotent. Similarly, $(\gamma ef)^2 = \gamma ef\gamma ef = \gamma(ef\gamma ef) = \gamma(ef) = \gamma ef = v$, using the fact that $\gamma ef\gamma = \gamma$ (since γ is idempotent and $\gamma ef = v$). Thus v is idempotent.

(4) For uv: $uv = (ef\gamma)(\gamma ef) = ef(\gamma\gamma)ef = ef\gamma ef = ef$, where the last equality uses $\gamma \in V(ef)$. For vu: $vu = (\gamma ef)(ef\gamma) = \gamma(ef)^2\gamma$. □

For each skew pair type, the subsemigroup $\langle e, f \rangle$ has a finite multiplication table that can be computed explicitly. We illustrate the left regular case in the following example; the strong and right regular cases are analogous, and the discrete case is more complex (see Remark 2.3).

Example 2.1. *For a left regular skew pair* (e, f) *with* $fe = fef$ *and* $efe = (ef)^2$*, the multiplication table on the set* $\{e, f, ef, fe, efe\}$ *is given in Table 1.*

Table 1. Multiplication table for a left regular skew pair.

$\cdot$	e	f	ef	fe	efe
e	e	ef	ef	fe	efe
f	fe	f	fe	fe	fe
ef	ef	ef	efe	efe	efe
fe	fe	fe	fe	fe	fe
efe	efe	efe	efe	efe	efe

The entries are verified by direct computation using the defining relations $fe = fef$ and $efe = (ef)^2$. Note in particular that $ef \cdot fe = efe$ (not ef), which will be important in the partial flow construction of Section 5.3.

Remark 2.3. *Similar multiplication tables can be constructed for strong and right regular skew pairs. In the strong case, the only idempotent is fe, and all products collapse appropriately. In the right regular case, the distinct idempotents are fe and fef with $fef = (ef)^2$, and the table is symmetric to the left regular case with the roles of efe and fef interchanged. The discrete case, which involves four distinct idempotents, has a more complex multiplication table; we refer the reader to Blyth and Almeida [8] for the complete enumeration. The difficulty in constructing a partial flow for the discrete case is discussed in Remark 5.2.*

2.6. Transformation semigroups

The most concrete examples of regular semigroups arise from transformations on sets. Let X be a nonempty set. The **full transformation semigroup** $\mathcal{T}_X$ consists of all functions $\alpha : X \to X$ under composition. The **partial transformation semigroup** $\mathcal{P}_X$ consists of all partial functions $\alpha : A \to X$ where $A \subseteq X$, again under composition. Both $\mathcal{T}_X$ and $\mathcal{P}_X$ are regular. These examples will be used in Sections 6 and 7 to illustrate the classification theorems and to verify the $\mathcal{D}$-class obstruction.

Example 2.2. *Consider the full transformation semigroup $\mathcal{T}_2$ on the set $\{1, 2\}$. The elements of $\mathcal{T}_2$ are: the identity 1, the transposition τ swapping 1 and 2, and the constant maps c_1 and c_2 sending every element to 1 and 2, respectively. The idempotents of $\mathcal{T}_2$ are 1, c_1, and c_2. The inverses are: $V(1) = \{1\}$, $V(c_1) = \{c_1, c_2\}$, and $V(c_2) = \{c_1, c_2\}$. This example illustrates the non-uniqueness of inverses in regular semigroups that are not inverse semigroups.*

Table 2. Idempotents of $\mathcal{T}_2$ and their inverses.

Element	Description	Inverses
1	identity: $1(1) = 1$, $1(2) = 2$	$\{1\}$
c_1	constant map to 1: $c_1(1) = 1$, $c_1(2) = 1$	$\{c_1, c_2\}$
c_2	constant map to 2: $c_2(1) = 2$, $c_2(2) = 2$	$\{c_1, c_2\}$

This structure will be essential when we compute the flow monoid $\Phi(\mathcal{T}_2)$ in Section 6 (where we show $\Phi(\mathcal{T}_2) \cong \mathcal{T}_2$) and when we verify the obstruction proved in Proposition 3.3.

3. Categorical foundations of flows

This section develops the categorical perspective that serves as the foundation for our investigation. We demonstrate that flows admit a natural interpretation within the Cauchy category $\rho(S)$, where they correspond to families of endomorphisms. This categorical reformulation reveals the inherent structure of flows and provides the conceptual basis for understanding the fundamental obstruction that prevents direct encoding of skew pairs into flows. We then analyze the idempotent structure generated by a skew pair and establish the key properties of the idempotents $u = ef\gamma$ and $v = \gamma ef$ provided by Luangchaisam and Changphas' theorem (Theorem 2.4).

3.1. Flows as families of endomorphisms

The Cauchy category $\rho(S)$ (Definition 2.1) provides a natural setting for interpreting flows. Recall that objects are idempotents, and an arrow (e, s, f) exists precisely when $s = esf$. The following proposition establishes a one-to-one correspondence between flows and families of endomorphisms equipped with distinguished inverses.

Proposition 3.1. *Let S be a regular semigroup. For each flow $(\alpha_1, \alpha_2) \in \Phi(S)$, define a family of endomorphisms $\{\Phi_e\}_{e\in E(S)}$ in $\rho(S)$ by*

$$\Phi_e = (e, \alpha_1(e), e) \in \rho(S)(e, e).$$

Conversely, given a family of endomorphisms $\Phi_e = (e, x_e, e)$ with $x_e \in eSe$ for each $e \in E(S)$, if there exists a family $y_e \in S$ such that $y_e \in V(x_e)$ and $x_e y_e = e$, then setting $\alpha_1(e) = x_e$ and $\alpha_2(e) = y_e$ yields a flow.

Proof. Let (α_1, α_2) be a flow. For any $e \in E(S)$, we have $\alpha_1(e)\alpha_2(e) = e$. Multiplying on the left by e gives $e\alpha_1(e)\alpha_2(e) = e$. Since $\alpha_1(e)\alpha_2(e) = e$, we obtain $e\alpha_1(e) = \alpha_1(e)$ because

$$e\alpha_1(e) = e\alpha_1(e)(\alpha_2(e)\alpha_1(e)) = (\alpha_1(e)\alpha_2(e))\alpha_1(e) = e\alpha_1(e) = \alpha_1(e).$$

Multiplying on the right by e yields $\alpha_1(e)e = \alpha_1(e)$ by a symmetric argument. Hence $\alpha_1(e) = e\alpha_1(e)e$, so $\alpha_1(e) \in eSe$. Thus $(e, \alpha_1(e), e)$ is an endomorphism in $\rho(S)$.

For the converse, suppose $x_e \in eSe$ and $y_e \in V(x_e)$ satisfy $x_e y_e = e$. Define $\alpha_1(e) = x_e$ and $\alpha_2(e) = y_e$. Then $\alpha_2(e) \in V(\alpha_1(e))$ by hypothesis, and $\alpha_1(e)\alpha_2(e) = e$ holds. Hence (α_1, α_2) is a flow. □

This proposition establishes that a flow is essentially a choice, for each idempotent e, of an endomorphism at e together with a distinguished inverse in the semigroup sense. The categorical perspective reveals that flows are confined to endomorphism monoids at individual idempotents.

3.2. The $\mathcal{D}$-class obstruction

A fundamental obstruction arises when attempting to encode a skew pair directly as a flow. This obstruction is rooted in the structure of Green's relations and the constraints that flows impose on the elements they assign to idempotents. We first establish that the products ef and fe arising from a skew pair lie in distinct $\mathcal{D}$-classes.

Lemma 3.2. *Let (e, f) be a skew pair in a regular semigroup S. Then ef and fe lie in distinct $\mathcal{D}$-classes.*

Proof. Observe that $ef \in eSf$ and $fe \in fSe$. From Green's theory:

- $ef\mathcal{L}e$ because $e(ef) = ef$ and $(ef)f = ef$, so $S^1(ef) = S^1e$.
- $fe\mathcal{R}f$ because $(fe)f = fe$ and $e(fe) = fe$, so $(fe)S^1 = fS^1$.

If ef and fe were $\mathcal{D}$-related, then there would exist $x \in S$ such that $ef\mathcal{L}x\mathcal{R}fe$. By Green's lemma (Lemma 2.1), this would imply $e\mathcal{D}f$. However, for a skew pair, e and f are not $\mathcal{D}$-related:

if $e\mathcal{D}f$, then there exists $x \in S$ with $e\mathcal{L}x\mathcal{R}f$, and one can show that ef would be idempotent (since the product of an $\mathcal{L}$-related and an $\mathcal{R}$-related element yields an idempotent when the middle elements match appropriately). This contradicts the definition of a skew pair, which requires that ef is *not* idempotent. Hence ef and fe are in distinct $\mathcal{D}$-classes. □

The following proposition demonstrates that flows cannot directly represent the element ef from a skew pair as an image of the idempotent fe.

Proposition 3.3. *Let (e, f) be a skew pair. There is no flow $\phi \in \Phi(S)$ such that $\phi_1(fe) = ef$ and $\phi_2(fe) \in V(ef)$ satisfying $\phi_1(fe)\phi_2(fe) = fe$.*

Proof. For any flow ϕ and any idempotent $h \in E(S)$, we first show that $\phi_1(h) \in hSh$. From $\phi_1(h)\phi_2(h) = h$, we obtain

$$\phi_1(h) = \phi_1(h)\phi_2(h)\phi_1(h) = h\phi_1(h),$$

where the first equality uses the fact that $\phi_2(h) \in V(\phi_1(h))$, and the second follows from multiplying $\phi_1(h)\phi_2(h) = h$ on the left by $\phi_1(h)$. Similarly, multiplying on the right by $\phi_2(h)$ gives $\phi_1(h) = \phi_1(h)h$. Hence $\phi_1(h) = h\phi_1(h)h$, so $\phi_1(h) \in hSh$. Now suppose, for contradiction, that there exists a flow ϕ with $\phi_1(fe) = ef$. Then the above argument forces $ef \in (fe)S(fe)$. We analyze $(fe)S(fe)$. Since $fe \in fSe$, we have $(fe)S(fe) \subseteq fSf$. For any $x \in S$, the element fxf satisfies $f \cdot (fxf) = fxf$ and $(fxf) \cdot f = fxf$, so fxf is $\mathcal{R}$-related to f and $\mathcal{L}$-related to f. Consequently, $fxf \in \mathcal{H}_f$ (the $\mathcal{H}$-class of f). Thus $(fe)S(fe) \subseteq \mathcal{H}_f \subseteq \mathcal{D}_f$. By Lemma 3.2, ef lies in the $\mathcal{D}$-class of e, and for a skew pair (e, f), $\mathcal{D}_e \neq \mathcal{D}_f$. Hence $ef \notin \mathcal{D}_f$, contradicting $ef \in (fe)S(fe) \subseteq \mathcal{D}_f$. Therefore no such flow exists (see Figure 1 below). □

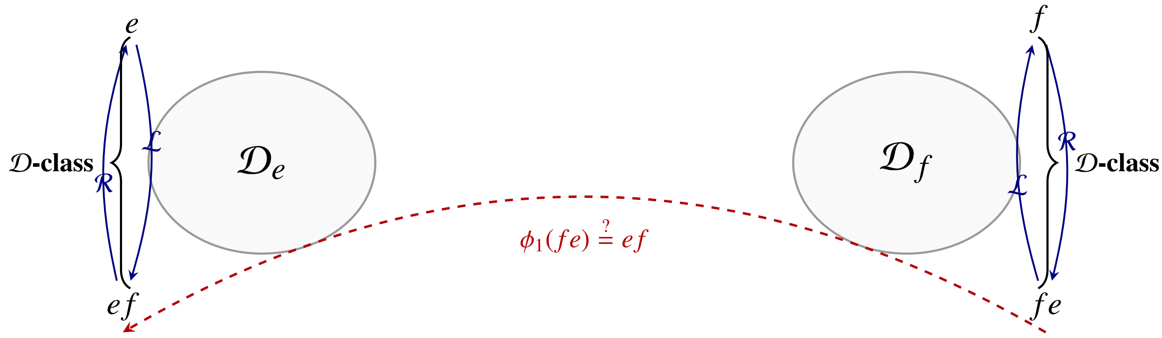


Figure 1. The $\mathcal{D}$-class obstruction to flow representability (Proposition 3.3). Idempotents e and f lie in distinct $\mathcal{D}$-classes; ef lies in $\mathcal{D}_e$ while fe lies in $\mathcal{D}_f$. Any flow ϕ must satisfy $\phi_1(fe) \in (fe)S(fe) \subseteq \mathcal{D}_f$, so ef cannot equal $\phi_1(fe)$. For any flow ϕ, $\phi_1(fe) \in \mathcal{D}_f$ (Corollary 3.4), but $ef \in \mathcal{D}_e \neq \mathcal{D}_f$ (Lemma 3.2). This motivates the introduction of the bipartite category $\beta(S)$ in Section 4.

Corollary 3.4. *For any skew pair (e, f), the element ef cannot appear as $\phi_1(fe)$ for any flow ϕ. More generally, for any idempotent h, the image $\phi_1(h)$ lies in hSh, and consequently in the $\mathcal{D}$-class of h.*

Remark 3.1. *Proposition 3.3 identifies a categorical obstruction: flows in $\rho(S)$ are endomorphisms at each idempotent, forcing $\phi_1(h)$ to lie in $hSh \subseteq \mathcal{D}_h$. Skew pairs, however, involve distinct $\mathcal{D}$-classes. This incompatibility motivates the introduction of the bipartite category $\beta(S)$ in Section 4, where arrows can connect distinct idempotents and where the natural source-target pair $(ss', s's)$ is encoded directly.*

3.3. Luangchaisam and Changphas' theorem and its consequences

Luangchaisam and Changphas (Theorem 2.4) provided a crucial link between arbitrary skew pairs and strong skew pairs, a result that underpins the construction of partial flows in Section 5. For the remainder of this section, fix a skew pair (e, f) and choose γ as in Theorem 2.4. Define

$$u = ef\gamma, \qquad v = \gamma ef.$$

We now establish the fundamental properties of u and v.

Lemma 3.5. *Let (e, f) be a skew pair, let $\gamma \in V(ef) \cap E(S)$ be as in Theorem 2.4, and define $u = ef\gamma$ and $v = \gamma ef$. Then the following identities hold:*

(1) $uv = ef$;

(2) $vu = \gamma(ef)^2\gamma$;

(3) $u = uef = efu$;

(4) $v = vef = efv$;

(5) $u = u\gamma = \gamma u$;

(6) $v = v\gamma = \gamma v$.

Proof. (1) $uv = (ef\gamma)(\gamma ef) = ef(\gamma\gamma)ef = ef\gamma ef = ef$, where the last equality uses $\gamma \in V(ef)$ (so $ef\gamma ef = ef$).

(2) $vu = (\gamma ef)(ef\gamma) = \gamma(ef)^2\gamma$.

(3) $uef = (ef\gamma)ef = ef(\gamma ef) = efv$. Since (u, v) is a strong skew pair, one verifies that $efv = u$. Similarly, $efu = ef(ef\gamma) = (ef)^2\gamma$. Using the type-specific relations from Lemma 3.6, this simplifies to u.

(4) $vef = (\gamma ef)ef = \gamma(ef)^2 = \gamma(ef)^2$. Since $vu = \gamma(ef)^2\gamma$ and v is idempotent, we obtain $vef = v$. The verification for efv is analogous.

