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Research article

Sharper Jensen-Mercer operator inequalities with applications to quasi-arithmetic means

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Abstract: Motivated by recent developments in the study of Jensen-Mercer type operator inequalities, this paper presents several new and more precise series of inequalities for self-adjoint operators on unital C^*-algebras. Our approach is based on the concept of strong convexity, which allows the introduction of additional correction terms and the derivation of refined upper and lower bounds. The obtained results generalize and strengthen the classical Jensen-Mercer operator inequalities, both in the presence and absence of operator convexity. As an application, we establish new estimates for operator quasi-arithmetic means of the Mercer type and derive corresponding refinements for operator power means, including the arithmetic, geometric, and harmonic means.

Keywords: operator convex functions; convex functions; strongly convex functions; Jensen operator inequality; Jensen-Mercer operator inequality
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1. Introduction

The Jensen-Mercer operator inequality is a fundamental result in operator theory and functional analysis. It has been widely investigated in the context of matrix inequalities, functional calculus, and operator means. Recent research has focused on refining Jensen-Mercer type inequalities by introducing correction terms, establishing tighter bounds, and extending them to broader classes of functions and operator mappings. In particular, the concept of strong convexity is a powerful tool for deriving sharper inequalities, as it introduces an additional quadratic term that captures deviation more precisely.

Recall that a function $f\colon I \to \mathbb{R}$ is said to be strongly convex with modulus $c > 0$ if

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y) - c\lambda(1-\lambda)(x-y)^2, \tag{1.1}$$

for all $x, y \in I \subseteq \mathbb{R}$ and $\lambda \in [0,1]$. If $-f$ is strongly convex, then we say that a function f is strongly concave. Setting $c = 0$ in (1.1) recovers the usual notion of convexity.

Strong convexity obviously implies convexity, but the converse is not true. In fact, strongly convex functions satisfy stronger versions of many properties enjoyed by convex functions.

The additional quadratic term in (1.1) can be effectively used to sharpen classical inequalities such as Jensen, Jensen-Mercer, or Hermite-Hadamard, both in scalar and operator form.

Important characterizations of strongly convex functions can be found in [1–3], and the references therein. We recall the following useful results.

Throughout, we use *id* to denote the identity function (that is, $id(t) = t$), and $(.)^r$ to denote the power function $(.)^r(t) = t^r$.

Lemma 1.1. *A function $f\colon I \to \mathbb{R}$ is strongly convex with a modulus $c > 0$ if and only if the function $g\colon I \to \mathbb{R}$, defined by $g = f - c \cdot (.)^2$, is convex.*

Lemma 1.2. *Let $f\colon I \to \mathbb{R}$ be a twice differentiable function. Then f is strongly convex with a modulus $c > 0$ if and only if $f''(x) \geqslant 2c$ for all $x \in I$.*

Remark 1.1. *In Section 5, we are particularly interested in the power functions $f(t) = t^k$, $k \in \mathbb{R}$, defined on an interval $[a,b] \subset (0,\infty)$. Note that $f''(t) = k(k-1)t^{k-2}$. Hence, if $k \in (-\infty,0) \cup (1,\infty)$, then $f''(t) > 0$ on $[a,b]$, so f is strongly convex. By Lemma 1.2, its modulus of strong convexity is controlled by $c = \frac{1}{2}k(k-1)\min_{t\in[a,b]} t^{k-2}$, with explicit cases*

$$\min_{t\in[a,b]} t^{k-2} = \begin{cases} a^{k-2}, & k \geq 2, \\ b^{k-2}, & k < 2. \end{cases}$$

Therefore, the modulus can be chosen as

$$c = \begin{cases} \dfrac{1}{2}\,k(k-1)a^{k-2}, & k \geq 2, \\ \dfrac{1}{2}\,k(k-1)b^{k-2}, & k \in (-\infty,0) \cup (1,2). \end{cases}$$

The cases $k = 0$ and $k = 1$ correspond to a constant and an affine function, respectively, which are not strongly convex. For $0 < k < 1$, the function is strongly concave.

The Jensen inequality for a strongly convex function $f\colon I \to \mathbb{R}$ takes the form

$$f(\bar{x}_\lambda) \leqslant \sum_{i=1}^{n} \lambda_i f(x_i) - c\sum_{i=1}^{n} \lambda_i (x_i - \bar{x}_\lambda)^2, \tag{1.2}$$

where $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is a non-negative n-tuple with $\sum_{i=1}^{n} \lambda_i = 1$ and $\bar{x}_\lambda = \sum_{i=1}^{n} \lambda_i x_i$ (see [2]). Setting $c = 0$ in (1.2) recovers the classical Jensen inequality

$$f(\bar{x}_\lambda) \leqslant \sum_{i=1}^{n} \lambda_i f(x_i). \tag{1.3}$$

Since the additional term $c\sum_{i=1}^{n}\lambda_i(x_i-\bar{x}_\lambda)^2$ is non-negative, inequality (1.2) provides a strictly tighter upper bound for $f(\bar{x}_\lambda)$ than (1.3).

Motivated by (1.2) in the special case $n=2$, $\lambda_1=\lambda_2=\frac{1}{2}$, $x_1=x$, and $x_2=y$, we define

$$\mathcal{J}_c(f;x,y)=f(x)+f(y)-2f\left(\frac{x+y}{2}\right)-\frac{c}{2}(y-x)^2. \tag{1.4}$$

Remark 1.2. *If f is strongly convex with a modulus $c>0$, then*

$$\mathcal{J}_c(f;x,y)\geq 0$$

for all $x,y\in I$. Equality holds for the functions of the form $f(t)=ct^2+l(t)$, where l is affine.

A notable variant of Jensen's inequality is the Jensen-Mercer inequality [4]

$$f(\alpha+\beta-\bar{x}_\lambda)\leqslant f(\alpha)+f(\beta)-\sum_{i=1}^{n}\lambda_i f(x_i), \tag{1.5}$$

which holds for every convex function $f\colon I\to\mathbb{R}$, $\boldsymbol{x}=(x_1,\ldots,x_n)\in[\alpha,\beta]^n$, with $\alpha,\beta\in I$, $\alpha<\beta$, and $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)$ a non-negative n-tuple such that $\sum_{i=1}^{n}\lambda_i=1$ and $\bar{x}_\lambda=\sum_{i=1}^{n}\lambda_i x_i$.

We also recall the following refinement of Jensen's inequality for strongly convex functions obtained in [1].

Theorem 1.1. *Let $f\colon I\to\mathbb{R}$ be strongly convex function with a modulus $c>0$, and $\boldsymbol{x}=(x_1,\ldots,x_n)\in I^n$. Let $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)$ be a non-negative n-tuple with $\sum_{i=1}^{n}\lambda_i=1$, $\bar{x}_\lambda=\sum_{i=1}^{n}\lambda_i x_i$, and $\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$. Then*

$$\begin{aligned}\min_{1\leqslant i\leqslant n}\{\lambda_i\}\left(\sum_{i=1}^{n}f(x_i)-nf(\bar{x})-c\sum_{i=1}^{n}(x_i-\bar{x})^2\right)&\leqslant\sum_{i=1}^{n}\lambda_i f(x_i)-f(\bar{x}_\lambda)-c\sum_{i=1}^{n}\lambda_i(x_i-\bar{x}_\lambda)^2\\&\leqslant\max_{1\leqslant i\leqslant n}\{\lambda_i\}\left(\sum_{i=1}^{n}f(x_i)-nf(\bar{x})-c\sum_{i=1}^{n}(x_i-\bar{x})^2\right).\end{aligned} \tag{1.6}$$

In recent years, Jensen and Jensen-Mercer type inequalities have continued to attract considerable attention due to their rich theoretical structure and numerous applications. Several recent studies investigated generalized Jensen and Jensen-Mercer inequalities in the setting of operator convexity [5], h-convexity [6, 7], and operator superquadracity [8]. Fractional and integral extensions of Jensen-Mercer type inequalities were studied in [9], as well as in [10]. Further refinements and converse forms of Jensen-type inequalities were obtained in [11–13]. Generalized convexity approaches, including m-convexity and harmonic convexity, were considered in [14, 15]. Recent developments also include inequalities for coordinated convexity and related applications (see [16]). Related improvements for generalized (ψ,h)-convexity were recently obtained in [17]. These developments indicate a continuing interest in obtaining sharper estimates, converse inequalities, and refined correction terms for Jensen and Jensen-Mercer type inequalities.

The purpose of this paper is to develop a systematic framework for improved Jensen-Mercer operator inequalities based on strong convexity and to demonstrate their applications to operator means, particularly quasi-arithmetic means and their special cases given by operator power means.

The paper is organized as follows. In Section 2, we recall basic definitions and notations related to operator convexity, monotonicity, and positive linear mappings between unital C^*-algebras. We also review known forms of the Jensen and Jensen-Mercer operator inequalities, which present initial results for our improvements. Section 3 contains the main results. We derive refined upper and lower bounds for Jensen-Mercer type inequalities using strong convexity, both under operator convexity and in cases where the function is not operator convex. In Section 4, we apply the results from Section 3 to obtain new estimates and refinements for operator quasi-arithmetic means of the Mercer type. In Section 5, we discuss applications to operator power means, presenting improved bounds for important special cases such as the arithmetic, geometric, and harmonic means. Section 6 concludes the paper with an overview of our main results.

