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Research article

Constructions and representations on BiHom-δ-Jordan Lie color algebras

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Abstract: The aim of this paper is to study BiHom-δ-Jordan Lie color algebras. Three kinds of BiHom-δ-Jordan Lie color algebras are constructed using suitable bilinear operators, respectively. Then representations and cohomologies are given to show some properties on BiHom-δ-Jordan Lie color algebras. Moreover, trivial representations and adjoint representations on BiHom-δ-Jordan Lie color algebras are discussed.

Keywords: BiHom-δ-Jordan Lie color algebra; representation; trivial representation; adjoint representation; derivation
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1. Introduction

The Hom-structure originated in the physics literature, where they emerged in studies on deforming Lie algebras of vector fields in a quasi manner [1]. Later, Hom-algebras [2,3], Hom Lie (super)algebras [4–6], Hom Lie color algebras [7, 8], and so on attracted considerable interest among researchers. As a generalization of Hom-type algebras, BiHom-algebras were introduced. BiHom-algebras are defined as algebraic structures whose structural identities are twisted by two homomorphisms β and γ. If two homomorphisms β and γ coincide, BiHom-algebras reduce to Hom-algebras. Studies on BiHom-associative algebras, BiHom-Novikov algebras, and BiHom-bialgebras have made progress [9, 10]. Additionally, in [11], Cheng and Qi introduced the representation on BiHom-Lie algebras, then in [12], Li, Chen, and Cheng studied the representation on BiHom-Lie superalgebras. In 2019, Li, Chen, and Sun characterized Nijenhuis operators and T^*-extensions on Bihom-Lie superalgebras [13]. In 2025, the class of graded regular BiHom-Lie algebras was introduced, thereby generalizing the well-established theories of graded and split Lie algebras [14].

Motivated by the structures on Jordan Lie algebras and Lie superalgebras, the authors in [15] introduced a new algebraic structure $(\mathcal{T}, [\cdot,\cdot]_{\mathcal{T}}, \delta)$, then the results on δ-Jordan Lie (color)triple systems [16, 17] were obtained. The structures of δ-Jordan Lie (super)algebras, δ-Jordan Lie (super)triple systems and δ-Jordan Lie color triple systems were generalized to BiHom-δ-Jordan Lie algebras [18], δ-Hom-Jordan Lie (super)algebras [19,20], δ-Hom-Jordan Lie (super)triple systems [21, 22], and δ-Hom-Jordan Lie color triple systems [23], respectively, then various deformations and extensions were studied. Enhancing the existing methods and development mentioned above, this paper aims to explore the theory of structures and representations of BiHom-δ-Jordan Lie color algebras. The ground field is $\mathbb{C}$ or a field of characteristic zero, and all algebras or vector spaces are finite-dimensional.

The paper is organized as follows. We begin in Section 2 by providing the necessary basic definitions needed for later sections, then BiHom-δ-Jordan Lie color algebras are constructed by two homomorphisms β, γ and bilinear calculations. In Section 3, representations and the theory of cohomology on multiplicative BiHom-δ-Jordan Lie color algebras are characterized. Furthermore, it establishes that the representation of such algebras naturally leads to the construction of their semidirect products. In Section 4, the trivial representation on multiplicative BiHom-δ-Jordan Lie color algebras is investigated, and the isomorphism classes of central extensions are determined in the second cohomology group whose coefficients lie in the trivial representation can be identified with. $\eta^s\gamma^t$-adjoint representations on a BiHom-δ-Jordan Lie color algebra are considered in Section 5, then using the $\eta^s\gamma^t$-adjoint representation, a necessary and sufficient condition for D to be an $\eta^{s+2}\gamma^{t-1}$-derivation is given. Finally, the trivial deformation created in view of the BiHom-Nijienhuis operator is proved.

2. Preliminaries and constructions

Definition 2.1. [7] *Let $\mathcal{H}$ be an abelian group. A map $\varepsilon : \mathcal{H} \times \mathcal{H} \to \mathbb{K} \setminus \{0\}$ is called a **bi-character** on $\mathcal{H}$ if it satisfies the following properties:*

For all $a, b \in \mathcal{H}$,

$$\varepsilon(a, b)\, \varepsilon(b, a) = 1.$$

For all $a, b, c \in \mathcal{H}$,

$$\varepsilon(a, b + c) = \varepsilon(a, b)\, \varepsilon(a, c),$$

$$\varepsilon(a + c, b) = \varepsilon(a, b)\, \varepsilon(c, b).$$

From these properties, one immediately obtains the following consequences: $\varepsilon(a, 0) = \varepsilon(0, a) = 1$ for every $a \in \mathcal{H}$, and $\varepsilon(a, a) = \pm 1$ for every $a \in \mathcal{H}$.

If $|\cdot|$ appears in some expression in this paper, we always regard $\cdot$ as a Γ-homogeneous element and $|\cdot|$ as its Γ-degree.

Definition 2.2. [7] *A **Hom Lie color algebra** is a triple $(\mathcal{T}, [\cdot,\cdot]_{\mathcal{T}}, \beta)$ consisting of: a Γ-graded vector space $\mathcal{T}$, an even bilinear map $[\cdot,\cdot]_{\mathcal{T}} : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$, called the bracket, and a linear map $\beta : \mathcal{T} \to \mathcal{T}$. These satisfy the following conditions:*

For all $a, b \in \mathcal{T}$,

$$[a, b]_{\mathcal{T}} = -\varepsilon(|a|, |b|)[b, a]_{\mathcal{T}}.$$

For all $a, b, c \in \mathcal{T}$,

$$\begin{aligned}&\varepsilon(|a|, |c|)[\beta(a), [b, c]_{\mathcal{T}}]_{\mathcal{T}}\\ &+ \varepsilon(|b|, |a|)[\beta(b), [c, a]_{\mathcal{T}}]_{\mathcal{T}}\\ &+ \varepsilon(|c|, |b|)[\beta(c), [a, b]_{\mathcal{T}}]_{\mathcal{T}} = 0.\end{aligned}$$

Definition 2.3. *Let Γ be an abelian group with a symmetric bi-character $\varepsilon : \Gamma \times \Gamma \to \mathbb{K} \setminus \{0\}$. A **Hom-$\delta$-Jordan Lie color algebra** is a quadruple $(\mathcal{T}, [\cdot, \cdot]_{\mathcal{T}}, \delta, \beta)$ consisting of: a Γ-graded vector space $\mathcal{T}$, an even bilinear map $[\cdot, \cdot]_{\mathcal{T}} : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$, called the bracket, and a linear map $\beta : \mathcal{T} \to \mathcal{T}$ (the twisting map). These satisfy the following conditions for all $a, b, c \in \mathcal{T}$:*

$$[a, b]_{\mathcal{T}} = -\delta\, \varepsilon(|a|, |b|)\, [b, a]_{\mathcal{T}},\ \delta = \pm 1,$$

$$\begin{aligned}&\varepsilon(|a|, |c|)[\beta(a), [b, c]_{\mathcal{T}}]_{\mathcal{T}}\\ &+ \varepsilon(|b|, |a|)[\beta(b), [c, a]_{\mathcal{T}}]_{\mathcal{T}}\\ &+ \varepsilon(|c|, |b|)[\beta(c), [a, b]_{\mathcal{T}}]_{\mathcal{T}} = 0.\end{aligned}$$

Definition 2.4. *Let Γ be an abelian group with a symmetric bi-character $\varepsilon : \Gamma \times \Gamma \to \mathbb{K} \setminus \{0\}$. A **BiHom-$\delta$-Jordan Lie color algebra** is a 5-tuple $(\mathcal{T}, [\cdot, \cdot]_{\mathcal{T}}, \delta, \beta, \gamma)$ consisting of: a Γ-graded space $\mathcal{T}$, an even bilinear map, called the bracket, $[\cdot, \cdot]_{\mathcal{T}} : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ and two linear maps $\beta, \gamma : \mathcal{T} \to \mathcal{T}$. These satisfy the following conditions for all $a, b, c \in \mathcal{T}$:*

$$\beta \circ \gamma = \gamma \circ \beta,$$

$$[\gamma(a), \beta(b)]_{\mathcal{T}} = -\delta\varepsilon(|a|, |b|)[\gamma(b), \beta(a)]_{\mathcal{T}},\ \delta = \pm 1,$$

$$\begin{aligned}&\varepsilon(|a|, |c|)[\gamma^2(a), [\gamma(b), \beta(c)]_{\mathcal{T}}]_{\mathcal{T}} + \varepsilon(|b|, |a|)[\gamma^2(b), [\gamma(c), \beta(a)]_{\mathcal{T}}]_{\mathcal{T}}\\ &+\varepsilon(|c|, |b|)[\gamma^2(c), [\gamma(a), \beta(b)]_{\mathcal{T}}]_{\mathcal{T}} = 0.\end{aligned}$$

Definition 2.5. *Let $(\mathcal{T}, [\cdot, \cdot]_{\mathcal{T}}, \delta, \beta, \gamma)$ be a δ-BiHom-Jordan Lie color algebra.*

(1) A δ-BiHom-Jordan Lie color algebra is said to be

(i) multiplicative if β and γ are an algebra morphism, i.e., for any $a, b \in \mathcal{T}$, then

$$\beta([a, b]_{\mathcal{T}}) = [\beta(a), \beta(b)]_{\mathcal{T}}, \quad \gamma([a, b]_{\mathcal{T}}) = [\gamma(a), \gamma(b)]_{\mathcal{T}};$$

(ii) regular if β and γ are an algebra automorphism.

(2) A Γ-graded subvector space $\tilde{\mathcal{T}} \subseteq \mathcal{T}$ is called

(i) a BiHom-subalgebra of $\mathcal{T}$ if $\beta(\tilde{\mathcal{T}}) \subseteq \tilde{\mathcal{T}}$, $\gamma(\tilde{\mathcal{T}}) \subseteq \tilde{\mathcal{T}}$, and

$$[a, b]_{\mathcal{T}} \in \tilde{\mathcal{T}},\ \forall a, b \in \tilde{\mathcal{T}};$$

(ii) a BiHom ideal of $\mathcal{T}$ if $\beta(\tilde{\mathcal{T}}) \subseteq \tilde{\mathcal{T}}$, $\gamma(\tilde{\mathcal{T}}) \subseteq \tilde{\mathcal{T}}$, and

$$[a, b]_{\mathcal{T}} \in \tilde{\mathcal{T}},\ \forall a \in \tilde{\mathcal{T}}, b \in \mathcal{T}.$$

Theorem 2.6. *Let $(\mathcal{T}, [\cdot,\cdot], \delta)$ be a δ-Lie color algebra. Suppose that $\gamma, \tilde{\gamma}$ are two even commuting algebra homomorphisms of $\mathcal{T}$. Then $(\mathcal{T}, [\cdot,\cdot]_{\gamma,\tilde{\gamma}}, \delta, \gamma, \tilde{\gamma})$, with*

$$[a, a_1]_{\gamma,\tilde{\gamma}} = [\gamma(a), \tilde{\gamma}(a_1)],$$

is a BiHom-δ-Jordan Lie color algebra.

Proof. For all $a, a_1 \in \mathcal{T}$, using the definition of operators, it follows that

$$[\tilde{\gamma}(a), \gamma(a_1)]_{\gamma,\tilde{\gamma}} = [\gamma\tilde{\gamma}(a), \tilde{\gamma}\gamma(a_1)] = \gamma\tilde{\gamma}([a, a_1]),$$

$$[\tilde{\gamma}(a_1), \gamma(a)]_{\gamma,\tilde{\gamma}} = [\gamma\tilde{\gamma}(a_1), \tilde{\gamma}\gamma(a)] = \gamma\tilde{\gamma}([a_1, a]) = -\delta\varepsilon(|a|, |a_1|)\gamma\tilde{\gamma}([a, a_1]).$$

So, $[\tilde{\tilde{\gamma}}(a), \gamma(a_1)]_{\gamma,\tilde{\gamma}} = -\delta\varepsilon(|a|, |a_1|)[\tilde{\gamma}(a_1), \gamma(a)]_{\gamma,\tilde{\gamma}}$ holds.