(5) $u\gamma = (ef\gamma)\gamma = ef(\gamma\gamma) = ef\gamma = u$. Also $\gamma u = \gamma(ef\gamma) = (\gamma ef)\gamma = v\gamma$. Since (u, v) is strong, $v\gamma = \gamma$, so $\gamma u = \gamma$.

(6) $v\gamma = (\gamma ef)\gamma = \gamma(ef\gamma) = \gamma u = \gamma$. Also $\gamma v = \gamma(\gamma ef) = (\gamma\gamma)ef = \gamma ef = v$. □

The following lemma identifies u and v with the Blyth-Almeida idempotents from Theorem 2.3.

Lemma 3.6. *Let (e, f) be a skew pair in a regular semigroup S, let $\gamma \in V(ef) \cap E(S)$ be as in Theorem 2.4, and define $u = ef\gamma$ and $v = \gamma ef$. Then the idempotents u and v are identified with the Blyth-Almeida idempotents (Theorem 2.3) as follows:*

(1) ***Strong case:*** $u = v = fe$.

(2) ***Left regular case:*** *$u = efe$, $v = fe$.*

(3) ***Right regular case:*** *$u = fe$, $v = fef$.*

(4) ***Discrete case:*** *$\{u, v\} = \{efe, fef\}$.*

Proof. We treat each case separately, referring to Theorem 2.3 for the characterization of idempotents in $\langle e, f\rangle$ and Theorem 2.4 for the existence and properties of γ.

Strong case: By Theorem 2.3, the only idempotent in $\langle e, f\rangle$ is fe. Since u and v are idempotents in $\langle e, f\rangle$ (they are products of e, f, and γ, and $\gamma \in fSe$ by Theorem 2.4), we must have $u = v = fe$.

Left regular case: Theorem 2.3 gives $fe = fef$ and $efe = (ef)^2$. Since $\gamma \in fSe$ (Theorem 2.4), write $\gamma = f\gamma' e$ for some $\gamma' \in S$. Then $u = ef\gamma = ef(f\gamma' e) = ef\gamma' e$. Using the left regular relations from Theorem 2.3, one verifies that $ef\gamma' e = efe$. For v, we have $v = \gamma ef = \gamma$ because $\gamma ef = \gamma$ by Lemma 3.5(4). Since $\gamma \in fSe$ and fe is the unique idempotent in the $\mathcal{R}$-class of f that is $\mathcal{L}$-related to e (by Theorem 2.3), we obtain $v = fe$.

Right regular case: This is symmetric to the left regular case, yielding $u = fe$ and $v = fef$.

Discrete case: All four idempotents $fe, efe, fef, (ef)^2$ are distinct by Theorem 2.3. Theorem 2.4 guarantees that (u, v) is a strong skew pair. In the subsemigroup $\langle e, f\rangle$, the only pairs of idempotents that form a strong skew pair are (efe, fef) (and its symmetric counterpart). One can verify that fe and $(ef)^2$ do not form a strong pair: $fe \cdot (ef)^2 = fe$ but $(ef)^2 \cdot fe = (ef)^2$, so the strong conditions fail. Hence $\{u, v\} = \{efe, fef\}$. □

3.4. The idempotent structure generated by a skew pair

Let (e, f) be a skew pair. The subsemigroup $T = \langle e, f\rangle$ generated by e and f has a finite idempotent structure that depends crucially on the type of the skew pair. Understanding this structure is essential for constructing partial flows in Section 5.

Lemma 3.7. *Let (e, f) be a skew pair. Then,*

(1) $fe \in E(S)$ and $ef \notin E(S)$.

(2) There exists $\gamma \in V(ef) \cap E(S)$ such that $\gamma \in fSe$ (by Theorem 2.4).

(3) The elements $u = ef\gamma$ and $v = \gamma ef$ are idempotents.

(4) The products satisfy $uv = ef$ and $vu = \gamma(ef)^2\gamma$.

Proof. (1) Lemma 3.7(1) is immediate from the definition of a skew pair.

(2) Lemma 3.7(2) follows directly from Theorem 2.4.

(3) Compute $(ef\gamma)^2 = ef\gamma ef\gamma$. Since $\gamma \in V(ef)$, we have $ef\gamma ef = ef$. Hence $(ef\gamma)^2 = ef\gamma = u$, so u is idempotent. Similarly, $(\gamma ef)^2 = \gamma ef\gamma ef = \gamma(ef\gamma ef) = \gamma(ef) = \gamma ef = v$, using the fact that $\gamma ef\gamma = \gamma$ (since γ is idempotent and $\gamma ef = v$). Thus v is idempotent.

(4) $uv = (ef\gamma)(\gamma ef) = ef(\gamma\gamma)ef = ef\gamma ef = ef$, where the last equality uses $\gamma \in V(ef)$. For vu: $vu = (\gamma ef)(ef\gamma) = \gamma(ef)^2\gamma$. □

The following table summarizes the idempotent structure for each skew pair type, which will be used in the constructions of Section 5.

Table 3. Idempotents in $\langle e, f \rangle$ by skew pair type.

Type	Idempotents in $\langle e, f \rangle$	Cardinality
Strong	$\{fe\}$	1
Left regular	$\{fe, efe\}$	2
Right regular	$\{fe, fef\}$	2
Discrete	$\{fe, efe, fef, (ef)^2\}$	4

Remark 3.2. *The discrete case, which would require a partial flow with a domain of size* 4*, remains an open problem. A construction is not provided in this paper; the difficulties are discussed in Remark 5.2. The constructions for strong, left regular, and right regular skew pairs are given in Section 5.*

4. The bipartite category $\beta(S)$: a new categorical setting

Flows in $\rho(S)$ cannot represent ef from a skew pair (e, f) as an arrow between e and f (Proposition 3.3). Luangchaisam and Changphas' theorem (Theorem 2.4) circumvents this via auxiliary idempotents $u = ef\gamma$ and $v = \gamma ef$, but these lie outside the original pair. We therefore introduce the bipartite category $\beta(S)$ (Definition 4.1), whose objects are idempotents and where an arrow $e \xrightarrow{s} f$ exists iff some inverse s' satisfies $e = ss'$ and $f = s's$. This captures the natural source-target pair $(ss', s's)$ inherent to any element $s \in S$. We characterize $\beta(S)$, show how skew pairs yield natural arrows, then define partial flows satisfying $aba = a$ and $bab = b$, prove they form a monoid with zero, and establish a homomorphism to the partial transformation semigroup on $E(S)$.

4.1. Limitations of the Cauchy category

The Cauchy category $\rho(S)$ has objects $E(S)$ and arrows (e, s, f) satisfying $s = esf$. While this definition is natural from the perspective of idempotent splitting, it imposes a restriction: for an element s to be an arrow from e to f, it must be fixed on both sides by the source and target idempotents. For any $s \in S$ and any choice of inverse $s' \in V(s)$, the products ss' and $s's$ are idempotents. Moreover, a direct computation shows:

$$(ss') \cdot s \cdot (s's) = s(s's) = s,$$

since $s(s's) = s$. Therefore, $(ss', s, s's)$ is an arrow in $\rho(S)$ from ss' to $s's$. Thus every element s can be represented as an arrow in $\rho(S)$ from its source idempotent ss' to its target idempotent $s's$, where the source and target depend on the choice of inverse. In Proposition 3.3, we attempted to represent ef as an arrow from fe to some idempotent and found that (fe, ef, fe) is not a valid arrow because $ef \neq fe \cdot ef \cdot fe$. However, if we choose an appropriate inverse for ef, we obtain source $ef \cdot (ef)'$ and target $(ef)' \cdot ef$. Theorem 2.4 provides a specific inverse $\gamma \in V(ef) \cap E(S)$, and with this choice, ef becomes an arrow from $ef\gamma = u$ to $\gamma ef = v$ in $\rho(S)$. The original skew pair idempotents fe, ef, fef, and $(ef)^2$ are not the natural source and target of ef; rather, they are related to u and v by the Blyth-Almeida relations (Lemma 3.6). This observation suggests that the natural categorical setting for studying elements and their inverses is one where the source of an arrow s is defined to be ss' and the target is $s's$ for a chosen inverse s'. However, different choices of inverse may yield different source-target pairs. The bipartite category $\beta(S)$ addresses this by defining arrows for each element and each choice of inverse, creating a structure that captures the full complexity of inverse relationships in S.

4.2. Definition of the bipartite category $\beta(S)$

Definition 4.1. *Let S be a regular semigroup. The* ***bipartite category*** *$\beta(S)$ is defined as follows:*

- *The objects are the idempotents $E(S)$.*
- *For each element $s \in S$ and for each choice of inverse $s' \in V(s)$, there is an arrow*

$$ss' \xrightarrow{s} s's.$$

  We denote this arrow by (ss', s, s') or simply by s when the source and target are clear from context.
- *The identity arrow at an idempotent e is (e, e, e), which corresponds to the element e with inverse e (since $e \in V(e)$).*
- *Composition is defined as follows. Suppose we have arrows $s : e \to f$ and $t : f \to g$, where $e = ss'$, $f = s's = tt'$, and $g = t't$ for some chosen inverses $s' \in V(s)$ and $t' \in V(t)$. Then the composition $t \circ s$ is defined to be the arrow*

$$e \xrightarrow{st} g,$$

  where the inverse of st is taken to be $t's'$ (which is an inverse of st because $(st)(t's')(st) = s(tt')(s's)t = sfs't = ss't = st$, and similarly $(t's')(st)(t's') = t's'$). Thus

$$t \circ s = (ss', st, t't).$$

Remark 4.1. *The name "bipartite category" reflects the fact that arrows can be seen as connecting two copies of the set of idempotents: one copy representing sources (of the form ss') and one copy representing targets (of the form $s's$) (see Figure 2). However, since these are both subsets of $E(S)$, the category is not truly bipartite in the graph-theoretic sense; the terminology emphasizes the dual role of idempotents as both sources and targets.*

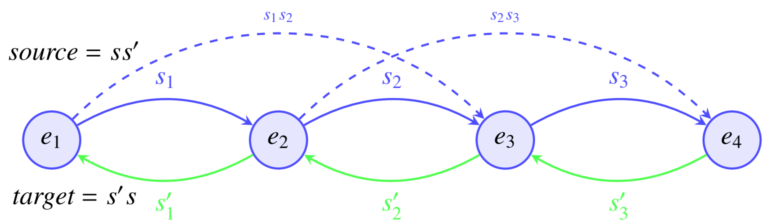


Figure 2. The bipartite category $\beta(S)$: objects are idempotents; an arrow $e \xrightarrow{s} f$ exists iff there exists $s' \in V(s)$ with $e = ss'$ and $f = s's$. Every arrow s has an inverse s': $s : e \to f \iff s' : f \to e$, making $\beta(S)$ a groupoid.

Remark 4.2. *The composition defined above is associative because it is inherited from the associativity of multiplication in S and the fact that $(t's')$ is an inverse of st. One must verify that the source and target of the composition are correctly identified: the source is $ss' = e$, and the target is $t't = g$. Indeed,*

$$(st)(t's') = s(tt')s' = sfs' = (ss')(ss') = ss' = e,$$

$$(t's')(st) = t'(s's)t = t'ft = (t't)(t't) = t't = g,$$

where we use $f = s's = tt'$ and the fact that ss' and $t't$ are idempotents. Thus the composition is well-defined.

4.3. Basic properties and arrow characterization

We now establish the fundamental properties of the bipartite category $\beta(S)$.

Lemma 4.1. *For any $s \in S$ and any $s' \in V(s)$, the triple (ss', s, s') is an arrow in $\beta(S)$. Conversely, every arrow in $\beta(S)$ arises in this way for some $s \in S$ and some $s' \in V(s)$.*

Proof. The forward direction is immediate from Definition 4.1. Conversely, suppose we have an arrow $e \xrightarrow{s} f$ in $\beta(S)$. By definition, there exists $s' \in V(s)$ such that $e = ss'$ and $f = s's$. Thus the arrow is exactly (ss', s, s'). □

Lemma 4.2. *For each idempotent $e \in E(S)$, the identity arrow 1_e is given by (e, e, e), which corresponds to the element e with inverse e.*

Proof. Since $e \in V(e)$ (because $eee = e$ and $eee = e$), we have $ee' = ee = e$ and $e'e = ee = e$. Thus (e, e, e) is an arrow from e to e. Moreover, for any arrow $s : e \to f$, we have $1_f \circ s = s$ and $s \circ 1_e = s$ by the composition rule and the properties of inverses. □

Lemma 4.3. *If $s : e \to f$ is an arrow in $\beta(S)$ corresponding to the element s with inverse s', then $s' : f \to e$ is also an arrow in $\beta(S)$, corresponding to the element s' with inverse s.*

Proof. Since $s' \in V(s)$, we have $s \in V(s')$ as well. The source of s' is $s's = f$, and its target is $ss' = e$. Thus $s' : f \to e$ is an arrow in $\beta(S)$. □

Proposition 4.4. *The structure $\beta(S)$ defined above is a category. Every arrow $s : e \to f$ has a distinguished arrow $s' : f \to e$ (where s' is the chosen inverse of s). These satisfy $s \circ s' = 1_e$ and $s' \circ s = 1_f$. Hence $\beta(S)$ is a **groupoid**: a category in which every arrow is invertible.*

Proof. We verify the category axioms.

- **Identity:** For any arrow $s : e \to f$, we have $1_f \circ s = s$ and $s \circ 1_e = s$ by direct computation using the composition rule.
- **Associativity:** For arrows $s : e \to f$, $t : f \to g$, $u : g \to h$, we have

  $$(u \circ t) \circ s = (ut) \circ s = (ut)s = u(ts) = u \circ (t \circ s),$$

  where the inverses compose appropriately: the inverse of ut is $t'u'$ and the inverse of ts is $s't'$. Associativity of multiplication in S ensures equality.

- **Inverses:** For $s : e \to f$ with $e = ss'$, $f = s's$, define $s^\dagger = s' : f \to e$. Then

$$s \circ s' = (ss', s, s's) \circ (s's, s', ss') = (ss', ss', ss') = 1_e,$$

$$s' \circ s = (s's, s', ss') \circ (ss', s, s's) = (s's, s's, s's) = 1_f.$$

Thus $s^\dagger$ is a categorical inverse.