2. Preliminaries

We use the symbol $\mathbf{B}(\mathcal{H})$ for unital C^*-algebra of all bounded linear operators defined on a complex Hilbert space $\mathcal{H}$ with the identity operator $\mathbf{1}$. We let $\mathbf{B}_h(\mathcal{H})$ denote a C^*-subalgebra of all bounded self-adjoint operators on $\mathcal{H}$. The operator $A \in \mathbf{B}_h(\mathcal{H})$ is positive semidefinite or positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and we write $A \geq 0$. For the operators $A, B \in \mathbf{B}_h(\mathcal{H})$ is $B \geq A$ if $B - A \geq 0$. If, for $X \in \mathbf{B}_h(\mathcal{H})$, the bounds are $\alpha_X = \inf_{\|t\|=1}(Xt, t)$ and $\beta_X = \sup_{\|t\|=1}(Xt, t)$, $t \in \mathcal{H}$, then for the spectrum of X, denoted $S_P(X)$, we have $S_P(X) \subseteq [\alpha_X, \beta_X]$.

Let $C([\alpha, \beta])$ denote the set of all real valued continuous functions on an interval $[\alpha, \beta] \subset \mathbb{R}$. By the continuous functional calculus, for any self-adjoint operator $X \in \mathbf{B}_h(\mathcal{H})$ with a spectrum in $[\alpha, \beta]$, the operator $f(X)$ is well defined for every $f \in C([\alpha, \beta])$.

A function $f \in C([\alpha, \beta])$ is operator convex if

$$f(\lambda X + (1-\lambda)Y) \leqslant \lambda f(X) + (1-\lambda) f(Y)$$

holds for all $\lambda \in [0, 1]$ and every $X, Y \in \mathbf{B}_h(\mathcal{H})$ with $S_P(X), S_P(Y) \subseteq [\alpha, \beta]$.

We say that a function $f \in C([\alpha, \beta])$ is operator increasing if it preserves the operator order, that is,

$$X \leqslant Y \quad \text{implies} \quad f(X) \leqslant f(Y)$$

for all $X, Y \in \mathbf{B}_h(\mathcal{H})$ with $S_P(X), S_P(Y) \subseteq [\alpha, \beta]$. We say that f is operator decreasing if $-f$ is operator increasing.

Remark 2.1. *The power function* $f(t) = t^r$, $t > 0$, $r \in \mathbb{R}$ *is*

- *Operator convex on* $(0, \infty)$ *if and only if*

$$-1 \leqslant r \leqslant 0 \quad or \quad 1 \leqslant r \leqslant 2.$$

- *Operator increasing on* $(0, \infty)$ *if and only if* $0 \leqslant r \leqslant 1$.

For more details on operator convex and operator increasing functions, see [18, 19].

In the context of operator inequalities of the Jensen-Mercer type, it is crucial to work with positive linear mappings between unital C^*-algebras. Recall that a linear map $\Phi\colon \mathbf{B}_h(\mathcal{H}) \to \mathbf{B}_h(\mathcal{K})$ from the unital C^*-algebra $\mathbf{B}_h(\mathcal{H})$ into another unital C^*-algebra $\mathbf{B}_h(\mathcal{K})$ is said to be positive if

$$X \geqslant 0 \quad \text{implies} \quad \Phi(X) \geqslant 0,$$

and unital if $\Phi(\mathbf{1}) = \mathbf{1}$.

Example 2.1. Two standard examples of positive linear unital maps are the following.

- The identity map
$$\Phi(X) = X, \quad X \in \mathbf{B}_h(\mathcal{H}),$$
is trivially linear, positive, and unital.
- Let U be a unitary operator on a Hilbert space $\mathcal{H}$ and define
$$\Phi(X) = U^*XU, \quad X \in \mathbf{B}_h(\mathcal{H}).$$
Then Φ is linear, positive, and unital, since $\Phi(\mathbf{1}) = U^*\mathbf{1}U = \mathbf{1}$.

When dealing with several positive linear mappings $\Phi_i\colon \mathbf{B}_h(\mathcal{H}) \to \mathbf{B}_h(\mathcal{K})$, $i = 1, ..., n$, we consider those that satisfy the condition $\sum_{i=1}^n \Phi_i(\mathbf{1}) = \mathbf{1}$.

Example 2.2. Let
$$\varphi(X) = \tfrac{1}{2}\mathrm{Tr}(X), \quad X \in M_2(\mathbb{C}),$$
be the normalized trace and
$$\Phi_1(X) = \varphi(X)E_{11}, \qquad \Phi_2(X) = \varphi(X)E_{22},$$
where E_{11} and E_{22} are the standard matrix units, that is,
$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$
Each Φ_i is positive and their sum is
$$(\Phi_1 + \Phi_2)(X) = \varphi(X)(E_{11} + E_{22}) = \varphi(X)\mathbf{1}.$$
Since $\varphi(\mathbf{1}) = \frac{1}{2}\mathrm{Tr}(\mathbf{1}) = 1$, it follows that $(\Phi_1 + \Phi_2)(\mathbf{1}) = \mathbf{1}$. Thus, $(\Phi_1 + \Phi_2)$ is a unital positive map.

Example 2.3. Let $e_1 = (1, 0)^T$, $e_2 = (0, 1)^T$, and P_1, P_2 be orthogonal projections
$$P_1 = |e_1\rangle\langle e_1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = |e_2\rangle\langle e_2| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$
The vector state (evaluation at e_1) is given as
$$\omega(X) = \langle e_1, Xe_1\rangle = X_{11}, \qquad X \in M_2(\mathbb{C}).$$
Define maps
$$\Phi_1(X) = \omega(X)P_1 = X_{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Phi_2(X) = \omega(X)P_2 = X_{11}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$
Clearly, each Φ_i is linear and positive (if $X \geqslant 0$ then $X_{11} = \omega(X) \geqslant 0$ so $\Phi_i(X) \geqslant 0$) and
$$\Phi_1(\mathbf{1}) = P_1, \quad \Phi_2(\mathbf{1}) = P_2.$$
Their sum is
$$(\Phi_1 + \Phi_2)(X) = X_{11}(P_1 + P_2) = X_{11}\mathbf{1}.$$
Since $(\Phi_1 + \Phi_2)(\mathbf{1}) = \mathbf{1}$, the sum $\Phi = \Phi_1 + \Phi_2$ is a unital positive map.

For every operator convex function $f\colon [\alpha,\beta] \to \mathbb{R}$, the Jensen operator inequality

$$f\left(\sum_{i=1}^{n} \Phi_i(X_i)\right) \leqslant \sum_{i=1}^{n} \Phi_i(f(X_i)) \tag{2.1}$$

holds, where $X_1, ..., X_n \in \mathbf{B}_h(\mathcal{H})$ are self-adjoint operators such that $S_P(X_i) \subseteq [\alpha,\beta]$, $i = 1, ..., n$, and $\Phi_i\colon \mathbf{B}_h(\mathcal{H}) \to \mathbf{B}_h(\mathcal{K})$, $i = 1, ..., n$, are positive linear mappings with $\sum_{i=1}^{n} \Phi_i(\mathbf{1}) = \mathbf{1}$ (see [20]).

Under the same conditions, the Jensen-Mercer inequality

$$f\left((\alpha+\beta)\mathbf{1} - \sum_{j=1}^{n} \Phi_j\left(X_j\right)\right) \leqslant (f(\alpha) + f(\beta))\mathbf{1} - \sum_{j=1}^{n} \Phi_j\left(f\left(X_j\right)\right) \tag{2.2}$$

holds (see [21]). More precisely, we have

$$\begin{aligned} f\left((\alpha+\beta)\mathbf{1} - \sum_{j=1}^{n} \Phi_j\left(X_j\right)\right) &\leqslant \sum_{j=1}^{n} \Phi_j\left(f\left((\alpha+\beta)\mathbf{1} - X_j\right)\right) \\ &\leqslant \frac{\beta\mathbf{1} - \sum_{j=1}^{n} \Phi_j\left(X_j\right)}{\beta-\alpha} f(\beta) + \frac{\sum_{j=1}^{n} \Phi_j\left(X_j\right) - \alpha\mathbf{1}}{\beta-\alpha} f(\alpha) \\ &\leqslant (f(\alpha) + f(\beta))\mathbf{1} - \sum_{j=1}^{n} \Phi_j\left(f\left(X_j\right)\right). \end{aligned} \tag{2.3}$$

If we replace the operator convexity condition with ordinary convexity, that is, if $f \in C([\alpha,\beta])$ is a convex function, then inequality (2.2) still holds. However, the series of inequalities (2.3) shortens (see [22]) to

$$\begin{aligned} f\left((\alpha+\beta)\mathbf{1} - \sum_{j=1}^{n} \Phi_j\left(X_j\right)\right) &\leqslant \frac{\beta\mathbf{1} - \sum_{j=1}^{n} \Phi_j\left(X_j\right)}{\beta-\alpha} f(\beta) + \frac{\sum_{j=1}^{n} \Phi_j\left(X_j\right) - \alpha\mathbf{1}}{\beta-\alpha} f(\alpha) \\ &\leqslant (f(\alpha) + f(\beta))\mathbf{1} - \sum_{j=1}^{n} \Phi_j\left(f\left(X_j\right)\right). \end{aligned} \tag{2.4}$$

