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Research article

The stability of the Nash equilibrium in a single-leader multi-follower game with strategic adjustment costs

Luping Liu[1], **Zhaonan Mu**[1,*] **and Chun Wang**[2]

[1] College of Big Data and Intelligent Engineering, Guizhou University of Commerce, Guiyang 550014, Guizhou, China

[2] Guizhou Open University and Guizhou Vocational Technology Institute, Guiyang 550023, Guizhou, China

* **Correspondence:** Email: 202420017@gzcc.edu.cn.

Abstract: In a single-leader multi-follower game (SLMFG), leaders or followers often incur additional costs while revising their decisions, such as suppliers versus retailers, central government versus local government, and conglomerates versus subsidiaries; however, these costs are usually ignored. To focus on this issue, using quadratic cost functions, these strategic adjustment costs are introduced into a dynamic SLMFG with adaptive expectations, thereby yielding a new dynamic model. Furthermore, we obtain some necessary and sufficient conditions for local asymptotic stability of the SLMFG. We also prove that both strategic adjustment costs and the leader have increased the stable region of the original Nash equilibrium. By comparing interval increments, the leader and strategy adjustment cost contribute equally to the Nash equilibrium, which, presented in the paper, develop corresponding conclusions to previous research.

Keywords: single-leader multi-follower game; quadratic cost functions; strategic adjustment costs; equilibrium stability analysis
Mathematics Subject Classification: 34D20, 91A25, 91A65

1. Introduction

The concept of a leader–follower game was first introduced by H. Von Stackelberg when he studied the asymmetric roles played by players in the decision-making process, which later became known as the Stackelberg game [1]. The leader has a leadership advantage and is able to take the first or favorable position in the Stackelberg game, and followers need to make decisions after the leader. It has attracted much attention from scholars and produced a large number of research results, which have been extensively used in economics [2], operations research [3, 4], stochastic differential game [5, 6], computer science [7–9], and other related fields. A single-leader multi-follower game (SLMFG) is a special case of the Stackelberg game, which is also known as a bilevel programming problem. The purpose of this paper is to study the stability of the Nash equilibrium for an SLMFG.

As is well-known, output adjustment occurs through an adjustment of the inputs. There is a great deal of literature on the topic, beginning with the paper by Gould [10] considering that the adjustment costs of capital are not linear. As described by Hayashi [11], in economics, most firms are modeled as perfectly rational. However, there is limited research on the bounded rationality of firms' decisions which recognizes the limits of information and cognition in real-world choices. Fortunately, the stability research of Cournot–Nash equilibria for firms that are bounded rationality, including adaptive expectations [12], relative profit maximization [13], decision-making methodology [14], delayed bounded rationality [15], and different heterogeneous expectations [16–18], is relatively well established. Additionally, because price and cost functions change over time in most cases, the concept of strategy adjustment cost is proposed. Meanwhile, strategic adjustment costs are introduced to a Cournot game [19–21] and a multiproduct Cournot game [22]. Recently, Wang et al. [23] further investigated the stability of the Cournot–Nash equilibrium by dynamically analyzing the effect of strategic adjustment costs. Additionally, Schoonbeek [24] introduced the strategic adjustment cost into a dynamic Stackelberg model and gave a necessary and sufficient condition for the stability of the Stackelberg game. Nie et al. [25] studied the dynamic Stackelberg game with leaders in turn under a feedback information structure and obtained the explicit solution for linear–quadratic systems in the case of an open-loop information structure. Very recently, Hu et al. [26] studied the dynamics of a quantum duopoly Stackelberg game with marginal costs and obtained the local stability conditions of the model. Ahmadi et al. [27] established a Stackelberg model with heterogeneous players and marginal costs and analyzed the stability of the duopoly Stackelberg game. They obtained chaotic solutions of the game by using a combination of state feedback and parameter adjustment approaches, which converge to its oligopoly equilibrium. Ahmed et al. [28] proposed a Stackelberg game-based dynamic resource allocation method to achieve optimal resource allocation while offering a high level of fairness in an edge-federated 5G network. Xu et al. [29] and Li et al. [30] developed optimal control schemes for unknown nonlinear systems using sliding–mode surfaces and reinforcement learning, respectively. Their studies offer promising tools for extending dynamic game models to more complex adjustment processes. Therefore, these methods above help to enhance the stabilities of game equilibrium.

Significantly, Wang et al. [23] studied a symmetric Cournot oligopoly where all firms are identical, there is no leader–follower hierarchy, and every firm shares the same adjustment cost parameter. Their stability condition depends only on the number of firms and the common adaptive expectation parameter, showing that strategic adjustment costs expand the stable region, but the effect is symmetric across firms. In contrast, this paper investigates a Stackelberg oligopoly with one leader and $N-1$ followers, allowing asymmetric adjustment costs. Its stability condition involves the number of firms, the adaptive expectation parameter, and both cost parameters. Moreover, it provides a novel comparative analysis revealing that followers' adjustment costs have a stronger stabilizing effect than the leader's cost. This result cannot be obtained from the symmetric Cournot framework [23]. Schoonbeek [24] analyzed a dynamic Stackelberg oligopoly with quadratic production adjustment costs, where followers use Cournot expectations (i.e., they assume rivals' current output equals their past output), and the leader correctly anticipates followers' reactions. In contrast, this paper extends the model by incorporating adaptive expectations for both leader and followers, which makes the adjustment process more realistic and introduces a new smoothing effect. Moreover, this paper provides a detailed comparative analysis of the stability region with respect to the number of firms, the adaptive expectation parameter, and the adjustment cost parameters for the leader and follower, and it reveals nonmonotonic effects, showing that followers' adjustment costs have a stronger stabilizing influence than the leader' s cost.

Inspired by the work described above, few studies have analyzed the extent to which the strategy adjustment cost and leader affect the stability of the Nash equilibrium. The paper focuses on the stability of Nash equilibria combining the literature on adaptive expectations opens by Bischi and Kopel [12] and the literature on output adjustment costs that comes back to the papers by Howroyd and Rickard [21] and by Wang et al. [23]. Consequently, this issue has been extensively analyzed over the past few years, and the focus of this paper is on examining the stability of the Nash equilibrium for the SLMFG with strategic adjustment costs, which represents a development and extension of previous work.