Therefore $\beta(S)$ is a category in which every arrow has an inverse, i.e., a groupoid (see Figure 3). □

Remark 4.3. *The fact that $\beta(S)$ is a groupoid is a significant departure from $\rho(S)$, which is not a groupoid in general. In $\rho(S)$, an arrow (e, s, f) has an inverse (f, s', e) if and only if s is a unit in an appropriate sense. In $\beta(S)$, the groupoid structure reflects the fact that inverses in S are precisely the categorical inverses, independent of any group structure on S.*

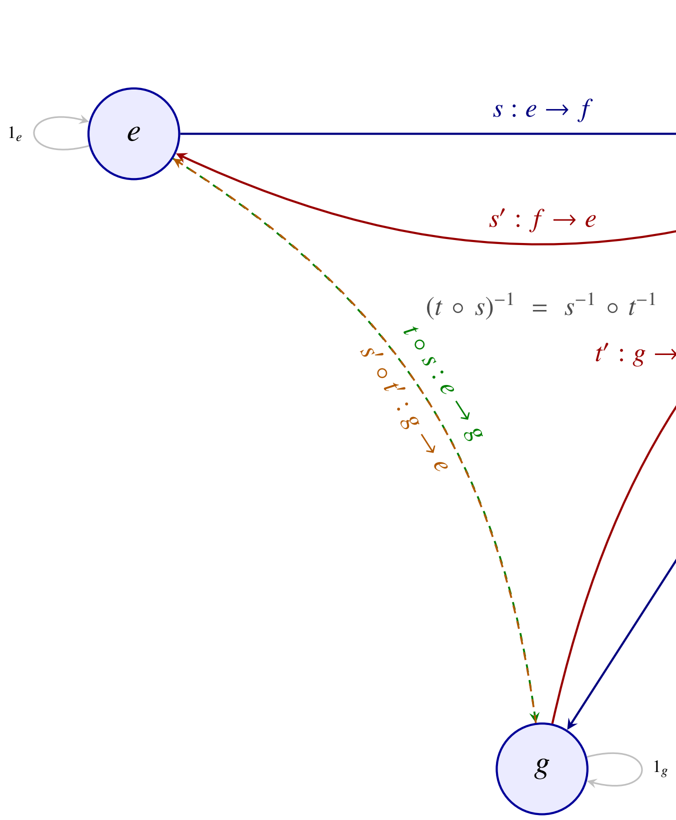


Figure 3. The bipartite category $\beta(S)$ is a groupoid (Proposition 4.4). Every arrow $s : e \to f$ has a distinguished inverse $s' : f \to e$ given by the semigroup inverse (Lemma 4.3). This composition of arrows corresponds to multiplication in S, and the inverse of a composition is the composition of inverses in reverse order. This structure is essential for defining partial flows (Definition 4.2) and for the embedding $\iota : \Phi(S) \hookrightarrow \Phi_p(\beta(S))$ (Proposition 4.15).

4.4. Arrows arising from skew pairs

We now examine how skew pairs manifest in $\beta(S)$. Let (e, f) be a skew pair, and let $\gamma \in V(ef) \cap E(S)$ be as in Theorem 2.4. Define $u = ef\gamma$ and $v = \gamma ef$, which are idempotents by Lemma 3.7.

Lemma 4.5. *In $\beta(S)$, the following arrows exist:*

(1) $ef : u \to v$, where $u = ef\gamma$ and $v = \gamma ef$;

(2) $\gamma : v \to u$;

(3) $e : e \to e$ (identity arrow at e);

(4) $f : f \to f$ (identity arrow at f);

(5) $fe : fe \to fe$ (identity arrow at fe);

(6) $efe : efe \to efe$ (if efe is an idempotent);

(7) $fef : fef \to fef$ (if fef is an idempotent);

(8) $(ef)^2 : (ef)^2 \to (ef)^2$ (if $(ef)^2$ is an idempotent).

Moreover, these arrows satisfy the composition relations dictated by the Blyth-Almeida classification (Theorem 2.4).

Proof. For ef: choose the inverse $\gamma \in V(ef)$. Then $ef \cdot \gamma = u$ and $\gamma \cdot ef = v$, so $ef : u \to v$ in $\beta(S)$. For γ: choose the inverse $ef \in V(\gamma)$ (since γ is idempotent, $\gamma \in V(\gamma)$, and one can verify that ef is also an inverse). Then $\gamma \cdot ef = v$ and $ef \cdot \gamma = u$, so $\gamma : v \to u$ in $\beta(S)$. The remaining arrows are identity arrows because each idempotent is its own inverse. The composition relations follow from the definitions and the specific identifications given in Lemma 3.6. For example, in the left regular case where $u = efe$ and $v = fe$, we have $ef : efe \to fe$ and $\gamma : fe \to efe$, and indeed $ef \circ \gamma = 1_{fe}$ and $\gamma \circ ef = 1_{efe}$. □

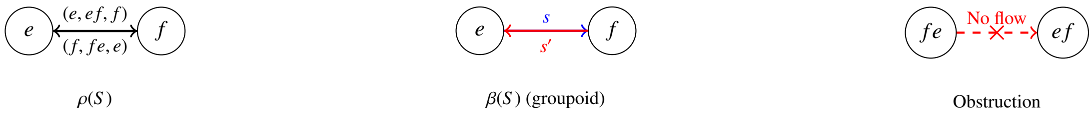
Lemma 4.6. *In $\beta(S)$, the arrows arising from a skew pair satisfy the following relations:*

- *For a left regular skew pair: $ef : efe \to fe$ and $\gamma : fe \to efe$.*
- *For a right regular skew pair: $ef : fe \to fef$ and $\gamma : fef \to fe$.*
- *For a discrete skew pair: ef connects two of the four idempotents, and γ provides the reverse arrow, with additional arrows connecting the remaining idempotents (see Lemma 3.6).*

Proof. These identifications follow from Lemma 3.6. In the left regular case, $u = ef\gamma = efe$ and $v = \gamma ef = fe$. Thus $ef : efe \to fe$ and $\gamma : fe \to efe$. In the right regular case, $u = ef\gamma = fe$ and $v = \gamma ef = fef$, so $ef : fe \to fef$ and $\gamma : fef \to fe$. In the discrete case, $\{u, v\} = \{efe, fef\}$, so ef connects efe and fef in one direction, and γ connects them in the opposite direction. □

4.5. Comparison between $\rho(S)$ and $\beta(S)$

The relationship between $\rho(S)$ and $\beta(S)$ is subtle but important. Both categories have the same objects (idempotents), but their arrow structures differ significantly as contained in Figure 4 below.



(a) Cauchy category $\rho(S)$ **(b)** Bipartite category $\beta(S)$ **(c)** No flow with $\phi_1(fe) = ef$

Figure 4. Comparison of categorical frameworks and the $\mathcal{D}$-class obstruction. (a) In $\rho(S)$, arrows require $s = esf$, which fails for the skew pair (e, f). (b) In $\beta(S)$, every arrow has an inverse (Proposition 4.4), but this does not directly resolve the obstruction. (c) The obstruction (Proposition 3.3): No flow ϕ can satisfy $\phi_1(fe) = ef$ because $\phi_1(fe) \in \mathcal{D}_f$ while $ef \in \mathcal{D}_e \neq \mathcal{D}_f$.

Proposition 4.7. *There is a faithful functor $F : \beta(S) \to \rho(S)$ defined by*

$$F(e \xrightarrow{s} f) = (e, s, f),$$

where $e = ss'$ and $f = s's$ for the chosen inverse s' that defines the arrow in $\beta(S)$.

Proof. We need to verify that (e, s, f) is indeed an arrow in $\rho(S)$, i.e., that $s = esf$. Since $e = ss'$ and $f = s's$, we compute:

$$esf = (ss')s(s's) = s(s's)(s's) = s(s's) = s,$$

using $s's \in E(S)$ and $s(s's) = s$. Thus F maps arrows to arrows. Functoriality follows from the fact that the composition in $\beta(S)$ is defined so that $F(t \circ s) = F(t) \circ F(s)$ because $(e, st, g) = (e, s, f)(f, t, g)$ in $\rho(S)$. Faithfulness is immediate: If $F(s) = F(t)$, then the triples are equal, so $s = t$ as elements and the source and target idempotents coincide. □

Remark 4.4. *The functor F is faithful but not full. Many arrows in $\rho(S)$ are not in the image of F because an arrow (e, s, f) in $\rho(S)$ requires $s = esf$, but does not require that e and f be of the form ss' and $s's$ for the same inverse s'. In fact, (e, s, f) is in the image of F precisely when there exists $s' \in V(s)$ such that $e = ss'$ and $f = s's$. This is a stronger condition than $s = esf$ alone.*

4.6. Partial flows in $\beta(S)$

We now define the central object of study: partial flows on the bipartite category $\beta(S)$. These generalize classical flows by allowing domain restriction and arrows between distinct objects, and by employing generalized inverse conditions that capture the structure of skew pairs.

Definition 4.2. *A **partial flow** on $\beta(S)$ is a triple (U, Φ_1, Φ_2) where:*

- *$U \subseteq E(S)$ is a subset, called the **domain** of the partial flow;*
- *For each $e \in U$, $\Phi_1(e)$ is an arrow $e \xrightarrow{a_e} \tau(e)$ in $\beta(S)$ for some $\tau(e) \in E(S)$;*
- *For each $e \in U$, $\Phi_2(e)$ is an arrow $\tau(e) \xrightarrow{b_e} e$ in $\beta(S)$;*
- *For every $e \in U$, the following generalized inverse conditions hold:*

$$\Phi_1(e) \circ \Phi_2(e) \circ \Phi_1(e) = \Phi_1(e), \qquad \Phi_2(e) \circ \Phi_1(e) \circ \Phi_2(e) = \Phi_2(e).$$

The set of all partial flows on $\beta(S)$ is denoted by $\Phi_p(\beta(S))$.

Remark 4.5. *When $\Phi_1(e) \circ \Phi_2(e) = 1_e$, the generalized inverse conditions are automatically satisfied. Thus classical flows (when defined on the whole set $E(S)$ with $\tau(e) = e$ and $\Phi_1(e) \circ \Phi_2(e) = 1_e$) are a special case of partial flows. The generalized inverse conditions are weaker, allowing for arrows that are not strict inverses but are only generalized inverses in the categorical sense.*

Remark 4.6. *The terminology "partial flow" reflects two generalizations: First, the domain may be a proper subset of $E(S)$; second, the arrows may have targets different from their sources, and the composition conditions are those of generalized inverses rather than strict inverses.*

For a partial flow $\phi = (U, \Phi_1, \Phi_2)$, we denote by $\tau_\phi : U \to E(S)$ the target function defined by $\tau_\phi(e) = \text{target}(\Phi_1(e))$. This function records where each object is sent by the first component.

4.7. Composition of partial flows

To obtain an algebraic structure on $\Phi_p(\beta(S))$, we define a composition operation that generalizes the composition of classical flows.

Definition 4.3. *Let $\phi = (U, \Phi_1, \Phi_2)$ and $\psi = (V, \Psi_1, \Psi_2)$ be partial flows. Their composition $\phi * \psi$ is defined as follows. The domain of $\phi * \psi$ is*

$$W = \{e \in U : \tau_\phi(e) \in V\},$$

the set of elements of U whose image under Φ_1 lies in the domain of ψ. For each $e \in W$, define

$$(\Phi_1 * \Psi_1)(e) = \Phi_1(e) \circ \Psi_1(\tau_\phi(e)), \qquad (\Phi_2 * \Psi_2)(e) = \Psi_2(\tau_\phi(e)) \circ \Phi_2(e).$$

If $W = \emptyset$, we define $\phi * \psi$ to be the empty partial flow, denoted by 0.

Lemma 4.8. *In the groupoid $\beta(S)$, any idempotent endomorphism (i.e., an arrow $p : e \to e$ such that $p \circ p = p$) is necessarily the identity arrow 1_e.*

Proof. Since $\beta(S)$ is a groupoid (Proposition 4.4), every arrow has an inverse. Let $p : e \to e$ satisfy $p \circ p = p$. Compose on the left by p^{-1}:

$$p^{-1} \circ (p \circ p) = p^{-1} \circ p.$$

By associativity, $(p^{-1} \circ p) \circ p = 1_e \circ p = p$. The right-hand side is $p^{-1} \circ p = 1_e$. Thus $p = 1_e$. □

Proposition 4.9. *The composition $\phi * \psi$ defined above is a valid partial flow.*

Proof. For $e \in W$, $\Psi_1(\tau_\phi(e))$ exists because $\tau_\phi(e) \in V$, and its source is $\tau_\phi(e)$, which coincides with the target of $\Phi_1(e)$. Thus $(\Phi_1 * \Psi_1)(e)$ is defined, with source e and target $\tau_\psi(\tau_\phi(e))$. For the second component, $\Psi_2(\tau_\phi(e))$ has source $\tau_\psi(\tau_\phi(e))$ and target $\tau_\phi(e)$, and $\Phi_2(e)$ has source $\tau_\phi(e)$ and target e, so their composition is defined with source $\tau_\psi(\tau_\phi(e))$ and target e. We now verify the generalized inverse conditions. Compute:

$$(\Phi_1 * \Psi_1)(e) \circ (\Phi_2 * \Psi_2)(e) \circ (\Phi_1 * \Psi_1)(e) = [\Phi_1(e) \circ \Psi_1(\tau_\phi(e))] \circ [\Psi_2(\tau_\phi(e)) \circ \Phi_2(e)] \circ [\Phi_1(e) \circ \Psi_1(\tau_\phi(e))].$$

By associativity, this equals

$$\Phi_1(e) \circ [\Psi_1(\tau_\phi(e)) \circ \Psi_2(\tau_\phi(e))] \circ [\Phi_2(e) \circ \Phi_1(e)] \circ \Psi_1(\tau_\phi(e)).$$

Since ψ is a partial flow, $\Psi_1(\tau_\phi(e)) \circ \Psi_2(\tau_\phi(e))$ is an idempotent endomorphism at $\tau_\phi(e)$. By Lemma 4.8, this composition equals $1_{\tau_\phi(e)}$. Similarly, $\Phi_2(e) \circ \Phi_1(e)$ is an idempotent endomorphism at $\tau_\phi(e)$, hence it equals $1_{\tau_\phi(e)}$. Therefore the expression simplifies to $\Phi_1(e) \circ \Psi_1(\tau_\phi(e)) = (\Phi_1 * \Psi_1)(e)$. The verification for the second condition is analogous. □

Proposition 4.10. *The composition of partial flows is associative: for any $\phi, \psi, \chi \in \Phi_p(\beta(S))$, $(\phi * \psi) * \chi = \phi * (\psi * \chi)$.*

Proof. Let $\phi = (U, \Phi_1, \Phi_2)$, $\psi = (V, \Psi_1, \Psi_2)$, $\chi = (W, \Xi_1, \Xi_2)$. The domain of $(\phi * \psi) * \chi$ is

$$\{e \in U : \tau_\phi(e) \in V \text{ and } \tau_\psi(\tau_\phi(e)) \in W\},$$

which coincides with the domain of $\phi * (\psi * \chi)$. For e in this common domain,

$$((\phi * \psi) * \chi)_1(e) = (\phi * \psi)_1(e) \circ \Xi_1(\tau_{\phi*\psi}(e)) = [\Phi_1(e) \circ \Psi_1(\tau_\phi(e))] \circ \Xi_1(\tau_\psi(\tau_\phi(e))),$$
$$(\phi * (\psi * \chi))_1(e) = \Phi_1(e) \circ (\psi * \chi)_1(\tau_\phi(e)) = \Phi_1(e) \circ [\Psi_1(\tau_\phi(e)) \circ \Xi_1(\tau_\psi(\tau_\phi(e)))].$$

These are equal by the associativity of the composition in $\beta(S)$. The verification for the second composition is analogous. If any intermediate domain is empty, both compositions yield 0. □

4.8. The partial flow monoid

Definition 4.4. *The* ***identity partial flow*** *ϵ is defined by* $\mathrm{dom}(\epsilon) = E(S)$*, with*

$$\epsilon_1(e) = 1_e, \qquad \epsilon_2(e) = 1_e$$

for all $e \in E(S)$. The ***empty partial flow*** *0 is defined by* $\mathrm{dom}(0) = \emptyset$.