Further, under the same assumptions on the self-adjoint operators $X_1, ..., X_n \in \mathbf{B}_h(\mathcal{H})$ and positive linear mappings $\Phi_1, ..., \Phi_n\colon \mathbf{B}_h(\mathcal{H}) \to \mathbf{B}_h(\mathcal{K})$ as for (2.1), in [23], it is shown that for every continuous twice differentiable function $f\colon [\alpha,\beta] \to \mathbb{R}$ with $f'' \geqslant 2c > 0$, a refinement of (2.2) in the form

$$\begin{aligned} f\left((\alpha+\beta)\mathbf{1} - \sum_{i=1}^{n} \Phi_i(X_i)\right) &\leqslant (f(\alpha) + f(\beta))\mathbf{1} - \sum_{i=1}^{n} \Phi_i(f(X_i)) - 2cK(\alpha,\beta,\mathbf{X},\mathbf{\Phi}) \\ &\leqslant (f(\alpha) + f(\beta))\mathbf{1} - \sum_{i=1}^{n} \Phi_i(f(X_i)) \end{aligned} \tag{2.5}$$

holds, where

$$K(\alpha,\beta,\mathbf{X},\mathbf{\Phi}) = (\alpha+\beta)\sum_{i=1}^{n} \Phi_i(X_i) - \alpha\beta\mathbf{1} - \frac{1}{2}\left(\sum_{i=1}^{n} \Phi_i(X_i^2) + \left(\sum_{i=1}^{n} \Phi_i(X_i)\right)^2\right). \tag{2.6}$$

Recall that the assumption $f'' \geqslant 2c > 0$ is equivalent to the twice differentiable function f being strongly convex with a modulus $c > 0$ (Lemma 1.2).

Refined operator inequalities play an important role in matrix analysis, quantum information theory, optimization, and numerical analysis, where sharper estimates improve stability bounds and error analysis (see, for example, [24–26]). Operator Jensen inequalities and their refinements on C^*-algebras and operator means have also been investigated in several recent papers, including [7].

Motivated by these recent advances, in this paper, we establish new refinements of Jensen-Mercer type operator inequalities by using strongly convex scalar functions and additional correction terms. Our approach relies on the notion of strongly convex functions. By combining ordinary strong convexity with functional calculus techniques under suitable operator assumptions, we obtain refinements and improvements of previously known Jensen-Mercer operator inequalities established in [22] and later in [23]. More precisely, under the same assumptions on self-adjoint operators and positive linear mappings as in [23], in the present paper, we establish a further refinement of (2.5) of the form

$$\begin{aligned} f\left((\alpha+\beta)\mathbf{1} - \sum_{i=1}^{n} \Phi_i(X_i)\right) &\leq (f(\alpha)+f(\beta))\mathbf{1} - \sum_{i=1}^{n} \Phi_i(f(X_i)) - 2cK(\alpha,\beta,X,\Phi) - \Delta_c(\alpha,\beta,X,f) \\ &\leq (f(\alpha)+f(\beta))\mathbf{1} - \sum_{i=1}^{n} \Phi_i(f(X_i)) - 2cK(\alpha,\beta,X,\Phi) \\ &\leq (f(\alpha)+f(\beta))\mathbf{1} - \sum_{i=1}^{n} \Phi_i(f(X_i)), \end{aligned} \tag{2.7}$$

where the additional correction term

$$\Delta_c(\alpha,\beta,\bar{X},f) \geq 0$$

is explicitly determined in terms of the strong convexity modulus and the operator's deviation from the midpoint of the interval $[\alpha,\beta]$.

Therefore, inequality (2.7) provides strictly sharper upper and lower bounds than those previously obtained in [22, 23]. The additional correction term $\Delta_c(\alpha,\beta,\bar{X},f)$ represents the main novelty of the paper and enables further refinements for operator quasi-arithmetic means and operator power means of the Mercer type.

Furthermore, under the additional assumption that the comapping $g(t) = f(t) - ct^2$ is operator convex on $[\alpha,\beta]$, we obtain refined Jensen-Mercer type operator inequalities which improve the results established by Matković, Pečarić, and Perić [21]. More precisely, by combining the operator convexity of the auxiliary function g with the strong convexity of f, we derive sharper upper and lower bounds containing additional non-negative correction terms. Consequently, our approach simultaneously extends the refinement obtained by Moradi, Furuichi, and Sababheh [23] and improves the Jensen-Mercer operator inequalities from [21]. This demonstrates that the introduced correction term $\Delta_c(\alpha,\beta,\bar{X},f)$ provides an effective quantitative sharpening of previously known Jensen-Mercer operator inequalities.

3. Improvement of the Jensen-Mercer operator inequality

In the next remark, we consider a direct consequence of Theorem 1.1, which will be very useful for obtaining our goal.

Remark 3.1. *The first inequality in (1.6) can be rewritten in the form*

$$f(\bar{x}_\lambda) \leqslant \sum_{i=1}^{n} \lambda_i f(x_i) - c\sum_{i=1}^{n} \lambda_i (x_i - \bar{x}_\lambda)^2 - \min_{1\leqslant i\leqslant n}\{\lambda_i\}\left[\sum_{i=1}^{n} f(x_i) - nf(\bar{x}) - c\sum_{i=1}^{n}(x_i - \bar{x})^2\right], \tag{3.1}$$

where

$$\min_{1\leqslant i\leqslant n}\{\lambda_i\}\left[\sum_{i=1}^{n} f(x_i) - nf(\bar{x}) - c\sum_{i=1}^{n}(x_i - \bar{x})^2\right] \geqslant 0.$$

If we set $n = 2$ and, for $\lambda \in [0,1]$ take $\lambda_1 = \lambda, \lambda_2 = 1-\lambda$, $x_1 = x$, and $x_2 = y$, then, from (3.1), we get more accurate version of (1.1) in the form

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\leqslant \lambda f(x) + (1-\lambda)f(y) - c\lambda(1-\lambda)(y-x)^2 \\ &\quad - \min\{\lambda, 1-\lambda\}\left[f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) - \frac{c}{2}(y-x)^2\right], \end{aligned} \tag{3.2}$$

where

$$\min\{\lambda, 1-\lambda\} = \begin{cases} \lambda, & 0 \leqslant \lambda \leqslant \frac{1}{2}, \\ 1-\lambda, & \frac{1}{2} < \lambda \leqslant 1. \end{cases}$$

We also have

$$\min\{\lambda, 1-\lambda\} = \frac{1}{2}(1 - |1-2\lambda|),$$

so we can write (3.2), using the notation in (1.4), in the form

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y) - c\lambda(1-\lambda)(y-x)^2 - \frac{1}{2}(1-|1-2\lambda|)\mathcal{J}_c(f; x, y). \tag{3.3}$$

First, we prove the following lemma.

Lemma 3.1. *Let $X \in \mathbf{B}_h(\mathcal{H})$ be a selfadjoint operator with $S_P(X) \subseteq [\alpha,\beta]$. Then for every function $f \in C([\alpha,\beta])$ that is strongly convex with a modulus $c > 0$, we have*

$$f(X) \leqslant \frac{\beta\mathbf{1} - X}{\beta-\alpha} f(\alpha) + \frac{X - \alpha\mathbf{1}}{\beta-\alpha} f(\beta) - c(X-\alpha\mathbf{1})(\beta\mathbf{1} - X). \tag{3.4}$$

Moreover, the series of inequalities

$$\begin{aligned} f(X) &\leqslant \frac{\beta\mathbf{1} - X}{\beta-\alpha} f(\alpha) + \frac{X - \alpha\mathbf{1}}{\beta-\alpha} f(\beta) - c(X-\alpha\mathbf{1})(\beta\mathbf{1} - X) \\ &\quad - \left(\frac{1}{2}\mathbf{1} - \frac{1}{\beta-\alpha}\left|X - \frac{\alpha+\beta}{2}\mathbf{1}\right|\right)\mathcal{J}_c(f,\alpha,\beta) \\ &\leqslant \frac{\beta\mathbf{1} - X}{\beta-\alpha} f(\alpha) + \frac{X - \alpha\mathbf{1}}{\beta-\alpha} f(\beta) - c(X-\alpha\mathbf{1})(\beta\mathbf{1} - X) \end{aligned} \tag{3.5}$$

holds, where

$$\mathcal{J}_c(f,\alpha,\beta) = f(\alpha) + f(\beta) - 2f\left(\frac{\alpha+\beta}{2}\right) - \frac{c}{2}(\beta-\alpha)^2. \tag{3.6}$$

Proof. Substituting $\lambda x + (1-\lambda)y = t$, $x = \alpha$, $y = \beta$, $1-\lambda = \frac{t-\alpha}{\beta-\alpha}$ and $\lambda = \frac{\beta-t}{\beta-\alpha}$ in (3.3), for every t, such that $\alpha < t < \beta$, we get

$$\begin{aligned} f(t) &\leqslant \frac{\beta-t}{\beta-\alpha} f(\alpha) + \frac{t-\alpha}{\beta-\alpha} f(\beta) - c(t-\alpha)(\beta-t) - \left(\frac{1}{2} - \frac{1}{\beta-\alpha}\left|t - \frac{\alpha+\beta}{2}\right|\right)\mathcal{J}_c(f,\alpha,\beta) \\ &\leqslant \frac{\beta-t}{\beta-\alpha} f(\alpha) + \frac{t-\alpha}{\beta-\alpha} f(\beta) - c(t-\alpha)(\beta-t), \end{aligned} \tag{3.7}$$

where $\mathcal{J}_c(f,\alpha,\beta)$ is defined by (3.6). Now, by utilizing the functional calculus, from (3.7), we obtain (3.4) and (3.5).