The remainder of this paper is organized as follows. Section 2 studies a static SLMFG and its Nash equilibria. Section 3 establishes the dynamic SLMFG model with adaptive expectation and strategic adjustment costs. Section 4 investigates the stability results of the Nash equilibria for different parameters, and some summaries are provided in the Section 5.

2. Nash equilibria of static SLMFGs

Considering that N firms produce a standardized product competing on output among producers in a market [21], the collection of all firms is defined by $\mathcal{F} = \{1, \cdots, N\}$. Assume that firm 1 is regarded as a leader, where other firms i are regarded as followers; we denote it briefly by $\mathcal{F}' = \{2, 3, \cdots, N\}$. The inverse demand function including the leader and followers in period t is expressed by

$$\mathcal{P}(\kappa_1, \cdots, \kappa_N) = a - \sum_{j\in\mathcal{F}} \kappa_j(t),$$

where $\kappa_j(t)$ denotes the output of firm j during period t, letting a denote a positive constant. Let $b(b > 0)$ be the same marginal cost of production for leader and followers. In an SLMFG, when the marginal costs of the leader and follower are identical, the leader's first-mover advantage leads to an oligopoly equilibrium outcome that differs significantly from the Cournot model [21, 23]. Then, we abbreviate the above game to **SLMFG**$= \{\mathcal{F}, a, b\}$.

Through backward induction, we first consider the optimal choice of all followers. Given κ_1, $\forall i \in \mathcal{F}'$, $\kappa_{-i} = \{\kappa_2, \cdots, \kappa_{i-1}, \kappa_{i+1}, \cdots, \kappa_N\}$, each follower i chooses κ_i to maximize its own profit function

$$\max_{\kappa_i\geq 0} \Pi_i(\kappa_1(t), \kappa_2(t), \cdots, \kappa_N(t)) = [\omega - \kappa_1(t) - \sum_{j\in\mathcal{F}'} \kappa_j(t)]\kappa_i(t),$$

where $\omega = a - b$ denotes the difference between maximum price and marginal cost, assuming a linear cost. Thus, the marginal profit of followers is

$$\frac{\partial \Pi_i}{\partial \kappa_i} = \omega - \kappa_1(t) - \sum_{j\in\mathcal{F}'} \kappa_j(t).$$

By $\partial\Pi_i/\partial\kappa_i = 0$, $\forall i \in \mathcal{F}'$, then we obtain the best responses of all followers:

$$\kappa_i(t) = \frac{1}{2}[\omega - \kappa_1(t) - \sum_{j\in\mathcal{F}'_{-i}} \kappa_j(t)], \forall i \in \mathcal{F}',$$

where $\mathcal{F}'_{-i} = \mathcal{F}' \setminus \{i\}$.

Otherwise, the leader predicts that all followers $i(i \in \mathcal{F}')$ will choose their outputs κ_i; then, the profit function of the leader in the first stage is denoted by

$$\max_{\kappa_1(t)\geq 0} \Pi_1(\kappa_1(t), s_i(\kappa_1(t))) = [\omega - \kappa_1(t) - \sum_{j\in\mathcal{F}'} \kappa_j(t)]\kappa_1(t).$$

Similarly, the marginal profit of the leader is

$$\frac{\partial\Pi_1}{\partial\kappa_1} = \omega - 2\kappa_1(t) - \sum_{j\in\mathcal{F}'} \kappa_j(t).$$

By $\partial\Pi_1/\partial\kappa_1 = 0$, we obtain the best response of the leader,

$$\kappa_1(t) = \frac{1}{2}(\omega - \sum_{j\in\mathcal{F}'} \kappa_j(t)).$$

Therefore, the best response for all players is to realize the Nash equilibria κ^*, with

$$\kappa^* = (\kappa_1^*, \kappa_i^*) = (\frac{1}{2}(\omega - \sum_{j\in\mathcal{F}'} \kappa_j), \frac{N+1}{N+3}\omega - \frac{1}{2}\sum_{j\in\mathcal{F}'_{-i}} \kappa_j), \forall i \in \mathcal{F}'.$$

3. Nash equilibrium of dynamic SLMFG

In a dynamic SLMFG, at time t, each firm's output is expressed by $\kappa_i(t)$. In order to increase profits, each firm adjusts its output gradually, and their adjustment rules are characterized by an adaptive expectation [12]. Furthermore, the cost of adjustment output is considered; that is, each firm is also required to additionally compute the square difference between the current period's output and the next period's output. And we show the strategic adjustment cost function of Firm 1 (the leader) by introducing a quadratic cost function during period t as follows:

$$C_1(\kappa_1(t), \kappa_1(t-1)) = L_1(\kappa_1(t) - \kappa_1(t-1))^2,$$

where L_1 denotes a non-negative constant. $C_1(\kappa_1(t), \kappa_1(t-1))$ represents the strategic quadratic production adjustment cost of leader resulting from the change in output bewteen period $t-1$ and period t. Furthermore, the strategic adjustment cost function for firms $i(\forall i \in \mathcal{F}')$ (followers) is defined as

$$C(\kappa_i(t) - \kappa_i(t-1)) = L(\kappa_i(t) - \kappa_i(t-1))^2,$$

where L is a non-negative constant. Additionally, at $t = 0$ period, each firm chooses an initial output $\kappa_i(0)$, and at each later $t = 1, 2, \cdots$ period, they initially predict the total production of other firms in period t by using the $s_i^e(t) = \sum_{j\in\mathcal{F}_{-i}} \kappa_j(t-1)$. Assume that at the estimation and size of $\kappa_i(t-1)$, follower

i is subsequently able to determine the maximum output $\kappa_i(t)$ of its own profits at t period. Then, we can obtain the profit function of the follower i by

$$\max_{\kappa_i(t)\geq 0} \Pi_i(t) = \kappa_i(t)[\omega - \kappa_i(t) - \sum_{j\in\mathcal{F}_{-i}} \kappa_j(t-1)] - C(\kappa_i(t) - \kappa_i(t-1)).$$