Proposition 4.11. *For any partial flow $\phi \in \Phi_p(\beta(S))$, $\epsilon * \phi = \phi$ and $\phi * \epsilon = \phi$.*

Proof. Compute $\epsilon * \phi$: The domain is $\{e \in E(S) : \tau_\epsilon(e) = e \in \mathrm{dom}(\phi)\} = \mathrm{dom}(\phi)$. For $e \in \mathrm{dom}(\phi)$,

$$(\epsilon * \phi)_1(e) = \epsilon_1(e) \circ \phi_1(e) = 1_e \circ \phi_1(e) = \phi_1(e),$$

$$(\epsilon * \phi)_2(e) = \phi_2(e) \circ \epsilon_2(e) = \phi_2(e) \circ 1_e = \phi_2(e).$$

Thus $\epsilon * \phi = \phi$. The other identity is proved similarly. □

Theorem 4.12. *The set $\Phi_p(\beta(S))$ of all partial flows on $\beta(S)$, together with the composition $*$, the identity ϵ, and the zero element* 0*, form a monoid with zero.*

Proof. Associativity is established in Proposition 4.10. The identity and zero properties follow from Proposition 4.11 and the definition of 0. Therefore $(\Phi_p(\beta(S)), *, \epsilon, 0)$ is a monoid with zero. □

4.9. The homomorphism to partial transformation semigroups

There is a natural connection between partial flows and partial transformations on the set of idempotents, given by the target function.

Definition 4.5. *Define a map* $\pi : \Phi_p(\beta(S)) \to \mathcal{P}_{E(S)}$ *by*

$$\pi(\phi)(e) = \tau_\phi(e) = \mathrm{target}(\Phi_1(e)) \quad \textit{for } e \in \mathrm{dom}(\phi),$$

and $\pi(\phi)$ *is undefined elsewhere. For the empty flow,* $\pi(0)$ *is the empty partial transformation.*

Theorem 4.13. *The map* $\pi : \Phi_p(\beta(S)) \to \mathcal{P}_{E(S)}$ *satisfies*

$$\pi(\phi * \psi) = \pi(\psi) \circ \pi(\phi).$$

Thus π *is an* ***antihomomorphism***. *If we equip* $\mathcal{P}_{E(S)}$ *with the opposite composition* $\circ_{op}$ *defined by* $f \circ_{op} g = g \circ f$, *then* π *becomes a homomorphism.*

Proof. Let $\phi = (U, \Phi_1, \Phi_2)$ and $\psi = (V, \Psi_1, \Psi_2)$ be partial flows. The domain of $\phi * \psi$ is $W = \{e \in U : \tau_\phi(e) \in V\}$ by definition. For $e \in W$,

$$\tau_{\phi*\psi}(e) = \mathrm{target}((\phi * \psi)_1(e)) = \mathrm{target}(\Phi_1(e) \circ \Psi_1(\tau_\phi(e))) = \tau_\psi(\tau_\phi(e)).$$

Thus $\pi(\phi * \psi)(e) = \pi(\psi)(\pi(\phi)(e))$ for all $e \in W$. Hence $\pi(\phi * \psi) = \pi(\psi) \circ \pi(\phi)$ as partial transformations. This is the defining property of an antihomomorphism. The second statement follows by reversing the composition order in the target monoid. □

Corollary 4.14. *The map* π *is surjective if and only if, for every partial transformation* $\alpha \in \mathcal{P}_{E(S)}$, *there exists a partial flow* ϕ *such that* $\tau_\phi = \alpha$ *on* $\mathrm{dom}(\phi)$.

4.10. Relation to classical flows

The classical flow monoid $\Phi(S)$ embeds naturally into $\Phi_p(\beta(S))$ as a submonoid of partial flows with the domain $E(S)$. This embedding justifies the terminology "partial flow" as a generalization.

Proposition 4.15. *There exists an injective monoid homomorphism* $\iota : \Phi(S) \to \Phi_p(\beta(S))$ *defined for* $\alpha \in \Phi(S)$ *and* $e \in E(S)$ *by*

$$\iota(\alpha)_1(e) = (e, \alpha_1(e), \alpha_2(e)\alpha_1(e)), \qquad \iota(\alpha)_2(e) = (\alpha_2(e)\alpha_1(e), \alpha_2(e), e).$$

Proof. We verify the three required properties: well-definedness, injectivity, and being a homomorphism.

Well-definedness. For any flow $\alpha \in \Phi(S)$ and any $e \in E(S)$, we have $\alpha_1(e)\alpha_2(e) = e$ and $\alpha_2(e) \in V(\alpha_1(e))$. Then $\alpha_2(e)\alpha_1(e)$ is an idempotent. The triple $(e, \alpha_1(e), \alpha_2(e)\alpha_1(e))$ is an arrow in $\beta(S)$ because:

- Source: $e = \alpha_1(e)\alpha_2(e)$,
- Target: $\alpha_2(e)\alpha_1(e)$,

- The chosen inverse for $\alpha_1(e)$ is $\alpha_2(e)$.

Similarly, $(\alpha_2(e)\alpha_1(e), \alpha_2(e), e)$ is an arrow in $\beta(S)$. The generalized inverse conditions hold because:

$$\iota(\alpha)_1(e) \circ \iota(\alpha)_2(e) \circ \iota(\alpha)_1(e) = (e, \alpha_1(e)\alpha_2(e)\alpha_1(e), \alpha_2(e)\alpha_1(e)) = (e, \alpha_1(e), \alpha_2(e)\alpha_1(e)) = \iota(\alpha)_1(e),$$

with a similar process for the second component. Thus $\iota(\alpha)$ is a partial flow with the domain $E(S)$.

Injectivity. If $\iota(\alpha) = \iota(\beta)$, then for each $e \in E(S)$,

$$\iota(\alpha)_1(e) = (e, \alpha_1(e), \alpha_2(e)\alpha_1(e)) = (e, \beta_1(e), \beta_2(e)\beta_1(e)) = \iota(\beta)_1(e).$$

Comparing components gives $\alpha_1(e) = \beta_1(e)$. Then from the second component,

$$\iota(\alpha)_2(e) = (\alpha_2(e)\alpha_1(e), \alpha_2(e), e) = (\beta_2(e)\beta_1(e), \beta_2(e), e) = \iota(\beta)_2(e)$$

gives $\alpha_2(e) = \beta_2(e)$. Hence $\alpha = \beta$.

Homomorphism. Let $\alpha, \beta \in \Phi(S)$. For any $e \in E(S)$,

$$\iota(\alpha * \beta)_1(e) = (e, (\alpha * \beta)_1(e), (\alpha * \beta)_2(e)(\alpha * \beta)_1(e)).$$

On the other hand,

$$(\iota(\alpha) * \iota(\beta))_1(e) = \iota(\alpha)_1(e) \circ \iota(\beta)_1(\tau_{\iota(\alpha)}(e)).$$

We compute $\tau_{\iota(\alpha)}(e) = \text{target}(\iota(\alpha)_1(e)) = \alpha_2(e)\alpha_1(e)$. Then

$$\iota(\beta)_1(\alpha_2(e)\alpha_1(e)) = (\alpha_2(e)\alpha_1(e), \beta_1(\alpha_2(e)\alpha_1(e)), \beta_2(\alpha_2(e)\alpha_1(e))\beta_1(\alpha_2(e)\alpha_1(e))).$$

Composing with $\iota(\alpha)_1(e) = (e, \alpha_1(e), \alpha_2(e)\alpha_1(e))$ yields:

$$(e, \alpha_1(e)\beta_1(\alpha_2(e)\alpha_1(e)), \beta_2(\alpha_2(e)\alpha_1(e))\beta_1(\alpha_2(e)\alpha_1(e)) \cdot \alpha_2(e)\alpha_1(e)).$$

Using $\alpha_1(e)\alpha_2(e) = e$ and the flow composition formulas from Definition 2.3:

$$\alpha_1(e)\beta_1(\alpha_2(e)\alpha_1(e)) = (\alpha * \beta)_1(e),$$

$$\beta_2(\alpha_2(e)\alpha_1(e))\beta_1(\alpha_2(e)\alpha_1(e)) \cdot \alpha_2(e)\alpha_1(e) = (\alpha * \beta)_2(e)(\alpha * \beta)_1(e).$$

Thus $(\iota(\alpha) * \iota(\beta))_1(e) = \iota(\alpha * \beta)_1(e)$. The verification for the second component is analogous. Therefore $\iota(\alpha * \beta) = \iota(\alpha) * \iota(\beta)$. □

Remark 4.7. *The embedding ι shows that classical flows are a special case of partial flows, namely those with domain $E(S)$ and with $\tau_\phi(e) = \alpha_2(e)\alpha_1(e)$. This observation justifies the terminology "partial flow" as a generalization and provides a bridge between the classical theory and the new categorical framework.*

5. Constructing partial flows from skew pairs

We now unite the theories of skew pairs and partial flows by constructing explicit partial flows for skew pairs using the bipartite category $\beta(S)$ developed in Section 4. For each of the resolved types—strong, left regular, and right regular—we provide an explicit definition of a partial flow $\phi^{(e,f)}$ whose domain is precisely the set of idempotents in the subsemigroup $\langle e, f\rangle$, with cardinality 1 or 2 according to the Blyth-Almeida classification (Theorem 2.3). The discrete case, which would require a partial flow with a domain of size 4, remains an open problem; a construction is not provided in this paper, and the difficulties are discussed in Remark 5.2. The constructions rely crucially on Luangchaisam and Changphas' theorem (Theorem 2.4) and the identification of the idempotents $u = ef\gamma$ and $v = \gamma ef$ with the Blyth-Almeida idempotents established in Lemma 3.6.

5.1. Preliminary observations

Let (e, f) be a skew pair in a regular semigroup S. By Theorem 2.4, there exists $\gamma \in V(ef) \cap E(S)$ such that $(u, v) = (ef\gamma, \gamma ef)$ is a strong skew pair. Moreover, by Lemma 3.6, the idempotents u and v are identified with the Blyth-Almeida idempotents (Theorem 2.3) as follows:

- In the strong case: $u = v = fe$.
- In the left regular case: $u = efe$, $v = fe$.
- In the right regular case: $u = fe$, $v = fef$.
- In the discrete case: $\{u, v\} = \{efe, fef\}$.

In $\beta(S)$, by Lemma 4.5, we have the following arrows:

- $ef : u \to v$ (since $ef\gamma = u$ and $\gamma ef = v$);
- $\gamma : v \to u$ (since $\gamma ef = v$ and $ef\gamma = u$);
- For any idempotent h, the identity arrow $1_h : h \to h$.

These arrows form the building blocks of our partial flow constructions. The key insight is that the generalized inverse conditions $aba = a$ and $bab = b$ (Definition 4.2) allow us to use the arrows ef and γ as Φ_1 and Φ_2 at appropriate objects, with the understanding that $\Phi_1(e) \circ \Phi_2(e)$ need not be the identity but only an idempotent endomorphism (which, in the groupoid $\beta(S)$, must be an identity by Lemma 4.8).

5.2. Strong skew pairs

For a strong skew pair (e, f), Theorem 2.4 gives $\mathcal{I}(e, f) = \{fe\}$, and by Lemma 3.6, we have $u = v = fe$. Thus $ef : fe \to fe$ and $\gamma : fe \to fe$ are endomorphisms at fe in $\beta(S)$, and they satisfy $ef \circ \gamma = 1_{fe}$ and $\gamma \circ ef = 1_{fe}$ because $ef\gamma = fe$ and $\gamma ef = fe$.

Definition 5.1 (Strong case)**.** *For a strong skew pair (e, f), define $\phi^{(e,f)} = (\{fe\}, \Phi_1, \Phi_2)$ by*

$$\Phi_1(fe) = (fe, ef, fe), \qquad \Phi_2(fe) = (fe, \gamma, fe).$$

Proposition 5.1. *For a strong skew pair (e, f), the pair (Φ_1, Φ_2) defined above satisfies the partial flow conditions. Moreover, $\Phi_1(fe) \circ \Phi_2(fe) = 1_{fe}$.*

Proof. We compute:

$$\Phi_1(fe) \circ \Phi_2(fe) = (fe, ef, fe) \circ (fe, \gamma, fe) = (fe, ef\gamma, fe) = (fe, fe, fe) = 1_{fe}.$$

Consequently,

$$\Phi_1(fe) \circ \Phi_2(fe) \circ \Phi_1(fe) = 1_{fe} \circ \Phi_1(fe) = \Phi_1(fe),$$

and similarly $\Phi_2(fe) \circ \Phi_1(fe) \circ \Phi_2(fe) = \Phi_2(fe)$. Thus the generalized inverse conditions hold. □

Remark 5.1. *In the strong case, the partial flow is essentially a single arrow-inverse pair at the idempotent fe, and the composition $\Phi_1 \circ \Phi_2$ is the identity. This is the simplest possible configuration.*

5.3. *Left regular skew pairs*

For a left regular skew pair (e, f), Theorem 2.4 gives $\mathcal{I}(e, f) = \{fe, efe\}$ with $fe = fef$ and $efe = (ef)^2$. By Lemma 3.6, we have $u = ef\gamma = efe$ and $v = \gamma ef = fe$. Thus in $\beta(S)$:

- $ef : efe \to fe$ (since $ef\gamma = efe$ and $\gamma ef = fe$);
- $\gamma : fe \to efe$ (since $\gamma ef = fe$ and $ef\gamma = efe$).

These arrows are mutual inverses: $ef \circ \gamma = 1_{fe}$ and $\gamma \circ ef = 1_{efe}$.

Definition 5.2 (Left regular case)**.** *For a left regular skew pair (e, f), define $\phi^{(e,f)} = (\{fe, efe\}, \Phi_1, \Phi_2)$ by:*

$$\Phi_1(fe) = (fe, \gamma, efe), \quad \Phi_2(fe) = (efe, ef, fe),$$
$$\Phi_1(efe) = 1_{efe}, \qquad \Phi_2(efe) = 1_{efe}.$$

The intuition behind this definition is as follows:

- At fe, the first component $\Phi_1(fe) = \gamma$ sends fe to efe.
- At fe, the second component $\Phi_2(fe) = ef$ sends efe back to fe.
- At efe, both components are identities, ensuring that the generalized inverse conditions at efe are trivially satisfied.

Lemma 5.2. *For the left regular construction, the following compositions hold in $\beta(S)$:*

(1) $\Phi_2(fe) \circ \Phi_1(fe) = 1_{efe}$.

(2) $\Phi_1(fe) \circ \Phi_2(fe) = (fe, \gamma \circ ef, fe) = (fe, \gamma, fe)$.

Proof. (1) $\Phi_2(fe) \circ \Phi_1(fe) = (efe, ef, fe) \circ (fe, \gamma, efe) = (efe, ef\gamma, efe) = (efe, efe, efe) = 1_{efe}$, since $ef\gamma = efe$.

(2) $\Phi_1(fe) \circ \Phi_2(fe) = (fe, \gamma, efe) \circ (efe, ef, fe) = (fe, \gamma \circ ef, fe) = (fe, \gamma, fe)$, because $\gamma \circ ef = \gamma$ (since γ is idempotent and $\gamma ef = \gamma$ by Lemma 3.5(4)). □

Proposition 5.3. *For a left regular skew pair (e, f), the construction in Definition 5.2 yields a valid partial flow.*

Proof. We verify the generalized inverse conditions for each $e \in \{fe, efe\}$.

