Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 20007–20028.
DOI:10.3934/math.2026812
Received: 13 May 2026
Revised: 28 June 2026
Accepted: 02 July 2026
Published: 08 July 2026

Research article

On the reciprocal augmented Sombor index

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Abstract: The reciprocal augmented Sombor (RASO) index is a recently introduced vertex-degree-based graph invariant whose mathematical properties have not yet been exclusively investigated. In this paper, we present the first comprehensive study of this index. We derive sharp bounds for the RASO index in terms of fundamental graph parameters, including the order, size, minimum degree, and maximum degree of a graph. We further characterize the graphs minimizing/maximizing the RASO index among all fixed-order (i) connected graphs, (ii) trees, and (iii) unicyclic graphs. In addition, we examine the effect of edge addition on the RASO index and establish conditions under which this operation increases its value.

Keywords: graph invariant; reciprocal augmented Sombor index; extremal problem; bound; tree (graph); unicyclic graph
Mathematics Subject Classification: 05C05, 05C07, 05C09, 05C35

1. Introduction

Topological indices are numerical invariants of graphs that capture structural information and play a fundamental role in mathematical chemistry, especially in chemical graph theory. They are widely used in quantitative structure-property relationships (QSPR) and quantitative structure-activity relationships (QSAR), where molecular properties are correlated with such indices; see, for example, [31, 32].

Among topological indices, those depending on vertex degrees have attracted considerable attention due to their simplicity and good predictive performance. Classical examples include the first Zagreb index [18, 20], the atom-bond connectivity index [13], and the Randić index [23, 28].

Currently, a well-focused and widely studied degree-based topological index is the Sombor index, introduced in [15]. For a given graph G possessing edge set $E(G)$, it is defined by

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}.$$

Here, d_v and d_u denote the degree of vertices v and u, respectively. (When there is a possibility of confusion about the graph G under consideration, we write $d_v(G)$ instead of d_v.) Owing to its simple definition and geometric background, the Sombor index has attracted considerable attention in both mathematical chemistry and graph theory (for example, see the reviews [24, 29]).

Following the success of the Sombor index, numerous modifications have been proposed, including the elliptic Sombor index [1, 19], Euler-Sombor index [16, 30], diminished Sombor index [2, 25], augmented Sombor index [12, 35], hyperbolic Sombor index [3, 9], and degree-ratio Sombor (DRSO) index [26]; a comprehensive list of Sombor-type indices is available in the recent survey [17].

We study in the present paper the reciprocal augmented Sombor (RASO) index, which appeared in [12]. It is defined for a graph G as

$$RASO(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u^2 + d_v^2}}.$$

This index can be viewed as a reciprocal analogue of the augmented Sombor (ASO) index [12] defined by

$$ASO(G) = \sum_{vu \in E(G)} \sqrt{\frac{d_v^2 + d_u^2}{d_v + d_u - 2}}.$$

A natural question is whether the reciprocal transformation used in the definition of the RASO index merely preserves the behavior (in the same or opposite direction) of the ASO index or leads to fundamentally different mathematical properties. Although the two indices are highly correlated on octane isomers (see [12]), they exhibit considerably different responses to graph operations. In particular, non-isolated edge addition always increases the ASO index (see [10]), whereas the RASO index may either increase or decrease, as discussed in Section 4. This phenomenon suggests that the RASO index possesses a richer structural sensitivity and provides the main motivation for the present study.

The study of extremal values of topological indices over various graph classes is a central topic in chemical graph theory. Another well-studied topic concerns inequalities and bounds for topological indices in terms of basic graph parameters such as order, size, minimum, and maximum degree. Furthermore, understanding how a topological index behaves under edge addition is of independent interest.

In this paper, we investigate several properties of the RASO index. We derive bounds for the RASO index in terms of standard graph parameters and related indices. We also determine graphs attaining the extremum values of the RASO index over the classes of fixed-order connected graphs, trees, and unicyclic graphs. In addition, we study the effect of edge addition on the RASO index and provide conditions under which this index increases. We provide an infinite family of graphs for which the RASO index decreases under edge addition.

The paper is organized as follows. In Section 2, we establish several bounds for the RASO index. Section 3 is devoted to extremal results for trees, connected graphs, and unicyclic graphs. In Section 4, we investigate the behavior of the RASO index under edge addition.

2. Bounds

This section is devoted to establishing a variety of bounds for the RASO index. We derive both lower and upper bounds expressed in terms of several standard graph invariants, such as size, order, maximum and minimum degrees, and related indices.

Theorem 2.1. *For every connected graph G of size $m \geq 3$,*

$$RASO(G) \leq \frac{m}{2}.$$

Equality holds if and only if $G \cong C_m$.

Proof. For every edge uv of G, by using the inequality

$$d_u^2 + d_v^2 \geq \frac{(d_u + d_v)^2}{2},$$

where the equality holds if and only if $d_v = d_u$, we get

$$\sqrt{\frac{d_u + d_v - 2}{d_u^2 + d_v^2}} \leq \sqrt{\frac{2(d_u + d_v - 2)}{(d_u + d_v)^2}}. \tag{2.1}$$

Since G is connected and its size is at least 3, every edge $uv \in E(G)$ satisfies $d_u + d_v \geq 3$. Also, the function g, defined as

$$g(t) = \frac{\sqrt{2(t-2)}}{t}, \qquad t \geq 3,$$

increases on the interval $[3, 4]$ and decreases on $[4, \infty)$, and therefore (2.1) yields

$$\sqrt{\frac{d_u + d_v - 2}{d_u^2 + d_v^2}} \leq g(d_u + d_v) \leq g(4) = \frac{1}{2}.$$

Therefore, summing over all m edges yields

$$RASO(G) \leq \frac{m}{2},$$

where the equality holds if and only if $d_u = d_v$ and $d_u + d_v = 4$ for every edge $uv \in E(G)$; that is, every edge joins two vertices of degree 2. □

Theorem 2.2. *For every connected graph G with size $m \geq 1$, maximum degree Δ and minimum degree δ,*

$$RASO(G) \leq \frac{m\sqrt{\Delta - 1}}{\delta}. \tag{2.2}$$

Equality in (2.2) *is achieved if and only if G is regular.*

Proof. For any edge uv of G, $\sqrt{d_u + d_v - 2} \leq \sqrt{2(\Delta - 1)}$ and $\sqrt{2}\delta \leq \sqrt{d_u^2 + d_v^2}$, where the first equality is achieved if and only if $d_v = \Delta = d_u$, whereas the second equality is achieved if and only if $d_v = \delta = d_u$. Therefore,

$$\sqrt{\frac{d_u + d_v - 2}{d_u^2 + d_v^2}} \leq \frac{\sqrt{\Delta - 1}}{\delta}.$$