In addition, the marginal profits of all followers are

$$\frac{\partial \Pi_i(t)}{\partial \kappa_i(t)} = \omega - (2+2L)\kappa_i(t) - \sum_{j\in\mathcal{F}_{-i}} \kappa_j(t-1) + 2L\kappa_i(t-1), \ \ \forall i \in \mathcal{F}'.$$

Let $\partial \Pi_i(t)/\partial \kappa_i(t) = 0$; then the best response of followers is obtained by

$$\kappa_i(t) = \frac{L}{1+L}\kappa_i(t-1) - \frac{1}{2+2L}\sum_{j\in\mathcal{F}_{-i}} \kappa_j(t-1) + \frac{1}{2+2L}\omega. \tag{3.1}$$

Assumption 3.1. *In all periods, all firms adjust their production based on the adaptive expectations and strategic adjustment cost rules, meaning that the output for the next period is influenced by both the prediction of the new best response and the previous period's output.*

$\forall i \in \mathcal{F}'$, we compute the equation of dynamic adjustment of the follower i as

$$\begin{aligned} \kappa_i(t) &= \vartheta\kappa_i(t-1) \\ &\quad + (1-\vartheta)[\frac{L}{1+L}\kappa_i(t-1) - \frac{1}{2+2L}\sum_{j\in\mathcal{F}_{-i}} \kappa_j(t-1) + \frac{\omega}{2+2L}], \\ &= \frac{L+\vartheta}{1+L}\kappa_i(t-1) - \frac{1-\vartheta}{2+2L}\sum_{j\in\mathcal{F}_{-i}} \kappa_j(t-1) + \frac{\omega(1-\vartheta)}{2+2L}, \end{aligned} \tag{3.2}$$

where $0 < \vartheta < 1$ signifies adaptive expectation parameter of the followers.

Next, we delve into the examination of the leader's behavior, necessitating the consideration of the fact that other firms are perceived as followers. Then, the prediction of the leader regarding the total production of all followers in period t is equivalent to

$$s_1^e(t) = \sum_{j\in\mathcal{F}'} \kappa_j(t).$$

By incorporating Eq (3.1) into the expression above, we obtain

$$s_1^e(t) = -\frac{N-1}{2+2L}\kappa_1(t-1) + (1 - \frac{N}{2+2L})\sum_{j\in\mathcal{F}'} \kappa_j(t-1) + \frac{(N-1)\omega}{2+2L}.$$

Therefore, the leader chooses $\kappa_1(t)$ to maximize the profit function in period t,

$$\max_{\kappa_1(t)\geq 0} \Pi_1(t) = \kappa_1(t)[\omega - \kappa_1(t) - s_1^e(t)] - C_1(\kappa_1(t), \kappa_1(t-1));$$

the marginal adjustment profit of the leader is

$$\frac{\partial \Pi_1(t)}{\partial \kappa_1(t)} = -4\kappa_1(t) + \left(\frac{N-1}{2+2L} + 2\right)\kappa_1(t-1) - \left(1 - \frac{N}{2+2L}\right)\sum_{j\in\mathcal{F}'} \kappa_j(t-1) + \left(1 - \frac{N-1}{2+2L}\right)\omega.$$

Setting $\partial\Pi_1(t)/\partial\kappa_1(t) = 0$, we obtain the following best response of the leader:

$$\kappa_1(t) = \left(\frac{2L_1 + \frac{N-1}{2+2L}}{2+2L_1}\right)\kappa_1(t-1) - \frac{1}{2+2L_1}\left(1 - \frac{N}{2+2L}\right)\sum_{j\in\mathcal{F}'}\kappa_j(t-1)$$
$$+ \frac{\omega - \frac{(N-1)\omega}{2+2L}}{2+2L_1}.$$

Under Assumption 3.1, the dynamic adjustment equation of the leader is denoted by

$$\begin{aligned}\kappa_1(t) &= \vartheta\kappa_1(t-1) + (1-\vartheta)[\frac{2L_1 + \frac{N-1}{2+2L}}{2+2L_1}\kappa_1(t-1) \\ &\quad - \frac{1}{2+2L_1}(1 - \frac{N}{2+2L})\sum_{j\in\mathcal{F}'}\kappa_j(t-1) + \frac{\omega - \frac{(N-1)\omega}{2+2L}}{2+2L_1}], \\ &= [1 - \frac{1-\vartheta}{1+L_1} + \frac{(N-1)(1-\vartheta)}{4(1+L_1)(1+L)}]\kappa_1(t-1) \\ &\quad - \frac{1-\vartheta}{2+2L_1}(1 - \frac{N}{2+2L})\sum_{j\in\mathcal{F}'}\kappa_j(t-1) + \frac{(\omega - \frac{(N-1)\omega}{2+2L})(1-\vartheta)}{2+2L_1}.\end{aligned} \tag{3.3}$$

We obtain the fixed-point equation of Eqs (3.2) and (3.3) by

$$\begin{cases}\kappa_1(t) = \frac{1}{2}(\omega - \sum_{j\in\mathcal{F}'}\kappa_j), \\ \kappa_2(t) = \frac{1}{2}[\omega - \kappa_1(t) - \sum_{j\in\mathcal{F}'_{-2}}\kappa_j(t)], \\ \quad\cdots\cdots \\ \kappa_N(t) = \frac{1}{2}[\omega - \kappa_1(t) - \sum_{j\in\mathcal{F}'_{-N}}\kappa_j(t)].\end{cases} \tag{3.4}$$

It is straightforward to verify that the unique solution to Eq (3.4) is the Nash equilibrium (κ_1^*, κ_i^*), $\forall i \in \mathcal{F}'$.