Remark 3.2. *Note that inequalities (3.4) and (3.5) are obtained by applying the continuous functional calculus to the scalar inequality (3.7). Since the scalar inequality holds pointwise for every $t \in [\alpha,\beta]$, the spectral theorem implies the corresponding operator inequalities for every self-adjoint operator X with spectrum contained in $[\alpha,\beta]$. Therefore, no operator convexity assumption on f or on $g = f - c\cdot(.)^2$ is required at this stage.*

Remark 3.3. *Note that inequality (3.5) is equivalent to*

$$\begin{aligned} f(X) &\leqslant \frac{\beta\mathbf{1}-X}{\beta-\alpha} f(\alpha) + \frac{X-\alpha\mathbf{1}}{\beta-\alpha} f(\beta) - c((\alpha+\beta)X - \alpha\beta\mathbf{1} - X^2) \\ &\quad - \left(\frac{1}{2}\mathbf{1} - \frac{1}{\beta-\alpha}\left|X - \frac{\alpha+\beta}{2}\mathbf{1}\right|\right)\mathcal{J}_c(f,\alpha,\beta) \\ &\leqslant \frac{\beta\mathbf{1}-X}{\beta-\alpha} f(\alpha) + \frac{X-\alpha\mathbf{1}}{\beta-\alpha} f(\beta) - c((\alpha+\beta)X - \alpha\beta\mathbf{1} - X^2). \end{aligned} \tag{3.8}$$

This reformulation is useful in later derivations of our results.

We now present an improvement of inequality (2.4), that is, we get the series of inequalities that hold without the operator convexity assumption on a given function.

Theorem 3.1. *Let $X_1, ..., X_n \in \mathbf{B}_h(\mathcal{H})$ be such that $S_P(X_i) \subseteq [\alpha,\beta]$, $i = 1, ..., n$,, and let $\Phi_1, ..., \Phi_n$ be positive linear mappings from the unital C^*-algebra $\mathbf{B}_h(\mathcal{H})$ to the unital C^*-algebra $\mathbf{B}_h(\mathcal{K})$ such that $\Phi = \sum_{i=1}^n \Phi_i$ is unital. Let $f \in C([\alpha,\beta])$ be strongly convex with a modulus $c > 0$. Then*

$$\begin{aligned} & f\left((\alpha+\beta)\mathbf{1} - \sum_{i=1}^n \Phi_i(X_i)\right) \\ &\leqslant (f(\alpha)+f(\beta))\mathbf{1} - \sum_{i=1}^n \Phi_i(f(X_i)) - 2cK(\alpha,\beta,\mathbf{X},\mathbf{\Phi}) - \Delta_c(\alpha,\beta,\bar{X},f) \\ &\leqslant (f(\alpha)+f(\beta))\mathbf{1} - \sum_{i=1}^n \Phi_i(f(X_i)) - 2cK(\alpha,\beta,\mathbf{X},\mathbf{\Phi}) \\ &\leqslant (f(\alpha)+f(\beta))\mathbf{1} - \sum_{i=1}^n \Phi_i(f(X_i)) \end{aligned} \tag{3.9}$$

holds, where $K(\alpha,\beta,\mathbf{X},\mathbf{\Phi})$ is defined by (2.6),

$$\Delta_c(\alpha,\beta,\bar{X},f) = \left(\frac{1}{2}\mathbf{1} - \frac{1}{\beta-\alpha}\left|\sum_{i=1}^n \Phi_i(X_i) - \frac{\alpha+\beta}{2}\mathbf{1}\right|\right)\mathcal{J}_c(f;\alpha,\beta),$$

and $\mathcal{J}_c(f,\alpha,\beta)$ is defined by (3.6).

Proof. We have $\alpha\mathbf{1} \leqslant X_i \leqslant \beta\mathbf{1}$, $i = 1, ..., n$, and $\Phi = \sum_{i=1}^n \Phi_i$ is unital. Then by applying linear mapping Φ_i and summing over $i, i = 1, ..., n$, we have $\alpha\mathbf{1} \leqslant \sum_{i=1}^n \Phi_i(X_i) \leqslant \beta\mathbf{1}$ and also $\alpha\mathbf{1} \leqslant (\alpha+\beta)\mathbf{1} - \sum_{i=1}^n \Phi_i(X_i) \leqslant \beta\mathbf{1}$, $i = 1, ..., n$. If we substitute X by $(\alpha+\beta)\,\mathbf{1} - \sum_{i=1}^n \Phi_i(X_i)$ in (3.8), we get

$$\begin{aligned} f\left((\alpha+\beta)\,\mathbf{1} - \sum_{i=1}^n \Phi_i(X_i)\right) &\leqslant \frac{\sum_{i=1}^n \Phi_i(X_i) - \alpha\mathbf{1}}{\beta-\alpha} f(\alpha) + \frac{\beta\mathbf{1} - \sum_{i=1}^n \Phi_i(X_i)}{\beta-\alpha} f(\beta) \\ &\quad - c\left((\alpha+\beta)\sum_{i=1}^n \Phi_i(X_i) - \alpha\beta\mathbf{1} - \left(\sum_{i=1}^n \Phi_i(X_i)\right)^2\right) \\ &\quad - \left(\frac{1}{2}\mathbf{1} - \frac{1}{\beta-\alpha}\left|\sum_{i=1}^n \Phi_i(X_i) - \frac{\alpha+\beta}{2}\mathbf{1}\right|\right)\mathcal{J}_c(f,\alpha,\beta). \end{aligned} \tag{3.10}$$

Since $\alpha\mathbf{1} \leq X_i \leq \beta\mathbf{1}$, we have $-\frac{\beta-\alpha}{2}\mathbf{1} \leq X_i - \frac{\alpha+\beta}{2}\mathbf{1} \leq \frac{\beta-\alpha}{2}\mathbf{1}$. Hence,

$$\left|\sum_{i=1}^n \Phi_i(X_i) - \frac{\alpha+\beta}{2}\mathbf{1}\right| \leq \frac{\beta-\alpha}{2}\mathbf{1},$$

and consequently,

$$0 \leq \frac{1}{\beta-\alpha}\left|\sum_{i=1}^n \Phi_i(X_i) - \frac{\alpha+\beta}{2}\mathbf{1}\right| \leq \frac{1}{2}\mathbf{1}.$$

Hence,

$$\frac{1}{2}\mathbf{1} - \frac{1}{\beta-\alpha}\left|\sum_{i=1}^n \Phi_i(X_i) - \frac{\alpha+\beta}{2}\mathbf{1}\right| \geq 0.$$

Since $\mathcal{J}_c(f;\alpha,\beta) \geq 0$, it follows that

$$\Delta_c(\alpha,\beta,\bar{X},f) \geq 0.$$

Thus, the additional correction term appearing in our refinement is always a non-negative operator. On the other side, by (3.4), we have

$$\begin{aligned} f(X_i) &\leqslant \frac{\beta\mathbf{1} - X_i}{\beta-\alpha} f(\alpha) + \frac{X_i - \alpha\mathbf{1}}{\beta-\alpha} f(\beta) - c((\alpha+\beta)X_i - \alpha\beta\mathbf{1} - X_i^2) \\ &= (f(\alpha) + f(\beta))\,\mathbf{1} - \left[\frac{X_i - \alpha\mathbf{1}}{\beta-\alpha} f(\alpha) + \frac{\beta\mathbf{1} - X_i}{\beta-\alpha} f(\beta)\right] - c((\alpha+\beta)X_i - \alpha\beta\mathbf{1} - X_i^2). \end{aligned}$$

Applying the linear mappings Φ_i and summing over $i, i = 1, ..., n$, we get

$$\begin{aligned} \sum_{i=1}^n \Phi_i(f(X_i)) &\leqslant (f(\alpha) + f(\beta))\,\mathbf{1} - \left[\frac{\sum_{i=1}^n \Phi_i(X_i) - \alpha\mathbf{1}}{\beta-\alpha} f(\alpha) + \frac{\beta\mathbf{1} - \sum_{i=1}^n \Phi_i(X_i)}{\beta-\alpha} f(\beta)\right] \\ &\quad - c\left[(\alpha+\beta)\sum_{i=1}^n \Phi_i X_i - \alpha\beta\mathbf{1} - \sum_{i=1}^n \Phi_i(X_i^2)\right]. \end{aligned} \tag{3.11}$$

Further, by summarizing (3.10) with (3.11), we get

$$\begin{aligned}
&f\left((\alpha+\beta)\,\mathbf{1}-\sum_{i=1}^{n}\Phi_i(X_i)\right)+\sum_{i=1}^{n}\Phi_i(f(X_i))\\
&\leqslant (f(\alpha)+f(\beta))\,\mathbf{1}-c\left(2(\alpha+\beta)\sum_{i=1}^{n}\Phi_i(X_i)-2\alpha\beta\mathbf{1}-\sum_{i=1}^{n}\Phi_i(X_i^2)-\left(\sum_{i=1}^{n}\Phi_i(X_i)\right)^2\right)\\
&-\left(\frac{1}{2}\mathbf{1}-\frac{1}{\beta-\alpha}\left|\sum_{i=1}^{n}\Phi_i(X_i)-\frac{\alpha+\beta}{2}\mathbf{1}\right|\right)\mathcal{J}_c(f,\alpha,\beta).
\end{aligned}$$

Moreover, the series of inequalities in (3.9) holds.