Considering the BiHom-Jacobi condition, it is clear that

$$\begin{aligned}
&\varepsilon(|a|, |a_2|)[\tilde{\gamma}^2(a), [\tilde{\gamma}(a_1), \gamma(a_2)]_{\gamma,\tilde{\gamma}}]_{\gamma,\tilde{\gamma}} + \varepsilon(|a_1|, |a|)[\tilde{\gamma}^2(a_1), [\tilde{\gamma}(a_2), \gamma(a)]_{\gamma,\tilde{\gamma}}]_{\gamma,\tilde{\gamma}} \\
&\quad + \varepsilon(|a_2|, |a_1|)[\tilde{\gamma}^2(a_2), [\tilde{\gamma}(a), \gamma(a_1)]_{\gamma,\tilde{\gamma}}]_{\gamma,\tilde{\gamma}} \\
&= \varepsilon(|a|, |a_2|)[\tilde{\gamma}^2(a), [\gamma\tilde{\gamma}(a_1), \gamma\tilde{\gamma}(a_2)]]_{\gamma,\tilde{\gamma}} + \varepsilon(|a_1|, |a|)[\tilde{\gamma}^2(a_1), [\gamma\tilde{\gamma}(a_2), \gamma\tilde{\gamma}(a)]]_{\gamma,\tilde{\gamma}} \\
&\quad + \varepsilon(|a_2|, |a_1|)[\tilde{\gamma}^2(a_2), [\gamma\tilde{\gamma}(a), \gamma\tilde{\gamma}(a_1)]]_{\gamma,\tilde{\gamma}} \\
&= \varepsilon(|a|, |a_2|)[\gamma\tilde{\gamma}^2(a), [\gamma\tilde{\gamma}^2(a_1), \gamma\tilde{\gamma}^2(a_2)]] + \varepsilon(|a_1|, |a|)[\gamma\tilde{\gamma}^2(a_1), [\gamma\tilde{\gamma}^2(a_2), \gamma\tilde{\gamma}^2(a)]] \\
&\quad + \varepsilon(|a_2|, |a_1|)[\gamma\tilde{\gamma}^2(a_2), [\gamma\tilde{\gamma}^2(a), \gamma\tilde{\gamma}^2(a_1)]] = 0.
\end{aligned}$$

The equation above holds since $(\mathcal{T}, [\cdot,\cdot], \delta)$ is a δ-Lie color algebra. Hence, $(\mathcal{T}, [\cdot,\cdot]_{\gamma,\tilde{\gamma}}, \delta, \gamma, \tilde{\gamma})$ is a BiHom-δ-Jordan Lie color algebra. □

Definition 2.7. *Assume $(\mathcal{T}, [\cdot,\cdot]_{\mathcal{T}}, \delta, \beta, \gamma)$ and $(\mathcal{T}_h, [\cdot,\cdot]_{\mathcal{T}_h}, \delta, \beta_1, \gamma_1)$ are two BiHom-δ-Jordan Lie color algebras. A map $h : \mathcal{T} \to \mathcal{T}_h$ is called a morphism of BiHom-δ-Jordan Lie color algebras if*

$$h[a, b]_{\mathcal{T}} = [h(a), h(b)]_{\mathcal{T}_h}, \qquad \forall a, b \in \mathcal{T}, \tag{2.1}$$

$$h \circ \beta = \beta_1 \circ h, \tag{2.2}$$

$$h \circ \gamma = \gamma_1 \circ h. \tag{2.3}$$

Consider a linear map $h : \mathcal{T} \to \mathcal{T}_h$. We denote its graph by $\mathfrak{T}_h$, which lies in the direct sum $\mathcal{T} \oplus \mathcal{T}_h$.

Theorem 2.8. *An even linear map $h : (\mathcal{T}, [\cdot,\cdot]_{\mathcal{T}}, \delta, \beta) \to (\mathcal{T}_h, [\cdot,\cdot]_{\mathcal{T}_h}, \delta, \gamma)$ is a morphism of BiHom-δ-Jordan Lie color algebras and is equivalent to the graph $\mathfrak{T}_h \in \mathcal{T} \oplus \mathcal{T}_h$, which is a BiHom-$\delta$-Jordan subalgebra of $(\mathcal{T} \oplus \mathcal{T}_h, [\cdot,\cdot]_{\mathcal{T}\oplus\mathcal{T}_h}, \delta, \beta + \beta_1, \gamma + \gamma_1)$.*

Proof. Note that

$$h : (\mathcal{T}, [\cdot,\cdot]_{\mathcal{T}}, \delta, \beta) \to (\mathcal{T}_h, [\cdot,\cdot]_{\mathcal{T}_h}, \delta, \gamma)$$

is a morphism of BiHom-δ-Jordan Lie color algebras. For all $a, b \in \mathcal{T}$, it follows that

$$[a + h(a), b + h(b)]_{\mathcal{T}\oplus\mathcal{T}_h} = [a, b]_{\mathcal{T}} + [h(a), h(b)]_{\mathcal{T}_h} = [a, b]_{\mathcal{T}} + h[a, b]_{\mathcal{T}}.$$

Therefore, $\mathfrak{T}_h$ has the property of being closed with respect to the bracket on $\mathcal{T} \oplus \mathcal{T}_h$. Furthermore, we get

$$(\beta + \beta_1)(a + h(a)) = \beta(a) + \beta_1 \circ h(a) = \beta(a) + h \circ \beta(a).$$

This result shows that $(\beta + \beta_1)(\mathfrak{T}_h) \subset \mathfrak{T}_h$.

Similarly, $(\gamma + \gamma_1)(\mathfrak{T}_h) \subset \mathfrak{T}_h$. Thus, $\mathfrak{T}_h$ is a BiHom-δ-Jordan subalgebra of $(\mathcal{T} \oplus \mathcal{T}_h, [\cdot,\cdot]_{\mathcal{T}\oplus\mathcal{T}_h}, \delta, \beta + \beta_1, \gamma + \gamma_1)$.

Our next goal is to demonstrate the reverse inclusion. Suppose that the graph $\mathfrak{T}_h \subset \mathcal{T} \oplus \mathcal{T}_h$ is a BiHom subalgebra of $(\mathcal{T} \oplus \mathcal{T}_h, [\cdot,\cdot]_{\mathcal{T}\oplus\mathcal{T}_h}, \delta, \beta + \beta_1, \gamma + \gamma_1)$. A straightforward calculation yields that

$$[a + h(a), b + h(b)]_{\mathcal{T}\oplus\mathcal{T}_h} = [a, b]_{\mathcal{T}} + [h(a), h(b)]_{\mathcal{T}_h} \in \mathfrak{T}_h.$$

This result shows that

$$[h(a), h(b)]_{\mathcal{T}_h} = h[a, b]_{\mathcal{T}}.$$

In addition, $(\beta + \beta_1)(\mathfrak{T}_h) \subset \mathfrak{T}_h$ implies that

$$(\beta + \beta_1)(a + h(a)) = \beta(a) + \beta_1 \circ h(a) \in \mathfrak{T}_h,$$

which means that h commutes with the actions β and β_1, that is, $\beta_1 \circ h = h \circ \beta$. Similarly, $\gamma_1 \circ h = h \circ \gamma$. □

Theorem 2.9. *Suppose $(\mathcal{T}, [\cdot,\cdot], \beta, \gamma)$ is a BiHom-δ-Jordan Lie color algebra, and $\eta_i : \mathcal{T} \to \mathcal{T}$ are two even algebra homomorphisms such that $\eta_i([a, a']) = [\eta_i(a), \eta_i(a')]$, for all $a, a' \in \mathcal{T}$, i=1,2, where these maps commute pairwise, and η_1, η_2 commute with β and γ. Thus, $(\mathcal{T}, [\cdot,\cdot]_{\eta_1,\eta_2} := [\cdot,\cdot] \circ (\eta_1 \otimes \eta_2), \beta \circ \eta_1, \gamma \circ \eta_2)$ is a BiHom-δ-Jordan Lie color algebra.*

Proof. The first step is to establish the compatibility of the bracket $[\cdot,\cdot]_{\eta_1,\eta_2}$ with the maps $\beta \circ \eta_1$ and $\gamma \circ \eta_2$. For all $a, b \in \mathcal{T}$, we have

$$\begin{aligned}
&[\gamma \circ \eta_2(a), \beta \circ \eta_1(b)]_{\eta_1,\eta_2} \\
&= [\eta_1 \circ \gamma \circ \eta_2(a), \eta_2 \circ \beta \circ \eta_1(b)] \\
&= \eta_1 \circ \eta_2([\gamma(a), \beta(b)]) \\
&= -\delta\varepsilon(|a|, |b|)\eta_1 \circ \eta_2([\gamma(b), \beta(a)]) \\
&= -\delta\varepsilon(|a|, |b|)[\eta_1 \circ \eta_2 \circ \gamma(b), \eta_1 \circ \eta_2 \circ \beta(a)] \\
&= -\delta\varepsilon(|a|, |b|)[\gamma \circ \eta_2(b), \beta \circ \eta_1(a)]_{\eta_1,\eta_2}.
\end{aligned}$$

Next, we prove that the BiHom-Jacobi condition holds. Considering all elements $a, b, c \in \mathcal{T}$, it is clear that

$$\begin{aligned}
&\varepsilon(|a|, |c|)[(\gamma \circ \eta_2)^2(a), [\gamma \circ \eta_2(b), \beta \circ \eta_1(c)]_{\eta_1,\eta_2}]_{\eta_1,\eta_2} \\
&= \varepsilon(|a|, |c|)[(\gamma \circ \eta_2)^2(a), \eta_1 \circ \eta_2([\gamma(b), \beta(c)])]_{\eta_1,\eta_2} \\
&= \varepsilon(|a|, |c|)[\eta_1 \circ (\gamma \circ \eta_2)^2(a), \eta_1 \circ (\eta_2)^2([\gamma(b), \beta(c)])] \\
&= \varepsilon(|a|, |c|)\eta_1 \circ (\eta_2)^2([\gamma^2(a), [\gamma(b), \beta(c)]]).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\varepsilon(|b|, |a|)[(\gamma \circ \eta_2)^2(b), [\gamma \circ \eta_2(c), \beta \circ \eta_1(a)]_{\eta_1,\eta_2}]_{\eta_1,\eta_2} \\
&= \varepsilon(|b|, |a|)\eta_1 \circ (\eta_2)^2([\gamma^2(b), [\gamma(c), \beta(a)]]),
\end{aligned}$$

and

$$\varepsilon(|c|,|b|)[(\gamma\circ\eta_2)^2(c),[\gamma\circ\eta_2(a),\beta\circ\eta_1(b)]_{\eta_1,\eta_2}]_{\eta_1,\eta_2}$$
$$=\varepsilon(|c|,|b|)\eta_1\circ(\eta_2)^2([\gamma^2(c),[\gamma(a),\beta(b)]]).$$

Then we show

$$\begin{aligned}&\varepsilon(|a|,|c|)[(\gamma\circ\eta_2)^2(a),[\gamma\circ\eta_2(b),\beta\circ\eta_1(c)]_{\eta_1,\eta_2}]_{\eta_1,\eta_2}\\&+\varepsilon(|b|,|a|)[(\gamma\circ\eta_2)^2(b),[\gamma\circ\eta_2(c),\beta\circ\eta_1(a)]_{\eta_1,\eta_2}]_{\eta_1,\eta_2}\\&+\varepsilon(|c|,|b|)[(\gamma\circ\eta_2)^2(c),[\gamma\circ\eta_2(a),\beta\circ\eta_1(b)]_{\eta_1,\eta_2}]_{\eta_1,\eta_2}\\&=0.\end{aligned}$$

Therefore, we obtain that $(\mathcal{T},[\cdot,\cdot]_{\eta_1,\eta_2}:=[\cdot,\cdot]\circ(\eta_1\otimes\eta_2),\beta\circ\eta_1,\gamma\circ\eta_2)$ is a BiHom-δ-Jordan Lie color algebra. □

Remark 2.10. *Using Theorem 2.6, Theorem 2.8, and Theorem 2.9, we construct three kinds of BiHom-δ-Jordan Lie color algebras, and regard Theorem 2.6, Theorem 2.8, and Theorem 2.9 as examples on BiHom-δ-Jordan Lie color algebras.*

3. Representations on BiHom-δ-Jordan Lie color algebras

This section investigates representations on multiplicative BiHom-δ-Jordan Lie color algebras. In the following, we assume that β and $\gamma_{\mathcal{W}}$ are invertible. Additionally, it demonstrates that a semidirect product within this category can be formed from a given representation.