By putting together Eqs (3.2) and (3.3), we obtain the first-order difference equation as follows:

$$\kappa(t) = \mathbf{H}_1\kappa(t-1) + \mathbf{h}, \tag{3.5}$$

where $\mathbf{h}$ is an $N \times 1$ constant vector and $\kappa(t) = (\kappa_1(t), \kappa_2(t), \cdots, \kappa_N(t))^{\mathrm{T}}$ denotes the $N \times 1$ vector. $\mathbf{H}_1$ denotes an $N \times N$ matrix, where all elements of $\mathbf{H}_1$ and $\mathbf{h}$ are constants. For analytical convenience, we introduce the following two symbols:

$$\alpha_1 = \frac{1}{1+L_1},\ \alpha = \frac{1}{1+L},$$

and note that $0 < \alpha_1 \le 1$, and $0 < \alpha \le 1$. Then, the Jacobian matrix $\mathbf{H}_1$ is written as

$$\mathbf{H}_1 = \begin{pmatrix} 1-(\alpha_1 - \frac{N-1}{4}\alpha_1\alpha)(1-\vartheta) & -\frac{\alpha_1}{2}(1-\frac{N\alpha}{2})(1-\vartheta) & \cdots & -\frac{\alpha_1}{2}(1-\frac{N\alpha}{2})(1-\vartheta) \\ -\frac{1}{2}\alpha(1-\vartheta) & 1-\alpha(1-\vartheta) & \cdots & -\frac{1}{2}\alpha(1-\vartheta) \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2}\alpha(1-\vartheta) & -\frac{1}{2}\alpha(1-\vartheta) & \cdots & 1-\alpha(1-\vartheta) \end{pmatrix}_{N\times N}.$$

We use several simplifying assumptions in our model. These include identical marginal costs, the same adjustment rules for all firms, one leader, and a linear demand function. These assumptions are common in oligopoly studies, but they may reduce the direct applicability of our findings. In future work, we can relax these assumptions by allowing heterogeneous firms, multiple leaders, or nonlinear demand. Such extensions would test the robustness of our stability conditions and offer deeper insights into the link between market structure and equilibrium stability.

4. Stability analysis

In this section, we analyze the stability of the Nash equilibrium with adaptive expectation and strategic adjustment costs, and the related Nash equilibrium of a Cournot game, respectively.

4.1. Stability analysis of Nash equilibrium with adaptive expectation and strategic adjustment costs

Next, we will study the stability on Eq (3.5). It is well-known that the equilibrium is globally asymptotically stable *iff* all eigenvalues of matrix $\mathbf{H}_1$ given in (3.5) have an absolute value smaller than unity; that is, all eigenvalues of $\mathbf{H}_1$ are inside the unit circle; see [31]. We will derive a necessary and sufficient condition for the stability of the game with regard to the parameters N, ϑ, L_1, and L. The Jacobian matrix $\mathbf{H}_1$ possesses a distinctive block structure arising from the model's symmetry: The leader's adjustment equation depends only on the aggregate output of all followers, whereas all followers share identical adjustment cost parameters and adaptive expectation rules. Consequently, $\mathbf{H}_1$ is invariant under any permutation of the follower indices. By the standard theory of symmetric matrices, this permutation invariance implies that the eigenspaces of $\mathbf{H}_1$ decompose into two orthogonal invariant subspaces: (1) the subspace where all followers have identical coordinates, that is, $p = (p_1, p_2, \cdots, p_2)$, and (2) the subspace where the leader's coordinate vanishes and the followers' coordinates sum to zero, that is, $p = (0, p_2, \cdots, p_N)^{\mathrm{T}}$ with $\sum_{i=2}^{N} p_i = 0$. This decomposition is a standard technique in the stability analysis of symmetric oligopoly models and has been employed in related works [21, 24]. Therefore, in order to analyze the eigenvalue problem related to Jacobian matrix $\mathbf{H}_1$, $\mathbf{H}_1 p = \lambda p$, where $p = p_i \neq 0$, and p represents an $N \times 1$ eigenvector, we consider the following two situations.

(1) Matrix $\mathbf{H}_1$ has eigenvectors of the type $p = (p_1, p_2, \cdots, p_2)$. According to $\mathbf{H}_1 p = \lambda p$, we have

$$(1 - \alpha_1(1-\vartheta) + \frac{(N-1)\alpha\alpha_1(1-\vartheta)}{4})p_1 + (N-1)\alpha_1(\frac{N\alpha}{4} - \frac{1}{2})(1-\vartheta)p_2 = \lambda p_1,$$

$$-\frac{\alpha(1-\vartheta)}{2}p_1 + (1 - \frac{N\alpha(1-\vartheta)}{2})p_2 = \lambda p_2.$$

A homogeneous linear system has a nonzero solution (i.e., a nontrivial solution) *iff* $|\mathbf{H}_1| = 0$ (i.e., the determinant of $\mathbf{H}_1$ is equal to 0), which also means that its rank is less than N, $\mathrm{rank}(\mathbf{H}_1) < N$ [32].

Therefore, we can find two nontrivial solutions for p_1 and p_2 *iff* λ satisfies the following equation:

$$\begin{aligned}&(1-\alpha_1(1-\vartheta)+\frac{(N-1)\alpha\alpha_1(1-\vartheta)}{4}-\lambda)(1-\frac{N\alpha(1-\vartheta)}{2}-\lambda)\\&+\frac{(N-1)\alpha\alpha_1(1-\vartheta)^2}{2}(\frac{N\alpha}{4}-\frac{1}{2})=0.\end{aligned}$$

We can derive

$$\begin{aligned}&\lambda^2+(-2+\alpha_1(1-\vartheta)-\frac{(N-1)\alpha\alpha_1(1-\vartheta)}{4})\lambda\\&+1-\alpha_1(1-\vartheta)-\frac{N}{2}\alpha(1-\vartheta)+\frac{(N-1)\alpha\alpha_1(1-\vartheta)}{4}+\frac{(N+1)\alpha\alpha_1(1-\vartheta)^2}{4}\\&=0.\end{aligned}$$