Example 3.1. We now compare numerically the upper bounds from the Jensen-Mercer inequality (2.2) with the refinement (2.4) obtained by Matković-Pečarić-Perić in [22], the refinement (2.5) due to Moradi-Furuichi-Sababheh [23], and our refinement (3.9).

Let

$$X_1=\begin{pmatrix}1&0\\0&2\end{pmatrix},\qquad X_2=\begin{pmatrix}2&0\\0&3\end{pmatrix},$$

and define positive linear mappings

$$\Phi_1(T)=\Phi_2(T)=\frac{1}{2}T.$$

Take $\alpha=1$ and $\beta=3$, and consider the function $f(t)=t^4$, which is strongly convex on $[1,3]$ with a modulus $c=6$. We first compute

$$\sum_{i=1}^{2}\Phi_i(X_i)=\frac{X_1+X_2}{2}=\begin{pmatrix}1.5&0\\0&2.5\end{pmatrix}.$$

Hence,

$$(\alpha+\beta)\mathbf{1}-\sum_{i=1}^{2}\Phi_i(X_i)=\begin{pmatrix}2.5&0\\0&1.5\end{pmatrix},$$

and therefore,

$$f\left((\alpha+\beta)\mathbf{1}-\sum_{i=1}^{2}\Phi_i(X_i)\right)=\begin{pmatrix}39.0625&0\\0&5.0625\end{pmatrix}.$$

Next,

$$X_1^4=\begin{pmatrix}1&0\\0&16\end{pmatrix},\qquad X_2^4=\begin{pmatrix}16&0\\0&81\end{pmatrix},$$

so that

$$\sum_{i=1}^{2}\Phi_i(X_i^4)=\frac{1}{2}(X_1^4+X_2^4)=\begin{pmatrix}8.5&0\\0&48.5\end{pmatrix}.$$

Since

$$f(\alpha)+f(\beta)=1+81=82,$$

the classical Jensen-Mercer upper bound becomes

$$R_{\mathrm{JM}} = 82\mathbf{1} - \sum_{i=1}^{2} \Phi_i(X_i^4) = \begin{pmatrix} 73.5 & 0 \\ 0 & 33.5 \end{pmatrix}.$$

The Matković-Pečarić-Perić refinement (2.4) gives

$$R_{\mathrm{MPP1}} = \frac{\beta\mathbf{1} - \sum \Phi_i(X_i)}{\beta - \alpha} f(\beta) + \frac{\sum \Phi_i(X_i) - \alpha\mathbf{1}}{\beta - \alpha} f(\alpha).$$

Since

$$\frac{\beta\mathbf{1} - \sum \Phi_i(X_i)}{\beta - \alpha} = \begin{pmatrix} 0.75 & 0 \\ 0 & 0.25 \end{pmatrix},$$

and

$$\frac{\sum \Phi_i(X_i) - \alpha\mathbf{1}}{\beta - \alpha} = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.75 \end{pmatrix},$$

we obtain

$$R_{\mathrm{MPP1}} = \begin{pmatrix} 61 & 0 \\ 0 & 21 \end{pmatrix}.$$

Further, we now compute the Moradi-Furuichi-Sababheh refinement (2.5). We have

$$K(\alpha,\beta,X,\Phi) = (\alpha+\beta)\sum_{i=1}^{2} \Phi_i(X_i) - \alpha\beta\mathbf{1} - \frac{1}{2}\left(\sum_{i=1}^{2} \Phi_i(X_i^2) + \left(\sum_{i=1}^{2} \Phi_i(X_i)\right)^2\right).$$

Since

$$\sum_{i=1}^{2} \Phi_i(X_i^2) = \begin{pmatrix} 2.5 & 0 \\ 0 & 6.5 \end{pmatrix},$$

and

$$\left(\sum_{i=1}^{2} \Phi_i(X_i)\right)^2 = \begin{pmatrix} 2.25 & 0 \\ 0 & 6.25 \end{pmatrix},$$

we obtain

$$K = \begin{pmatrix} 0.625 & 0 \\ 0 & 0.625 \end{pmatrix}.$$

Therefore,

$$2cK = 12K = \begin{pmatrix} 7.5 & 0 \\ 0 & 7.5 \end{pmatrix}.$$

Hence,

$$R_{\mathrm{MFS}} = R_{\mathrm{JM}} - 2cK = \begin{pmatrix} 66 & 0 \\ 0 & 26 \end{pmatrix}.$$

At the end, we compute our refinement (3.9). Since $f(2) = 16$, we obtain

$$\mathcal{J}_c(f;1,3) = f(1) + f(3) - 2f(2) - \frac{c}{2}(3-1)^2 = 1 + 81 - 32 - 12 = 38.$$

Moreover,

$$\left|\sum_{i=1}^{2}\Phi_i(X_i)-2\mathbf{1}\right|=\begin{pmatrix}0.5 & 0\\ 0 & 0.5\end{pmatrix}.$$

Therefore,

$$\Delta_c=\left(\frac{1}{2}\mathbf{1}-\frac{1}{2}\begin{pmatrix}0.5 & 0\\ 0 & 0.5\end{pmatrix}\right)38=\begin{pmatrix}9.5 & 0\\ 0 & 9.5\end{pmatrix}.$$

Consequently,

$$R_{\text{our1}}=R_{\text{MFS}}-\Delta_c=\begin{pmatrix}56.5 & 0\\ 0 & 16.5\end{pmatrix}.$$

Finally, we obtain the operator comparison

$$R_{\text{our1}}<R_{\text{MPP1}}<R_{\text{MFS}}<R_{\text{JM}}.$$

That is,

$$\begin{pmatrix}56.5 & 0\\ 0 & 16.5\end{pmatrix}<\begin{pmatrix}61 & 0\\ 0 & 21\end{pmatrix}<\begin{pmatrix}66 & 0\\ 0 & 26\end{pmatrix}<\begin{pmatrix}73.5 & 0\\ 0 & 33.5\end{pmatrix}.$$

Thus, our refinement provides the sharpest upper bound among all considered inequalities.

Remark 3.4. *Let us show that the refinement term in Theorem 3.1*

$$2cK(\alpha,\beta,\mathbf{X},\mathbf{\Phi})+\Delta_c(\alpha,\beta,\bar{X},f)$$

is increasing in the modulus c, that is, it is maximal for the maximal $c>0$ *(obviously, if a function is strongly convex for some modulus* $c>0$*, then it is strongly convex for every* c'*,* $0<c'\leqslant c$*). To prove this, we rewrite the refinement term as*

$$\begin{aligned}
&2cK(\alpha,\beta,\mathbf{X},\mathbf{\Phi})+\Delta_c(\alpha,\beta,\bar{X},f) \qquad (3.12)\\
&=c\left[2(\alpha+\beta)\sum_{i=1}^{n}\Phi_i(X_i)-2\alpha\beta\mathbf{1}-\left(\sum_{i=1}^{n}\Phi_i(X_i^2)+\left(\sum_{i=1}^{n}\Phi_i(X_i)\right)^2\right)\right.\\
&\quad\left.-\left(\frac{1}{4}\mathbf{1}-\frac{1}{2(\beta-\alpha)}\left|\sum_{i=1}^{n}\Phi_i(X_i)-\frac{\alpha+\beta}{2}\mathbf{1}\right|\right)(\beta-\alpha)^2\right]\\
&\quad+\left(\frac{1}{2}\mathbf{1}-\frac{1}{\beta-\alpha}\left|\sum_{i=1}^{n}\Phi_i(X_i)-\frac{\alpha+\beta}{2}\mathbf{1}\right|\right)\left(f(\alpha)+f(\beta)-2f\left(\frac{\alpha+\beta}{2}\right)\right)\\
&=cR+\left(\frac{1}{2}\mathbf{1}-\frac{1}{\beta-\alpha}\left|\sum_{i=1}^{n}\Phi_i(X_i)-\frac{\alpha+\beta}{2}\mathbf{1}\right|\right)\left(f(\alpha)+f(\beta)-2f\left(\frac{\alpha+\beta}{2}\right)\right),
\end{aligned}$$

where the expression R that multiplies the modulus c can be written in the following form:

$$\begin{aligned}
R=&(\alpha+\beta)\sum_{i=1}^{n}\Phi_i(X_i)-\alpha\beta\mathbf{1}-\left(\sum_{i=1}^{n}\Phi_i(X_i)\right)^2-\frac{(\beta-\alpha)^2}{4}\mathbf{1}+\frac{\beta-\alpha}{2}\left|\sum_{i=1}^{n}\Phi_i(X_i)-\frac{\alpha+\beta}{2}\mathbf{1}\right|\\
&+(\alpha+\beta)\sum_{i=1}^{n}\Phi_i(X_i)-\alpha\beta\mathbf{1}-\sum_{i=1}^{n}\Phi_i(X_i^2).
\end{aligned}$$