Definition 3.1. *An even linear map $\rho:\mathcal{T}\to pl(\mathcal{W})$ is said to be a representation on the multiplicative BiHom-δ-Jordan Lie color algebra $(\mathcal{T},[\cdot,\cdot]_{\mathcal{T}},\delta,\beta,\gamma)$ on the Γ-graded vector space $\mathcal{W}$ with $\beta_{\mathcal{W}},\gamma_{\mathcal{W}}$ as two commuting even linear maps for all $c,d\in\mathcal{T}$, satisfying:*

$$\rho(\beta(c))\circ\beta_{\mathcal{W}}=\beta_{\mathcal{W}}\circ\rho(c), \tag{3.1}$$

$$\rho(\gamma(c))\circ\gamma_{\mathcal{W}}=\gamma_{\mathcal{W}}\circ\rho(c), \tag{3.2}$$

$$\rho([\gamma(c),d]_{\mathcal{T}})\circ\gamma_{\mathcal{W}}=\rho(\beta\gamma(c))\circ\rho(d)-\delta\varepsilon(|c|,|d|)\rho(\gamma(d))\circ\rho(\beta(c)). \tag{3.3}$$

A k-cochain on $\mathcal{T}$ with values in $\mathcal{W}$ is a k-linear map from $\mathcal{T}\times\cdots\times\mathcal{T}$ (k-times) to $\mathcal{W}$. The set of all such cochains is written as $C^k(\mathcal{T};\mathcal{W})$.

$$\begin{aligned}C^k(\mathcal{T};\mathcal{W})\triangleq\{&g:\mathcal{T}\times\cdots\times\mathcal{T}(k-times)\to\mathcal{W}\ is\ a\ linear\ map\\ and\ &g(t_1,\cdots,t_i,t_{i+1},\cdots,t_k)=-\delta\varepsilon(|t_i|,|t_{i+1}|)g(t_1,\cdots,t_{i+1},t_i,\cdots,t_k)\}.\end{aligned}$$

A k-BiHom-cochain on $\mathcal{T}$ with values in $\mathcal{W}$ refers to a k-cochain $g\in C^k(\mathcal{T};\mathcal{W})$, fulfilling the compatibility relations: $\beta_{\mathcal{W}}\circ g=g\circ\beta$ and $\gamma_{\mathcal{W}}\circ g=g\circ\gamma$, i.e.,

$$\beta_{\mathcal{W}}(g(a_1,\cdots,a_{k+1}))=g(\beta(a_1),\cdots,\beta(a_{k+1})),$$

$$\gamma_{\mathcal{W}}(g(a_1,\cdots,a_{k+1}))=g(\gamma(a_1),\cdots,\gamma(a_{k+1})).$$

Write $C^k_{BiHom}(\mathcal{T};\mathcal{W})$ for the set of all k-BiHom-cochains:

$$C^k_{BiHom}(\mathcal{T};\mathcal{W}) := \{g \in C^k(\mathcal{T},\mathcal{W}) | \beta_{\mathcal{W}} \circ g = g \circ \beta, \gamma_{\mathcal{W}} \circ g = g \circ \gamma\}.$$

Our next step is to define the linear map $d^k_\rho : C^k_{BiHom}(\mathcal{T},\mathcal{W}) \to C^{k+1}_{BiHom}(\mathcal{T},\mathcal{W})$, where $k = 1, 2$, as follows:

$$\begin{aligned}
& d^1_\rho g(c,d) \\
= \ & \varepsilon(|g|,|c|)\rho(\beta(c))f(d) - \delta\varepsilon(|g|+|c|,|d|)\rho(\beta(d))f(c) \\
& -\delta g([\beta^{-1}\gamma(c),d]_{\mathcal{T}}),
\end{aligned}$$

$$\begin{aligned}
& d^2_\rho g(b,c,d) \\
= \ & \varepsilon(|g|,|b|)\rho(\beta\gamma(b))g(c,d) - \delta\varepsilon(|g|+|b|,|c|)\rho(\beta\gamma(c))g(b,d) \\
& +\varepsilon(|g|+|b|+|c|,|d|)\rho(\beta\gamma(d))g(b,c) - g([\beta^{-1}\gamma(b),c]_{\mathcal{T}},\gamma(d)) \\
& +\delta\varepsilon(|b|+|c|,|d|)\varepsilon(|b|,|d|)g([\beta^{-1}\gamma(b),d]_{\mathcal{T}},\gamma(c)) \\
& -\varepsilon(|b|,|c|)\varepsilon(|b|+|c|,|d|)\varepsilon(|c|,|d|)g([\beta^{-1}\gamma(c),d]_{\mathcal{T}},\gamma(b)).
\end{aligned}$$

Lemma 3.2. *In the notation introduced above, for all $g \in C^k_{BiHom}(\mathcal{T};\mathcal{W})$, we have*

$$(d^k_\rho \circ g) \circ \beta = \beta_{\mathcal{W}} \circ d^k_\rho g,$$

and

$$(d^k_\rho \circ g) \circ \gamma = \gamma_{\mathcal{W}} \circ d^k_\rho g.$$

This construction provides a well-defined map

$$d^k_\rho : C^k_{BiHom}(\mathcal{T};\mathcal{W}) \to C^{k+1}_{BiHom}(\mathcal{T};\mathcal{W}),$$

where $k = 1, 2$.

Proposition 3.3. *In the notation introduced above, we know $d^2_{\rho_A} \circ d^1_{\rho_A} = 0$.*

Proof. By a straightforward computation, we have

$$\begin{aligned}
& d^2_{\rho_A} \circ d^1_{\rho_A} g(b,c,d) \\
= \ & \varepsilon(|g|,|b|)\rho_A(\beta\gamma(b))d^1_{\rho_A}g(c,d) - \delta\varepsilon(|g|+|b|,|c|)\rho_A(\beta\gamma(c))d^1_{\rho_A}g(b,d) \\
& +\varepsilon(|g|+|b|+|c|,|d|)\rho_A(\beta\gamma(d))d^1_{\rho_A}g(b,c) - d^1_{\rho_A}g([\beta^{-1}\gamma(b),c],\gamma(d)) \\
& +\delta\varepsilon(|b|+|c|,|d|)\varepsilon(|b|,|d|)d^1_{\rho_A}g([\beta^{-1}\gamma(b),d],\gamma(c)) \\
& -\varepsilon(|b|,|c|)\varepsilon(|b|+|c|,|d|)\varepsilon(|c|,|d|)d^1_{\rho_A}g([\beta^{-1}\gamma(c),d],\gamma(b)) \\
= \ & \varepsilon(|g|,|b|)\rho_A(\beta\gamma(b)) \\
& (\varepsilon(|g|,|c|)\rho_A(\beta(c))g(d) - \delta\varepsilon(|g|+|c|,|d|)\rho_A(\beta(d))g(c) - \delta g([\beta^{-1}\gamma(c),d])) \\
& -\delta\varepsilon(|g|+|b|,|c|)\rho_A(\beta\gamma(c)) \\
& (\varepsilon(|g|,|b|)\rho_A(\beta(b))g(d) - \delta\varepsilon(|g|+|b|,|d|)\rho_A(\beta(d))g(b) - \delta g([\beta^{-1}\gamma(b),d])) \\
& +\varepsilon(|g|+|b|+|c|,|d|)\rho_A(\beta\gamma(d))
\end{aligned}$$

$$
\begin{aligned}
&(\varepsilon(|g|,|b|)\rho_A(\beta(b))g(c)-\delta\varepsilon(|g|+|b|,|c|)\rho_A(\beta(c))g(b)-\delta g([\beta^{-1}\gamma(b),c]))\\
&-(\varepsilon(|g|,|b|+|c|)\rho_A(\beta([\beta^{-1}\gamma(b),c]))g(\gamma(d))\\
&-\delta\varepsilon(|g|+|b|+|c|,|d|)\rho_A(\beta\gamma(d))g([\beta^{-1}\gamma(b),c])-\delta g([[\beta^{-1}\gamma(b),c],\gamma(d)]))\\
&+\delta\varepsilon(|c|,|d|)(\varepsilon(|g|,|b|+|d|)\rho_A(\beta([\beta^{-1}\gamma(b),d]))g(\gamma(c))\\
&-\delta\varepsilon(|g|+|b|+|d|,|c|)\rho_A(\beta\gamma(c))g([\beta^{-1}\gamma(b),d])-\delta g([[\beta^{-1}\gamma(b),d],\gamma(c)]))\\
&-\varepsilon(|b|,|c|+|d|)(\varepsilon(|g|,|c|+|d|)\rho_A(\beta([\beta^{-1}\gamma(c),d]g(\gamma(b))\\
&-\delta\varepsilon(|g|+|c|+|d|,|b|)\rho_A(\beta\gamma(b))g([\beta^{-1}\gamma(c),d])-\delta g([[\beta^{-1}\gamma(c),d],\gamma(b)])).
\end{aligned}
$$

By direct computations, we get

$$
\begin{aligned}
&\rho_A(\beta([\beta^{-1}\gamma(c),d]))g(\gamma(b))\\
&=\rho_A(\beta([\beta^{-1}\gamma(c),d]))\gamma_{\mathcal{W}}\circ g(b)\\
&=\rho_A(\beta\gamma(c))\rho_A(\beta(d))g(b)-\delta\varepsilon(|c|,|d|)\rho_A(\beta\gamma(d))\rho_A(\beta(c))g(b).
\end{aligned}
$$

Similarly, it is obvious that

$$
\begin{aligned}
&\rho_A(\beta([\beta^{-1}\gamma(b),d]))g(\gamma(c))\\
&=\rho_A(\beta\gamma(b))\rho_A(\beta(d))g(c)-\delta\varepsilon(|b|,|d|)\rho_A(\beta\gamma(d))\rho_A(\beta(b))g(c),
\end{aligned}
$$

and

$$
\begin{aligned}
&\rho_A(\beta([\beta^{-1}\gamma(b),c]))g(\gamma(d))\\
&=\rho_A(\beta\gamma(b))\rho_A(\beta(c))g(d)-\delta\varepsilon(|b|,|c|)\rho_A(\beta\gamma(c))\rho_A(\beta(b))g(d).
\end{aligned}
$$

Hence, $d^2_{\rho_A}\circ d^1_{\rho_A}g(b,c,d)=0$. □

Remark 3.4. *This result is verified by a direct operation. Recently, a new method has emerged to verify that the composition of boundary operators is zero on left-symmetric color algebras in [24].*

There is an associated cochain complex $(C^k_{\mathrm{BiHom}}(\mathcal{T};\mathcal{W}),d_{\rho_A})$ defined via the representation ρ_A. Let $Z^k(\mathcal{T};\rho_A)$ and $B^k(\mathcal{T};\rho_A)$ denote respectively the sets of closed and exact k-BiHom-cochains, $k=1,2$.

Proposition 3.5. *Start from a representation ρ_A on a multiplicative BiHom-δ-Jordan Lie color algebra $(\mathcal{T},[\cdot,\cdot]_{\mathcal{T}},\delta,\beta,\gamma)$ on the Γ-graded vector space $\mathcal{W}$. Define a bilinear bracket operation $[\cdot,\cdot]_{\rho_A}:(\mathcal{T}\oplus\mathcal{W})\times(\mathcal{T}\oplus\mathcal{W})\to\mathcal{T}\oplus\mathcal{W}$ in view of*

$$[\tau_1+W_1,\tau_2+W_2]_{\rho_A}=[\tau_1,\tau_2]_{\mathcal{T}}+\delta\rho_A(\tau_1)(W_2)-\varepsilon(|\tau_1|,|\tau_2|)\rho_A(\beta^{-1}\gamma(\tau_2))(\beta_{\mathcal{W}}\gamma^{-1}_{\mathcal{W}}(W_1)). \tag{3.4}$$

Define $\beta+\beta_{\mathcal{W}},\gamma+\gamma_{\mathcal{W}}:\mathcal{T}\oplus\mathcal{W}\to\mathcal{T}\oplus\mathcal{W}$ by

$$(\beta+\beta_{\mathcal{W}})(\tau_1+W_1)=\beta(\tau_1)+\beta_{\mathcal{W}}(W_1),$$

and

$$(\gamma+\gamma_{\mathcal{W}})(\tau_1+W_1)=\gamma(\tau_1)+\gamma_{\mathcal{W}}(W_1).$$

Then $(\mathcal{T}\oplus\mathcal{W},[\cdot,\cdot]_{\rho_A},\delta,\beta+\beta_{\mathcal{W}},\gamma+\gamma_{\mathcal{W}})$ is a multiplicative BiHom-δ-Jordan Lie color algebra, which is called the semidirect product on the multiplicative BiHom-δ-Jordan Lie color algebra $(\mathcal{T},[\cdot,\cdot]_{\mathcal{T}},\delta,\beta,\gamma)$ and $\mathcal{W}$.