Let the above equation be denoted by

$$\lambda^2+c_1\lambda+c_2=0, \tag{4.1}$$

where

$$\begin{aligned}c_1&=-2+\alpha_1(1-\vartheta)-\frac{(N-1)\alpha\alpha_1(1-\vartheta)}{4}+\frac{N}{2}\alpha(1-\vartheta),\\c_2&=(1-\alpha_1(1-\vartheta)+\frac{(N-1)\alpha\alpha_1(1-\vartheta)}{4})(1-\frac{N\alpha(1-\vartheta)}{2})+\frac{(N-1)\alpha\alpha_1(1-\vartheta)^2}{2}(\frac{N\alpha}{4}-\frac{1}{2})\\&=1-\alpha_1(1-\vartheta)-\frac{N}{2}\alpha(1-\vartheta)+\frac{(N-1)\alpha\alpha_1(1-\vartheta)}{4}+\frac{(N+1)\alpha\alpha_1(1-\vartheta)^2}{4}.\end{aligned}$$

The roots (λ_1 and λ_2) of Eq (4.1) are the eigenvalues of the Jacobian matrix $\mathbf{H}_1$, and the corresponding eigenvectors are of the required type. The absolute value of the two eigenvalues is less than unity *iff* the following three inequalities hold [31]:

$$\begin{aligned}1-c_1+c_2&>0\\1+c_1+c_2&>0\\c_2&<1.\end{aligned} \tag{4.2}$$

The first inequality of (4.2) is equivalent to

$$4-2\alpha_1(1-\vartheta)-\alpha[\frac{4N\alpha_1\vartheta+4N+\alpha_1-3N\alpha_1-4N\vartheta-N\alpha_1\vartheta^2-\alpha_1\vartheta^2}{4}]>0,$$

which holds when and only when

$$\alpha<\frac{16-8\alpha_1(1-\vartheta)}{N(4-4\vartheta+4\alpha_1\vartheta-3\alpha_1-\alpha_1\vartheta^2)+\alpha_1(1-\vartheta^2)}. \tag{4.3}$$

It becomes evident that the validity of the second and third inequalities outlined in Eq (4.2) are true; see Appendix. Hence, we demonstrate that the absolute values of eigenvalues λ_1 and λ_2 are smaller than unity *iff* the inequality (4.3) holds.

(2) Matrix $\mathbf{H}_1$ has eigenvectors of the type $p = (0, p_2, \cdots, p_N)^{\mathrm{T}}$. Assume that p is indeed of this form; then, we need only to study the equation system $\mathbf{H}_1 p = \lambda p$; that is,

$$\begin{aligned} \alpha_1(\frac{N\alpha}{4} - \frac{1}{2})(1-\vartheta)(p_2 + p_3 + \cdots + p_N) = &\quad 0, \\ (1-\alpha(1-\vartheta))p_2 - \frac{\alpha(1-\vartheta)}{2}p_3 - \cdots - \frac{\alpha(1-\vartheta)}{2}p_N = &\quad \lambda p_2, \\ \cdots \qquad \cdots \qquad \cdots & \\ -\frac{\alpha(1-\vartheta)}{2}p_2 - \frac{\alpha(1-\vartheta)}{2}p_3 - \cdots - \frac{\alpha(1-\vartheta)}{2}p_{N-1} + (1-\alpha(1-\vartheta))p_N = &\quad \lambda p_N. \end{aligned} \tag{4.4}$$

In the second situation, the eigenvector is of the form $p = (0, p_2, \cdots, p_N)^{\mathrm{T}}$ with the constraint $p_2 + p_3 + \cdots + p_N = 0$. Under this condition, the first equation of (4.4) is automatically satisfied. For the remaining $N-1$ equations, we observe that if we set $\lambda = 1 - (1/2)\alpha(1-\theta)$, then for any $j \in \{2, \cdots, N\}$, the left-hand side becomes,

$$(1-\alpha(1-\vartheta))p_i - \frac{\alpha(1-\vartheta)}{2} \sum_{j=2, j\neq i}^{N} p_j.$$

Because $p_2 + p_3 + \cdots + p_N = 0$, we have $\sum_{j\neq i} p_j = -p_i$. Substituting this yields

$$(1-\alpha(1-\vartheta))p_i + \frac{\alpha(1-\vartheta)}{2}p_i = (1 - \frac{\alpha(1-\vartheta)}{2})p_i = \lambda p_i.$$

Thus, each of the last $N-1$ equations is identically satisfied. This confirms that $\lambda = 1-(1/2)\alpha(1-\theta)$ is indeed an eigenvalue. The choice is not arbitrary but is suggested by the structure of the equations and then rigorously verified. Moreover, because the subspace defined by $p_1 = 0$, and $p_2 + p_3 + \cdots + p_N = 0$ has dimension $N-2$, this eigenvalue has geometric multiplicity at least $N-2$.

Based on the above analysis, it is easy to see that $p_2 + p_3 + \cdots + p_N = 0$. In particular, we choose a collection of $N-2$ vectors, and the first element p_1 is equal to 0; the last element p_N is equal to -1; all other elements are 0 except for p_2, p_3, and so on until p_{N-1} is equal to $+1$. In summary, it can be deduced that $\lambda = 1-(1/2)\alpha(1-\vartheta)$ is an eigenvalue of the Jacobian matrix $\mathbf{H}_1$ possessing a multiplicity of $N-2$. According to $0 < 1-(1/2)\alpha(1-\vartheta) < 1$, all the $N-2$ eigenvalues discussed above are located within the unit circle.

Observing the above analysis and rewriting (4.3) in terms of the parameters N, ϑ, L_1, and L, we have the following theorem.

Theorem 4.1. *By Assumption 3.1, the unique Nash equilibrium $\kappa^* = (\kappa_1^*, \kappa_2^*, \cdots, \kappa_N^*)^{\mathrm{T}}$ of the dynamic SLMFG system* (3.5)*, with parameters $\vartheta \in (0,1)$ and $L_1 \geq 0$, $L \geq 0$, is locally asymptotically stable iff the inequality*

$$N < \frac{(8 + 16L_1 - 8\vartheta)(1+L) - 1 + \vartheta^2}{4L_1(1-\vartheta) + 1 - \vartheta^2} := f(\vartheta, L_1, L) \tag{4.5}$$

holds.