Summing over all m edges gives the desired inequality. □

Theorem 2.3. *For every connected graph G with maximum degree $\Delta \geq 1$, size m, and minimum degree δ,*

$$RASO(G) \geq \frac{1}{\sqrt{\Delta}} \cdot ABS(G) \geq m\sqrt{\frac{\delta - 1}{\delta\Delta}}, \tag{2.3}$$

where $ABS(G)$ is the atom-bond sum-connectivity index (see, for example, [8]) defined as

$$ABS(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}}.$$

Either of the equalities in (2.3) *is achieved if and only if G is regular.*

Proof. For any edge uv of G, we have $d_u^2 + d_v^2 \leq \Delta(d_u + d_v)$, where the equality is achieved if and only if $d_v = \Delta = d_u$. Therefore, the required conclusion follows from

$$\sqrt{\frac{d_u + d_v - 2}{d_u^2 + d_v^2}} \geq \sqrt{\frac{d_u + d_v - 2}{\Delta(d_u + d_v)}} \geq \sqrt{\frac{\delta - 1}{\Delta\delta}}$$

because $d_u + d_v \geq 2\delta$. □

Lemma 2.1. *Let G be a connected graph possessing $n \geq 5$ vertices, and let $uv \in E(G)$. Then,*

$$\frac{d_v + d_u - 2}{d_v^2 + d_u^2} \geq \frac{n - 2}{(n-1)^2 + 1}, \tag{2.4}$$

where equality holds if and only if either $(d_v, d_u) = (n - 1, 1)$ or $(d_v, d_u) = (1, n - 1)$.

Proof. Define a function g on the closed interval $[1, d_v + d_u - 1]$ by

$$g(t) = \frac{d_v + d_u - 2}{t^2 + (d_v + d_u - t)^2}.$$

Then, its derivative function is calculated as follows:

$$g'(t) = \frac{2(d_v + d_u - 2)(d_v + d_u - 2t)}{(t^2 + (d_v + d_u - t)^2)^2}.$$

So, g attains the minimum value over the interval $[1, d_v + d_u - 1]$ at either $t = 1$ or $t = d_v + d_u - 1$. Therefore,

$$\frac{d_v + d_u - 2}{t^2 + (d_v + d_u - t)^2} = g(t) \geq g(1) = g(d_v + d_u - 1) = \frac{d_v + d_u - 2}{(d_v + d_u - 1)^2 + 1}. \tag{2.5}$$

where equalities throughout in (2.5) hold if and only if either $t = 1$ or $t = d_v + d_u - 1$.

First, we consider the case where $d_u + d_v \leq n$. Define a function h on the closed interval $[3, n]$ by

$$h(x) = \frac{x-2}{(x-1)^2+1}.$$

Then,

$$h'(x) = -\frac{x(x-4)+2}{(x(x-2)+2)^2}.$$

Hence, h is strictly increasing on the interval $\left[3, 2+\sqrt{2}\right]$ and strictly decreasing on $\left[2+\sqrt{2}, n\right]$, and therefore, its minimum over the interval $[3, n]$ is attained at either $x = 3$ or $x = n$. But, $h(3) = \frac{1}{5} > h(n)$ for $n \geq 5$. So, $h(x) \geq h(n)$ with equality if and only if $x = n$. Consequently, we have

$$\frac{d_v + d_u - 2}{(d_v + d_u - 1)^2 + 1} = h(d_v + d_u) \geq h(n) = \frac{n-2}{(n-1)^2+1}, \tag{2.6}$$

where equalities throughout in (2.6) hold if and only if $d_v + d_u = n$. Now, from (2.5) and (2.6), it follows that

$$\frac{d_v + d_u - 2}{t^2 + (d_v + d_u - t)^2} \geq \frac{n-2}{(n-1)^2+1}, \tag{2.7}$$

with equality if and only if $d_v + d_u = n$ and either $t = 1$ or $t = d_v + d_u - 1$. Finally, setting $t = d_v$ or $t = d_u$ in (2.7) yields the desired conclusion for the considered case $d_u + d_v \leq n$.

It remains to prove the result for the case where $n \leq d_u + d_v \leq 2(n-1)$; thus, in the rest of the proof, we assume this condition. Also, without loss of generality, we assume that $d_u \geq d_v$. We note that

$$(d_u - 1)^2 \leq (n-2)(d_u - 1), \qquad (d_v - 1)^2 \leq (n-2)(d_v - 1),$$

whose addition yields

$$d_u^2 + d_v^2 \leq n(d_u + d_v) - 2n + 2. \tag{2.8}$$

We observe that equality in (2.8) holds if and only if $(d_v, d_u) \in \{(1, n-1), (n-1, n-1)\}$. We claim that

$$n(d_u + d_v) - 2n + 2 \leq \frac{(n-1)^2+1}{n-2}(d_u + d_v - 2), \tag{2.9}$$

which is equivalent to

$$d_u + d_v - n \geq 0. \tag{2.10}$$

Equality in (2.10), and hence in (2.9), holds if and only if $d_u + d_v = n$. Thus, from (2.8) and (2.9), we get

$$d_u^2 + d_v^2 \leq \frac{(n-1)^2+1}{n-2}(d_u + d_v - 2),$$

or

$$\frac{d_u + d_v - 2}{d_u^2 + d_v^2} \geq \frac{n-2}{(n-1)^2+1},$$

where the equality holds if and only if $(d_v, d_u) \in \{(1, n-1), (n-1, n-1)\}$ and $d_u + d_v = n$ simultaneously; that is, if and only if $(d_v, d_u) = (1, n-1)$. □

The next result follows from Lemma 2.1.

Theorem 2.4. *For every connected graph G possessing size m and order n ≥ 5,*

$$RASO(G) \geq m\sqrt{\frac{n-2}{(n-1)^2+1}},$$

where the equality is attained if and only if G is the star S_n.

In order to derive our next bound on the RASO index, we require the following lemma.

Lemma 2.2. *Let vu be any edge in an n-order non-trivial graph G. Then,*

$$\sqrt{\frac{d_v+d_u-2}{d_v^2+d_u^2}} \leq \frac{1}{2}\sqrt{\frac{n-2}{n-1}}\left(\frac{1}{\sqrt{d_v}}+\frac{1}{\sqrt{d_u}}\right),$$

where the equality holds if and only if $d_v = n-1 = d_u$.

Proof. We note that

$$\frac{d_v+d_u-2}{d_v+d_u} \leq \frac{n-2}{n-1}, \tag{2.11}$$

and hence

$$\sqrt{\frac{d_v+d_u-2}{d_v^2+d_u^2}} \leq \sqrt{\frac{n-2}{n-1}}\sqrt{\frac{d_v+d_u}{d_v^2+d_u^2}}, \tag{2.12}$$

where equality in (2.11) (and hence in (2.12)) holds if and only if $d_v = n-1 = d_u$. Since the function f defined by $f(t) = t^{-1/2}$ for $t > 0$ is continuous and convex, Jensen's inequality yields

$$\frac{1}{2}\left(\frac{1}{\sqrt{d_v}}+\frac{1}{\sqrt{d_u}}\right) \geq \sqrt{\frac{2}{d_v+d_u}}. \tag{2.13}$$

Also, $(d_v+d_u)^2 \leq 2(d_v^2+d_u^2)$ implies

$$\sqrt{\frac{d_v+d_u}{d_v^2+d_u^2}} \leq \sqrt{\frac{2}{d_v+d_u}}. \tag{2.14}$$

Combining (2.13) and (2.14) yields

$$\sqrt{\frac{d_v+d_u}{d_v^2+d_u^2}} \leq \frac{1}{2}\left(\frac{1}{\sqrt{d_v}}+\frac{1}{\sqrt{d_u}}\right). \tag{2.15}$$

We note that the equality in either of the inequalities (2.13) and (2.14) (and hence, in (2.15)) holds if and only if $d_v = d_u$.