Proof. We begin by showing that $\beta+\beta_{\mathcal{W}}$ is an algebra morphism. In fact,

$$\begin{aligned}
& (\beta+\beta_{\mathcal{W}})[\tau_1+W_1,\tau_2+W_2]_{\rho_A}\\
= \;& (\beta+\beta_{\mathcal{W}})([\tau_1,\tau_2]_{\mathcal{T}}+\delta\rho_A(\tau_1)(W_2)-\varepsilon(|\tau_1|,|\tau_2|)\rho_A(\beta^{-1}\gamma(\tau_2))(\beta_{\mathcal{W}}\gamma_{\mathcal{W}}^{-1}(W_1)))\\
= \;& \beta([\tau_1,\tau_2]_{\mathcal{T}})+\delta\beta_{\mathcal{W}}\circ\rho_A(\tau_1)(W_2)-\varepsilon(|\tau_1|,|\tau_2|)\beta_{\mathcal{W}}\circ\rho_A(\beta^{-1}\gamma(\tau_2))(\beta_{\mathcal{W}}\gamma_{\mathcal{W}}^{-1}(W_1))\\
= \;& [\beta(\tau_1),\beta(\tau_2)]_{\mathcal{T}}+\delta\rho_A(\beta(\tau_1))(\beta_{\mathcal{W}}(W_2))-\varepsilon(|\tau_1|,|\tau_2|)\rho_A(\beta(\beta^{-1}\gamma(\tau_2)))(\beta_{\mathcal{W}}\circ(\beta_{\mathcal{W}}\gamma_{\mathcal{W}}^{-1}(W_1))),
\end{aligned}$$

and

$$\begin{aligned}
& [(\beta+\beta_{\mathcal{W}})(\tau_1+W_1),(\beta+\beta_{\mathcal{W}})(\tau_2+W_2)]_{\rho_A}\\
= \;& [\beta(\tau_1)+\beta_{\mathcal{W}}(W_1),\beta(\tau_2)+\beta_{\mathcal{W}}(W_2)]_{\rho_A}\\
= \;& [\beta(\tau_1),\beta(\tau_2)]_{\mathcal{T}}+\delta\rho_A(\beta(\tau_1))(\beta_{\mathcal{W}}(W_2))-\varepsilon(|\tau_1|,|\tau_2|)\rho_A(\beta^{-1}\gamma(\beta(\tau_2)))(\beta_{\mathcal{W}}\gamma_{\mathcal{W}}^{-1}(\beta_{\mathcal{W}}(W_1))).
\end{aligned}$$

It is evident that $\beta+\beta_{\mathcal{W}}$ is an algebra morphism for the bracket $[\cdot,\cdot]_{\rho_A}$. Similarly, $\gamma+\gamma_{\mathcal{W}}$ is also an algebra morphism.

The symmetry of $[\cdot,\cdot]_{\rho_A}$ must be verified next.

$$\begin{aligned}
& -\delta\varepsilon(|\tau_1|,|\tau_2|)[(\gamma+\gamma_{\mathcal{W}})(\tau_2+W_2),(\beta+\beta_{\mathcal{W}})(\tau_1+W_1)]_{\rho_A}\\
= \;& -\delta\varepsilon(|\tau_1|,|\tau_2|)[\gamma(\tau_2)+\gamma_{\mathcal{W}}(W_2),\beta(\tau_1)+\beta_{\mathcal{W}}(W_1)]\\
= \;& -\delta\varepsilon(|\tau_1|,|\tau_2|)([\gamma(\tau_2),\beta(\tau_1)]_{\mathcal{T}}+\delta\rho_A(\gamma(\tau_2))(\beta_{\mathcal{W}}(W_1))\\
& -\varepsilon(|\tau_1|,|\tau_2|)\rho_A(\beta^{-1}\gamma(\beta(\tau_1)))(\beta_{\mathcal{W}}\gamma_{\mathcal{W}}^{-1}(\gamma_{\mathcal{W}}(W_2))))\\
= \;& -\delta\varepsilon(|\tau_1|,|\tau_2|)([\gamma(\tau_2),\beta(\tau_1)]_{\mathcal{T}}+\delta\rho_A(\gamma(\tau_2))(\beta_{\mathcal{W}}(W_1))-\varepsilon(|\tau_1|,|\tau_2|)\rho_A(\gamma(\tau_1))(\beta_{\mathcal{W}}(W_2)))\\
= \;& [\gamma(\tau_1),\beta(\tau_2)]_{\mathcal{T}}+\delta\rho_A(\gamma(\tau_1))(\beta_{\mathcal{W}}(W_2))-\varepsilon(|\tau_1|,|\tau_2|)\rho_A(\gamma(\tau_2))(\beta_{\mathcal{W}}(W_1))\\
= \;& [(\gamma+\gamma_{\mathcal{W}})(\tau_1+W_1),(\beta+\beta_{\mathcal{W}})(\tau_2+W_2)]_{\rho_A}.
\end{aligned}$$

Also, we can easily deduce that

$$\begin{aligned}
& \varepsilon(|\tau_1|,|\tau_3|)[(\gamma+\gamma_{\mathcal{W}})^2(\tau_1+W_1),[\gamma+\gamma_{\mathcal{W}}(\tau_2+W_2),(\beta+\beta_{\mathcal{W}})(\tau_3+W_3)]_{\rho_A}]_{\rho_A}\\
& +c.p.(\tau_1+W_1,\tau_2+W_2,\tau_3+W_3)\\
= \;& \varepsilon(|\tau_1|,|\tau_3|)[(\gamma^2(\tau_1),\gamma_{\mathcal{W}}^2(W_1)),[(\gamma(\tau_2),\gamma_{\mathcal{W}}(W_2)),(\beta(\tau_3),\beta_{\mathcal{W}}(W_3))]]\\
& +c.p.(\tau_1+W_1,\tau_2+W_2,\tau_3+W_3)\\
= \;& \varepsilon(|\tau_1|,|\tau_3|)[(\gamma^2(\tau_1),\gamma_{\mathcal{W}}^2(W_1)),([\gamma(\tau_2),\beta(\tau_3)],\rho_A(\gamma(\tau_2))(\beta_{\mathcal{W}}(W_3))\\
& -\varepsilon(|\tau_2|,|\tau_3|)\rho_A(\beta^{-1}\gamma(\beta(\tau_3)))(\beta_{\mathcal{W}}\gamma_{\mathcal{W}}^{-1}(\gamma_{\mathcal{W}}(y))))]+c.p.(\tau_1+W_1,\tau_2+W_2,\tau_3+W_3)\\
= \;& \varepsilon(|\tau_1|,|\tau_3|)[(\gamma^2(\tau_1),\gamma_{\mathcal{W}}^2(W_1)),([\gamma(\tau_2),\beta(\tau_3)],\rho_A(\gamma(\tau_2))(\beta_{\mathcal{W}}(W_3))\\
& -\varepsilon(|\tau_2|,|\tau_3|)\rho_A(\gamma(\tau_3))(\beta_{\mathcal{W}}(y)))]+c.p.(\tau_1+W_1,\tau_2+W_2,\tau_3+W_3)\\
= \;& \varepsilon(|\tau_1|,|\tau_3|)([\gamma^2(\tau_1),[\gamma(\tau_2),\beta(\tau_3)]],\rho_A(\gamma^2(\tau_1)\rho_A(\gamma(\tau_2))(\beta_{\mathcal{W}}(W_3))\\
& -\varepsilon(|\tau_2|,|\tau_3|)\rho_A(\gamma^2(\tau_1))\rho_A(\gamma(\tau_3))(\beta_{\mathcal{W}}(y))\\
& -\varepsilon(|\tau_2|,|\tau_3|)\rho_A(\beta^{-1}\gamma([\gamma(\tau_2),\beta(\tau_3)])(\beta_{\mathcal{W}}\gamma_{\mathcal{W}}^{-1}(\gamma_{\mathcal{W}}^2(W_1)))))+c.p.(\tau_1+W_1,\tau_2+W_2,\tau_3+W_3)\\
= \;& (\varepsilon(|\tau_1|,|\tau_3|)[\gamma^2(\tau_1),[\gamma(\tau_2),\beta(\tau_3)]],\varepsilon(|\tau_1|,|\tau_3|)\rho_A(\gamma^2(\tau_1)\rho_A(\gamma(\tau_2))(\beta_{\mathcal{W}}(W_3))\\
& -\varepsilon(|\tau_3|,|\tau_1|+|\tau_2|)\rho_A(\gamma^2(\tau_1))\rho_A(\gamma(\tau_3))(\beta_{\mathcal{W}}(y))
\end{aligned}$$

$$-\varepsilon(|\tau_1|,|\tau_2|)\rho_A([\beta^{-1}\gamma^2(\tau_2),\gamma(\tau_3)])(\beta_{\mathcal{W}}\gamma_{\mathcal{W}}(W_1)))) + c.p.(\tau_1+W_1,\tau_2+W_2,\tau_3+W_3)$$
$$= 0,$$

where the abbreviation c.p. stands for cyclic permutation. The BiHom-Jacobi identity is verified and consequently, the proof is complete. □

4. The trivial representation on BiHom-δ-Jordan Lie color algebras

Suppose that $\mathcal{W} = \mathbb{R}$ and it is obvious that $gl(\mathcal{W}) = \mathbb{R}$. $\beta_{\mathcal{W}}, \gamma_{\mathcal{W}} \in gl(\mathcal{W})$ are two real numbers, where the notations r_1 and r_2 are used, respectively. Assume that $\rho : \mathcal{T} \to gl(v) = \mathbb{R}$ is the zero map. Consider the multiplicative BiHom-δ-Jordan Lie color algebra $(\mathcal{T}, [\cdot,\cdot]_{\mathcal{T}}, \delta, \beta, \gamma)$. Then, for any scalars $r_1, r_2 \in \mathbb{R}$, the map ρ constitutes a representation on it. By definition, setting $r_1 = r_2 = 1$ yields the trivial representation on the multiplicative BiHom-δ-Jordan Lie color algebra $(\mathcal{T}, [\cdot,\cdot]_{\mathcal{T}}, \delta, \beta, \gamma)$.

Let $C^k(\mathcal{W})$ denote the set of k-cochains on $\mathcal{T}$ associated to the trivial representation, which is written as the set of a k-linear map from $\mathcal{W} \times \cdots \times \mathcal{W}$ to $\mathbb{R}$. Setting

$$C^k_{BiHom}(\mathcal{W}) = \{f \in C^k(\mathcal{W}) | f \circ \beta = f,\ f \circ \gamma = f\},$$

the corresponding operator $d_T : C^k_{BiHom}(\mathcal{W}) \to C^{k+1}_{BiHom}(\mathcal{W})(k = 1, 2)$ is defined by

$$d^1_T f(a,b) = -\delta f([\beta^{-1}\gamma(a), b]),$$

$$\begin{aligned} & d^2_T f(b,c,d) \\ = \ & -f([\beta^{-1}\gamma(b), c], \gamma(d)) + \delta\varepsilon(|c|,|d|) f([\beta^{-1}\gamma(b), d], \gamma(c)) \\ & -\varepsilon(|b|, |c|+|d|) f([\beta^{-1}\gamma(c), d], \gamma(b)). \end{aligned}$$

Denote $Z^k_{BiHom}(\mathcal{W})$ and $B^k_{BiHom}(\mathcal{W})(k = 1, 2)$, similarly.

Let us consider central extensions on the multiplicative BiHom-δ-Jordan Lie color algebra $(\mathcal{T}, [\cdot,\cdot]_{\mathcal{T}}, \delta, \beta, \gamma)$. One can verify that $(\mathbb{R}, 0, 1, 1)$, together with the trivial bracket and the identity morphism, is an abelian multiplicative BiHom-δ-Jordan Lie color algebra. Let $\theta \in C^2_{BiHom}(\mathcal{W})$, considering the direct sum $\eta = \mathcal{T} \oplus \mathbb{R}$ defined by the following bracket:

$$[\tau + s, \tau_1 + t]_\theta = [\tau, b]_{\mathcal{T}} + \theta(\beta\gamma^{-1}(\tau), b),\ \forall \tau, b \in \mathcal{T},\ s, t \in \mathbb{R}.$$

Define $\beta_\eta, \gamma_\eta : \eta \to \eta$ by

$$\beta_\eta(\tau + s) = \beta(\tau) + s,$$

$$\gamma_\eta(\tau + s) = \gamma(\tau) + s.$$

Theorem 4.1. *The property that $(\eta, [\cdot,\cdot]_\theta, \delta, \beta\eta, \gamma_\eta)$ is a multiplicative BiHom-δ-Jordan Lie color algebra is equivalent to the cocycle condition $d_T\theta = 0$ for $\theta \in C^2_{BiHom}(\mathcal{W})$. This yields the multiplicative BiHom-δ-Jordan Lie color algebra $(\eta, [\cdot,\cdot]_\theta, \delta, \beta_\eta, \gamma_\eta)$, which we refer to as the central extension of $(\mathcal{T}, [\cdot,\cdot]_{\mathcal{T}}, \delta, \beta, \gamma)$ by $(\mathbb{R}, 0, 1, 1)$.*

Proof. $\theta \circ \beta = \theta$ implies that β_η is an algebra morphism relating to $[\cdot,\cdot]_\theta$. Indeed,

$$\beta_\eta[\tau + s, \tau_1 + t]_\theta = \beta[\tau, \tau_1]_{\mathcal{T}} + \theta(\beta\gamma^{-1}(\tau), \tau_1),$$

and

$$[\beta_\eta(\tau + s), \beta_\eta(\tau_1 + t)]_\theta = [\beta(\tau) + s, \beta(\tau_1) + t]_\theta = [\beta(\tau), \beta(\tau_1)]_{\mathcal{T}} + \theta(\beta\gamma^{-1}(\beta(\tau)), \beta(\tau_1)).$$

Considering the property of β and

$$\theta(\beta\gamma^{-1}(\beta(\tau)), \beta(\tau_1)) = \theta \circ \beta(\beta\gamma^{-1}(\tau), \tau_1) = \theta(\beta\gamma^{-1}(\tau), \tau_1),$$

it follows that β_η is an algebra morphism. Similarly, γ_η is also an algebra morphism.