Proof. By Ineq (4.3), it is not difficult to deduce the condition (4.5). □

Here, local asymptotic stability signifies that there exists a neighborhood of κ^* such that for any initial state $\kappa(0)$ within this neighborhood, the solution trajectory $\kappa(t)$ of the system (3.5) converges to κ^* as $t \to \infty$. The stability condition (4.5) reveals a direct dependence on the number of competing firms N. Specifically, as N increases, the right-hand side of (4.5) decreases, implying that the equilibrium becomes more susceptible to instability. This phenomenon can be attributed to the aggregate feedback effect inherent in oligopolistic competition. With a larger number of firms, the total output response to individual deviations is amplified, thereby reinforcing the propagation of perturbations throughout the system. Consequently, the system is more prone to oscillatory or divergent behavior. The threshold $f(\vartheta, L_1, L)$ provides a quantitative upper bound on the number of firms that can be accommodated while preserving local stability, given the prevailing adjustment parameters. This result is consistent with the classical instability condition for Cournot oligopolies established by Theocharis [33], and our analysis extends this insight to the Stackelberg framework with strategic adjustment costs and adaptive expectations. From a policy perspective, this finding implies that in markets with a large number of competitors, stronger adjustment costs, particularly on the part of followers, or more inertial adaptive expectations (i.e., larger ϑ) are necessary to maintain equilibrium stability.

By Theorem 4.1, we can draw some conclusions regarding the SLMFG (3.5). By stability conditions (4.5), we can solve the stable region S of Eq (3.5) by

$$S = \left\{ (N, L_1, L, \vartheta) : N < \frac{(8 + 16L_1 - 8\vartheta)(1 + L) - 1 + \vartheta^2}{4L_1(1 - \vartheta) + 1 - \vartheta^2} := f(\vartheta, L_1, L) \right\}.$$

We now analyze the effects of ϑ, L, and L_1 on the stability threshold $f(\vartheta, L_1, L)$, which determines the maximum number of firms N_{max} for local stability. Analytical and numerical results are presented, with economic interpretations and a comparison of the leader's versus followers' adjustment costs.

4.2. *Effect of the adaptive expectation parameter ϑ*

For the asymptotic analysis of ϑ, we have

$$\lim_{\vartheta \to 1^-} \frac{(8 + 16L_1 - 8\vartheta)(1 + L) - 1 + \vartheta^2}{4L_1(1 - \vartheta) + 1 - \vartheta^2} = +\infty;$$

that is, the stability condition (4.5) is easily satisfied; see Figure 1.

By Figure 1, when $\vartheta = 0.1$, the maximum number of firms $N_{max} \approx 3.2$; when $\vartheta = 0.3$, the maximum number of firms $N_{max} \approx 5.1$; when $\vartheta = 0.5$, the maximum number of firms $N_{max} \approx 9.8$; when $\vartheta = 0.7$, the maximum number of firms $N_{max} \approx 24.5$; when $\vartheta = 0.9$, the maximum number of firms $N_{max} \approx 199.0$; when $\vartheta = 0.99$, the maximum number of firms $N_{max} \approx 481.48$. Thus, when $\vartheta \to 1$, the denominator approaches 0, causing the threshold f to increase sharply. The stability in the limiting case $\vartheta \to 1$ arises because firms rely almost exclusively on past output, resulting in negligible adjustments and thereby rendering the system inherently stable regardless of the number of firms. When ϑ decreases, the threshold becomes relatively smaller, demanding stricter stability requirements. As the right-hand side of the inequality is increasing with ϑ, the stable region expands with the asymmetry in the adaptive expectation parameter ϑ.

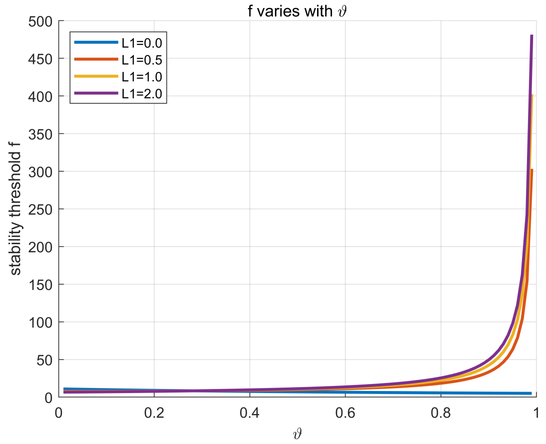


Figure 1. The stability threshold f varies with ϑ when $L = 0.5$.

4.3. Effect of the followers' strategy adjustment cost parameter L

Because

$$f(\vartheta, L_1, L) = \frac{(8 + 16L_1 - 8\vartheta)(1 + L) - 1 + \vartheta^2}{4L_1(1 - \vartheta) + 1 - \vartheta^2} := B(\vartheta, L_1) \cdot L + A(\vartheta, L_1),$$

where

$$B = \frac{8 + 16L_1 - 8\vartheta}{4L_1(1 - \vartheta) + 1 - \vartheta^2},$$

$$A = \frac{8 + 16L_1 - 8\vartheta - 1 + \vartheta^2}{4L_1(1 - \vartheta) + 1 - \vartheta^2},$$

and the slope of B is positive, and the intercept A can be either positive or negative, we conclude that f is a linear function of L; see Figure 2.

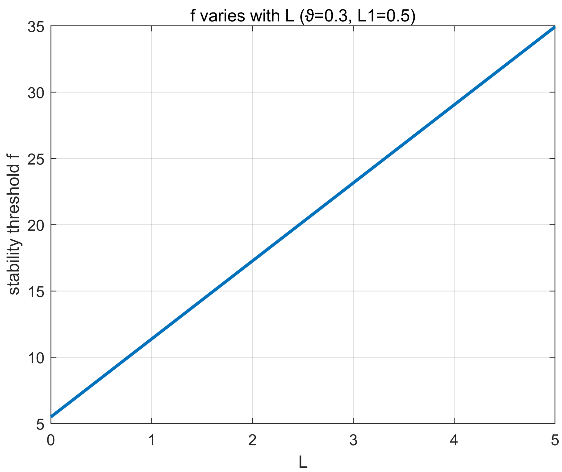


Figure 2. The stability threshold f varies with L when $L_1 = 0.5$, and $\vartheta = 0.3$.