Now, the desired conclusion holds because of (2.12) and (2.15). □

Since the function $f(t) = \sqrt{t}$ is concave on the interval $(0, \infty)$, Jensen's inequality gives

$$\frac{1}{n}\sum_{v\in V(G)}\sqrt{d_v} \leq \sqrt{\frac{1}{n}\sum_{v\in V(G)} d_v} = \sqrt{\frac{2m}{n}}.$$

Therefore,

$$\sum_{v\in V(G)}\sqrt{d_v} \leq \sqrt{2mn}.$$

Consequently, the result given next holds by Lemma 2.2.

Theorem 2.5. *For any non-trivial n-order graph G possessing size m,*

$$RASO(G) \leq \frac{1}{2}\sqrt{\frac{n-2}{n-1}}\cdot {}^{0}R(G) \leq \sqrt{\frac{mn(n-2)}{2(n-1)}},$$

where the first equality (second equality, respectively) holds if and only if G is K_n (regular, respectively). Here, ${}^{0}R(G)$ represents the zeroth-order Randić index [11, 21] defined by

$${}^{0}R(G) = \sum_{vu\in E(G)}\left(\frac{1}{\sqrt{d_v}} + \frac{1}{\sqrt{d_u}}\right) = \sum_{v\in V(G)}\sqrt{d_v}.$$

Theorem 2.6. *For every connected graph G possessing minimum degree $\delta \geq 1$ and size m,*

$$RASO(G) \leq \sqrt{RMS(G)\cdot Pl(G)} \leq \frac{1}{\delta\sqrt{2}}\sqrt{m\cdot Pl(G)}.$$

where the first equality is attained if and only if G is either regular or semiregular bipartite, whereas the second equality is achieved if and only if G is regular. Here, $Pl(G)$ and $RMS(G)$ represent, respectively, the Platt index [4, 27] and reciprocal mean square index [5, 6], defined as

$$Pl(G) = \sum_{uv\in E(G)}(d_u + d_v - 2) \quad \textit{and} \quad RMS(G) = \sum_{uv\in E(G)}\frac{1}{d_u^2 + d_v^2}.$$

Proof. By the Cauchy–Schwarz inequality,

$$(RASO(G))^2 = \left(\sum_{uv\in E(G)}\sqrt{\frac{d_u + d_v - 2}{d_u^2 + d_v^2}}\right)^2 \leq \left(\sum_{uv\in E(G)}(d_u + d_v - 2)\right)\left(\sum_{uv\in E(G)}\frac{1}{d_u^2 + d_v^2}\right)$$
$$= Pl(G)\cdot RMS(G), \tag{2.16}$$

where the equality throughout in (2.16) is achieved if and only if there exists a positive integer ℓ provided that

$$(d_u^2 + d_v^2)(d_u + d_v - 2) = \ell$$

for every $uv \in E(G)$. Since $d_u^2 + d_v^2 \geq 2\delta^2$, the following is obtained:

$$RMS(G) = \sum_{e\in E(G)}\frac{1}{d_u^2 + d_v^2} \leq \frac{m}{2\delta^2}, \tag{2.17}$$

where the last equality is achieved if and only if G is regular. Now, from (2.16) and (2.17), we get the required chain of inequalities.

It remains to show that $(d_u + d_v - 2)(d_u^2 + d_v^2)$ is constant for all $uv \in E(G)$ if and only if G is either regular or semiregular bipartite. For this, we define

$$F(x, y) := (x + y - 2)(x^2 + y^2), \text{ where } x \geq 1 \text{ and } y \geq 1.$$

First, we assume that there exists a positive integer ℓ such that $F(d_u, d_v) = \ell$ for all $vu \in E(G)$. Since F is strictly increasing in either of its variables, for any two adjacent edges $uv, vw \in E(G)$, we have

$$F(d_v, d_u) = \ell = F(d_v, d_w) \quad \Longrightarrow \quad d_u = d_w.$$

Hence, every vertex is adjacent only to vertices of one fixed degree. Take any edge $ab \in E(G)$, and set $d_a = r$ and $d_b = s$. Since every neighbor of the vertex a has degree s, and every neighbor of b has degree r, and since G is connected, we conclude that every vertex of G has degree either r or s. If $r = s$, then G is s-regular. If $r \neq s$, then let $V_r := \{u \in V(G) : d_u = r\}$ and $V_s := \{v \in V(G) : d_v = s\}$. So, every neighbor of a vertex in V_r belongs to V_s, and every neighbor of a vertex in V_s belongs to V_r. Thus, no two vertices in either of the sets V_r and V_s are adjacent. Hence, G is semiregular bipartite.

Conversely, assume that G is either regular or semiregular bipartite. Then, in either case, $F(d_u, d_v)$ is constant for all $uv \in E(G)$. □

Theorem 2.7. *For every connected graph G possessing size $m \geq 2$,*

$$RASO(G) \geq \frac{m^2}{ASO(G)}, \tag{2.18}$$

with equality if and only if there exists $\lambda > 0$ provided that $d_u^2 + d_v^2 = \lambda(d_u + d_v - 2)$ for every $uv \in E(G)$. Particularly, if G is either regular or semiregular bipartite, then the equality is attained.

Proof. By the Cauchy-Schwarz inequality,

$$\left(\sum_{uv \in E(G)} \sqrt{\frac{d_u^2 + d_v^2}{d_u + d_v - 2}}\right)\left(\sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u^2 + d_v^2}}\right) \geq m^2,$$

or

$$RASO(G) \geq \frac{m^2}{ASO(G)},$$

where the equality holds if and only if there exists a constant $\lambda > 0$ such that

$$\frac{d_u^2 + d_v^2}{d_u + d_v - 2} = \lambda \qquad \text{for every } uv \in E(G).$$

□

The next result, which follows directly from Theorem 2.7, demonstrates that imposing a condition on the minimum degree leads to a substantially clearer understanding of the extremal configurations. Under this condition, the class of graphs that attain equality in (2.18) can be described in a much more explicit and structured manner, revealing the underlying simplicity of the extremal cases.