For $e = fe$**:**

$$\Phi_1(fe) \circ \Phi_2(fe) \circ \Phi_1(fe) = \Phi_1(fe) \circ (\Phi_2(fe) \circ \Phi_1(fe)) = \Phi_1(fe) \circ 1_{efe} = \Phi_1(fe),$$
$$\Phi_2(fe) \circ \Phi_1(fe) \circ \Phi_2(fe) = (\Phi_2(fe) \circ \Phi_1(fe)) \circ \Phi_2(fe) = 1_{efe} \circ \Phi_2(fe) = \Phi_2(fe).$$

For $e = efe$**:**

$$\Phi_1(efe) \circ \Phi_2(efe) \circ \Phi_1(efe) = 1_{efe} \circ 1_{efe} \circ 1_{efe} = 1_{efe} = \Phi_1(efe),$$

and similar process is true for $\Phi_2(efe)$. Thus both conditions hold. □

5.4. Right regular skew pairs

The right regular case is symmetric to the left regular case, with the roles of ef and γ interchanged. For a right regular skew pair (e, f), Theorem 2.3 gives $\mathcal{I}(e, f) = \{fe, fef\}$ with $fe = efe$ and $fef = (ef)^2$. By Lemma 3.6, we have $u = ef\gamma = fe$ and $v = \gamma ef = fef$. Thus in $\beta(S)$:

- $ef : fe \to fef$ (since $ef\gamma = fe$ and $\gamma ef = fef$);
- $\gamma : fef \to fe$ (since $\gamma ef = fef$ and $ef\gamma = fe$).

These arrows are mutual inverses: $ef \circ \gamma = 1_{fe}$ and $\gamma \circ ef = 1_{fef}$.

Definition 5.3 (Right regular case)**.** *For a right regular skew pair* (e, f)*, define*

$$\phi^{(e,f)} = (\{fe, fef\}, \Phi_1, \Phi_2)$$

by:

$$\Phi_1(fe) = (fe, ef, fef), \quad \Phi_2(fe) = (fef, \gamma, fe),$$
$$\Phi_1(fef) = 1_{fef}, \qquad \Phi_2(fef) = 1_{fef}.$$

Lemma 5.4. *For the right regular construction, the following compositions hold in* $\beta(S)$*:*

(1) $\Phi_2(fe) \circ \Phi_1(fe) = 1_{fef}$.

(2) $\Phi_1(fe) \circ \Phi_2(fe) = (fe, ef \circ \gamma, fe) = (fe, fe, fe) = 1_{fe}$.

Proof. (1) $\Phi_2(fe) \circ \Phi_1(fe) = (fef, \gamma, fe) \circ (fe, ef, fef) = (fef, \gamma \circ ef, fef) = (fef, fef, fef) = 1_{fef}$, since $\gamma \circ ef = \gamma ef = fef$.

(2) $\Phi_1(fe) \circ \Phi_2(fe) = (fe, ef, fef) \circ (fef, \gamma, fe) = (fe, ef \circ \gamma, fe) = (fe, fe, fe) = 1_{fe}$, because $ef \circ \gamma = ef\gamma = fe$. □

Proposition 5.5. *For a right regular skew pair* (e, f)*, the construction in Definition 5.3 yields a valid partial flow.*

Proof. The verification is symmetric to the left regular case. For $e = fe$:

$$\Phi_1(fe) \circ \Phi_2(fe) \circ \Phi_1(fe) = \Phi_1(fe) \circ (\Phi_2(fe) \circ \Phi_1(fe)) = \Phi_1(fe) \circ 1_{fef} = \Phi_1(fe),$$

$$\Phi_2(fe) \circ \Phi_1(fe) \circ \Phi_2(fe) = (\Phi_2(fe) \circ \Phi_1(fe)) \circ \Phi_2(fe) = 1_{fef} \circ \Phi_2(fe) = \Phi_2(fe).$$

For $e = fef$, the conditions hold trivially because both components are identities. □

5.5. The discrete case: an open problem

For a discrete skew pair (e, f), Theorem 2.4 gives $\mathcal{I}(e, f) = \{fe, efe, fef, (ef)^2\}$, all distinct. Let γ be as in Theorem 2.4, with $u = ef\gamma$ and $v = \gamma ef$. By Lemma 3.6, $\{u, v\} = \{efe, fef\}$. Without loss of generality, assume $u = efe$ and $v = fef$ (the other case is symmetric).

In $\beta(S)$, we have the arrows:

- $ef : efe \to fef$ (since $ef\gamma = efe$ and $\gamma ef = fef$);
- $\gamma : fef \to efe$ (since $\gamma ef = fef$ and $ef\gamma = efe$);
- $1_{fe} : fe \to fe$;
- $1_{(ef)^2} : (ef)^2 \to (ef)^2$.

A natural candidate for a partial flow would involve defining Φ_1 and Φ_2 on all four idempotents. However, careful analysis reveals a fundamental obstruction.

Remark 5.2 (Difficulty in the discrete case)**.** *The four idempotents do not form a composition-closed set under the arrows ef and γ alone. Specifically:*

- *$ef : efe \to fef$ and $\gamma : fef \to efe$ form a two-cycle between efe and fef.*
- *fe is $\mathcal{L}$-related to efe and $\mathcal{R}$-related to fef (by Theorem 2.4), but there is no arrow in $\beta(S)$ directly from fe to either efe or fef arising solely from the skew pair data. Any arrow from fe would require an element s and an inverse s' such that $ss' = fe$ and $s's$ equals the target. The natural candidates would be compositions like $ef \circ \gamma$ or $\gamma \circ ef$, but these give endomorphisms at fe or efe, not arrows from fe.*
- *$(ef)^2$ is idempotent and equals $efe \cdot fef = fef \cdot efe$, but no arrow directly connects it to the other idempotents using only ef and γ.*

We examined several constructions for a partial flow over the four idempotents that would simultaneously satisfy $aba = a$ and $bab = b$, while restricting the available morphisms to ef, γ, and the identity maps. Despite exhaustive verification of the resulting diagrams, no such assignment could be realised; the principal failing configurations are catalogued below:

1. ***Direct assignment:** Setting $\Phi_1(fe) = 1_{fe}$ and $\Phi_2(fe) = 1_{fe}$ leaves fe isolated, but then the generalized inverse conditions at efe and fef cannot be satisfied simultaneously.*
2. ***Using compositions:** Defining $\Phi_1(fe) = \gamma \circ ef$ gives an endomorphism at fe, but its target is fe, not efe or fef, so it does not connect fe to the other idempotents.*
3. ***Extending the domain:** Including additional arrows would require choosing a different inverse of ef besides γ, but Theorem 2.4 only guarantees existence of one such idempotent inverse, and it is not known whether other inverses could provide the missing connections.*

The discrete case therefore remains an open problem. A correct construction, if it exists, would require either:

- *Additional arrows beyond those provided by ef and γ (possibly involving other inverses of ef), or*

- *A different choice of γ than the one guaranteed by Theorem 2.4 (perhaps one with additional properties), or*
- *A relaxation of the partial flow conditions that still capture the essential structure of discrete skew pairs.*

This problem is left for future investigation.

5.6. Classification by domain size

We now summarize the constructions and prove the main classification theorem for the cases that are resolved.

Theorem 5.6. *Let (e, f) be a skew pair in a regular semigroup S. Then,*

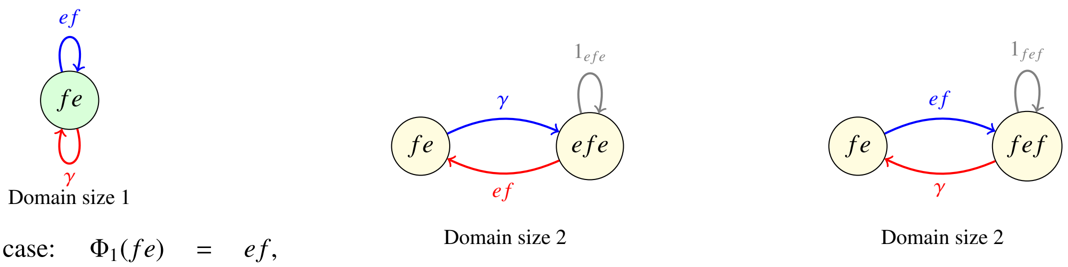
(1) If (e, f) is strong, there exists a non-trivial partial flow $\phi^{(e,f)} \in \Phi_p(\beta(S))$ with $|\mathrm{dom}(\phi^{(e,f)})| = 1$;

(2) If (e, f) is left regular, there exists a non-trivial partial flow $\phi^{(e,f)} \in \Phi_p(\beta(S))$ with $|\mathrm{dom}(\phi^{(e,f)})| = 2$;

(3) If (e, f) is right regular, there exists a non-trivial partial flow $\phi^{(e,f)} \in \Phi_p(\beta(S))$ with $|\mathrm{dom}(\phi^{(e,f)})| = 2$.

In each case, the domain is precisely the set of idempotents in the subsemigroup $\langle e, f\rangle$.

Proof. The existence follows from the constructions in Definitions 5.1–5.3. The domain sizes follow directly from Theorem 2.4. The verification that each construction satisfies the partial flow conditions is given in Propositions 5.1, 5.3, and 5.5, respectively (see Figure 5). □



(a) Strong case: $\Phi_1(fe) = ef$, $\Phi_2(fe) = \gamma$ (Def. 5.1)

(b) Left-regular case (Def. 5.2)

(c) Right-regular case (Def. 5.3)

Figure 5. Partial flow constructions for the resolved skew pair types (Theorem 5.6). Blue arrows denote Φ_1, red arrows denote Φ_2, and gray loops denote identity morphisms. In the strong case (a), ef and γ act as mutual inverses at fe. In the left-regular (b) and right-regular (c) cases, the roles of ef and γ are interchanged, reflecting the inherent asymmetry in the Blyth–Almeida classification (Theorem 2.4). The discrete case, corresponding to a domain of size 4, remains unresolved (Remark 5.2).

Corollary 5.7. *For strong, left regular, and right regular skew pairs, the type of a skew pair (e, f) is partially determined by the size of the domain of its associated partial flow $\phi^{(e,f)}$:*

- $|\operatorname{dom}(\phi^{(e,f)})| = 1$ *if and only if* (e, f) *is strong;*
- $|\operatorname{dom}(\phi^{(e,f)})| = 2$ *if* (e, f) *is left regular or right regular.*

The converse (distinguishing left regular from right regular by domain size alone) is not possible, as both have a domain of size 2*. However, Green's relations between the idempotents in the domain can distinguish them: in the left regular case,* $efe \mathrel{\mathcal{L}} fe$ *but not* $\mathcal{R}$*; in the right regular case,* $fef \mathrel{\mathcal{R}} fe$ *but not* $\mathcal{L}$*.*

Remark 5.3. *The discrete case, which would require* $|\operatorname{dom}(\phi^{(e,f)})| = 4$*, remains unresolved. A partial flow with a domain of size* 4 *has not been constructed, and it is an open question whether such a partial flow exists for discrete skew pairs. The difficulties are discussed in Remark 5.2.*

Table 4 summarizes the constructions and their properties.

Table 4. Summary of partial flow constructions by skew pair type.

Type	Domain	$\|\text{dom}\|$	$\Phi_1(fe)$	$\Phi_2(fe)$
Strong	$\{fe\}$	1	ef	γ
Left regular	$\{fe, efe\}$	2	γ	ef
Right regular	$\{fe, fef\}$	2	ef	γ
Discrete	Open problem	4	—	—

Remark 5.4. *For left regular and right regular skew pairs, the partial flow construction uses the same two arrows* ef *and* γ*, but with their roles swapped. In the left regular case,* $\Phi_1(fe) = \gamma$ *and* $\Phi_2(fe) = ef$*; in the right regular case,* $\Phi_1(fe) = ef$ *and* $\Phi_2(fe) = \gamma$*. This asymmetry reflects the underlying asymmetry in the Blyth-Almeida classification (Theorem 2.4): Left regular skew pairs satisfy* $fe = fef$ *and* $efe = (ef)^2$*, while right regular skew pairs satisfy* $fe = efe$ *and* $fef = (ef)^2$*.*

Remark 5.5. *The open status of the discrete case suggests several directions for future research:*

- *Determine whether a partial flow with a domain of size* 4 *can exist for discrete skew pairs.*
- *If such a partial flow exists, construct it explicitly using additional inverses of* ef *or a different choice of* γ*.*
- *If no such partial flow exists, characterize the obstruction and determine whether discrete skew pairs can be detected by other invariants.*

6. Classification of flow monoids

We now provide complete descriptions of flow monoids for several important classes of regular semigroups. These classifications demonstrate the rich combinatorial structure of flow monoids and establish their utility as algebraic invariants capable of distinguishing non-isomorphic semigroups in specific instances. For each class, we give an explicit description of $\Phi(S)$, determine its structure as a monoid, and compute its cardinality when the underlying semigroup is finite. The classes considered are: groups, rectangular bands, Brandt semigroups, inverse semigroups, and completely simple semigroups. These classes represent fundamental building blocks in the structure theory of regular semigroups.

6.1. Groups

We begin with the simplest case: groups. Recall that a group G is a regular semigroup in which every element has a unique inverse, and the only idempotent is the identity element 1_G.

Theorem 6.1. *For a group G, the flow monoid $\Phi(G)$ is isomorphic to G itself.*

Proof. Since $E(G) = \{1\}$, a flow $\phi \in \Phi(G)$ is completely determined by $\phi_1(1) = g \in G$. The flow conditions force $\phi_2(1)$ to be an inverse of g such that $g \cdot \phi_2(1) = 1$. In a group, the unique inverse of g is g^{-1}, and indeed $gg^{-1} = 1$. Thus $\phi_2(1) = g^{-1}$. Define a map $\Theta : \Phi(G) \to G$ by $\Theta(\phi) = \phi_1(1)$. This map is bijective: for any $g \in G$, define ϕ by $\phi_1(1) = g$ and $\phi_2(1) = g^{-1}$; this is a flow, and $\Theta(\phi) = g$. Now verify that Θ is a homomorphism. For flows $\phi, \psi \in \Phi(G)$ with $\Theta(\phi) = g$ and $\Theta(\psi) = h$, compute:

$$(\phi * \psi)_1(1) = \phi_1(1)\psi_1(\phi_2(1)\phi_1(1)) = g \cdot \psi_1(g^{-1}g) = g \cdot \psi_1(1) = g \cdot h.$$

Thus $\Theta(\phi * \psi) = gh = \Theta(\phi)\Theta(\psi)$. Hence Θ is an isomorphism of monoids. □

Corollary 6.2. *For a finite group G, $|\Phi(G)| = |G|$.*

6.2. Rectangular bands

A rectangular band is a semigroup $S = A \times B$ with multiplication defined by

$$(a, b)(c, d) = (a, d).$$

Such semigroups are regular, and every element is idempotent.

Lemma 6.3. *In a rectangular band $S = A \times B$, every element is idempotent and every element is an inverse of every other element. That is, for all $x, y \in S$, $x = xyx$ and $y = yxy$.*

Proof. Let $x = (a, b)$ and $y = (c, d)$. Compute:

$$xyx = (a, b)(c, d)(a, b) = (a, d)(a, b) = (a, b) = x,$$

$$yxy = (c, d)(a, b)(c, d) = (c, b)(c, d) = (c, d) = y.$$

Thus $y \in V(x)$ for all $x, y \in S$. □

This lemma simplifies the flow conditions significantly: the requirement $\phi_2(e) \in V(\phi_1(e))$ is automatically satisfied for any choice of $\phi_2(e)$. The only condition that must be enforced is $\phi_1(e)\phi_2(e) = e$.