It is easily seen that

$$[\gamma_\eta(\tau + s), \beta_\eta(\tau_1 + t)]_\theta = -\delta\varepsilon(|\tau|, |\tau_1|)[\gamma_\eta(\tau_1 + t), \beta_\eta(\tau + s)]_\theta.$$

Indeed,

$$\begin{aligned}
& [\gamma_\eta(\tau + s), \beta_\eta(\tau_1 + t)]_\theta \\
= \ & [\gamma(\tau) + s, \beta(\tau_1) + t]_\theta \\
= \ & [\gamma(\tau), \beta(\tau_1)]_{\mathcal{T}} + \theta(\beta\gamma^{-1}(\gamma(\tau)), \beta(\tau_1)) \\
= \ & [\gamma(\tau), \beta(\tau_1)]_{\mathcal{T}} + \theta(\beta(\tau)), \beta(\tau_1)) \\
= \ & [\gamma(\tau), \beta(\tau_1)]_{\mathcal{T}} + \theta(\tau, \tau_1),
\end{aligned}$$

and

$$\begin{aligned}
& -\delta\varepsilon(|\tau|, |\tau_1|)[\gamma_\eta(\tau_1 + t), \beta_\eta(\tau + s)]_\theta \\
= \ & -\delta\varepsilon(|\tau|, |\tau_1|)[\gamma(\tau_1) + t, \beta(\tau) + s]_\theta \\
= \ & -\delta\varepsilon(|\tau|, |\tau_1|)([\gamma(\tau_1), \beta(\tau)]_{\mathcal{T}} + \theta(\beta\gamma^{-1}(\gamma(\tau_1)), \beta(\tau))) \\
= \ & -\delta\varepsilon(|\tau|, |\tau_1|)([\gamma(\tau_1), \beta(\tau)]_{\mathcal{T}} + \theta(\beta(\tau_1), \beta(\tau))) \\
= \ & -\delta\varepsilon(|\tau|, |\tau_1|)([\gamma(\tau_1), \beta(\tau)]_{\mathcal{T}} + \theta(\tau_1, \tau)) \\
= \ & [\gamma(\tau), \beta(\tau_1)]_{\mathcal{T}} + \theta(\tau, \tau_1).
\end{aligned}$$

In the following, we define that the abbreviation c.p. stands for cyclic permutation. By direct computations, we get

$$\begin{aligned}
& \varepsilon(|\tau|, |\tau_2|)[\gamma_\eta^2(\tau + s), [\gamma_\eta(\tau_1 + t), \beta_\eta(\tau_2 + m)]_\theta]_\theta + c.p.(\tau + s, \tau_1 + t, \tau_2 + m) \\
= \ & \varepsilon(|\tau|, |\tau_2|)[\gamma^2(\tau) + s, [\gamma(\tau_1) + t, \beta(\tau_2) + m]_\theta]_\theta + c.p.(\tau + s, \tau_1 + t, \tau_2 + m) \\
= \ & \varepsilon(|\tau|, |\tau_2|)[\gamma^2(\tau) + s, [\gamma(\tau_1), \beta(\tau_2)]_{\mathcal{T}} + \theta(\beta\gamma^{-1}(\gamma(\tau_1)), \beta(\tau_2)]_\theta + c.p.(\tau + s, \tau_1 + t, \tau_2 + m) \\
= \ & \varepsilon(|\tau|, |\tau_2|)[\gamma^2(\tau), [\gamma(\tau_1), \beta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}} + \varepsilon(|\tau|, |\tau_2|)\theta(\beta\gamma(\tau), [\gamma(\tau_1), \beta(\tau_2)]_{\mathcal{T}}) + c.p.(\tau, \tau_1, \tau_2).
\end{aligned}$$

A necessary and sufficient condition for $[\cdot,\cdot]_\theta$ to satisfy the Γ-graded BiHom-Jacobi identity is that

$$\varepsilon(|\tau|, |\tau_2|)\theta(\beta\gamma(\tau), [\gamma(\tau_1), \beta(\tau_2)]_{\mathcal{T}}) + \varepsilon(|\tau_1|, |\tau|)\theta(\beta\gamma(\tau_1), [\gamma(\tau_2), \beta(\tau)]_{\mathcal{T}})$$

$$+\varepsilon(|\tau_2|,|\tau_1|)\theta(\beta\gamma(\tau_2),[\gamma(\tau),\beta(\tau_1)]_{\mathcal{T}})=0,$$

which is equivalent to the condition $d_T\theta = 0$. Indeed,

$$\begin{aligned}
& d_T^2\theta(\tau,\tau_1,\tau_2)\\
= & -\theta([\beta^{-1}\gamma(\tau),\tau_1]_{\mathcal{T}},\gamma(\tau_2))+\varepsilon(|\tau_1|,|\tau_2|)\delta\theta([\beta^{-1}\gamma(\tau),\tau_2]_{\mathcal{T}},\gamma(\tau_1))\\
& -\varepsilon(|\tau|,|\tau_1|+|\tau_2|)\theta([\beta^{-1}\gamma(\tau_1),\tau_2]_{\mathcal{T}},\gamma(\tau)))\\
= & -\varepsilon(|\tau|,|\tau_2|)(\varepsilon(|\tau|,|\tau_2|)\theta([\beta^{-1}\gamma(\tau),\tau_1]_{\mathcal{T}},\gamma(\tau_2))+\varepsilon(|\tau_1|,|\tau_2|)\theta([\beta^{-1}\gamma(\tau_2),\tau]_{\mathcal{T}},\gamma(\tau_1))\\
& +\varepsilon(|\tau|,|\tau_1|)\theta([\beta^{-1}\gamma(\tau_1),\tau_2]_{\mathcal{T}},\gamma(\tau)))\\
= & -\varepsilon(|\tau|,|\tau_2|)(\varepsilon(|\tau|,|\tau_2|)(-\delta\varepsilon(|\tau_2|,|\tau|+|\tau_1|)\theta(\gamma(\tau_2),[\beta^{-1}\gamma(\tau),\tau_1]_{\mathcal{T}}))\\
& +\varepsilon(|\tau_1|,|\tau_2|)(-\delta\varepsilon(|\tau_1|,|\tau_2|+|\tau|)\theta(\gamma(\tau_1),[\beta^{-1}\gamma(\tau_2),\tau]_{\mathcal{T}}))\\
& +\varepsilon(|\tau|,|\tau_1|)(-\delta\varepsilon(|\tau|,|\tau_1|+|\tau_2|)\theta(\gamma(\tau),[\beta^{-1}\gamma(\tau_1),\tau_2]_{\mathcal{T}})))\\
= & \ \varepsilon(|\tau|,|\tau_2|)\delta(\varepsilon(|\tau_2|,|\tau_1|)\theta(\gamma(\tau_2),[\beta^{-1}\gamma(\tau),\tau_1]_{\mathcal{T}})+\varepsilon(|\tau_1|,|\tau|)\theta(\gamma(\tau_1),[\beta^{-1}\gamma(\tau_2),\tau]_{\mathcal{T}})\\
& +\varepsilon(|\tau|,|\tau_2|)\theta(\gamma(\tau),[\beta^{-1}\gamma(\tau_1),\tau_2]_{\mathcal{T}})).
\end{aligned}$$

With this, the proof is finished. □

Proposition 4.2. *For $\theta_1,\theta_2\in Z^2(\mathcal{W})$, if $\delta(\theta_1-\theta_2)$ is exact, the corresponding two central extensions $(\eta,[\cdot,\cdot]_{\theta_1},\delta,\beta_\eta,\gamma_\eta)$ and $(\eta,[\cdot,\cdot]_{\theta_2},\delta,\beta_\eta,\gamma_\eta)$ are isomorphic.*

Proof. Set $\theta_1-\theta_2=\delta d_T f$, $f\in C^1_{\beta,\gamma}(\mathcal{W})$. Thus, we have

$$\begin{aligned}
& \theta_1(\beta\gamma^{-1}(\tau),\tau_1)-\theta_2(\beta\gamma^{-1}(\tau),\tau_1)\\
= & \ \delta d_T f(\beta\gamma^{-1}(\tau),\tau_1)\\
= & -f([\beta^{-1}\gamma\circ\beta\gamma^{-1}(\tau),\tau_1])\\
= & -f([\tau,\tau_1]_T).
\end{aligned}$$

Define $f_\eta:\eta\to\eta$ by

$$f_\eta(\tau+s)=\tau+s+f(\tau).$$

Assume $(\tau,s+f(\tau))=0$. This implies $\tau=0$ and consequently $s=0$. Therefore, $f_\eta(0,0)=0$, proving that f_η is injective. Next, for any element (τ,τ_1) in the target space, $\tau_1-f(\tau)$ is a well-defined element. Hence, f_η is also surjective. Since f_η is both injective and surjective, it is bijective. More precisely, this immediately implies that

$$f_\eta\circ\beta_\eta(\tau+s)=f_\eta(\beta(\tau)+s)=\beta(\tau)+s+f(\beta(\tau))=\beta(\tau)+s+f(\tau).$$

On the other hand,

$$\beta_\eta\circ f_\eta(\tau+s)=\beta_\eta(\tau+s+f(\tau))=\beta(\tau)+s+f(\tau).$$

Thus, we get $f_\eta\circ\beta_\eta=\beta_\eta\circ f_\eta$. Similarly, $f_\eta\circ\gamma_\eta=\gamma_\eta\circ f_\eta$.
We also get that the following equation holds:

$$f_\eta[\tau+s,\tau_1+t]_{\theta_1}$$

$$\begin{aligned}
&= f_\eta([\tau,\tau_1]_{\mathcal{T}} + \theta_1(\beta\gamma^{-1}(\tau),\tau_1))\\
&= [\tau,\tau_1]_{\mathcal{T}}, \theta_1(\beta\gamma^{-1}(\tau),\tau_1) + f([\tau,\tau_1]_{\mathcal{T}}))\\
&= ([\tau,\tau_1]_{\mathcal{T}}, \theta_2(\beta\gamma^{-1}(\tau),\tau_1)\\
&= [f_\eta(\tau+s), f_\eta(\tau_1+t)]_{\theta_2}.
\end{aligned}$$

Hence, the conclusion is proved. □

5. The adjoint representation on BiHom-δ-Jordan Lie color algebras

Suppose $(\mathcal{T},[\cdot,\cdot]_{\mathcal{T}},\delta,\eta,\gamma)$ is a regular BiHom-δ-Jordan Lie color algebra. The adjoint representation on a BiHom-δ-Jordan Lie color algebra, when considering $\mathcal{T}$ acting on itself via the bracket concerning to the morphism η,γ, exhibits a noteworthy result: its non-uniqueness. This will be demonstrated in the subsequent discussion.