Corollary 2.1. *Let G be a connected graph with size $m \geq 2$ and minimum degree* 1*. Then, the equality in* (2.18) *holds if and only if either G is a star, or*

$$(d_v, d_u) \in \{(1, 2), (1, 3), (2, 4), (3, 4)\}$$

for every edge $vu \in E(G)$ with $d_u \geq d_v$. It is notable that all these graphs attaining the equality are necessarily bipartite. All such graphs for the case where $m = 10$ are depicted in Figure 1.

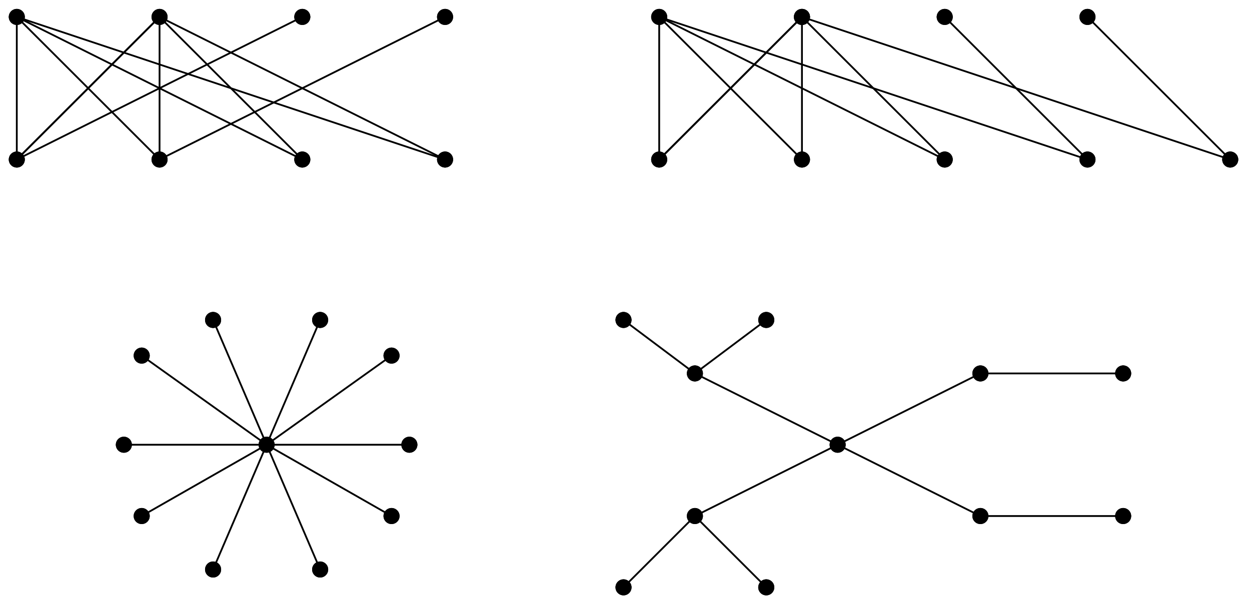

Figure 1. All those non-isomorphic connected graphs of size 10 and minimum degree 1 that attain the equality in (2.18).

Proof. By Theorem 2.7, the equality in (2.18) holds if and only if there exists a constant $\lambda > 0$ such that $d_u^2 + d_v^2 = \lambda(d_u + d_v - 2)$ for every edge $vu \in E(G)$. In the rest of the proof, we use this criterion.

First, we assume that the equality holds. Then, it is sufficient to assume that G is not a star. Choose a pendent vertex $v \in V(G)$, and let u be its unique neighbor. Set $d_u = s$. Since $uv \in E(G)$, the equality condition gives

$$\lambda = \frac{s^2 + 1}{s - 1}. \tag{2.19}$$

Since G is not a star, u has at least one non-pendent neighbor, say w. Write $d_w = x$. Then, $s^2 + x^2 = \lambda(s + x - 2)$, which together with (2.19) yields

$$x = s + \frac{2}{s - 1}. \tag{2.20}$$

Since $x \geq 2$ and $s \geq 2$ are integers, from (2.20) it follows that $s - 1$ must divide 2. Hence, $s \in \{2, 3\}$. Consequently, by utilizing (2.19) and (2.20), we obtained $\lambda = 5$ and $x = 4$. Thus, every edge $xy \in E(G)$, with $d_x \geq d_y$, must satisfy the Diophantine equation $d_x^2 + d_y^2 = 5(d_x + d_y - 2)$. The only integer solutions of this Diophantine equation are $(d_y, d_x) \in \{(1, 2), (1, 3), (2, 4), (3, 4)\}$.

Conversely, we assume that either $(d_v, d_u) \in \{(1,2),(1,3),(2,4),(3,4)\}$ for every $vu \in E(G)$ with $d_u \geq d_v$ or $G \cong S_n$. Then, for every $vu \in E(G)$,

$$\sqrt{\frac{d_u^2 + d_v^2}{d_u + d_v - 2}} = \begin{cases} \sqrt{\dfrac{1 + (n-1)^2}{n-2}} & \text{if } G \text{ is } S_n, \\ \sqrt{5} & \text{otherwise.} \end{cases}$$

Therefore,

$$RASO(G) = \frac{m^2}{ASO(G)}.$$

This completes the proof. □

In the following result, which arises as a consequence of Theorem 2.7, we show that when the minimum degree is at least 2, the structure of the extremal graphs achieving equality in (2.18) becomes significantly more transparent.

We say that an edge in a graph is of type (p, r) if one of its end vertex has degree p and the other has degree r.

Corollary 2.2. *Let G be a connected graph with size $m \geq 2$ and minimum degree larger than 1. Then, the equality in (2.18) holds if and only if one of the following holds:*

(i). G is regular; or

(ii). G is bipartite with bipartition (A, B) such that all vertices in one part, say A, have the same degree r, while every vertex in the other part B has degree p or q, where p, q, and r are some integers satisfying $2 \leq p \leq q < r$, and

$$\frac{p^2 + r^2}{p + r - 2} = \frac{q^2 + r^2}{q + r - 2}.$$

For the case where $m = 12$, all those graphs mentioned in (ii) that are different from $K_{2,6}$ and $K_{3,4}$ are depicted in Figure 2.

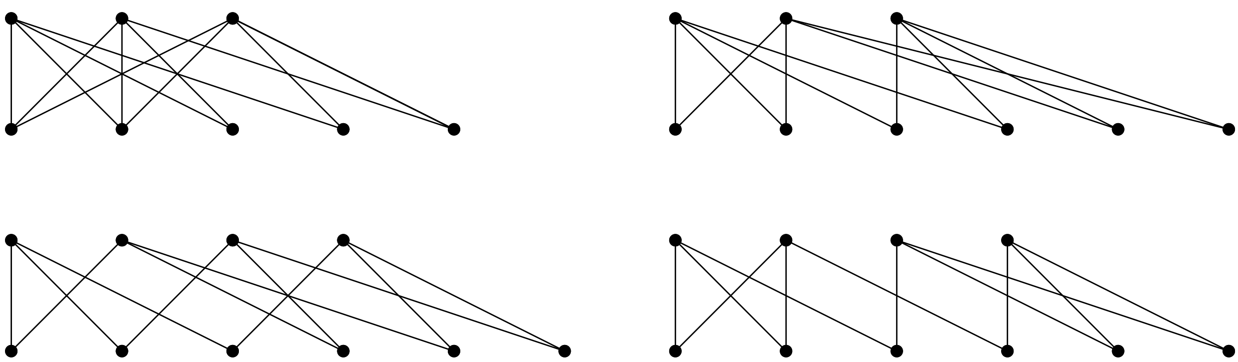

Figure 2. All those non-isomorphic non-regular connected graphs of size 12 and minimum degree larger than 1 different from $K_{2,6}$ and $K_{3,4}$ that attain the equality in (2.18).