As shown in Figure 2, L appears only in the $(1 + L)$ term of the molecule. As L increases, the threshold increases linearly, making the system more easily stabilized.

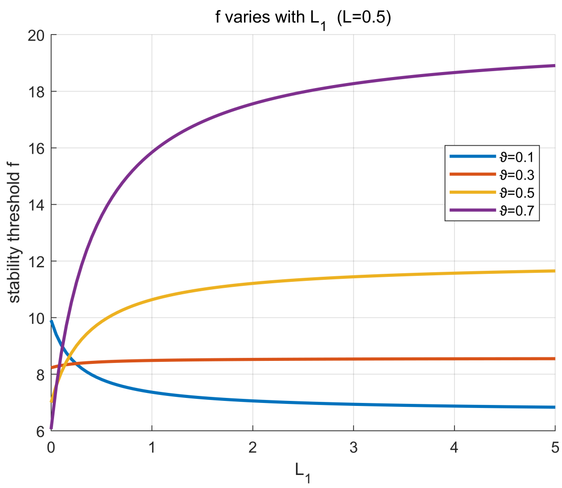


Figure 3. The stability threshold f varies with L_1 when $L = 0.5$.

4.4. Effect of the leader's strategy adjustment cost parameter L_1

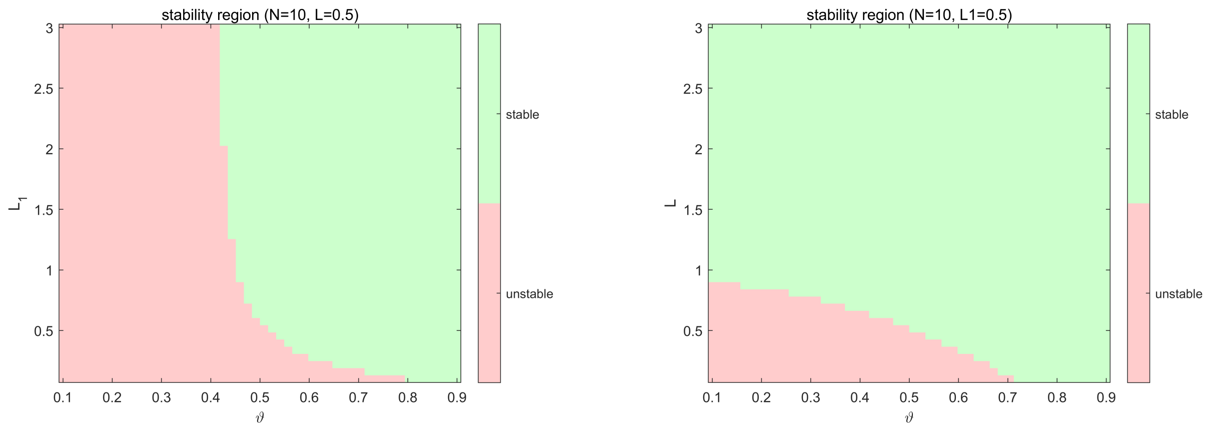
We introduce the partial derivative of f with respect to L_1:

$$\frac{\partial f}{\partial L_1} = \frac{16(1 + L)[4L_1(1 - \vartheta) + 1 - \vartheta^2] - 4(1 - \vartheta)[(8 + 16L_1 - 8\vartheta)(1 - L) - 1 + \vartheta^2]}{(4L_1(1 - \vartheta) + 1 - \vartheta^2)^2}.$$

As L_1 increases, the numerator term $16(1 + L)$ grows, which raises the stability threshold; however, the denominator term $4L_1(1 - \vartheta)$ also increases, which lowers the threshold. Thus, the effect of L_1 is nonmonotonic, and an optimal range exists; see Figure 3. Due to the local asymptotic stability condition, the stable region is defined as $N < f$. As shown in Figure 3, the stable region of the game gradually expands as L increases.

4.5. Comparative analysis of ϑ with L_1 and L

Assuming $N = 10$, we analyze the stability regions of ϑ with respect to L_1 and L, respectively. N must be less than the threshold f. By Figure 4(a), ϑ and L_1 are inversely related. A larger ϑ requires a larger L_1, whereas a smaller ϑ broadens the admissible interval of L_1, allowing for lower values. When ϑ is below 0.42, stability cannot be achieved even if L_1 is set to the maximum value of the image. When L_1 exceeds 0.79, the stability region is achieved even if ϑ is set to the minimum value. By Figure 4(b), ϑ and L are also inversely related. As ϑ increases, L must also increase. When ϑ exceeds approximately 0.5, the system is essentially stable, which is lower than the value of 0.7 shown in Figure 4(a). In Figure 4(b), even when ϑ is very small, L must still be greater than approximately 0.8 for the system to be stable. As shown in Figure 4, strategic adjustment costs for both leaders and followers can expand the system's stable region, but the contribution of followers' adjustment costs to system stability is greater than that of the leaders' adjustment costs.

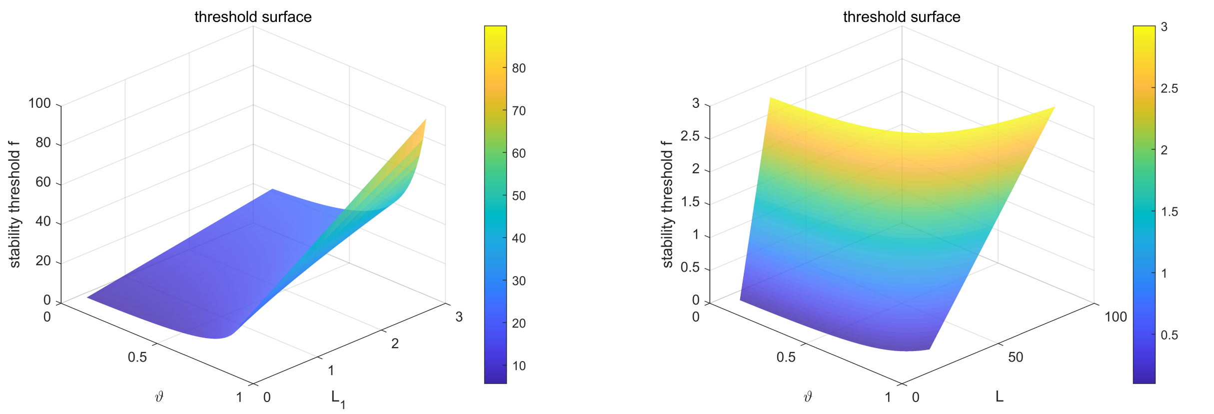


(a) **(b)**

Figure 4. (a) The stability region of the adaptive parameter ϑ and the strategy adjustment cost parameter L_1 of leader when $N = 10$, and $L = 0.5$; (b) The stability region of the adaptive parameter ϑ and the strategy adjustment cost parameter L of followers when $N = 10$, and $L_1 = 0.5$.