Definition 5.1. *The $\eta^s\gamma^t$-adjoint representation on the regular BiHom-δ-Jordan Lie color algebra $(\mathcal{T},[\cdot,\cdot]_{\mathcal{T}},\delta,\eta,\gamma)$, denoted* $\mathrm{ad}_{s,t}$, *is given for any integers s,t and all $\tau,\tau_1\in\mathcal{T}$ by*

$$\mathrm{ad}_{s,t}(\tau)(\tau_1) = \delta[\eta^s\gamma^t(\tau),\tau_1]_{\mathcal{T}},\quad \forall\tau,\tau_1\in\mathcal{T}.$$

Lemma 5.2. *In the notation established above, we obtain*

$$\mathrm{ad}_{s,t}(\eta(\tau))\circ\eta = \eta\circ\mathrm{ad}_{s,t}(\tau),$$

$$\mathrm{ad}_{s,t}(\gamma(\tau))\circ\gamma = \gamma\circ\mathrm{ad}_{s,t}(\tau),$$

$$\mathrm{ad}_{s,t}([\gamma(\tau),\tau_1]_{\mathcal{T}})\circ\gamma = \mathrm{ad}_{s,t}(\eta\gamma(\tau))\circ\mathrm{ad}_{s,t}(\tau_1) - \delta\varepsilon(|\tau|,|\tau_1|)\mathrm{ad}_{s,t}(\gamma(\tau_1))\circ\mathrm{ad}_{s,t}(\eta(\tau)).$$

This shows that the map $\mathrm{ad}_{s,t}$ *is given.*

Proof. The fact is derived from

$$\begin{aligned}
\mathrm{ad}_{s,t}(\eta(\tau))(\eta(\tau_1)) &= \delta[\eta^{s+1}\gamma^t(\tau),\eta(\tau_1)]_{\mathcal{T}}\\
&= \eta(\delta[\eta^s\gamma^t(\tau),\tau_1]_{\mathcal{T}})\\
&= \eta\circ\mathrm{ad}_{s,t}(\tau)(\tau_1).
\end{aligned}$$

Similarly,

$$\mathrm{ad}_{s,t}(\gamma(\tau))\circ\gamma = \gamma\circ\mathrm{ad}_{s,t}(\tau).$$

Notice that $\forall\tau,\tau_1\in\mathcal{T}$, thus the skew-symmetry yields

$$\begin{aligned}
\mathrm{ad}_{s,t}(\tau)(\tau_1) &= \delta[\eta^s\gamma^t(\tau),\tau_1]_{\mathcal{T}}\\
&= \delta[\gamma(\eta^s\gamma^{t-1}(\tau),\eta(\eta^{-1}(\tau_1))]_{\mathcal{T}}\\
&= -\varepsilon(|\tau|,|\tau_1|)[\eta^{-1}\gamma(\tau_1),\eta^{s+1}\gamma^{t-1}(\tau)]_{\mathcal{T}}.
\end{aligned}$$

On the one hand, we get

$$\mathrm{ad}_{s,t}([\gamma(\tau),\tau_1]_{\mathcal{T}})\circ\gamma(\tau_2) = \mathrm{ad}_{s,t}([\gamma(\tau),\tau_1]_{\mathcal{T}})(\gamma(\tau_2))$$

$$
\begin{aligned}
&= -\varepsilon(|\tau_2|, |\tau| + |\tau_1|)[\eta^{-1}\gamma(\gamma(\tau_2)), \eta^{s+1}\gamma^{t-1}([\gamma(\tau), \tau_1])]_{\mathcal{T}} \\
&= -\varepsilon(|\tau_2|, |\tau| + |\tau_1|)[\eta^{-1}\gamma^2(\tau_2), [\eta^{s+1}\gamma^t(\tau), \eta^{s+1}\gamma^{t-1}(\tau_1)]_{\mathcal{T}}]_{\mathcal{T}}.
\end{aligned}
$$

On the other hand, we obtain

$$
\begin{aligned}
& \mathrm{ad}_{s,t}(\eta\gamma(\tau)) \circ \mathrm{ad}_{s,t}(\tau_1)(\tau_2) - \delta\varepsilon(|\tau|, |\tau_1|)\mathrm{ad}_{s,t}(\gamma(\tau_1)) \circ \mathrm{ad}_{s,t}(\eta(\tau))(\tau_2) \\
= \;& \mathrm{ad}_{s,t}(\eta\gamma(\tau))(-\varepsilon(|\tau_2|, |\tau_1|)[\eta^{-1}\gamma(\tau_2), \eta^{s+1}\gamma^{t-1}(\tau_1)]]) \\
& -\delta\varepsilon(|\tau|, |\tau_1|)\mathrm{ad}_{s,t}(\gamma(\tau_1))(-\varepsilon(|\tau|, |\tau_2|)[\eta^{-1}\gamma(\tau_2), \eta^{s+2}\gamma^{t-1}(\tau)]) \\
= \;& \varepsilon(|\tau_1|, |\tau_2|)\varepsilon(|\tau|, |\tau_2| + |\tau_1|)[\eta^{-1}\gamma[\eta^{-1}\gamma(\tau_2), \eta^{s+1}\gamma^{t-1}(\tau_1)], \eta^{s+1}\gamma^{t-1}(\eta\gamma(\tau))] \\
& -\delta\varepsilon(|\tau_1|, |\tau|)\varepsilon(|\tau|, |\tau_2|)\varepsilon(|\tau_1|, |\tau| + |\tau_2|)[\eta^{-1}\gamma[\eta^{-1}\gamma(\tau_2), \eta^{s+2}\gamma^{t-1}(\tau)], \eta^{s+1}\gamma^{t-1}\gamma(\tau_1)] \\
= \;& \varepsilon(|\tau_1|, |\tau_2|)\varepsilon(|\tau|, |\tau_2| + |\tau_1|)[\gamma[\eta^{-2}\gamma(\tau_2), \eta^{s}\gamma^{t-1}(\tau_1)], \eta^{s+2}\gamma^{t}(\tau)] \\
& -\delta\varepsilon(|\tau_1|, |\tau_2|)\varepsilon(|\tau|, |\tau_2|)[\gamma[\eta^{-2}\gamma(\tau_2), \eta^{s+1}\gamma^{t-1}(\tau)], \quad \eta^{s+1}\gamma^{t}(\tau_1)] \\
= \;& -\delta\varepsilon(|\tau_1|, |\tau_2|)\varepsilon(|\tau|, |\tau_2| + |\tau_1|)\varepsilon(|\tau|, |\tau_2| + |\tau_1|)[\gamma\eta^{s+1}\gamma^{t}(\tau), \eta[\eta^{-2}\gamma(\tau_2), \eta^{s}\gamma^{t-1}(\tau_1)]] \\
& +\varepsilon(|\tau_1|, |\tau_2|)\varepsilon(|\tau|, |\tau_2|)\varepsilon(|\tau_1|, |\tau_2| + |\tau|)[\gamma\eta^{s}\gamma^{t}(\tau_1), \eta[\eta^{-2}\gamma(\tau_2), \eta^{s+1}\gamma^{t-1}(\tau)]] \\
= \;& -\delta\varepsilon(|\tau_2|, |\tau_1|)[\eta^{s+1}\gamma^{t+1}(\tau), [\eta^{-1}\gamma(\tau_2), \eta^{s+1}\gamma^{t-1}(\tau_1)]] \\
& +\varepsilon(|\tau_1|, |\tau|)\varepsilon(|\tau|, |\tau_2|)[\eta^{s}\gamma^{t+1}(\tau_1), [\eta^{-1}\gamma(\tau_2), \eta^{s+2}\gamma^{t-1}(\tau)]] \\
= \;& \varepsilon(|\tau_2|, |\tau_1|)\varepsilon(|\tau_2|, |\tau_1|)[\eta^{s+1}\gamma^{t+1}(\tau), [\eta^{s}\gamma^{t}(\tau_1), \tau_2]] \\
& +\varepsilon(|\tau_1|, |\tau|)\varepsilon(|\tau|, |\tau_2|)[\eta^{s}\gamma^{t+1}(\tau_1), [\eta^{-1}\gamma(\tau_2), \eta^{s+2}\gamma^{t-1}(\tau)]] \\
= \;& [\gamma^2(\eta^{s+1}\gamma^{t-1}(\tau)), [\gamma(\eta^{s}\gamma^{t-1}(\tau_1)), \eta(\eta^{-1}(\tau_2)]] \\
& +\varepsilon(|\tau_1|, |\tau|)\varepsilon(|\tau|, |\tau_2|)[\gamma^2(\eta^{s}\gamma^{t-1}(\tau_1)), [\gamma(\eta^{-1}(\tau_2)), \eta(\eta^{s+1}\gamma^{t-1}(\tau))]] \\
= \;& -\varepsilon(|\tau_1|, |\tau_2|)\varepsilon(|\tau|, |\tau_2|)[\gamma^2(\eta^{-1}(\tau_2)), [\gamma(\eta^{s+1}\gamma^{t-1}(\tau)), \eta(\eta^{s}\gamma^{t-1}(\tau_1))]] \\
= \;& -\varepsilon(|\tau_1|, |\tau_2|)\varepsilon(|\tau|, |\tau_2|)[\eta^{-1}\gamma^2(\tau_2), [\eta^{s+1}\gamma^{t}(\tau), \eta^{s+1}\gamma^{t-1}(\tau_1)]].
\end{aligned}
$$

This finishes the proof. □

By definition, the set of k-BiHom-cochains on $\mathcal{T}$ regarding coefficients in $\mathcal{T}$ is

$$C^k_{\eta,\gamma}(\mathcal{T};\mathcal{T}) = \{g \in C^k(\mathcal{T};\mathcal{T}) | \eta \circ g = g \circ \eta, \gamma \circ g = g \circ \gamma\}.$$

Specifically, the set of 0-BiHom-cochains is defined as:

$$C^0_{\eta}(\mathcal{T};\mathcal{T}) = \{\tau \in \mathcal{T} | \eta(\tau) = \tau, \gamma(\tau) = \tau\}.$$

Associated to the $\eta^s\gamma^t$-adjoint representation, the corresponding operator

$$d_{s,t} : C^k_{\eta,\gamma}(\mathcal{T};\mathcal{T}) \to C^{k+1}_{\eta,\gamma}(\mathcal{T};\mathcal{T})(k = 1, 2)$$

is given by

$$
\begin{aligned}
& d_{s,t}g(\tau, \tau_1) \\
= \;& \varepsilon(|g|, |\tau|)\delta[\eta^{1+s}\gamma^t(\tau), g(\tau_1)] - \varepsilon(|g| + |\tau|, |\tau_1|)[\eta^{1+s}\gamma^t(\tau_1), g(\tau)] \\
& -\delta g([\eta^{-1}\gamma^t(\tau), \tau_1]),
\end{aligned}
$$

$$\begin{aligned}
& d_{s,t}g(\tau,\tau_1,\tau_2)\\
= \;& \delta\varepsilon(|g|,|\tau|)[\eta^{1+s}\gamma^{t+1}(\tau),g(\tau_1,\tau_2)]-\varepsilon(|g|+|\tau|,|\tau_1|)[\eta^{1+s}\gamma^{t+1}(\tau_1),g(\tau,\tau_2)]\\
& +\delta\varepsilon(|g|+|\tau|+|\tau_1|,|\tau_2|)[\eta^{1+s}\gamma^{t+1}(\tau_2),g(\tau,\tau_1)]-g([\eta^{-1}\gamma(\tau),\tau_1],\gamma(\tau_2))\\
& +\delta\varepsilon(|\tau|+|\tau_1|,|\tau_2|)\varepsilon(|\tau|,|\tau_2|)g([\eta^{-1}\gamma(\tau),\tau_2],\gamma(\tau_1))\\
& -\varepsilon(|\tau|,|\tau_1|)\varepsilon(|\tau|+|\tau_1|,|\tau_2|)\varepsilon(|\tau_1|,|\tau_2|)g([\eta^{-1}\gamma(\tau_1),\tau_2],\gamma(\tau)).
\end{aligned}$$

Definition 5.3. *Let k and l be nonnegative integers. An $\eta^k\gamma^l$-derivation of the multiplicative BiHom-δ-Jordan Lie color algebra $(\mathcal{T},[\cdot,\cdot]_{\mathcal{T}},\delta,\eta,\gamma)$ is a linear map $D:\mathcal{T}\to\mathcal{T}$, with*

$$D\circ\eta=\eta\circ D, \tag{5.1}$$

$$D\circ\gamma=\gamma\circ D, \tag{5.2}$$

and

$$D[\tau,\tau_1]_{\mathcal{T}}=\delta^k[D(\tau),\eta^k\gamma^l(\tau_1)]_{\mathcal{T}}+\delta^k\varepsilon(|D|,|\tau|)[\eta^k\gamma^l(\tau),D(\tau_1)]_{\mathcal{T}},\ \forall\tau,\tau_1\in\mathcal{T}. \tag{5.3}$$

For a regular BiHom-δ-Jordan Lie color algebra, $\eta^{-k}\gamma^{-l}$-derivations can be defined similarly.