Proof. We use the same criterion as utilized in Corollary 2.1; that is, the equality in (2.18) holds if and only if there exists a constant $\lambda > 0$ such that $d_u^2 + d_v^2 = \lambda(d_u + d_v - 2)$ for every edge $vu \in E(G)$.
Step 1. Assume that equality holds. We distinguish two cases.
Case 1. There exists an edge joining two vertices of the same degree.

Suppose $vu \in E(G)$ and $d_v = r = d_u$. Then, $r \geq 2$ as the minimum degree of G is larger than 1, and hence $\lambda = \frac{r^2}{r-1}$. We claim that every neighbor of u and v has degree r. Indeed, let $uw \in E(G)$, and write $d_w = s$. Since uw also satisfies the same equality condition,

$$\frac{r^2 + s^2}{r + s - 2} = \frac{r^2}{r - 1},$$

which yields $(r - s)((r - 1)(s - 1) - 1) = 0$. Thus, either $(r - 1)(s - 1) = 1$ (which gives $r = s = 2$ as $r, s \geq 2$) or $r = s$. Therefore, every neighbor of v and u has degree r.

Because G is connected, it follows that every vertex of G has degree r. Hence, G is regular, which means condition (i) of the corollary holds.

Case 2. No edge joins two vertices of the same degree.

In this case, G is not regular. Let r be the maximum degree of G. Let x be the degree of a neighbor of a maximum-degree vertex. Then, $r^2 + x^2 = \lambda(r + x - 2)$. For fixed r, this is a quadratic equation in x: $x^2 - \lambda x + (r^2 - \lambda r + 2\lambda) = 0$. Hence, there are at most two possible values of x. Thus, every vertex of degree r is adjacent only to vertices whose degrees belong to either $\{p, q\}$ or $\{p\}$, where p and q are some positive integers.

We now show that vertices of degree p or q can be adjacent only to vertices of degree r. Take one of these values, say p, and consider an edge $ab \in E(G)$ of type (p, r), where $d_a = p$ and $d_b = r$. Let y be the degree of a neighbor, say c, of the vertex $a \in V(G)$. Since $ab \in E(G)$ is an edge of type (p, r), r is a root of the following quadratic equation in y: $y^2 - \lambda y + (p^2 - \lambda p + 2\lambda) = 0$. By the Vietas formulas, the other root of this quadratic equation equals $\lambda - r := t$. Since

$$\lambda = \frac{p^2 + r^2}{p + r - 2},$$

we have

$$t = \frac{p^2 - pr + 2r}{p + r - 2}.$$

We claim that $t < 2$. Indeed,

$$t < 2 \iff p^2 - pr - 2p + 4 < 0.$$

Since $r \geq p + 1$ (because r is the maximum degree and $p \neq r$), we have

$$p^2 - pr - 2p + 4 \leq p^2 - p(p + 1) - 2p + 4 = 4 - 3p < 0$$

because $p \geq 2$. Hence, $t < 2$, as claimed.

But, the minimum degree of G is at least 2, so no vertex of G can have degree $t < 2$. Therefore, every vertex of degree p is adjacent only to vertices of degree r. The same argument applies to q.

Consequently, every edge of G joins a vertex of degree r to a vertex of degree p or q. Hence, G is bipartite: one bipartition class A consists of all vertices of degree r, and the other class B consists of all vertices of degrees p or q. Since edges of both types (p, r) and (q, r) (whenever present) satisfy the same equality condition, we have

$$\frac{p^2 + r^2}{p + r - 2} = \frac{q^2 + r^2}{q + r - 2} = \lambda.$$

This means condition (ii) of the corollary holds.

Step 2. Assume that either condition (i) or condition (ii) holds.

If G is regular, then

$$RASO(G) = \frac{m^2}{ASO(G)}.$$

Now, suppose G satisfies (ii). Then, every edge of G has degree pair either (p, r) or (q, r), and by assumption,

$$\frac{p^2 + r^2}{p + r - 2} = \frac{q^2 + r^2}{q + r - 2}.$$

Hence, $(d_u^2 + d_v^2)/(d_u + d_v - 2)$ is constant for all edges $uv \in E(G)$. Therefore,

$$RASO(G) = \frac{m^2}{ASO(G)}.$$

This completes the proof. □

Theorem 2.8. *For every non-trivial connected graph G possessing maximum degree* Δ,

$$RASO(G) \leq \sqrt{\frac{2(\Delta - 1)}{\Delta}}\chi(G), \tag{2.21}$$

where $\chi(G)$ *represents the sum-connectivity index [37] defined by*

$$\chi(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}}.$$

Equality in (2.21) *is achieved if and only if G is regular.*

Proof. For an edge uv of G, since $2(d_u^2 + d_v^2) \geq (d_u + d_v)^2$ with equality if and only if $d_v = d_u$, we get

$$\begin{aligned} \sqrt{\frac{d_u + d_v - 2}{d_u^2 + d_v^2}} \leq \frac{\sqrt{2(d_u + d_v - 2)}}{d_u + d_v} &= \sqrt{2\left(1 - \frac{2}{d_u + d_v}\right)} \cdot \frac{1}{\sqrt{d_u + d_v}} \\ &\leq \sqrt{\frac{2(\Delta - 1)}{\Delta}} \cdot \frac{1}{\sqrt{d_u + d_v}}. \end{aligned} \tag{2.22}$$

Summing over all edges in (2.22) yields (2.21). We note that the equalities throughout in (2.22) hold if and only if $d_v = d_u = \Delta$ for every edge $uv \in E(G)$. □

Theorem 2.9. *For every connected non-trivial graph G possessing maximum degree* Δ,

$$RASO(G) \leq \sqrt{\Delta - 1}\, H(G),$$

where equality holds if and only if G is regular. Here, $H(G)$ *is the harmonic index [14] defined as*

$$H(G) = \sum_{vu \in E(G)} \frac{2}{d_v + d_u}.$$

Proof. For any edge $vu \in E(G)$, using (2.22) and $\sqrt{d_u + d_v} \le \sqrt{2\Delta}$, we obtain

$$\sqrt{\frac{d_u + d_v - 2}{d_u^2 + d_v^2}} \le \sqrt{\frac{2(\Delta - 1)}{\Delta}} \cdot \frac{1}{\sqrt{d_u + d_v}} \le \sqrt{\frac{2(\Delta - 1)}{\Delta}} \cdot \frac{\sqrt{2\Delta}}{d_u + d_v}, \tag{2.23}$$

where equalities throughout in (2.23) hold if and only if $d_v = \Delta = d_u$. Summing over all edges in (2.23) yields the desired conclusion. □

We know that (for instance, see [36]) for any non-trivial connected graph, $H(G) \le n/2$, where the equality holds if and only if G is regular. Hence, Theorem 2.9 has the following consequence.