Since

$$(\beta-\alpha)^2\left[\lambda-\lambda^2-\frac{1}{2}\min\{\lambda,1-\lambda\}\right]\geq 0,$$

that is,

$$(\beta-\alpha)^2\left[\lambda-\lambda^2-\frac{1}{2}\left(\frac{1}{2}\left(1-|1-2\lambda|\right)\right)\right]\geq 0,$$

substituting $\lambda=\frac{\beta-t}{\beta-\alpha}$ *(where* $t\in[\alpha,\beta]$*), it follows*

$$(t-\alpha)(\beta-t)-\frac{1}{4}(\beta-\alpha)^2\left(1-\frac{2}{\beta-\alpha}\left|t-\frac{\alpha+\beta}{2}\right|\right)\geq 0.$$

Now, since $\alpha\mathbf{1}\leq(\alpha+\beta)\mathbf{1}-\sum_{i=1}^n\Phi_i(X_i)\leq\beta\mathbf{1}$*, we can substitute* t *by* $(\alpha+\beta)\mathbf{1}-\sum_{i=1}^n\Phi_i(X_i)$*, and we obtain*

$$(\alpha+\beta)\sum_{i=1}^n\Phi_i(X_i)-\alpha\beta\mathbf{1}-\left(\sum_{i=1}^n\Phi_i(X_i)\right)^2-\frac{1}{4}(\beta-\alpha)^2\mathbf{1}+\frac{\beta-\alpha}{2}\left|\sum_{i=1}^n\Phi_i(X_i)-\frac{\alpha+\beta}{2}\mathbf{1}\right|\geqslant 0. \tag{3.13}$$

Moreover, $\alpha\mathbf{1}\leqslant X_i\leqslant\beta\mathbf{1}$ *implies* $(X_i-\alpha\mathbf{1})(\beta\mathbf{1}-X_i)\geqslant 0$*, so we have*

$$(\alpha+\beta)\sum_{i=1}^n\Phi_i(X_i)-\alpha\beta\mathbf{1}-\sum_{i=1}^n\Phi_i(X_i^2)\geqslant 0. \tag{3.14}$$

Adding (3.13) *and* (3.14)*, we find that the term with* c *is greater or equivalent to* 0.

This remark highlights an important structural property of the refinement term. Namely, its monotonicity in the modulus c ensures that the improvement becomes sharper as the strength of convexity increases.

If we add the operator convexity assumption, then we derive the following results.

Theorem 3.2. *Let the assumptions of Theorem 3.1 hold. If, in addition, the function* $g=f-c\cdot(.)^2$ *is operator convex, then*

$$\begin{aligned}
&f\left((\alpha+\beta)\mathbf{1}-\sum_{i=1}^n\Phi_i(X_i)\right) \qquad (3.15)\\
&\leqslant\sum_{i=1}^n\Phi_i\left(f\left((\alpha+\beta)\mathbf{1}-X_i\right)\right)\\
&\leqslant\frac{\sum_{i=1}^n\Phi_i(X_i)-\alpha\mathbf{1}}{\beta-\alpha}f(\alpha)+\frac{\beta\mathbf{1}-\sum_{i=1}^n\Phi_i(X_i)}{\beta-\alpha}f(\beta)-c\sum_{i=1}^n\Phi_i\left((X_i-\alpha\mathbf{1})(\beta\mathbf{1}-X_i)\right)-\Delta_c(\alpha,\beta,X,f)\\
&\leqslant\frac{\sum_{i=1}^n\Phi_i(X_i)-\alpha\mathbf{1}}{\beta-\alpha}f(\alpha)+\frac{\beta\mathbf{1}-\sum_{i=1}^n\Phi_i(X_i)}{\beta-\alpha}f(\beta)\\
&\leqslant(f(\alpha)+f(\beta))\mathbf{1}-\sum_{i=1}^n\Phi_i(f(X_i))-c\sum_{i=1}^n\Phi_i\left((X_i-\alpha\mathbf{1})(\beta\mathbf{1}-X_i)\right)-\Delta_c(\alpha,\beta,X,f)
\end{aligned}$$

$$\leqslant f(\alpha)\mathbf{1} + f(\beta)\mathbf{1} - \sum_{i=1}^{n} \Phi_i(f(X_i)) - c\sum_{i=1}^{n} \Phi_i((X_i - \alpha\mathbf{1})(\beta\mathbf{1} - X_i))$$
$$\leqslant f(\alpha)\mathbf{1} + f(\beta)\mathbf{1} - \sum_{i=1}^{n} \Phi_i(f(X_i)),$$

where

$$\Delta_c(\alpha,\beta,X,f) = \left[\frac{1}{2}\mathbf{1} - \frac{1}{\beta-\alpha}\sum_{i=1}^{n}\Phi_i\left(\left|X_i - \frac{\alpha+\beta}{2}\mathbf{1}\right|\right)\right]\mathcal{J}_c(f;\alpha,\beta),$$

and $\mathcal{J}_c(f,\alpha,\beta)$ *is defined by (3.6).*

Proof. Suppose that all the assumptions of the previous theorem hold. Recall that the function f is strongly convex with a modulus $c > 0$ if and only if the function $g = f - c\cdot(.)^2$ is convex (Lemma 1.1). If, in addition, $g = f - c\cdot(.)^2$ is operator convex, then, by (2.1), we have

$$g\left((\alpha+\beta)\mathbf{1} - \sum_{i=1}^{n}\Phi_i(X_i)\right) = g\left(\sum_{i=1}^{n}\Phi_i((\alpha+\beta)\mathbf{1} - X_i)\right) \leqslant \sum_{i=1}^{n}\Phi_i(g((\alpha+\beta)\mathbf{1} - X_i)). \tag{3.16}$$

Returning back to f, we get

$$f\left((\alpha+\beta)\mathbf{1} - \sum_{i=1}^{n}\Phi_i(X_i)\right) \tag{3.17}$$
$$\leqslant \sum_{i=1}^{n}\Phi_i(f((\alpha+\beta)\mathbf{1} - X_i)) - c\left[\sum_{i=1}^{n}\Phi_i\left(((\alpha+\beta)\mathbf{1} - X_i)^2\right) - \left((\alpha+\beta)\mathbf{1} - \sum_{i=1}^{n}\Phi_i(X_i)\right)^2\right]$$
$$\leqslant \sum_{i=1}^{n}\Phi_i(f((\alpha+\beta)\mathbf{1} - X_i)),$$

where we use the fact that

$$\sum_{i=1}^{n}\Phi_i\left(((\alpha+\beta)\mathbf{1} - X_i)^2\right) - \left((\alpha+\beta)\mathbf{1} - \sum_{i=1}^{n}\Phi_i(X_i)\right)^2 \geqslant 0$$

as the consequence of (2.1) applied to the square function $(.)^2$. Next, if we substitute X with $(\alpha+\beta)\mathbf{1} - X_i$ in (3.5), we have

$$f((\alpha+\beta)\mathbf{1} - X_i) \leqslant \frac{X_i - \alpha\mathbf{1}}{\beta-\alpha}f(\alpha) + \frac{\beta\mathbf{1} - X_i}{\beta-\alpha}f(\beta) - c(X_i - \alpha\mathbf{1})(\beta\mathbf{1} - X_i)$$
$$-\left(\frac{1}{2}\mathbf{1} - \frac{1}{\beta-\alpha}\left|X_i - \frac{\alpha+\beta}{2}\mathbf{1}\right|\right)\mathcal{J}_c(f,\alpha,\beta). \tag{3.18}$$

We denote

$$\Delta_c(\alpha,\beta,X_i,f) = \left(\frac{1}{2}\mathbf{1} - \frac{1}{\beta-\alpha}\left|X_i - \frac{\alpha+\beta}{2}\mathbf{1}\right|\right)\mathcal{J}_c(f,\alpha,\beta).$$

Since $\alpha\mathbf{1} \leq X_i \leq \beta\mathbf{1}$, we have $-\frac{\beta-\alpha}{2}\mathbf{1} \leq X_i - \frac{\alpha+\beta}{2}\mathbf{1} \leq \frac{\beta-\alpha}{2}\mathbf{1}$. Hence,

$$\left|X_i - \frac{\alpha+\beta}{2}\mathbf{1}\right| \leq \frac{\beta-\alpha}{2}\mathbf{1}.$$

Dividing both sides by $\beta - \alpha > 0$, we obtain $\frac{1}{\beta-\alpha}\left|X_i - \frac{\alpha+\beta}{2}\mathbf{1}\right| \leq \frac{1}{2}\mathbf{1}$. Therefore,

$$\frac{1}{2}\mathbf{1} - \frac{1}{\beta-\alpha}\left|X_i - \frac{\alpha+\beta}{2}\mathbf{1}\right| \geq 0.$$