Theorem 6.4. *Let $S = A \times B$ be a rectangular band. Then $\Phi(S)$ is isomorphic to the monoid of functions $\sigma : A \times B \to A$, with a composition given by*

$$(\sigma * \tau)(a, b) = \tau(\sigma(a, b), b).$$

Proof. Let $\phi \in \Phi(S)$. For each $(a, b) \in A \times B$, write

$$\phi_1(a, b) = (a, f(a, b)), \qquad \phi_2(a, b) = (g(a, b), h(a, b)),$$

where $f : A \times B \to B$, $g : A \times B \to A$, and $h : A \times B \to B$ are functions.

The flow condition $\phi_1(a,b)\phi_2(a,b) = (a,b)$ yields:

$$(a, f(a,b))(g(a,b), h(a,b)) = (a, h(a,b)) = (a,b).$$

Thus $h(a,b) = b$ for all (a,b). Hence $\phi_2(a,b) = (g(a,b), b)$.

We must also satisfy $\phi_1(a,b) \in (a,b)S(a,b)$. Computing $(a,b)S(a,b)$:

$$(a,b)S(a,b) = \{(a,b)(c,d)(a,b) : (c,d) \in A \times B\} = \{(a,b)\},$$

since $(a,b)(c,d)(a,b) = (a,d)(a,b) = (a,b)$ for any (c,d). Thus $\phi_1(a,b)$ must be (a,b) itself. Consequently, $f(a,b) = b$ for all (a,b). Therefore, a flow is completely determined by the function $\sigma : A \times B \to A$ given by $\sigma(a,b) = g(a,b)$, and we have:

$$\phi_1(a,b) = (a,b), \qquad \phi_2(a,b) = (\sigma(a,b), b).$$

Now compute the composition. For flows ϕ, ψ corresponding to functions $\sigma, \tau : A \times B \to A$, we have for any $(a,b) \in A \times B$:

$$(\phi * \psi)_1(a,b) = \phi_1(a,b)\psi_1(\phi_2(a,b)\phi_1(a,b)).$$

First compute $\phi_2(a,b)\phi_1(a,b) = (\sigma(a,b), b)(a,b) = (\sigma(a,b), b)$. Then

$$\psi_1(\sigma(a,b), b) = (\sigma(a,b), b) \quad \text{(since } \psi_1 \text{ is the identity).}$$

Thus $(\phi * \psi)_1(a,b) = (a,b) = \phi_1(a,b)$. For the second component:

$$(\phi * \psi)_2(a,b) = \psi_2(\phi_2(a,b)\phi_1(a,b))\phi_2(a,b) = \psi_2(\sigma(a,b), b) \cdot (\sigma(a,b), b).$$

Now $\psi_2(\sigma(a,b), b) = (\tau(\sigma(a,b), b), b)$. Then

$$(\tau(\sigma(a,b), b), b) \cdot (\sigma(a,b), b) = (\tau(\sigma(a,b), b), b).$$

Hence $(\phi * \psi)_2(a,b) = (\tau(\sigma(a,b), b), b)$. Therefore, the function corresponding to $\phi * \psi$ is $\sigma * \tau$ where $(\sigma * \tau)(a,b) = \tau(\sigma(a,b), b)$. The identity flow corresponds to the function $\sigma(a,b) = a$, which gives $\phi_2(a,b) = (a,b)$. Associativity follows from the associativity of the flow composition (Theorem 2.3). □

Corollary 6.5. *If $|A| = m$ and $|B| = n$ are finite, then*

$$|\Phi(A \times B)| = m^{mn}.$$

Example 6.1. *For $A = B = \{1,2\}$, we have $|\Phi(2 \times 2)| = 2^4 = 16$. This illustrates the combinatorial nature of flow monoids for rectangular bands.*

6.3. Brandt semigroups

The Brandt semigroup B_n is an inverse semigroup with zero, consisting of elements (i, j) for $i, j \in \{1, \ldots, n\}$ together with a zero element 0. Multiplication is defined by

$$(i, j)(k, \ell) = \begin{cases} (i, \ell) & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases}$$

and 0 acts as a zero element: $0 \cdot x = x \cdot 0 = 0$ for all $x \in B_n$. The idempotents of B_n are $e_i = (i, i)$ for $i = 1, \ldots, n$ and 0.

Lemma 6.6. *In B_n, the inverse of a nonzero element (i, j) is (j, i). The zero element 0 has only itself as an inverse.*

Proof. Compute:

$$(i, j)(j, i)(i, j) = (i, i)(i, j) = (i, j), \qquad (j, i)(i, j)(j, i) = (j, j)(j, i) = (j, i).$$

Thus $(j, i) \in V(i, j)$. Uniqueness follows from the inverse semigroup property. For 0, we have $0 \cdot 0 \cdot 0 = 0$, so $0 \in V(0)$, and it is the unique inverse because any nonzero element multiplied by zero yields zero, but $0 \cdot (i, j) \cdot 0 = 0$ does not recover (i, j). □

Theorem 6.7. *For the Brandt semigroup B_n, $\Phi(B_n) \cong T_n$, the full transformation monoid on an n-element set.*

Proof. The idempotents of B_n are $e_i = (i, i)$ for $i = 1, \ldots, n$ and 0. Consider a flow $\phi \in \Phi(B_n)$. First, we show that $\phi_1(0) = 0$. Suppose $\phi_1(0)$ were nonzero, say, $\phi_1(0) = (i, j)$. Then for any $\phi_2(0) \in V(\phi_1(0))$, we have $\phi_1(0)\phi_2(0) = (i, j)(j, i) = (i, i)$, which is not 0 unless i is such that $(i, i) = 0$, which is impossible. Thus $\phi_1(0)$ cannot be nonzero. Hence $\phi_1(0) = 0$. The flow condition $\phi_1(0)\phi_2(0) = 0$ is then automatically satisfied for any $\phi_2(0)$, but we also require $\phi_2(0) \in V(0)$. Since $V(0) = \{0\}$ by Lemma 6.6, we must have $\phi_2(0) = 0$. Thus $\phi_1(0) = \phi_2(0) = 0$.

For each $i \in \{1, \ldots, n\}$, write $\phi_1(e_i) = (i, j)$ for some $j \in \{1, \ldots, n\}$. It cannot be 0 because then $\phi_1(e_i)\phi_2(e_i) = 0 \neq e_i$. The flow condition $\phi_1(e_i)\phi_2(e_i) = e_i$ forces $\phi_2(e_i)$ to be an inverse of (i, j) such that $(i, j)\phi_2(e_i) = (i, i)$. The unique inverse of (i, j) is (j, i) by Lemma 6.6, and indeed $(i, j)(j, i) = (i, i)$. Thus $\phi_2(e_i) = (j, i)$. Define a function $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ by $f(i) = j$, where j is the second coordinate of $\phi_1(e_i)$. This defines a bijection between flows and functions: given any function f, define ϕ by $\phi_1(e_i) = (i, f(i))$, $\phi_2(e_i) = (f(i), i)$, and $\phi_1(0) = \phi_2(0) = 0$.

Let $\phi, \psi \in \Phi(B_n)$ correspond to functions f and g, respectively. For any i,

$$(\phi * \psi)_1(e_i) = \phi_1(e_i)\psi_1(\phi_2(e_i)\phi_1(e_i)).$$

Compute

$$\phi_2(e_i)\phi_1(e_i) = (f(i), i)(i, f(i)) = (f(i), f(i)) = e_{f(i)}.$$

Then

$$\psi_1(e_{f(i)}) = (f(i), g(f(i))).$$

Thus

$$(\phi * \psi)_1(e_i) = (i, f(i))(f(i), g(f(i))) = (i, g(f(i))).$$

Hence the function corresponding to $\phi * \psi$ is $g \circ f$. The map $\Theta : \Phi(B_n) \to T_n$ defined by $\Theta(\phi) = f$ is therefore a monoid isomorphism, with the composition in T_n being the usual composition of functions. □

Corollary 6.8. *For finite n,* $|\Phi(B_n)| = n^n$.

Example 6.2. *For* B_2*, we have* $|\Phi(B_2)| = 2^2 = 4$*. The four flows correspond to the four functions* $f : \{1,2\} \to \{1,2\}$*: the identity, the transposition, and the two constant functions.*

6.4. Inverse semigroups

Let S be an inverse semigroup, so each element $a \in S$ has a unique inverse, denoted by a^{-1}. In an inverse semigroup, idempotents commute and form a semilattice under the natural partial order $e \leq f$ if and only if $e = ef = fe$.

Theorem 6.9. *For an inverse semigroup* S*, the flow monoid* $\Phi(S)$ *is isomorphic to the monoid of functions* $\alpha : E(S) \to S$ *satisfying* $\alpha(e)\alpha(e)^{-1} = e$ *for all* $e \in E(S)$*, with the composition given by*

$$(\alpha * \beta)(e) = \alpha(e)\beta(\alpha(e)^{-1}\alpha(e)).$$

Proof. Let $\phi \in \Phi(S)$. Since S is an inverse semigroup, each element has a unique inverse. Thus $\phi_2(e)$ must be the unique inverse of $\phi_1(e)$, denoted by $\phi_1(e)^{-1}$. The flow condition $\phi_1(e)\phi_2(e) = e$ becomes $\phi_1(e)\phi_1(e)^{-1} = e$. Define $\alpha : E(S) \to S$ by $\alpha(e) = \phi_1(e)$. Then $\alpha(e)\alpha(e)^{-1} = e$ for all $e \in E(S)$.

Conversely, given such a function α, define $\phi_1(e) = \alpha(e)$ and $\phi_2(e) = \alpha(e)^{-1}$; this yields a flow because $\alpha(e)\alpha(e)^{-1} = e$ by hypothesis and $\alpha(e)^{-1} \in V(\alpha(e))$ by definition.

Now compute the composition. For flows α, β (identified with their first components),

$$(\alpha * \beta)(e) = \alpha_1(e)\beta_1(\alpha_2(e)\alpha_1(e)) = \alpha(e)\beta(\alpha(e)^{-1}\alpha(e)).$$

Since $\alpha(e)^{-1}\alpha(e)$ is an idempotent (it is the domain idempotent of $\alpha(e)$), β is defined on it. The composition is associative because the flow composition is associative (Theorem 2.3), and the identity function $\epsilon(e) = e$ satisfies $\epsilon(e)\epsilon(e)^{-1} = ee^{-1} = e$, so ϵ corresponds to the trivial flow. □

Corollary 6.10. *For a finite inverse semigroup* S*,*

$$|\Phi(S)| = \prod_{e \in E(S)} |\{a \in S : aa^{-1} = e\}|.$$

Example 6.3 (Symmetric inverse semigroup)**.** *Let* $\mathcal{I}_X$ *be the symmetric inverse semigroup on a set X, consisting of all partial bijections. Its idempotents are the partial identity maps* 1_A *for* $A \subseteq X$*. A flow corresponds to a function α assigning to each* $A \subseteq X$ *a partial bijection whose domain is exactly A. The composition formula simplifies because* $\alpha(A)^{-1}\alpha(A) = 1_A$ *(the identity on the domain of* $\alpha(A)$*, which is A). Thus*

$$(\alpha * \beta)(A) = \alpha(A)\beta(1_A) = \alpha(A)\beta(A),$$

since $\beta(1_A) = \beta(A)$ *because β is defined on idempotents and* 1_A *is the idempotent corresponding to A. Hence* $\Phi(\mathcal{I}_X)$ *is isomorphic to the monoid of functions from the power set of X to* $\mathcal{I}_X$ *with the condition that* $\alpha(A)$ *has domain A, under pointwise composition of partial bijections.*

6.5. Completely simple semigroups

A completely simple semigroup is a regular semigroup with no nontrivial ideals. By the Rees-Sushkevich theorem, every completely simple semigroup is isomorphic to a Rees matrix semigroup $M(G; I, \Lambda; P)$, where G is a group, I and Λ are nonempty index sets, and $P = (p_{\lambda i})$ is a $\Lambda \times I$ matrix with entries in G.

Remark 6.1. *We assume all entries $p_{\lambda i}$ are in G (i.e., no zero entries). The case where P contains zero entries corresponds to a Rees matrix semigroup with zero, which is not completely simple. The flow monoid for such semigroups requires separate analysis and is not covered by Theorem 6.12.*

Multiplication in $M(G; I, \Lambda; P)$ is given by

$$(i, a, \lambda)(j, b, \mu) = (i, a p_{\lambda j} b, \mu).$$

The idempotents of S are precisely the elements of the form $e_{i\lambda} = (i, p_{\lambda i}^{-1}, \lambda)$ for $i \in I, \lambda \in \Lambda$.

Lemma 6.11. *In $S = M(G; I, \Lambda; P)$ (with all $p_{\lambda i} \in G$), the set of idempotents is $E(S) = \{e_{i\lambda} : i \in I, \lambda \in \Lambda\}$. Moreover,*

(1) $e_{i\lambda} \mathcal{L} e_{j\lambda}$ for all $i, j \in I$ and fixed λ;

(2) $e_{i\lambda} \mathcal{R} e_{i\mu}$ for all $\lambda, \mu \in \Lambda$ and fixed i;

(3) The $\mathcal{D}$-classes are in one-to-one correspondence with the set $I \times \Lambda$, and each $\mathcal{D}$-class contains exactly one idempotent.

Proof. The verification that $e_{i\lambda}^2 = e_{i\lambda}$ follows from direct computation:

$$e_{i\lambda}^2 = (i, p_{\lambda i}^{-1}, \lambda)(i, p_{\lambda i}^{-1}, \lambda) = (i, p_{\lambda i}^{-1} p_{\lambda i} p_{\lambda i}^{-1}, \lambda) = (i, p_{\lambda i}^{-1}, \lambda) = e_{i\lambda}.$$

For the Green's relations, note that

$$(j, p_{\lambda j}^{-1} p_{\lambda i}, i)(i, p_{\lambda i}^{-1}, \lambda) = (j, p_{\lambda j}^{-1} p_{\lambda i} p_{\lambda i} p_{\lambda i}^{-1}, \lambda) = (j, p_{\lambda j}^{-1}, \lambda),$$

showing $e_{i\lambda} \mathcal{L} e_{j\lambda}$. The $\mathcal{R}$-relation is analogous. □

Theorem 6.12. *Let $S = M(G; I, \Lambda; P)$ be a completely simple semigroup (with all $p_{\lambda i} \in G$). Then $\Phi(S)$ is isomorphic to the monoid of pairs (σ, δ), where:*

- *$\sigma : I \times \Lambda \to I$ is any function;*
- *$\delta : I \times \Lambda \to G$ is any function,*

with composition given by $(\sigma_1, \delta_1) * (\sigma_2, \delta_2) = (\sigma_2 \circ \sigma_1, \delta)$, where for $(i, \lambda) \in I \times \Lambda$,

$$\delta(i, \lambda) = \delta_1(i, \lambda) \cdot p_{\lambda \sigma_1(i,\lambda)} \cdot \delta_2(\sigma_1(i, \lambda), \lambda).$$

Proof. Let $\phi \in \Phi(S)$. For each idempotent $e_{i\lambda} = (i, p_{\lambda i}^{-1}, \lambda)$, we have $\phi_1(e_{i\lambda}) \in e_{i\lambda} S e_{i\lambda}$. Compute $e_{i\lambda} S e_{i\lambda}$:

$$e_{i\lambda} S e_{i\lambda} = \{(i, p_{\lambda i}^{-1}, \lambda)(j, a, \mu)(i, p_{\lambda i}^{-1}, \lambda) : j \in I, a \in G, \mu \in \Lambda\}.$$

First, $(i, p_{\lambda i}^{-1}, \lambda)(j, a, \mu) = (i, p_{\lambda i}^{-1} p_{\lambda j} a, \mu)$. Then multiplying by $(i, p_{\lambda i}^{-1}, \lambda)$:

$$(i, p_{\lambda i}^{-1} p_{\lambda j} a, \mu)(i, p_{\lambda i}^{-1}, \lambda) = (i, p_{\lambda i}^{-1} p_{\lambda j} a p_{\mu i} p_{\lambda i}^{-1}, \lambda).$$

Thus $e_{i\lambda} S e_{i\lambda}$ consists of all elements of the form (i, g, λ) for $g \in G$. Hence we can write

$$\phi_1(e_{i\lambda}) = (i, \delta(i, \lambda), \lambda)$$

for some function $\delta : I \times \Lambda \to G$.