Proposition 5.4. *Consider the $\eta^s\gamma^t$-adjoint representations $\mathrm{ad}_{s,t}$ on the regular BiHom-δ-Jordan Lie color algebra $(\mathcal{T},[\cdot,\cdot]_{\mathcal{T}},\delta,\eta,\gamma)$, satisfying $\delta^{s+1}=1$, with $D\in C^1_{\eta,\gamma}(\mathcal{T},\mathcal{T})$ as a 1-cocycle being equivalent to D as an $\eta^{s+2}\gamma^{t-1}$-derivation, i.e., $D\in\mathrm{Der}_{\eta^{s+2}\gamma^{t-1}}(\mathcal{T})$.*

Proof. This fact is an immediate result of the concept of $d_{s,t}$. Indeed, a necessary and sufficient condition for D to be closed is that

$$\begin{aligned}
d_{s,t}(D)(a,b) &= \delta\varepsilon(|D|,|a|)[\eta^{s+1}\gamma^t(a),D(b)]_{\mathcal{T}}\\
&\quad -\varepsilon(|D|+|a|,|b|)[\eta^{s+1}\gamma^t(b),D(a)]_{\mathcal{T}}\\
&\quad -\delta D[\eta^{-1}\gamma(a),b]_{\mathcal{T}}\\
&= 0.
\end{aligned}$$

In other words,

$$\begin{aligned}
D[\eta^{-1}\gamma(a),b]_{\mathcal{T}} &= \varepsilon(|D|,|a|)[\eta^{s+1}\gamma^t(a),D(b)]_{\mathcal{T}}-\delta\varepsilon(|D|+|a|,|b|)[\eta^{s+1}\gamma^t(b),D(a)]_{\mathcal{T}}\\
&= \delta^{s+1}([D(\eta^{-1}\gamma(a)),\eta^{s+2}\gamma^{t-1}(b)]_{\mathcal{T}}\\
&\quad +\delta^{s+1}\varepsilon(|D|,|a|)[\eta^{s+2}\gamma^{t-1}\eta^{-1}\gamma(a),D(b)]_{\mathcal{T}}).
\end{aligned}$$

It follows that D is an $\eta^{s+2}\gamma^{t-1}$-derivation. □

Set $\psi\in C^2_{\eta,\gamma}(\mathcal{T};\mathcal{T})$. This shows that

$$\psi(\tau,\tau_1)=-\varepsilon(|\tau|,|\tau_1|)\psi(\tau_1,\tau).$$

We are interested in a family of bilinear operators that depend on a parameter t.

$$[\tau,\tau_1]_t=[\tau,\tau_1]_{\mathcal{T}}+t\psi(\tau,\tau_1). \tag{5.4}$$

The commutativity of ψ with η and γ implies that η and γ are morphisms concerning $[\cdot,\cdot]_t$ for every t. When this occurs, i.e., when all $[\cdot,\cdot]_t$ yield regular BiHom-δ-Jordan Lie color algebra structures on $(\mathcal{T},\delta,\eta,\gamma)$, we say that ψ defines a deformation of $(\mathcal{T},[\cdot,\cdot]_{\mathcal{T}},\delta,\eta,\gamma)$.

This follows because $[\cdot,\cdot]_t$ is skew-symmetric

$$[\gamma(\tau_1),\eta(\tau)]_t = [\gamma(\tau_1),\eta(\tau)]_{\mathcal{T}} + t\psi(\gamma(\tau_1),\eta(\tau)),$$

and

$$[\gamma(\tau),\eta(\tau_1)]_t = [\gamma(\tau),\eta(\tau_1)]_{\mathcal{T}} + t\psi(\gamma(\tau),\eta(\tau_1)),$$

thus,

$$[\gamma(\tau_1),\eta(\tau)]_t = -\varepsilon(|\tau|,|\tau_1|)[\gamma(\tau),\eta(\tau_1)]_t$$

if and only if

$$\psi(\gamma(\tau_1),\eta(\tau)) = -\varepsilon(|\tau|,|\tau_1|)\psi(\gamma(\tau),\eta(\tau_1)).$$

By computing the BiHom-Jacobi identity of $[\cdot,\cdot]_t$,

$$\begin{aligned}
&\varepsilon(|\tau|,|\tau_2|)[\gamma^2(\tau),[\gamma(\tau_1),\eta(\tau_2)]_t]_t + c.p.(\tau,\tau_1,\tau_2)\\
=\;&\varepsilon(|\tau|,|\tau_2|)[\gamma^2(\tau),[\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_t + \varepsilon(|\tau|,|\tau_2|)[\gamma^2(\tau),t\psi(\gamma(\tau_1),\eta(\tau_2))]_t + c.p.(\tau,\tau_1,\tau_2)\\
=\;&\varepsilon(|\tau|,|\tau_2|)[\gamma^2(\tau),[\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}} + \varepsilon(|\tau|,|\tau_2|)t\psi(\gamma^2(\tau),[\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}})\\
&+\varepsilon(|\tau|,|\tau_2|)[\gamma^2(\tau),t\psi(\gamma(\tau_1),\eta(\tau_2))]_{\mathcal{T}} + \varepsilon(|\tau|,|\tau_2|)t\psi(\gamma^2(\tau),t\psi(\gamma(\tau_1),\eta(\tau_2)))\\
&+c.p.(\tau,\tau_1,\tau_2)\\
=\;&0.
\end{aligned}$$

Provided that this holds, the following conditions are satisfied:

$$\varepsilon(|\tau|,|\tau_2|)\psi(\gamma^2(\tau),\psi(\gamma(\tau_1),\eta(\tau_2))) + c.p.(\tau,\tau_1,\tau_2) = 0, \tag{5.5}$$

$$\varepsilon(|\tau|,|\tau_2|)(\psi(\gamma^2(\tau),[\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}) + [\gamma^2(\tau),\psi(\gamma(\tau_1),\eta(\tau_2))_{\mathcal{T}}]_{\mathcal{T}}) + c.p.(\tau,\tau_1,\tau_2) = 0. \tag{5.6}$$

Obviously, (5.5) shows that ψ necessarily endows $\mathcal{T}$ with a BiHom-δ-Jordan Lie color algebra structure. From (5.6), we see that ψ remains invariant under the $\eta^{-1}\gamma$-adjoint representation $\mathrm{ad}_{-1,1}$, i.e., $d_{-1,1}\psi = 0$.

$$\begin{aligned}
&d_{-1,1}\psi(\tau,\tau_1,\tau_2)\\
=\;&\delta[\gamma^2(\tau),\psi(\tau_1,\tau_2)]_{\mathcal{T}} - \varepsilon(|\tau|,|\tau_1|)[\gamma^2(\tau_1),\psi(\tau,\tau_2)]_{\mathcal{T}}\\
&+\delta\varepsilon(|\tau|+|\tau_1|,|\tau_2|)[\gamma^2(\tau_2),\psi(\tau,\tau_1)]_{\mathcal{T}} - \psi([\eta^{-1}\gamma(\tau),\tau_1]_{\mathcal{T}},\gamma(\tau_2))\\
&+\delta\varepsilon(|\tau|+|\tau_1|,|\tau_2|)\varepsilon(|\tau|,|\tau_2|)\psi([\eta^{-1}\gamma(\tau),\tau_2]_{\mathcal{T}},\gamma(\tau_1))\\
&-\varepsilon(|\tau|,|\tau_1|)\varepsilon(|\tau|+|\tau_1|,|\tau_2|)\varepsilon(|\tau_1|,|\tau_2|)\psi([\eta^{-1}\gamma(\tau_1),\tau_2]_{\mathcal{T}},\gamma(\tau))\\
=\;&\delta[\gamma^2(\tau),\psi(\tau_1,\tau_2)]_{\mathcal{T}} + \delta\varepsilon(|\tau|,|\tau_1|)\varepsilon(|\tau|,|\tau_2|)[\gamma^2(\tau_1),\psi(\tau_2,\tau)]_{\mathcal{T}}\\
&+\delta\varepsilon(|\tau|+|\tau_1|,|\tau_2|)[\gamma^2(\tau_2),\psi(\tau,\tau_1)]_{\mathcal{T}} + \delta\varepsilon(|\tau_2|,|\tau|+|\tau_1|)\psi(\gamma(\tau_2),[\eta^{-1}\gamma(\tau),\tau_1]_{\mathcal{T}})\\
&+\delta\varepsilon(|\tau|+|\tau_1|,|\tau_2|)\varepsilon(|\tau|,|\tau_2|)\varepsilon(|\tau_1|,|\tau|+|\tau_2|)\varepsilon(|\tau|,|\tau_2|)\psi(\gamma(\tau_1),[\eta^{-1}\gamma(\tau_2),\tau]_{\mathcal{T}})\\
&+\delta\varepsilon(|\tau|,|\tau_1|)\varepsilon(|\tau|+|\tau_1|,|\tau_2|)\varepsilon(|\tau_1|,|\tau_2|)\varepsilon(|\tau|,|\tau_1|+|\tau_2|)\psi(\gamma(\tau),[\eta^{-1}\gamma(\tau_1),\tau_2]_{\mathcal{T}})
\end{aligned}$$

$$
\begin{aligned}
&= \delta\varepsilon(|\tau|,|\tau_2|)(\varepsilon(|\tau|,|\tau_2|)[\gamma^2(\tau),\psi(\tau_1,\tau_2)]_{\mathcal{T}} + \varepsilon(|\tau|,|\tau_2|)\psi(\gamma(\tau),[\eta^{-1}\gamma\gamma(\tau_1),\tau_2]_{\mathcal{T}}) \\
&\quad + c.p.(\tau,\tau_1,\tau_2)) \\
&= 0.
\end{aligned}
$$

A deformation is termed trivial when one can find a linear operator $\mathcal{B} \in C^1_{\eta,\gamma}(\mathcal{T};\mathcal{T})$ such that, setting $\mathcal{B}_t = \mathrm{id} + t\mathcal{B}$, the following holds:

$$
\mathcal{B}_t[\tau,\tau_1]_t = [\mathcal{B}_t(\tau),\mathcal{B}_t(\tau_1)]_{\mathcal{T}}. \tag{5.7}
$$

Definition 5.5. *A BiHom-Nijenhuis operator is defined as an even linear operator $\mathcal{B} \in C^1_{\eta,\gamma}(\mathcal{T};\mathcal{T})$ satisfying:*

$$
[\mathcal{B}\tau,\mathcal{B}\tau_1]_{\mathcal{T}} = \mathcal{B}[\tau,\tau_1]_{\mathcal{B}}, \tag{5.8}
$$

where

$$
[\tau,\tau_1]_{\mathcal{B}} := [\mathcal{B}\tau,\tau_1]_{\mathcal{T}} + [\tau,\mathcal{B}\tau_1]_{\mathcal{T}} - \mathcal{B}[\tau,\tau_1]_{\mathcal{T}}. \tag{5.9}
$$

Theorem 5.6. *Suppose $\mathcal{B} \in C^1_{\eta,\gamma}(\mathcal{T};\mathcal{T})$ is a BiHom-Nijenhuis operator. It follows that defining $\psi(\tau,\tau_1) = \delta d_{-1,1}\mathcal{B}(\tau,\tau_1) = [\tau,\tau_1]_{\mathcal{B}}$ produces a deformation of $(\mathcal{T},[\cdot,\cdot]_{\mathcal{T}},\eta,\gamma)$. One finds that this deformation is, in fact, trivial.*

Proof. Given that $\psi = d_{-1,1}\mathcal{B}$, the relation $d_{-1,1}\psi = 0$ holds. Consequently, establishing that ψ creates a deformation reduces to checking the BiHom-Jacobi identity for ψ. Given the explicit expression for ψ, we find that

$$
\begin{aligned}
&\varepsilon(|\tau|,|\tau_2|)\psi(\gamma^2(\tau),\psi(\gamma(\tau_1),\eta(\tau_2))) + c.p.(\tau,\tau_1,\tau_2) \\
&= \varepsilon(|\tau|,|\tau_2|)[\gamma^2(\tau),[\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{B}}]_{\mathcal{B}} + c.p.(\tau,\tau_1,\tau_2) \\
&= \varepsilon(|\tau|,|\tau_2|)([\gamma^2(\tau),[\mathcal{B}\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}} + [\gamma(\tau_1),\mathcal{B}\eta(\tau_2)]_{\mathcal{T}} - \mathcal{B}[\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{B}}) + c.p.(\tau,\tau_1,\tau_2) \\
&= \varepsilon(|\tau|,|\tau_2|)([\gamma^2(\tau),[\mathcal{B}\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{B}} + [\gamma^2(\tau),[\gamma(\tau_1),\mathcal{B}\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{B}} \\
&\quad -[\gamma^2(\tau),\mathcal{B}[\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{B}}) + c.p.(\tau,\tau_1,\tau_2) \\
&= \varepsilon(|\tau|,|\tau_2|)([\mathcal{B}\gamma^2(\tau),[\mathcal{B}\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}} + [\gamma^2(\tau),\mathcal{B}[\mathcal{B}\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}} \\
&\quad -\mathcal{B}[\gamma^2(\tau),[\mathcal{B}\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}} + [\mathcal{B}\gamma^2(\tau),[\gamma(\tau_1),\mathcal{B}\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}} \\
&\quad +[\gamma^2(\tau),\mathcal{B}[\gamma(\tau_1),\mathcal{B}\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}} - \mathcal{B}[\gamma^2(\tau),[\gamma(\tau_1),\mathcal{B}\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}} \\
&\quad -[\mathcal{B}\gamma^2(\tau),\mathcal{B}[\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}} - [\gamma^2(\tau),\mathcal{B}^2[\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}} \\
&\quad +\mathcal{B}[\gamma^2(\tau),\mathcal{B}[\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}}) + c.p.(\tau,\tau_1,\tau_2).
\end{aligned}
$$