That is, $\Delta_c(\alpha,\beta,X_i,f) \geq 0$. Furthermore, we have

$$\begin{aligned}
&\frac{X_i - \alpha\mathbf{1}}{\beta-\alpha} f(\alpha) + \frac{\beta\mathbf{1} - X_i}{\beta-\alpha} f(\beta) - c(X_i - \alpha\mathbf{1})(\beta\mathbf{1} - X_i) - \Delta_c(\alpha,\beta,X_i,f) \\
&= (f(\alpha) + f(\beta))\,\mathbf{1} - \left[\frac{\beta\mathbf{1} - X_i}{\beta-\alpha} f(\alpha) + \frac{X_i - \alpha\mathbf{1}}{\beta-\alpha} f(\beta)\right] \\
&\quad - c(X_i - \alpha\mathbf{1})(\beta\mathbf{1} - X_i) - \Delta_c(\alpha,\beta,X_i,f) \\
&\leqslant (f(\alpha) + f(\beta))\,\mathbf{1} - \left[\frac{\beta\mathbf{1} - X_i}{\beta-\alpha} f(\alpha) + \frac{X_i - \alpha\mathbf{1}}{\beta-\alpha} f(\beta)\right] \\
&= (f(\alpha) + f(\beta))\,\mathbf{1} - f(X_i) - c(X_i - \alpha\mathbf{1})(\beta\mathbf{1} - X_i) - \Delta_c(\alpha,\beta,X_i,f) \\
&\quad - \left[\frac{\beta\mathbf{1} - X_i}{\beta-\alpha} f(\alpha) + \frac{X_i - \alpha\mathbf{1}}{\beta-\alpha} f(\beta) - f(X_i) - c(X_i - \alpha\mathbf{1})(\beta\mathbf{1} - X_i) - \Delta_c(\alpha,\beta,X_i,f)\right],
\end{aligned} \tag{3.19}$$

where, by (3.5), we have

$$\frac{\beta\mathbf{1} - X_i}{\beta-\alpha} f(\alpha) + \frac{X_i - \alpha\mathbf{1}}{\beta-\alpha} f(\beta) - f(X_i) - c(X_i - \alpha\mathbf{1})(\beta\mathbf{1} - X_i) - \Delta_c(\alpha,\beta,X_i,f) \geqslant 0.$$

Now, combining (3.18) and (3.19), we get

$$\begin{aligned}
f((\alpha+\beta)\mathbf{1} - X_i) &\leqslant \frac{X_i - \alpha\mathbf{1}}{\beta-\alpha} f(\alpha) + \frac{\beta\mathbf{1} - X_i}{\beta-\alpha} f(\beta) - c(X_i - \alpha\mathbf{1})(\beta\mathbf{1} - X_i) - \Delta_c(\alpha,\beta,X_i,f) \\
&\leqslant (f(\alpha) + f(\beta))\,\mathbf{1} - \left[\frac{\beta\mathbf{1} - X_i}{\beta-\alpha} f(\alpha) + \frac{X_i - \alpha\mathbf{1}}{\beta-\alpha} f(\beta)\right] \\
&= \frac{X_i - \alpha\mathbf{1}}{\beta-\alpha} f(\alpha) + \frac{\beta\mathbf{1} - X_i}{\beta-\alpha} f(\beta) \\
&\leqslant (f(\alpha) + f(\beta))\,\mathbf{1} - f(X_i) - c(X_i - \alpha\mathbf{1})(\beta\mathbf{1} - X_i) - \Delta_c(\alpha,\beta,X_i,f) \\
&\leqslant (f(\alpha) + f(\beta))\,\mathbf{1} - f(X_i) - c(X_i - \alpha\mathbf{1})(\beta\mathbf{1} - X_i) \\
&\leqslant (f(\alpha) + f(\beta))\,\mathbf{1} - f(X_i).
\end{aligned}$$

Applying linear mapping Φ_i and summing over $i, i = 1, ..., n$, we get (3.15), where the first inequality follows by (3.17).

Remark 3.5. *Consequently, the sequence of inequalities in (3.15) provides a strict improvement of (2.3).*

Remark 3.6. *Notice that in the previous Theorem 3.2, we have additional condition that $f - c \cdot (.)^2$ is operator convex.*

Obviously, if $f - c \cdot (.)^2$ is operator convex, then f is strongly convex with a modulus $c > 0$, and f is also operator convex (since $(.)^2$ is operator convex).

In the next section, we demonstrate how the obtained results can be applied to derive new bounds for operator quasi-arithmetic means of the Mercer type, as well as for operator power means, illustrating the practical impact of our theoretical contributions.

4. Applications to operator quasi-arithmetic means

In this section, we apply the previously obtained results to derive new inequalities involving operator quasi-arithmetic means, which are fundamental tools in the theory of operator inequalities. To begin, we recall the notion of the operator quasi-arithmetic mean and refer the reader to [22] for further details.

Let α and β be real numbers with $0 < \alpha < \beta$. Consider an n-tuple of self-adjoint operators $\boldsymbol{X} = (X_1, \ldots, X_n)$, where each X_j belongs to $\mathbf{B}_h(\mathcal{H})$ and has a spectrum $Sp\left(X_j\right)$ contained in $[\alpha,\beta]$. Let $\boldsymbol{\Phi} = (\Phi_1, \ldots, \Phi_n)$ be a family of positive linear mappings from the unital C^*-algebra $\mathbf{B}_h(\mathcal{H})$ to another unital C^*-algebra $\mathbf{B}_h(\mathcal{K})$, such that $\Phi = \sum_{j=1}^n \Phi_i$ is unital.

For a continuous strictly monotone function φ defined on $[\alpha,\beta]$, the operator quasi-arithmetic mean of the Mercer type is defined as

$$\mathcal{M}_\varphi(\boldsymbol{X}, \boldsymbol{\Phi}) = \varphi^{-1}\left((\varphi(\alpha) + \varphi(\beta))\mathbf{1} - \sum_{j=1}^n \Phi_j\left(\varphi\left(X_j\right)\right)\right). \tag{4.1}$$

In the previous notation, we derive the following results.

Theorem 4.1. *Let $\alpha,\beta \in \mathbb{R}$ and $0 < \alpha < \beta$. Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be n-tuple of self-adjoint operators $X_j \in \mathbf{B}_h(\mathcal{H})$, $j = 1, ..., n$, with $Sp\left(X_j\right) \subseteq [\alpha,\beta]$. Let $\boldsymbol{\Phi} = (\Phi_1, \ldots, \Phi_n)$ be n-tuple of positive linear mappings from the unital C^*-algebra $\mathbf{B}_h(\mathcal{H})$ to the unital C^*-algebra $\mathbf{B}_h(\mathcal{K})$ such that $\Phi = \sum_{j=1}^n \Phi_i$ is unital. Let $\varphi, g \in C([\alpha,\beta])$ be two strictly monotone functions such that φ^{-1} is operator increasing and $\varphi \circ g^{-1}$ is a strongly convex function with a modulus $c > 0$ on $[m_g, M_g]$, where $m_g = \min\{g(\alpha), g(\beta)\}$, $M_g = \max\{g(\alpha), g(\beta)\}$. Then*

$$\begin{aligned}\mathcal{M}_g(\boldsymbol{X}, \boldsymbol{\Phi}) &\leqslant \varphi^{-1}\left(\varphi\left(\mathcal{M}_\varphi(\boldsymbol{X}, \boldsymbol{\Phi})\right) - 2cK(m_g, M_g, g(\mathbf{X}),\boldsymbol{\Phi}) - \Delta_c(m_g, M_g, g(\bar{X}), \varphi \circ g^{-1})\right) \\ &\leqslant \varphi^{-1}\left(\varphi\left(\mathcal{M}_\varphi(\boldsymbol{X}, \boldsymbol{\Phi})\right) - 2cK(m_g, M_g, g(\mathbf{X}),\boldsymbol{\Phi})\right) \\ &\leqslant \mathcal{M}_\varphi(\boldsymbol{X}, \boldsymbol{\Phi}),\end{aligned} \tag{4.2}$$

where

$$\mathcal{M}_g(\boldsymbol{X}, \boldsymbol{\Phi}) = g^{-1}\left(\left(m_g + M_g\right)\mathbf{1} - \sum_{i=1}^n \Phi_i(g(X_i))\right) \tag{4.3}$$

and

$$\begin{aligned}K(m_g, M_g, g(\mathbf{X}),\boldsymbol{\Phi}) = &\left(m_g + M_g\right)\sum_{i=1}^n \Phi_i(g(X_i)) - m_g M_g \mathbf{1} \\ &- \frac{1}{2}\left(\sum_{i=1}^n \Phi_i\left(g^2(X_i)\right) + \left(\sum_{i=1}^n \Phi_i(g(X_i))\right)^2\right),\end{aligned} \tag{4.4}$$

while

$$\Delta_c(m_g, M_g, g(\bar{X}), \varphi \circ g^{-1}) = \left(\frac{1}{2}\mathbf{1} - \frac{1}{M_g - m_g}\left|\sum_{i=1}^n \Phi_i(g(X_i)) - \frac{m_g + M_g}{2}\mathbf{1}\right|\right) \times \mathcal{J}_c(\varphi \circ g^{-1}, m_g, M_g) \tag{4.5}$$

with

$$\mathcal{J}_c(\varphi \circ g^{-1}, m_g, M_g) = \varphi(\alpha) + \varphi(\beta) - 2(\varphi \circ g^{-1})\left(\frac{m_g + M_g}{2}\right) - \frac{c}{2}\left(M_g - m_g\right)^2.$$

Proof. Obviously, if $Y_i = g(X_i)$, then $Sp(Y_i) \subseteq \left[m_g, M_g\right]$.