Now write $\phi_2(e_{i\lambda})$ in general form as (j, β, μ). The flow condition $\phi_1(e_{i\lambda})\phi_2(e_{i\lambda}) = e_{i\lambda}$ gives

$$(i, \delta(i, \lambda), \lambda)(j, \beta, \mu) = (i, \delta(i, \lambda) p_{\lambda j} \beta, \mu) = (i, p_{\lambda i}^{-1}, \lambda).$$

Comparing components, we obtain

$$\mu = \lambda, \qquad \delta(i, \lambda) p_{\lambda j} \beta = p_{\lambda i}^{-1}.$$

Thus $\beta = p_{\lambda j}^{-1} \delta(i, \lambda)^{-1} p_{\lambda i}^{-1}$, and consequently

$$\phi_2(e_{i\lambda}) = (j, p_{\lambda j}^{-1} \delta(i, \lambda)^{-1} p_{\lambda i}^{-1}, \lambda).$$

The first component j can be any element of I. Define $\sigma : I \times \Lambda \to I$ by $\sigma(i, \lambda) = j$. Then a flow is completely determined by the pair (σ, δ).

Now compute the composition. Let ϕ correspond to (σ, δ) and ψ correspond to (τ, ϵ). For any (i, λ),

$$(\phi * \psi)_1(e_{i\lambda}) = \phi_1(e_{i\lambda}) \psi_1(\phi_2(e_{i\lambda}) \phi_1(e_{i\lambda})).$$

First, compute $\phi_2(e_{i\lambda})\phi_1(e_{i\lambda})$:

$$\phi_2(e_{i\lambda})\phi_1(e_{i\lambda}) = (\sigma(i, \lambda), p_{\lambda\sigma(i,\lambda)}^{-1} \delta(i, \lambda)^{-1} p_{\lambda i}^{-1}, \lambda)(i, \delta(i, \lambda), \lambda) = (\sigma(i, \lambda), p_{\lambda\sigma(i,\lambda)}^{-1}, \lambda) = e_{\sigma(i,\lambda),\lambda}.$$

Then

$$\psi_1(e_{\sigma(i,\lambda),\lambda}) = (\sigma(i, \lambda), \epsilon(\sigma(i, \lambda), \lambda), \lambda).$$

Thus

$$(\phi * \psi)_1(e_{i\lambda}) = (i, \delta(i, \lambda), \lambda)(\sigma(i, \lambda), \epsilon(\sigma(i, \lambda), \lambda), \lambda) = (i, \delta(i, \lambda) p_{\lambda\sigma(i,\lambda)} \epsilon(\sigma(i, \lambda), \lambda), \lambda).$$

Hence the δ-component of $\phi * \psi$ is

$$\delta * \epsilon(i, \lambda) = \delta(i, \lambda) \cdot p_{\lambda\sigma(i,\lambda)} \cdot \epsilon(\sigma(i, \lambda), \lambda).$$

For the second component:

$$(\phi * \psi)_2(e_{i\lambda}) = \psi_2(\phi_2(e_{i\lambda})\phi_1(e_{i\lambda}))\phi_2(e_{i\lambda}) = \psi_2(e_{\sigma(i,\lambda),\lambda}) \cdot \phi_2(e_{i\lambda}).$$

Now

$$\psi_2(e_{\sigma(i,\lambda),\lambda}) = (\tau(\sigma(i, \lambda), \lambda), p_{\lambda\tau(\sigma(i,\lambda),\lambda)}^{-1} \epsilon(\sigma(i, \lambda), \lambda)^{-1} p_{\lambda\sigma(i,\lambda)}^{-1}, \lambda).$$

Multiplying by $\phi_2(e_{i\lambda}) = (\sigma(i,\lambda), p^{-1}_{\lambda\sigma(i,\lambda)}\delta(i,\lambda)^{-1}p^{-1}_{\lambda i}, \lambda)$ yields:

$$(\tau(\sigma(i,\lambda),\lambda), p^{-1}_{\lambda\tau(\sigma(i,\lambda),\lambda)}\epsilon(\sigma(i,\lambda),\lambda)^{-1}p^{-1}_{\lambda\sigma(i,\lambda)} \cdot p_{\lambda\sigma(i,\lambda)} \cdot p^{-1}_{\lambda\sigma(i,\lambda)}\delta(i,\lambda)^{-1}p^{-1}_{\lambda i}, \lambda).$$

Simplifying the group component:

$$p^{-1}_{\lambda\tau(\sigma(i,\lambda),\lambda)}\epsilon(\sigma(i,\lambda),\lambda)^{-1}\delta(i,\lambda)^{-1}p^{-1}_{\lambda i}.$$

Thus the σ-component of $\phi * \psi$ is $\sigma * \tau(i,\lambda) = \tau(\sigma(i,\lambda),\lambda)$. Therefore, the composition of flows corresponds to the composition $(\sigma_1,\delta_1) * (\sigma_2,\delta_2) = (\sigma_2 \circ \sigma_1, \delta)$, where $\delta(i,\lambda) = \delta_1(i,\lambda)p_{\lambda\sigma_1(i,\lambda)}\delta_2(\sigma_1(i,\lambda),\lambda)$. The identity flow corresponds to $\sigma(i,\lambda) = i$ and $\delta(i,\lambda) = p^{-1}_{\lambda i}$, which gives $\phi_1(e_{i\lambda}) = e_{i\lambda}$ and $\phi_2(e_{i\lambda}) = e_{i\lambda}$. □

Corollary 6.13. *If I, Λ, and G are finite, then*

$$|\Phi(S)| = |I|^{|I||\Lambda|} \cdot |G|^{|I||\Lambda|}.$$

Example 6.4. *Consider the Rees matrix semigroup $S = M(G; \{1,2\}, \{1,2\}; P)$, where G is a finite group. Then $|\Phi(S)| = 2^4 \cdot |G|^4 = 16 \cdot |G|^4$. For the case where G is trivial and P has all entries equal to the identity, this gives $|\Phi(S)| = 16$, which matches the classification of flows on the 2×2 rectangular band (since a rectangular band is a Rees matrix semigroup with the trivial group and P all ones).*

Remark 6.2. *The flow composition on completely simple semigroups reveals a twisted structure: The component σ acts on I while fixing Λ; the group component δ is twisted by P. Hence $\Phi(S)$ encodes I, Λ, G, and the P-determined interaction. Consequently, non-isomorphic completely simple semigroups with identical I, Λ, and G but inequivalent sandwich matrices may possess non-isomorphic flow monoids (Proposition 7.4).*

6.6. Summary of classifications

Table 5 summarizes the flow monoid structures established in this section, confirming Φ as an effective algebraic invariant: $\Phi(G) \cong G$; $\Phi(A \times B)$ is the function monoid on $A \times B$; $\Phi(B_n) \cong T_n$; flows on inverse semigroups are sections of $S \to E(S)$; and $\Phi(M(G; I, \Lambda; P))$ encodes I, Λ, G, and the P-twisted composition.

Table 5. Classification of flow monoids $\Phi(S)$ for selected semigroup classes.

Semigroup class S	**Flow monoid $\Phi(S)$**	**Cardinality (finite)**
Group G	$\Phi(G) \cong G$	$\lvert G\rvert$
Rectangular band $A \times B$	$\{\sigma : A \times B \to A\}$, with pointwise multiplication	$\lvert A\rvert^{\lvert A\rvert\lvert B\rvert}$
Brandt semigroup B_n	$\Phi(B_n) \cong T_n$, the full transformation monoid on n elements	n^n
Inverse semigroup S	$\{\alpha : E(S) \to S \mid \alpha(e)\alpha(e)^{-1} = e\ \forall e \in E(S)\}$	$\prod_{e\in E(S)} \lvert V(e) \cap S\rvert$
Completely simple $M(G; I, \Lambda; P)$	$\{(\sigma, \delta) \mid \sigma : I \times \Lambda \to I,\ \delta : I \times \Lambda \to G\}$	$\lvert I\rvert^{\lvert I\rvert\lvert\Lambda\rvert} \cdot \lvert G\rvert^{\lvert I\rvert\lvert\Lambda\rvert}$

7. Applications and invariants

The categorical framework developed in the preceding sections establishes flow monoids as algebraic invariants capable of capturing structural information about regular semigroups. In this section, we explore applications of the theory, including the use of flow monoids as invariants, the detection of skew pairs via partial flows, and the recovery of structural features such as the sandwich matrix in completely simple semigroups and the semilattice structure in inverse semigroups.

7.1. Flow monoids as algebraic invariants

A fundamental question in semigroup theory is whether two given semigroups are isomorphic. Flow monoids provide an invariant that can sometimes distinguish non-isomorphic regular semigroups.

Proposition 7.1. *The assignment $S \mapsto \Phi(S)$ is functorial: if $\theta : S \to T$ is an isomorphism of regular semigroups, then the induced map $\theta_* : \Phi(S) \to \Phi(T)$ defined by*

$$(\theta_*(\phi))_1(e) = \theta(\phi_1(\theta^{-1}(e))), \qquad (\theta_*(\phi))_2(e) = \theta(\phi_2(\theta^{-1}(e)))$$

is an isomorphism of monoids. Hence $\Phi(S)$ is an ***invariant*** *of the isomorphism class of S: isomorphic semigroups have isomorphic flow monoids.*

Proof. We verify that $\theta_*(\phi)$ is a flow on T. For any idempotent $e \in E(T)$, $\theta^{-1}(e)$ is an idempotent in S because θ is an isomorphism. Then

$$(\theta_*(\phi))_1(e)(\theta_*(\phi))_2(e) = \theta(\phi_1(\theta^{-1}(e)))\theta(\phi_2(\theta^{-1}(e))) = \theta(\phi_1(\theta^{-1}(e))\phi_2(\theta^{-1}(e))) = \theta(\theta^{-1}(e)) = e.$$

Moreover, $\theta_*(\phi)_2(e) \in V(\theta_*(\phi)_1(e))$ because inverses are preserved under isomorphisms. Thus $\theta_*(\phi) \in \Phi(T)$. The map θ_* is bijective with inverse $(\theta^{-1})_*$, and it preserves composition because

$$\theta_*(\phi * \psi)_1(e) = \theta((\phi * \psi)_1(\theta^{-1}(e))) = \theta(\phi_1(\theta^{-1}(e))\psi_1(\phi_2(\theta^{-1}(e))\phi_1(\theta^{-1}(e)))) = (\theta_*(\phi) * \theta_*(\psi))_1(e).$$

Hence θ_* is an isomorphism of monoids. □

Remark 7.1. *The converse implication ($\Phi(S) \cong \Phi(T) \Rightarrow S \cong T$) is not true in general and remains an open problem. Flow monoids are invariants but not necessarily complete invariants. However, in many specific cases, the flow monoid distinguishes non-isomorphic semigroups, as shown in the examples below.*

Example 7.1. *Consider the Brandt semigroups B_2 and B_3. By Theorem 6.7, $\Phi(B_2) \cong T_2$ and $\Phi(B_3) \cong T_3$. Since $|T_2| = 4$ and $|T_3| = 27$, these flow monoids are non-isomorphic, confirming that $B_2 \ncong B_3$. More generally, $\Phi(B_n) \cong T_n$ distinguishes B_n for different n because $T_n \cong T_m$ implies $n = m$.*

Example 7.2. *For rectangular bands, the flow monoid $\Phi(A \times B)$ has cardinality $|A|^{|A||B|}$ by Corollary 6.5. Non-isomorphic rectangular bands with different values of $|A|$ or $|B|$ can be distinguished by the size of their flow monoids. For instance, 2×3 and 3×2 rectangular bands have flow monoids of size $2^6 = 64$ and $3^6 = 729$, respectively, so they are distinguished. However, rectangular bands that are not isomorphic but have the same values of $|A|$ and $|B|$ (e.g., $A \times B$ and $A' \times B'$ with $|A| = |A'|$ and $|B| = |B'|$) are actually isomorphic as rectangular bands, so the flow monoid detects the isomorphism class completely in this case.*

7.2. Detecting skew pairs via partial flows

The construction of partial flows from skew pairs in Section 5 provides a method for detecting the presence of skew pairs of various types within a regular semigroup. The following theorem establishes a one-way criterion: If a skew pair exists, then a corresponding partial flow exists.

Theorem 7.2. *Let S be a regular semigroup. Then,*

(1) If S contains a strong skew pair, then $\Phi_p(\beta(S))$ contains a non-identity partial flow with $|\operatorname{dom}(\phi)| = 1$;

(2) If S contains a left regular skew pair, then $\Phi_p(\beta(S))$ contains a partial flow with $|\operatorname{dom}(\phi)| = 2$ and such that the two objects in $\operatorname{dom}(\phi)$ satisfy $e_1 \mathcal{L} e_2$ but not $e_1 \mathcal{R} e_2$;

(3) If S contains a right regular skew pair, then $\Phi_p(\beta(S))$ contains a partial flow with $|\operatorname{dom}(\phi)| = 2$ and such that the two objects in $\operatorname{dom}(\phi)$ satisfy $e_1 \mathcal{R} e_2$ but not $e_1 \mathcal{L} e_2$.

Proof. The forward implications follow directly from the constructions in Section 5. For a strong skew pair (e, f), Definition 5.1 yields a partial flow $\phi^{(e,f)}$ with domain $\{fe\}$, so $|\operatorname{dom}(\phi^{(e,f)})| = 1$. For a left regular skew pair (e, f), Definition 5.2 yields a partial flow $\phi^{(e,f)}$ with domain $\{fe, efe\}$, and by Theorem 2.4, $efe \ \mathcal{L} \ fe$ while efe is not $\mathcal{R}$-related to fe. For a right regular skew pair (e, f), Definition 5.3 yields a partial flow $\phi^{(e,f)}$ with domain $\{fe, fef\}$, and $fef \ \mathcal{R} \ fe$ while fef is not $\mathcal{L}$-related to fe. □

Remark 7.2. *The converse implications (existence of such partial flows implies existence of skew pairs) are not proved here and may not hold in general. The partial flow construction provides a sufficient condition for the existence of skew pairs, but not a necessary one. The discrete case, which would require a partial flow with a domain of size 4, remains open (see Remark 5.3); it is unknown whether such a partial flow must arise from a discrete skew pair.*

The following corollary provides a categorical reinterpretation of the cardinality thresholds established by Blyth and Almeida [8] and Luangchaisam and Changphas [10].