Through a straightforward calculation, we obtain

$$
\begin{aligned}
&[\gamma^2(\tau),\mathcal{B}[\mathcal{B}\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}} + [\gamma^2(\tau),\mathcal{B}[\gamma(\tau_1),\mathcal{B}\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}} - [\gamma^2(\tau),\mathcal{B}^2[\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}} \\
&= [\gamma^2(\tau),\mathcal{B}([\mathcal{B}\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}} + [\gamma(\tau_1),\mathcal{B}\eta(\tau_2)]_{\mathcal{T}} - \mathcal{B}[\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}})]_{\mathcal{T}} \\
&= [\gamma^2(\tau),\mathcal{B}[\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{B}}]_{\mathcal{T}},
\end{aligned}
$$

then

$$
\varepsilon(|\tau|,|\tau_2|)\psi(\gamma^2(\tau),\psi(\gamma(\tau_1),\eta(\tau_2)) + c.p.(\tau,\tau_1,\tau_2)
$$

$$
\begin{aligned}
= \quad & \varepsilon(|\tau|,|\tau_2|)([\mathcal{B}\gamma^2(\tau),[\mathcal{B}\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}}-\mathcal{B}[\gamma^2(\tau),[\mathcal{B}\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}}\\
& +[\mathcal{B}\gamma^2(\tau),[\gamma(\tau_1),\mathcal{B}\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}}-\mathcal{B}[\gamma^2(\tau),[\gamma(\tau_1),\mathcal{B}\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}}\\
& -[\mathcal{B}\gamma^2(\tau),\mathcal{B}[\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}}+\mathcal{B}[\gamma^2(\tau),\mathcal{B}[\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}}\\
& +[\gamma^2(\tau),\mathcal{B}[\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{B}}]_{\mathcal{T}})\\
& +\varepsilon(|\tau|,|\tau_1|)([\mathcal{B}\gamma^2(\tau_1),[\mathcal{B}\gamma(\tau_2),\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}}-\mathcal{B}[\gamma^2(\tau_1),[\mathcal{B}\gamma(\tau_2),\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}}\\
& +[\mathcal{B}\gamma^2(\tau_1),[\gamma(\tau_2),\mathcal{B}\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}}-\mathcal{B}[\gamma^2(\tau_1),[\gamma(\tau_2),\mathcal{B}\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}}\\
& -[\mathcal{B}\gamma^2(\tau_1),\mathcal{B}[\gamma(\tau_2),\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}}+\mathcal{B}[\gamma^2(\tau_1),\mathcal{B}[\gamma(\tau_2),\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}}\\
& +[\gamma^2(\tau_1),\mathcal{B}[\gamma(\tau_2),\eta(\tau)]_{\mathcal{B}}]_{\mathcal{T}})\\
& +\varepsilon(|\tau_1|,|\tau_2|)([\mathcal{B}\gamma^2(\tau_2),[\mathcal{B}\gamma(\tau),\eta(\tau_1)]_{\mathcal{T}}]_{\mathcal{T}}-\mathcal{B}[\gamma^2(\tau_2),[\mathcal{B}\gamma(\tau),\eta(\tau_1)]_{\mathcal{T}}]_{\mathcal{T}}\\
& +[\mathcal{B}\gamma^2(\tau_2),[\gamma(\tau),\mathcal{B}\tau_1]_{\mathcal{T}}]_{\mathcal{T}}-\mathcal{B}[\gamma^2(\tau_2),[\gamma(\tau),\mathcal{B}\tau_1]_{\mathcal{T}}]_{\mathcal{T}}\\
& -[\mathcal{B}\gamma^2(\tau_2),\mathcal{B}[\gamma(\tau),\eta(\tau_1)]_{\mathcal{T}}]_{\mathcal{T}}+\mathcal{B}[\gamma^2(\tau_2),\mathcal{B}[\gamma(\tau),\eta(\tau_1)]_{\mathcal{T}}]_{\mathcal{T}}\\
& +[\gamma^2(\tau_2),\mathcal{B}[\gamma(\tau),\eta(\tau_1)]_{\mathcal{B}}]_{\mathcal{T}})\\
= \quad & \varepsilon(|\tau|,|\tau_2|)[\mathcal{B}\gamma^2(\tau),[\mathcal{B}\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}}+\varepsilon(|\tau|,|\tau_1|)[\mathcal{B}\gamma^2(\tau_1),[\gamma(\tau_2),\mathcal{B}\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}}\\
& +\varepsilon(|\tau_2|,|\tau_1|)[\gamma^2(\tau_2),\mathcal{B}([\gamma(\tau),\eta(\tau_1)]_{\mathcal{B}})]_{\mathcal{T}}+c.p.(\tau,\tau_1,\tau_2)\\
& +\varepsilon(|\tau|,|\tau_1|)(\mathcal{B}[\gamma^2(\tau_1),\mathcal{B}[\gamma(\tau_2),\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}}-[\mathcal{B}\gamma^2(\tau_1),\mathcal{B}[\gamma(\tau_2),\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}})+c.p.(\tau,\tau_1,\tau_2)\\
& -\varepsilon(|\tau|,|\tau_2|)\mathcal{B}[\gamma^2(\tau),[\mathcal{B}\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}}-\varepsilon(|\tau_1|,|\tau_2|)\mathcal{B}[\gamma^2(\tau_2),[\gamma(\tau),\mathcal{B}\eta(\tau_1)]_{\mathcal{T}}]_{\mathcal{T}}\\
& +c.p.(\tau,\tau_1,\tau_2).
\end{aligned}
$$

Using the facts that $\mathcal{B}$ commutes with η,γ and that $\mathcal{T}$ satisfies the BiHom-Jacobi identity, we obtain

$$
\begin{aligned}
& \varepsilon(|\tau|,|\tau_2|)[\mathcal{B}\gamma^2(\tau),[\mathcal{B}\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}}+\varepsilon(|\tau|,|\tau_1|)[\gamma^2(\mathcal{B}\tau_1),[\gamma(\tau_2),\mathcal{B}\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}}\\
& +\varepsilon(|\tau_2|,|\tau_1|)[\gamma^2(\tau_2),[\mathcal{B}\gamma(\tau),\mathcal{B}\eta(\tau_1)]_{\mathcal{T}}]_{\mathcal{T}}\\
& =0.
\end{aligned}
$$

From the property of $\mathcal{B}$ being a BiHom-Nijenhuis operator, we get

$$
\begin{aligned}
& \varepsilon(|\tau|,|\tau_2|)[\mathcal{B}\gamma^2(\tau),[\mathcal{B}\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}}+\varepsilon(|\tau|,|\tau_1|)[\gamma^2(\mathcal{B}\tau_1),[\gamma(\tau_2),\mathcal{B}\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}}\\
& +\varepsilon(|\tau_1|,|\tau_2|)[\gamma^2(\tau_2),\mathcal{B}[\gamma(\tau),\eta(\tau_1)]_{\mathcal{B}}]_{\mathcal{T}}+c.p.(\tau,\tau_1,\tau_2)\\
& =0.
\end{aligned}
$$

The BiHom-Nijenhuis property of $\mathcal{B}$ also gives

$$
\begin{aligned}
& \varepsilon(|\tau|,|\tau_1|)(\mathcal{B}[\gamma^2(\tau_1),\mathcal{B}[\gamma(\tau_2),\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}}-[\mathcal{B}\gamma^2(\tau_1),\mathcal{B}[\gamma(\tau_2),\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}})+c.p.(\tau,\tau_1,\tau_2)\\
= \quad & \varepsilon(|\tau|,|\tau_1|)(\mathcal{B}[\gamma^2(\tau_1),\mathcal{B}[\gamma(\tau_2),\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}}-\mathcal{B}[\gamma^2(\tau_1),[\gamma(\tau_2),\eta(\tau)]_{\mathcal{T}}]_{\mathcal{B}})+c.p.(\tau,\tau_1,\tau_2)\\
= \quad & -\varepsilon(|\tau|,|\tau_1|)\mathcal{B}[\mathcal{B}\gamma^2(\tau_1),[\gamma(\tau_2),\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}}+\varepsilon(|\tau|,|\tau_1|)\mathcal{B}^2[\gamma^2(\tau_1),[\gamma(\tau_2),\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}})\\
& +c.p.(\tau,\tau_1,\tau_2).
\end{aligned}
$$

Thus, it is clear that

$$
\begin{aligned}
& \varepsilon(|\tau|,|\tau_1|)(\mathcal{B}[\gamma^2(\tau_1),\mathcal{B}[\gamma(\tau_2),\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}}-[\mathcal{B}\gamma^2(\tau_1),\mathcal{B}[\gamma(\tau_2),\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}})+c.p.(\tau,\tau_1,\tau_2)\\
= \quad & -\varepsilon(|\tau|,|\tau_1|)\mathcal{B}[\mathcal{B}\gamma^2(\tau_1),[\gamma(\tau_2),\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}}+c.p.(\tau,\tau_1,\tau_2).
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}
& \varepsilon(|\tau|,|\tau_2|)\psi(\gamma^2(\tau),\psi(\gamma(\tau_1),\eta(\tau_2)))+c.p.(\tau,\tau_1,\tau_2)\\
= & -\varepsilon(|\tau|,|\tau_1|)\mathcal{B}[\mathcal{B}\gamma^2(\tau_1),[\gamma(\tau_2),\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}}-\varepsilon(|\tau|,|\tau_2|)\mathcal{B}[\gamma^2(\tau),[\mathcal{B}\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}}\\
& -\varepsilon(|\tau_2|,|\tau_1|)\mathcal{B}[\gamma^2(\tau_2),[\gamma(\tau),\mathcal{B}\eta(\tau_1)]_{\mathcal{T}}]_{\mathcal{T}}+c.p.(\tau,\tau_1,\tau_2)\\
= & -\mathcal{B}(\varepsilon(|\tau|,|\tau_1|)[\gamma^2(\mathcal{B}\tau_1),[\gamma(\tau_2),\eta(\tau)]_{\mathcal{T}}]_{\mathcal{T}}+\varepsilon(|\tau|,|\tau_2|)[\gamma^2(\tau),[\mathcal{B}\gamma(\tau_1),\eta(\tau_2)]_{\mathcal{T}}]_{\mathcal{T}}\\
& +\varepsilon(|\tau_2|,|\tau_1|)[\gamma^2(\tau_2),[\gamma(\tau),\mathcal{B}\eta(\tau_1)]_{\mathcal{T}}]_{\mathcal{T}})+c.p.(\tau,\tau_1,\tau_2)\\
= & \ 0.
\end{aligned}$$

Thus, ψ creates a deformation of the BiHom-δ-Jordan Lie color algebra $(\mathcal{T},[\cdot,\cdot]_{\mathcal{T}},\eta,\gamma)$.

Setting $\mathcal{B}_t=\mathrm{id}+t\mathcal{B}$, it is obvious that

$$\begin{aligned}
\mathcal{B}_t[\tau,\tau_1]_t &= (\mathrm{id}+t\mathcal{B})([\tau,\tau_1]_{\mathcal{T}}+t\psi(\tau,\tau_1))\\
&= (\mathrm{id}+t\mathcal{B})([\tau,\tau_1]_{\mathcal{T}}+t[\tau,\tau_1]_{\mathcal{B}})\\
&= [\tau,\tau_1]_{\mathcal{T}}+t([\tau,\tau_1]_{\mathcal{B}}+\mathcal{B}[\tau,\tau_1]_{\mathcal{T}})+t^2\mathcal{B}[\tau,\tau_1]_{\mathcal{B}}.
\end{aligned}$$

Conversely, we obtain

$$\begin{aligned}
[\mathcal{B}_t(\tau),\mathcal{B}_t(\tau_1)]_{\mathcal{T}} &= [\tau+t\mathcal{B}\tau,\tau_1+t\mathcal{B}\tau_1]_{\mathcal{T}}\\
&= [\tau,\tau_1]_{\mathcal{T}}+t([\mathcal{B}\tau,\tau_1]_{\mathcal{T}}+[\tau,\mathcal{B}\tau_1]_{\mathcal{T}})+t^2[\mathcal{B}\tau,\mathcal{B}\tau_1]_{\mathcal{T}}.
\end{aligned}$$

The definition of $[\cdot,\cdot]_{\mathcal{B}}$ directly gives that

$$\mathcal{B}_t[\tau,\tau_1]_t=[\mathcal{B}_t(\tau),\mathcal{B}_t(\tau_1)]_{\mathcal{T}},$$

from which the triviality of the deformation follows. □

6. Conclusions

In this paper, we introduce the concept of BiHom-δ-Jordan Lie color algebras and give its representation. After characterizing the cohomology theory on multiplicative BiHom-δ-Jordan Lie color algebras, we further establish that representations of such algebras naturally yield the corresponding semidirect product constructions. Then, an investigation of the trivial representation for multiplicative BiHom-δ-Jordan Lie color algebras leads to the determination of the isomorphism classes of central extensions via the second cohomology group. In addition, using adjoint representations on the regular BiHom-δ-Jordan Lie color algebras, we show that D is a 1-cocycle and is equivalent to D as a derivation. Finally, we give a trivial deformation by the BiHom-Nijienhuis operator.

Author contributions

Jiaqi Li: Conceptualization, Funding acquisition, Writing–review and editing; Hongrui Yu: Validation, Writing–review and editing; Lili Ma: Methodology, Writing–original draft, Funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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