Corollary 2.3. *For every connected non-trivial n-order graph G possessing maximum degree Δ,*

$$RASO(G) \le \frac{n\sqrt{\Delta - 1}}{2},$$

where equality holds if and only if G is regular.

Theorem 2.10. *For every connected non-trivial graph G possessing maximum degree Δ and minimum degree δ,*

$$\sqrt{\frac{2\Delta\delta(\delta - 1)}{\Delta^2 + \delta^2}}\, R(G) \le RASO(G) \le \sqrt{\Delta - 1}\, R(G),$$

where the equality in either case holds if and only if G is regular. Here, $R(G)$ is the Randić index [23,28] defined as

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}.$$

Proof. Since $H(G) \le R(G)$ with equality if and only if G is regular (for example, see [36]), the upper bound follows from Theorem 2.9.

For the lower bound, using the inequalities $d_u + d_v - 2 \ge 2(\delta - 1)$ and

$$d_u^2 + d_v^2 \le \frac{\Delta^2 + \delta^2}{\Delta\delta}\, d_u d_v,$$

we have

$$\sqrt{\frac{d_u + d_v - 2}{d_u^2 + d_v^2}} \ge \sqrt{\frac{2(\delta - 1)}{\frac{\Delta^2+\delta^2}{\Delta\delta}\, d_u d_v}} = \sqrt{\frac{2\Delta\delta(\delta - 1)}{\Delta^2 + \delta^2}}\, \frac{1}{\sqrt{d_u d_v}}. \tag{2.24}$$

Summing over all edges in (2.24) yields

$$RASO(G) \ge \sqrt{\frac{2\Delta\delta(\delta - 1)}{\Delta^2 + \delta^2}}\, R(G).$$

We note that the equalities throughout in (2.24) hold if and only if $d_v = \Delta = \delta = d_u$ for any edge $vu \in E(G)$. □

3. Extremal results

In this section, we investigate extremal problems for the RASO index over several well-studied graph classes. In particular, we characterize graphs that minimize or maximize the RASO index among fixed-order (i) trees, (ii) unicyclic graphs, and (iii) connected graphs.

Theorem 3.1. *Among all n-order connected graphs, S_n and K_n uniquely minimize and maximize, respectively, the RASO index for every $n \geq 5$, and these minimum and maximum values are given, respectively, as follows:*

$$(n-1)\sqrt{\frac{n-2}{(n-1)^2+1}} \quad \textit{and} \quad \frac{n\sqrt{n-2}}{2}.$$

Proof. Let G be an n-order connected graph with $n \geq 5$. Let Δ and m be the maximum degree and size of G. Then, by Theorem 2.4 and Corollary 2.3, we have

$$(n-1)\sqrt{\frac{n-2}{(n-1)^2+1}} \leq m\sqrt{\frac{n-2}{(n-1)^2+1}} \leq RASO(G) \leq \frac{n\sqrt{\Delta-1}}{2} \leq \frac{n\sqrt{n-2}}{2},$$

where the left two equalities hold simultaneously if and only if G is S_n, whereas the right two equalities hold simultaneously if and only if G is K_n. Here, we remark that the conclusion regarding K_n can also be obtained using Theorem 2.5. □

By Theorem 3.1, S_n uniquely minimizes the RASO index in the class of all n-order trees for each $n \geq 5$; in order to characterize trees maximizing the RASO index over this class, we need to establish three lemmas first.

Lemma 3.1. *Let vu be a non-pendent edge in a graph G. Then,*

$$\sqrt{\frac{d_v+d_u-2}{d_v^2+d_u^2}} \leq \frac{1}{2},$$

where the equality holds if and only if $d_v = d_u = 2$.

Proof. Since $\min(d_u, d_v) \geq 2$ and

$$d_v^2+d_u^2 \geq \frac{(d_v+d_u)^2}{2}, \tag{3.1}$$

we obtain

$$\sqrt{\frac{d_v+d_u-2}{d_v^2+d_u^2}} \leq \frac{\sqrt{2(d_v+d_u-2)}}{d_v+d_u}, \tag{3.2}$$

where the equality in (3.1), and hence in (3.2), holds if and only if $d_v = d_u$. We note that the expression given in the right-hand side of (3.2) strictly decreases in $d_v + d_u$ for $d_v + d_u \geq 4$, and hence, this expression attains its maximum (which is $1/2$) at $d_v + d_u = 4$. □

Lemma 3.2. *Let vu be a pendent edge in a graph G such that at least one of v and u is non-pendent. Then,*

$$\sqrt{\frac{d_v+d_u-2}{d_v^2+d_u^2}} \leq \frac{1}{\sqrt{5}},$$

where the equality holds if and only if either $\{d_v, d_u\} = \{1, 2\}$ or $\{d_v, d_u\} = \{1, 3\}$.

Proof. Without loss of generality, we assume $d_v = 1$. Then, the expression

$$\sqrt{\frac{d_v + d_u - 2}{d_v^2 + d_u^2}} = \sqrt{\frac{d_u - 1}{d_u^2 + 1}},$$

strictly decreases for $d_u \geq 3$. Also, the mentioned expression has the same value (which is $1/\sqrt{5}$) for each $d_u \in \{2, 3\}$. Hence, the desired result holds. □

Lemma 3.3. *Let G be a graph without any isolated edge (that is, an edge with both pendent end vertices). If G has m edges, from which p are pendent ones, then*

$$RASO(G) \leq \frac{p}{\sqrt{5}} + \frac{m-p}{2},$$

where the equality holds if and only if every non-pendent edge has both end vertices of degree 2 *and every pendent edge is incident with either a vertex of degree* 2 *or a vertex of degree* 3.

Proof. The required conclusion is deduced directly from Lemmas 3.1 and 3.2. □

Theorem 3.2. *Among all n-order trees, only the path P_n maximizes the RASO index, where $n \geq 4$. Also,*

$$RASO(P_n) = \frac{n-3}{2} + \frac{2}{\sqrt{5}},$$

Proof. Let T be an n-order tree possessing p pendent vertices, where $n \geq 4$. Then, $p \geq 2$, and hence, by Lemma 3.3, we obtain

$$RASO(T) \leq \frac{p}{\sqrt{5}} + \frac{n-1-p}{2} \leq \frac{n-3}{2} + \frac{2}{\sqrt{5}}, \tag{3.3}$$

where the equalities throughout in (3.3) hold iff

- $p = 2$,
- every non-pendent edge has both end vertices of degree 2, and
- every pendent edge is incident with either a vertex of degree 2 or a vertex of degree 3.