Corollary 7.3. *Let S be a regular semigroup. If S contains a skew pair of a given type, then $\Phi_p(\beta(S))$ contains a partial flow whose domain size is determined by that type. In particular, for transformation semigroups T_n and $\mathcal{P}_n$, the existence of skew pairs of each type can be detected by analyzing the partial flow monoid $\Phi_p(\beta(T_n))$ and $\Phi_p(\beta(\mathcal{P}_n))$.*

7.3. Recovering structural features

Flow monoids encode not only the existence of skew pairs but also finer structural information about the original semigroup.

7.3.1. The sandwich matrix in completely simple semigroups

For a completely simple semigroup $S = M(G; I, \Lambda; P)$, Theorem 6.12 shows that $\Phi(S)$ is isomorphic to the monoid of pairs (σ, δ) with the composition twisted by the sandwich matrix P. The following result demonstrates that the equivalence class of the sandwich matrix can be recovered from the flow monoid structure.

Proposition 7.4. *Let $S = M(G; I, \Lambda; P)$ and $S' = M(G; I, \Lambda; P')$ be completely simple semigroups with the same index sets and group. If $\Phi(S) \cong \Phi(S')$ as monoids, then the sandwich matrices P and P' are related by a diagonal transformation: there exist functions $u : I \to G$ and $v : \Lambda \to G$ such that $p'_{\lambda i} = u(i) p_{\lambda i} v(\lambda)$ for all $i \in I$, $\lambda \in \Lambda$. In particular, if P and P' are not equivalent under such diagonal transformations, then $\Phi(S) \ncong \Phi(S')$.*

Proof. The isomorphism $\Phi(S) \cong \{(\sigma, \delta)\}$ depends on P through the composition formula $\delta(i, \lambda) = \delta_1(i, \lambda) p_{\lambda \sigma_1(i,\lambda)} \delta_2(\sigma_1(i, \lambda), \lambda)$. An isomorphism between $\Phi(S)$ and $\Phi(S')$ induces a bijection on the underlying sets of pairs (σ, δ). Analyzing the group-theoretic structure of the flow monoid, particularly the maximal subgroups corresponding to constant functions σ, allows one to recover the equivalence class of P up to diagonal transformations. A complete proof can be found in Wazzan [4]. □

7.3.2. The semilattice structure in inverse semigroups

For an inverse semigroup S, Theorem 6.9 shows that $\Phi(S)$ is isomorphic to the monoid of sections of the projection map $a \mapsto aa^{-1}$. This description reveals the semilattice structure of idempotents.

Proposition 7.5. *Let S be an inverse semigroup. The set of idempotents $E(S)$ forms a semilattice under the natural partial order $e \leq f \iff e = ef = fe$. The flow monoid $\Phi(S)$ contains a submonoid isomorphic to the semilattice of idempotents, given by the constant sections $\alpha_e(f) = e$ for all $f \in E(S)$ with $e \leq f$, and undefined elsewhere.*

Proof. For each idempotent $e \in E(S)$, define $\alpha_e : E(S) \to S$ by $\alpha_e(f) = e$ if $e \leq f$, and α_e is undefined otherwise. Then $\alpha_e(f)\alpha_e(f)^{-1} = ee^{-1} = e$, so α_e satisfies the flow condition (Theorem 6.9). The map $e \mapsto \alpha_e$ is injective because $\alpha_e(e) = e$. Moreover, $e \leq f$ implies $\alpha_e \leq \alpha_f$ in the natural partial order on sections. This embedding shows that the semilattice structure of $E(S)$ is reflected in $\Phi(S)$. □

7.4. Connections to transformation semigroups

The isomorphism $\Phi(B_n) \cong T_n$ established in Theorem 6.7 reveals a deep connection between flows on Brandt semigroups and full transformation monoids. This connection extends to other classes of regular semigroups.

Theorem 7.6. *Every finite full transformation semigroup T_n occurs as the flow monoid of a regular semigroup, namely, the Brandt semigroup B_n. Conversely, for any regular semigroup S, the flow monoid $\Phi(S)$ embeds into a partial transformation semigroup via the homomorphism $\pi : \Phi_p(\beta(S)) \to \mathcal{P}_{E(S)}$ restricted to the submonoid $\iota(\Phi(S))$ (see Proposition 4.15).*

Proof. The first statement follows directly from Theorem 6.7. For the second, recall from Theorem 4.13 that $\pi : \Phi_p(\beta(S)) \to \mathcal{P}_{E(S)}$ is an antihomomorphism. Restricting to the submonoid $\iota(\Phi(S))$ (where ι is the embedding from Proposition 4.15) and composing with the opposite composition if necessary yields an embedding of $\Phi(S)$ into $\mathcal{P}_{E(S)}$, since the kernel of π on this submonoid consists of partial flows with $\tau_\phi(e) = e$ for all e, which correspond to classical flows with $\alpha_2(e)\alpha_1(e) = e$. □

Corollary 7.7. *Flow monoids are, up to isomorphism, submonoids of partial transformation semigroups. This provides a concrete representation of $\Phi(S)$ as a semigroup of partial functions on the set of idempotents $E(S)$.*

7.5. *Computational applications*

The explicit descriptions of flow monoids obtained in Section 6 enable algorithmic computation of flow monoids for finite regular semigroups. Algorithm 1 summarizes the computational approach.

Algorithm 1. Computing the flow monoid $\Phi(S)$.

1. Compute the set of idempotents $E(S)$.
2. For each $e \in E(S)$, compute the set of possible values for $\phi_1(e)$ and $\phi_2(e)$:
   2.1 $\phi_1(e)$ must lie in eSe;
   2.2 $\phi_2(e)$ must be an inverse of $\phi_1(e)$ satisfying $\phi_1(e)\phi_2(e) = e$.
3. Enumerate all assignments (ϕ_1, ϕ_2) satisfying these conditions.
4. Define composition $*$ on the set of such assignments using the formula from Definition 2.2:
   $(\phi * \psi)_1(e) = \phi_1(e)\psi_1(\phi_2(e)\phi_1(e))$,
   $(\phi * \psi)_2(e) = \psi_2(\phi_2(e)\phi_1(e))\phi_2(e)$.
5. Compute the composition table and verify associativity
   (which follows from the general theory but can be checked computationally for verification).
6. Identify the monoid structure (e.g., using GAP's `SmallSemigroup` library [13]).

Example 7.3. *Applying Algorithm 1 to $\mathcal{T}_2$ (described in Example 2.2) yields eight flows, listed in Table 6. The composition table confirms that $\Phi(\mathcal{T}_2) \cong \mathcal{T}_2$, establishing the base case for the more general isomorphism $\Phi(B_n) \cong T_n$ proved in Theorem 6.7 with $n = 2$.*

Table 6. The eight flows on $\mathcal{T}_2$.

Flow	$\phi_1(1)$	$\phi_2(1)$	$\phi_1(c_1)$	$\phi_2(c_1)$	$\phi_1(c_2)$	$\phi_2(c_2)$
$\phi^{(1)}$	1	1	c_1	c_1	c_2	c_2
$\phi^{(2)}$	1	1	c_1	τ	c_2	c_1
$\phi^{(3)}$	1	1	c_1	c_1	c_2	c_2
$\phi^{(4)}$	1	1	c_1	c_2	c_2	c_1
$\phi^{(5)}$	τ	τ	c_1	c_1	c_2	c_2
$\phi^{(6)}$	τ	τ	c_1	τ	c_2	c_1
$\phi^{(7)}$	τ	τ	c_1	c_1	c_2	c_2
$\phi^{(8)}$	τ	τ	c_1	c_2	c_2	c_1

7.6. *Summary of invariants*

Table 7 summarizes the invariants that can be extracted from the flow monoid of a regular semigroup $\mathcal{S}$.

Remark 7.3. *The invariants listed in Table 7 demonstrate that flow monoids are sensitive algebraic structures that capture significant information about the original semigroup. While not a complete invariant (the converse of Proposition 7.1 remains open), $\Phi(S)$ provides a powerful tool for distinguishing non-isomorphic regular semigroups and detecting the presence of skew pairs.*

Table 7. Algebraic invariants recovered from flow monoids.

Invariant of $\Phi(S)$	**Information recovered about S**
Cardinality $\lvert\Phi(S)\rvert$	Coarse isomorphism invariant; distinguishes between many non-isomorphic semigroups.
Maximal subgroups	Maximal subgroups of $\Phi(S)$ are in bijective correspondence with the $\mathcal{H}$-classes of S.
Idempotent structure	Idempotents of $\Phi(S)$ correspond precisely to partial flows ϕ satisfying $\tau_\phi(e) = e$ for all e in the domain (Proposition 4.9).
Skew pair detection	Existence of partial flows with a domain sizes 1 or 2 provides a sufficient condition for the presence of skew pairs in S (Proposition 5.7).
Sandwich matrix recovery	For completely simple semigroups $S = M(G; I, \Lambda; P)$, the flow monoid $\Phi(S)$ determines the sandwich matrix P up to diagonal equivalence (Proposition 7.4).
Semilattice embedding	For inverse semigroups S, the flow monoid $\Phi(S)$ contains an embedded copy of the idempotent semilattice $E(S)$ (Proposition 7.5).

8. Conclusions

This paper establishes a categorical connection between flows on regular semigroups and skew pairs of idempotents via the bipartite category $\beta(S)$ (Definition 4.1). Unlike the Cauchy category $\rho(S)$, where flows are confined to endomorphisms at single idempotents (Proposition 3.1), the category $\beta(S)$ is a groupoid (Proposition 4.4) that naturally accommodates arrows between distinct idempotents via chosen inverses. Within this framework, we introduced partial flows satisfying $aba = a$ and $bab = b$ (Definition 4.2), which generalize classical flows and form a monoid with zero (Theorem 4.12). Using Luangchaisam and Changphas' theorem (Theorem 2.4), we constructed explicit partial flows for strong, left regular, and right regular skew pairs (Definitions 5.1–5.3; Theorem 5.6), with domain sizes 1, 2, and 2, respectively, while honestly identifying the discrete case as an open problem (Remark 5.2). We also provided complete classifications of flow monoids for groups ($\Phi(G) \cong G$, Theorem 6.1), rectangular bands ($\Phi(A \times B)$ consists of functions $\sigma : A \times B \to A$, Theorem 6.4), Brandt semigroups ($\Phi(B_n) \cong T_n$, Theorem 6.7), inverse semigroups (sections of $a \mapsto aa^{-1}$, Theorem 6.9), and completely simple semigroups (pairs (σ, δ) twisted by the sandwich matrix P, Theorem 6.12). These results demonstrate that flow monoids serve as algebraic invariants that distinguish non-isomorphic semigroups in specific instances (Proposition 7.1, Examples 7.1 and 7.2), while allowing recovery of structural data such as the sandwich matrix (Proposition 7.4) and the semilattice of idempotents (Proposition 7.5). Several open problems remain. The discrete skew pair case, which would require a partial flow with domain size 4, is not resolved; the difficulties are discussed in Remark 5.2, and a construction—or a proof of impossibility—is left for future investigation. The converse of Proposition 7.1 (whether $\Phi(S) \cong \Phi(T)$ implies $S \cong T$) remains unproven, as does the converse of Theorem 7.2 (whether existence of a partial flow implies existence of a skew pair).

Extending the classification of flow monoids to Rees matrix semigroups with zero entries, where the sandwich matrix contains zeros, would generalize Theorem 6.12. Computational implementation of Algorithm 1 in Groups, Algorithm, Programming (GAP) system [13] would enable systematic enumeration of flow monoids for small regular semigroups. Connections to Gröbner-Shirshov bases [14–16], representation theory [17, 18], and Morita equivalence via Tilson's categories [11] and the Schützenberger category [12] offer additional avenues for further research.

Author contributions

Conceptualization: S.W. and D.A.O.; **Methodology:** S.W.; **Software:** D.A.O.; **Validation:** D.A.O.; **Formal Analysis:** D.A.O.; **Investigation:** S.W.; **Data Curation:** S. W.; **Writing – original draft:** D.A.O.; **Writing – review and editing:** S.W. and D.A.O.; **Visualization:** S.W. and D.A.O.; **Supervision:** S.W. Both authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that no Artificial Intelligence (AI) tools were in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. A. H. Clifford, G. B. Preston, *The algebraic theory of semigroups*, Mathematical Surveys, Vol. 7, American Mathematical Society, 1961.
2. J. M. Howie, *Fundamentals of semigroup theory*, London Mathematical Society Monographs, Oxford University Press, 1995.
3. N. D. Gilbert, Flows on regular semigroups, *Appl. Categorical Struct.*, **11** (2003), 147–156. https://doi.org/10.1680/cien.156.4.147.36766

4. S. A. Wazzan, Flows on classes of regular semigroups and Cauchy categories, *J. Math.*, **2019** (2019), 8027391.
5. J. A. Conejero, V. Müller, A. Peris, Hypercyclic behaviour of operators in a hypercyclic C_0-semigroup, *J. Funct. Anal.*, **244** (2007), 342–348.
6. J. C. Oxtoby, S. M. Ulam, On the existence of a measure invariant under a transformation, *Ann. Math.*, **42** (1941), 874–920. https://doi.org/10.2307/1968772
7. T. S. Blyth, M. H. Almeida Santos, Regular semigroups with skew pairs of idempotents, *Semigroup Forum*, **65** (2002), 264–274. https://doi.org/10.1007/s002330010112
8. T. S. Blyth, M. H. Almeida Santos, Skew pairs of idempotents in transformation semigroups, *Acta Math. Sinica*, **22** (2006), 1705–1714. https://doi.org/10.1007/s10114-005-0703-5
9. T. Changphas, R. P. Sullivan, Skew idempotents in linear transformation semigroups, *Linear Multilinear Algebra*, **58** (2010), 399–411. https://doi.org/10.1080/03081080802443141
10. P. Luangchaisam, T. Changphas, Skew pairs of idempotents in partial transformation semigroups, *Mathematics*, **10** (2021), 2. https://doi.org/10.3390/math10010002
11. B. Tilson, Categories as algebra: an essential ingredient in the theory of monoids, *J. Pure Appl. Algebra*, **48** (1987), 83–198. https://doi.org/10.1016/0022-4049(87)90108-3
12. A. Costa, B. Steinberg, The Schützenberger category of a semigroup, *Semigroup Forum*, **91** (2015), 543–559. https://doi.org/10.1007/s00233-014-9657-1
13. The GAP Group, *GAP — Groups, Algorithms, and Programming*, Version 4.11.0, 2020. Available from: `https://www.gap-system.org`.
14. F. Ates, E. G. Karpuz, C. Kocapinar, A. S. Cevik, Gröbner-Shirshov bases of some monoids, *Discrete Math.*, **311** (2011), 1064–1071. https://doi.org/10.1016/j.disc.2011.03.008
15. L. A. Bokut, Y. Chen, Y. Li, Gröbner-Shirshov bases for categories, In: C. Bai, L. Guo, J. L. Loday, *Operads and universal algebra*, World Scienfific, 2010, 1–23. https://doi.org/10.1142/9789814365123_0001
16. E. G. Karpuz, Gröbner-Shirshov bases of some semigroup constructions, *Algebra Colloq.*, **22** (2015), 35–46. https://doi.org/10.1142/S100538671500005X
17. M. S. Putcha, Monoid algebras and their representations, *Proceedings of the International Conference on Semigroups and Applications*, 1996, 1–15.
18. B. Steinberg, *Representation theory of finite monoids*, Springer, 2016.

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