4.6. Threshold surface visualization

Under the asymptotic stability condition ($N < f$), the stable region can be expanded by adjusting parameters, particularly how the threshold f varies with L_1 and L and how the stable region changes accordingly; see Figure 5.



(a) **(b)**

Figure 5. (a) Visualization of the threshold interface between f and ϑ, L_1; (b) Visualization of the threshold interface between f and ϑ, L.

As shown in Figure 5(a), the value of threshold f increases with increasing ϑ and L_1, meaning that the stable region expands as ϑ and L_1 increase. By Figure 5(b), the value of threshold f increases with increasing ϑ and L, meaning that the stable region expands as ϑ and L increase. By comparing

Figures 5(a) and 5(b), the pair ϑ-L_1 has a significantly greater impact on the stable region than the pair ϑ-L. This also indicates that when the number of firms is sufficiently large, adjusting strategy parameters can increase the stable region. Although a single leader possesses a first-mover advantage, its effect on expanding the stable region is limited compared to the followers' adjustment costs. The analysis of multiple leaders is beyond the scope of this paper, but it represents a promising avenue for future investigation.

Therefore, this precise analysis demonstrates that the stability of the SLMFG depends not only on market structure (N) but also on the adaptive expectation mechanism (ϑ) and cost adjustment parameters (L_1, L), providing a new analytical framework for industrial organization theory and competition policy.

5. Conclusions

The problems of the SLMFGs are of great significance and have been thoroughly explored by numerous researchers. However, relatively few efforts have been made to investigate the stability of the Nash equilibrium for SLMFG in the presence of strategic adjustment costs. In this paper, we have found that all firms pay an additional strategy adjustment cost apart from production costs when updating their output, which is the square difference between the current period's output and the next period's output. Thus, the stability of the Nash equilibrium for the SLMFG is re-examined under suitable conditions, taking into account possible costs, which may be incurred when firms change their production output from one business period to the next. We obtain the following results.

(i) By introducing a quadratic cost function, we have constructed a new model of dynamic SLMFG with adaptive expectation and strategic adjustment costs, which is different from the models in [24].

(ii) We have obtained some necessary and sufficient conditions for the asymptotic stability of different the SLMFG on Eq (3.5); see Theorem 4.1, which includes the results of [23]. Next, we have calculated that for the interval increment caused by the strategic adjustment cost and leader on the dynamic SLMFG, it is easy to check that the stability effects of the leader and the strategy adjustment cost promoting the Nash equilibrium are the different. These new results extend the conclusions of [19, 23].

In the coming years, we plan to investigate the influence of different cost parameters L and L_1 on the stability of the Nash equilibrium. Also, we will discuss the scenario of extending a leader to multiple leaders to provide policy recommendations for high-quality development of firms or society. In addition, the study of stability based on strategy adjustment cost from the perspective of bounded rationality will be applied to a multi-leader–follower game, mean field game, consensus game, and discontinuous game, and then, we will closely examine the influence of different dynamics.

Author contributions

Liu L.P.: Writing - review & editing, Writing - original draft, Methodology, Formal analysis, Conceptualization. **Mu Z.N.**: Writing - review & editing, Formal analysis. **Wang C.**: Writing - review & editing, Formal analysis. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that no generative AI tools were used in the writing or research of this paper.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

$$\begin{aligned}
1+c_1+c_2 &= 1+(-2+\alpha_1(1-\vartheta)-\frac{(N-1)\alpha\alpha_1(1-\vartheta)}{4}+\frac{N}{2}\alpha(1-\vartheta)) \\
&\quad +(1-\alpha_1(1-\vartheta)-\frac{N}{2}\alpha(1-\vartheta)+\frac{(N-1)\alpha\alpha_1(1-\vartheta)}{4}+\frac{(N+1)\alpha\alpha_1(1-\vartheta)^2}{4}) \\
&= \frac{(N+1)\alpha\alpha_1(1-\vartheta)^2}{4} > 0\ (\because\ 0<\vartheta<1 \text{ and } 0<\alpha,\alpha_1\le 1). \\
1-c_2 &= 1-(1-\alpha_1(1-\vartheta)-\frac{N}{2}\alpha(1-\vartheta)+\frac{(N-1)\alpha\alpha_1(1-\vartheta)}{4}+\frac{(N+1)\alpha\alpha_1(1-\vartheta)^2}{4}) \\
&= \alpha_1(1-\vartheta)+\frac{N}{2}\alpha(1-\vartheta)-\frac{(N-1)\alpha\alpha_1(1-\vartheta)}{4}-\frac{(N+1)\alpha\alpha_1(1-\vartheta)^2}{4} \\
&= \alpha_1(1-\vartheta)+\frac{N}{2}\alpha(1-\vartheta)-\frac{N(2-\vartheta)\alpha\alpha_1(1-\vartheta)}{4}+\frac{\vartheta\alpha\alpha_1(1-\vartheta)}{4} \\
&= \alpha_1(1-\vartheta)+\frac{N\alpha(1-\vartheta)}{4}(2-(2-\vartheta)\alpha_1)+\frac{\vartheta\alpha\alpha_1(1-\vartheta)}{4} \\
&> 0\ (\because\ N\ge 2, 0<\vartheta<1 \text{ and } 0<\alpha,\alpha_1\le 1).
\end{aligned}$$

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