If $p = 2$, then no pendent edge can be incident with a vertex of degree 3. Therefore, equalities throughout in (3.3) hold if and only if T is P_n. □

We now turn to the problem of characterizing fixed-order unicyclic graphs that minimize/maximize the RASO index. Let S_n^+ represent the graph obtained from the star S_n by inserting an edge.

Theorem 3.3. *Let C_n denote the class of all n-order unicyclic graphs with constraint $n \geq 6$. Then, only C_n (S_n^+, respectively) maximizes (minimizes, respectively) the RASO index in C_n. Also,*

$$RASO(S_n^+) = \frac{(n-3)\sqrt{n-2}}{\sqrt{(n-1)^2+1}} + \frac{2\sqrt{n-1}}{\sqrt{(n-1)^2+4}} + \frac{1}{2} \quad \textit{and} \quad RASO(C_n) = \frac{n}{2}.$$

Proof. The maximal case of the result is a direct consequence of Theorem 2.1, whereas the minimal case was proved in [7]; see Table 2.1 in [7]. □

4. Edge-addition effects

In this section, we investigate whether the RASO index exhibits monotonic behavior under the addition of edges. More precisely, we consider the following natural question: given a graph G and two non-adjacent vertices $u, v \in V(G)$ such that at least one of them has a positive degree, does the inequality

$$RASO(G+uv) > RASO(G) \tag{4.1}$$

always hold? Here, $G + uv$ denotes the graph obtained from G by adding the edge uv.

We first note that if both u and v are isolated vertices, then adding the edge uv produces a single edge whose each end vertex has degree one. In this case, the contribution of this edge to the RASO index is zero, and therefore $RASO(G + uv) = RASO(G)$. Consequently, the assumption that at least one of u and v has a degree greater than zero is necessary when investigating strict monotonicity.

To gain insight into this question, we performed an exhaustive computational search over all non-isomorphic graphs of order up to 10. The results obtained indicate that inequality (4.1) holds for all tested graphs whenever at least one of the vertices u and v has a positive degree. Therefore, at first glance, it was natural to expect that inserting an edge would not decrease the value of the RASO index; nevertheless, repeated attempts to prove this statement were unsuccessful. Indeed, we present an example demonstrating that inequality (4.1) does not hold in general.

Example 4.1. *Let $K_{2,q}$ be the complete bipartite graph with degree set $\{2, q\}$, where $q \geq 3$. Let u and v be the two vertices in the partite set of size 2. Then,*

$$RASO(K_{2,q}+uv) - RASO(K_{2,q}) = \frac{\sqrt{q}}{q+1} + 2q\left(\sqrt{\frac{q+1}{(q+1)^2+4}} - \sqrt{\frac{q}{q^2+4}}\right).$$

We claim that

$$RASO(K_{2,q}+uv) > RASO(K_{2,q}) \quad \text{for } 3 \leq q \leq 39,$$

while

$$RASO(K_{2,q}+uv) < RASO(K_{2,q}) \quad \text{for } q \geq 40.$$

Since the proofs for both cases are similar, we prove the claim only for the case where $q \geq 40$. Let

$$F(q) = \frac{\sqrt{q}}{q+1} + 2q\left(\sqrt{\frac{q+1}{(q+1)^2+4}} - \sqrt{\frac{q}{q^2+4}}\right) \text{ and } q \geq 40.$$

Since

$$\frac{q}{q^2+4} - \frac{q+1}{(q+1)^2+4} = \frac{q^2+q-4}{(q^2+4)((q+1)^2+4)} > 0,$$

we have

$$\sqrt{\frac{q}{q^2+4}} > \sqrt{\frac{q+1}{(q+1)^2+4}}.$$

Therefore, $F(q) < 0$ is equivalent to

$$\left(\sqrt{\frac{q}{q^2+4}} - \sqrt{\frac{q+1}{(q+1)^2+4}}\right)^2 > \left(\frac{\sqrt{q}}{2q(q+1)}\right)^2,$$

that is,

$$2\sqrt{\frac{q(q+1)}{(q^2+4)((q+1)^2+4)}} < \frac{q}{q^2+4} + \frac{q+1}{(q+1)^2+4} - \frac{1}{4q(q+1)^2}$$
$$= \frac{8q^6+28q^5+67q^4+98q^3+59q^2+8q-20}{4q(q+1)^2\,(q^2+4)\,(q^2+2q+5)},$$

which, after squaring and simplification, yields

$$8q^9 - 279q^8 - 1420q^7 - 3290q^6 - 4420q^5 - 4543q^4 - 4256q^3 - 2296q^2 - 320q + 400 > 0. \tag{4.2}$$

Hence, $F(q) < 0$ if and only if (4.2) *holds. But,* (4.2) *is equivalent to*

$$\begin{aligned} &8(q-40)^9 + 2601(q-40)^8 + 370100(q-40)^7 + 30107910(q-40)^6 + 1532037980(q-40)^5 \\ &+ 49961751457(q-40)^4 + 1021099748864(q-40)^3 + 12005191074184(q-40)^2 \\ &+ 63374944379200(q-40) + 22104445850000 > 0, \end{aligned}$$

which holds because $q \geq 40$.

Next, we show that if one of the vertices u, v is isolated and the other is not isolated, then inequality (4.1) holds. For a vertex v in a graph G, let $N_G(v)$ denote the set of neighbors of v. In the rest of this section, we use the following notation:

$$f(a,b) := \sqrt{\frac{a+b-2}{a^2+b^2}}, \qquad a, b \geq 1.$$

Proposition 4.1. *Let G be a graph, and let $u, v \in V(G)$ be two non-adjacent vertices such that $d_u = 0$ and $d_v = k \geq 1$. Then,*

$$RASO(G+uv) > RASO(G).$$

Proof. Let $N_G(v) = \{x_1, \ldots, x_k\}$ and $d_{x_i}(G) = a_i \geq 1$ for $i = 1, \ldots, k$. Then,

$$RASO(G+uv) - RASO(G) = f(1, k+1) + \sum_{i=1}^{k}\Big(f(k+1, a_i) - f(k, a_i)\Big). \tag{4.3}$$

If $f(k+1, a_i) \geq f(k, a_i)$ for every $i \in \{1, 2, \ldots, k\}$, then the desired inequality is immediate. In what follows, we assume that $f(k+1, a_i) < f(k, a_i)$ for at least one $i \in \{1, 2, \ldots, k\}$. In the following, for the sake of simplicity, we write $f(\alpha, \beta)^2$ in place of $[f(\alpha, \beta)]^2$. Here,

$$f(k, a_i) - f(k+1, a_i) = \frac{f(k,a_i)^2 - f(k+1,a_i)^2}{f(k,a_i) + f(k+1,a_i)} < \frac{f(k,a_i)^2 - f(k+1,a_i)^2}{2f(k+1,a_i)}$$

and

$$f(k,a_i)^2 - f(k+1,a_i)^2 = \frac{-a_i^2 + (2k+1)a_i + k^2 - 3k - 2}{(k^2+a_i^2)((k+1)^2+a_i^2)}.$$

Since $(2k+1)a_i \leq k^2 + 3a_i^2$, we have $-a_i^2 + (2k+1)a_i + k^2 - 3k - 2 < 2(k^2 + a_i^2)$, and hence

$$f(k,a_i)^2 - f(k+1,a_i)^2 < \frac{2}{(k+1)^2 + a_i^2}.$$

Consequently,

$$\begin{aligned} f(k,a_i)-f(k+1,a_i) &< \frac{1}{((k+1)^2+a_i^2)f(k+1,a_i)} \\ &= \frac{1}{\sqrt{(k+a_i-1)[(k+1)^2+a_i^2]}} \\ &\le \frac{1}{\sqrt{k[(k+1)^2+1]}} = \frac{f(1,k+1)}{k}. \end{aligned}$$

Hence, for any $j \in \{1,2,\ldots,k\}$, whether $f(k+1,a_j) \ge f(k,a_j)$ or $f(k+1,a_j) < f(k,a_j)$, in either case, we have

$$f(k+1,a_j)-f(k,a_j) > -\frac{f(1,k+1)}{k},$$

which together with (4.3) yields the desired inequality. □

Proposition 4.2. *Let G be a graph possessing an order larger than 2. Let $v,u \in V(G)$ be non-adjacent vertices such that $1 \le d_v \le \lfloor 2d_{v_i}/5 \rfloor$ for every $v_i \in N_G(v)$ and $1 \le d_u \le \lfloor 2d_{u_i}/5 \rfloor$ for every $u_i \in N_G(u)$, then*

$$RASO(G+uv) > RASO(G).$$

Proof. Let $N_G(u) = \{u_1,\ldots,u_a\}$ and $N_G(v) = \{v_1,\ldots,v_b\}$. We set

$$\alpha_i := d_{u_i}(G) \quad (1 \le i \le a), \qquad \beta_j := d_{v_j}(G) \quad (1 \le j \le b).$$

Then,

$$\begin{aligned} RASO(G+uv) - RASO(G) = &\sum_{i=1}^{a}\Big(f(a+1,\alpha_i)-f(a,\alpha_i)\Big) + \sum_{j=1}^{b}\Big(f(b+1,\beta_j)-f(b,\beta_j)\Big) \\ &+ f(a+1,b+1), \end{aligned}$$

By the given constraints, we have $a \le \lfloor 2\alpha_i/5 \rfloor$ for each $i \in \{1,\ldots,a\}$ and $b \le \lfloor 2\beta_j/5 \rfloor$ for every $j \in \{1,\ldots,b\}$. Under these conditions, we note that $f(a,\alpha_i)$ ($f(b,\beta_j)$, respectively) strictly increases in a (b, respectively). Hence, $RASO(G+uv) - RASO(G) > f(a+1,b+1) > 0$, as required. □

If both u,v are pendent, each of whose neighbors has degree at least 3, then inequality (4.1) holds by Proposition 4.2. Next, we establish a similar result without imposing any restriction on the neighbors of u and v.

Proposition 4.3. *Let G be a graph, and let u,v be two non-adjacent pendent vertices of G. Then,*

$$RASO(G+uv) > RASO(G).$$

Proof. Let $N_G(u) = \{u'\}$ and $N_G(v) = \{v'\}$. Since, for every $t \ge 1$,

$$[f(2,t)]^2 - [f(1,t)]^2 = \frac{\left(t-\frac{3}{2}\right)^2+\frac{7}{4}}{(t^2+4)(t^2+1)} > 0,$$

we have $RASO(G+uv) - RASO(G) = [f(2,d_{u'})-f(1,d_{u'})] + [f(2,d_{v'})-f(1,d_{v'})] + f(2,2) > 0$. □

Proposition 4.4. *Let G be a graph, and let $u, v \in V(G)$ be non-adjacent vertices such that these vertices and all their neighbors have the same degree $r \geq 1$. Then,*

$$RASO(G + uv) > RASO(G).$$

Proof. If $r = 1$, then the required conclusion holds due to Proposition 4.3. In the remainder of the proof, let $r \geq 2$. Here,

$$RASO(G + uv) - RASO(G) = \frac{\sqrt{r}}{r+1} + 2r\left(\frac{\sqrt{2r-1}}{\sqrt{2r^2+2r+1}} - \frac{\sqrt{r-1}}{r}\right). \tag{4.4}$$

We note that

$$\frac{r-1}{r^2} - \frac{2r-1}{2r^2+2r+1} = \frac{r^2-r-1}{r^2(2r^2+2r+1)} < \frac{r(r-1)}{r^2(2r^2+2r+1)} = \frac{r-1}{r(2r^2+2r+1)}.$$

Hence,

$$\frac{\sqrt{r-1}}{r} - \frac{\sqrt{2r-1}}{\sqrt{2r^2+2r+1}} = \frac{\frac{r-1}{r^2} - \frac{2r-1}{2r^2+2r+1}}{\frac{\sqrt{r-1}}{r} + \frac{\sqrt{2r-1}}{\sqrt{2r^2+2r+1}}} < \frac{\frac{r-1}{r(2r^2+2r+1)}}{\frac{\sqrt{r-1}}{r}} = \frac{\sqrt{r-1}}{2r^2+2r+1},$$

which yields

$$2r\left(\frac{\sqrt{r-1}}{r} - \frac{\sqrt{2r-1}}{\sqrt{2r^2+2r+1}}\right) < \frac{2r\sqrt{r-1}}{2r^2+2r+1} < \frac{2r\sqrt{r-1}}{2(r+1)\sqrt{r(r-1)}} = \frac{\sqrt{r}}{r+1} \tag{4.5}$$

because $2r^2 + 2r + 1 > 2(r+1)\sqrt{r(r-1)}$. Now, the required inequality follows from (4.4) and (4.5). □

5. Conclusions

In this paper, we initiated the systematic study of the RASO index. We have obtained sharp bounds for this index in terms of several basic graph parameters, including the order, size, minimum degree, and maximum degree. We have also determined the extremal graphs with respect to the RASO index among fixed-order connected graphs, trees, and unicyclic graphs. Furthermore, we have investigated the behavior of the RASO index under edge addition and derived sufficient conditions ensuring that this operation increases the index. These results provide a foundation for the further study of the RASO index and suggest several possible directions for future research. For example, establishing extremal results regarding this index for n-order connected graphs of a given size $m \geq n+1$, or deriving relationships between the RASO index and other graph invariants, including the subpath number [22] and invariants associated with subtrees [33,34], seem to be interesting directions for future research on this index.

Author contributions

Conceptualization, A. M. A. and A. A.; Validation, A. M. A. and A. A.; Writing original draft, A. M. A. and A. A.; Writing review and editing, A. M. A. and A. A.; Funding acquisition, A. M. A. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgment

The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2025/01/34159).

Conflict of interest

The authors declare no conflicts of interest.

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