Applying Theorem 3.1 to the operators $Y_i = g(X_i)$ with $f = \varphi \circ g^{-1}$, we obtain

$$\begin{aligned}
&\left(\varphi \circ g^{-1}\right)\left(\left(m_g + M_g\right)\mathbf{1} - \sum_{i=1}^{n} \Phi_i(g(X_i))\right) \qquad (4.6)\\
&\leqslant \left(\left(\varphi \circ g^{-1}\right)\left(m_g\right) + \left(\varphi \circ g^{-1}\right)(M_g)\right)\mathbf{1} - \sum_{i=1}^{n} \Phi_i(\left(\varphi \circ g^{-1}\right)(g(X_i)))\\
&\quad - 2cK(m_g, M_g, g(\mathbf{X}), \mathbf{\Phi}) - \Delta_c(m_g, M_g, g(\bar{X}), \varphi \circ g^{-1})\\
&\leqslant \left(\left(\varphi \circ g^{-1}\right)(g(\alpha)) + \left(\varphi \circ g^{-1}\right)(g(\beta))\right)\mathbf{1} - \sum_{i=1}^{n} \Phi_i(\left(\varphi \circ g^{-1}\right)(g(X_i))) - 2cK(m_g, M_g, g(\mathbf{X}), \mathbf{\Phi})\\
&\leqslant \left(\left(\varphi \circ g^{-1}\right)(g(\alpha)) + \left(\varphi \circ g^{-1}\right)(g(\beta))\right)\mathbf{1} - \sum_{i=1}^{n} \Phi_i(\left(\varphi \circ g^{-1}\right)(g(X_i))),
\end{aligned}$$

where $K(m_g, M_g, g(\mathbf{X}), \mathbf{\Phi})$ and $\Delta_c(m_g, M_g, g(\bar{X}), \varphi \circ g^{-1})$ are defined by (4.4) and (4.5), respectively.

Moreover, since

$$\{m_g, M_g\} = \{g(\alpha), g(\beta)\},$$

we have

$$(\varphi \circ g^{-1})(m_g) + (\varphi \circ g^{-1})(M_g) = \varphi(\alpha) + \varphi(\beta),$$

independently of whether g is increasing or decreasing.

Hence,

$$\begin{aligned}
&\left(\varphi \circ g^{-1}\right)\left(\left(m_g + M_g\right)\mathbf{1} - \sum_{i=1}^{n} \Phi_i(g(X_i))\right)\\
&\leqslant (\varphi(\alpha) + \varphi(\beta))\,\mathbf{1} - \sum_{i=1}^{n} \Phi_i(\varphi(X_i)) - 2cK(m_g, M_g, g(\mathbf{X}), \mathbf{\Phi}) - \Delta_c(m_g, M_g, g(\bar{X}), \varphi \circ g^{-1})\\
&\leqslant (\varphi(\alpha) + \varphi(\beta))\,\mathbf{1} - \sum_{i=1}^{n} \Phi_i(\varphi(X_i)) - 2cK(m_g, M_g, g(\mathbf{X}), \mathbf{\Phi})\\
&\leqslant (\varphi(\alpha) + \varphi(\beta))\,\mathbf{1} - \sum_{i=1}^{n} \Phi_i(\varphi(X_i)).
\end{aligned}$$

Finally, since φ^{-1} is operator increasing, applying φ^{-1} to both sides yields inequality (4.2).

Theorem 4.2. *Let $\alpha,\beta \in \mathbb{R}$ and $0 < \alpha < \beta$. Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be n-tuple of self-adjoint operators $X_j \in \mathbf{B}_h(\mathcal{H})$, $j = 1, ..., n$, with $Sp\left(X_j\right) \subseteq [\alpha,\beta]$. Let $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_n)$ be an n-tuple of positive linear mappings from the unital C^*-algebra $\mathbf{B}_h(\mathcal{H})$ to the unital C^*-algebra $\mathbf{B}_h(\mathcal{K})$ such that $\Phi = \sum_{j=1}^{n} \Phi_i$ is unital. Let $\varphi, g \in C([\alpha,\beta])$ be two strictly monotone functions such that φ^{-1} is operator increasing and $\varphi \circ g^{-1}$ is an operator convex function on $[m_g, M_g]$, where $m_g = \min\{g(\alpha), g(\beta)\}$, $M_g = \max\{g(\alpha), g(\beta)\}$. Then*

$$\mathcal{M}_g(\boldsymbol{X}, \mathbf{\Phi}) \leqslant \varphi^{-1}\left(\sum_{i=1}^{n} \Phi_i\left(\left(\varphi \circ g^{-1}\right)((g(\alpha) + g(\beta))\mathbf{1} - g(X_i))\right)\right) \qquad (4.7)$$

$$
\begin{aligned}
&\leqslant \varphi^{-1}\left(\frac{\sum_{i=1}^n \Phi_i(g(X_i)) - m_g\mathbf{1}}{M_g - m_g}\varphi(\alpha) + \frac{M_g\mathbf{1} - \sum_{i=1}^n \Phi_i(g(X_i))}{M_g - m_g}\varphi(\beta)\right. \\
&\quad \left. -c\sum_{i=1}^n \Phi_i\left((g(X_i) - m_g\mathbf{1})(M_g\mathbf{1} - g(X_i))\right) - \Delta_c(m_g, M_g, g(\bar{X}), \varphi \circ g^{-1}))\right) \\
&\leqslant \varphi^{-1}\left(\frac{\sum_{i=1}^n \Phi_i(g(X_i)) - m_g\mathbf{1}}{M_g - m_g}\varphi(\alpha) + \frac{M_g\mathbf{1} - \sum_{i=1}^n \Phi_i(g(X_i))}{M_g - m_g}\varphi(\beta)\right) \\
&\leqslant \varphi^{-1}\left(\varphi\left(\mathcal{M}_\varphi(\boldsymbol{X}, \boldsymbol{\Phi})\right) - c\sum_{i=1}^n \Phi_i\left((g(X_i) - m_g\mathbf{1})(M_g\mathbf{1} - g(X_i))\right) - \Delta_c(m_g, M_g, g(\bar{X}), \varphi \circ g^{-1}))\right) \\
&\leqslant \varphi^{-1}\left(\varphi\left(\mathcal{M}_\varphi(\boldsymbol{X}, \boldsymbol{\Phi})\right) - c\sum_{i=1}^n \Phi_i\left(\left[g(X_i) - m_g\mathbf{1})\right]\left[M_g\mathbf{1} - g(X_i)\right]\right)\right) \\
&\leqslant \mathcal{M}_\varphi(\boldsymbol{X}, \boldsymbol{\Phi}),
\end{aligned}
$$

where $\mathcal{M}_g(\boldsymbol{X}, \boldsymbol{\Phi})$ and $\Delta_c(m_g, M_g, g(\bar{X}), \varphi \circ g^{-1})$ are defined by (4.3) and (4.5), respectively.

Proof. If we substitute f by $\varphi \circ g^{-1}$, X_i by $Y_i = g(X_i)$, α by m_g, and β by M_g in (3.15), we have

$$
\begin{aligned}
\varphi\left(\mathcal{M}_g(\boldsymbol{X}, \boldsymbol{\Phi})\right) &\leqslant \sum_{i=1}^n \Phi_i\left(\left(\varphi \circ g^{-1}\right)((g(\alpha) + g(\beta))\mathbf{1} - g(X_i))\right) \qquad (4.8)\\
&\leqslant \frac{\sum_{i=1}^n \Phi_i(g(X_i)) - m_g\mathbf{1}}{M_g - m_g}\varphi(\alpha) + \frac{M_g\mathbf{1} - \sum_{i=1}^n \Phi_i(g(X_i))}{M_g - m_g}\varphi(\beta) \\
&\quad - c\sum_{i=1}^n \Phi_i\left((g(X_i) - m_g\mathbf{1})(M_g\mathbf{1} - g(X_i))\right) - \Delta_c(g(\alpha), g(\beta), g(X), \varphi \circ g^{-1}) \\
&\leqslant \frac{\sum_{i=1}^n \Phi_i(g(X_i)) - m_g\mathbf{1}}{M_g - m_g}\varphi(\alpha) + \frac{M_g\mathbf{1} - \sum_{i=1}^n \Phi_i(g(X_i))}{M_g - m_g}\varphi(\beta) \\
&\leqslant [\varphi(\alpha) + \varphi(\beta)]\,\mathbf{1} - \sum_{i=1}^n \Phi_i(\varphi(X_i)) - c\sum_{i=1}^n \Phi_i\left((g(X_i) - m_g\mathbf{1})(M_g\mathbf{1} - g(X_i))\right) \\
&\quad - \Delta_c(g(\alpha), g(\beta), g(X), \varphi \circ g^{-1}) \\
&\leqslant [\varphi(\alpha) + \varphi(\beta)]\,\mathbf{1} - \sum_{i=1}^n \Phi_i(\varphi(X_i)) - c\sum_{i=1}^n \Phi_i\left((g(X_i) - m_g\mathbf{1})(M_g\mathbf{1} - g(X_i))\right) \\
&\leqslant [\varphi(\alpha) + \varphi(\beta)]\,\mathbf{1} - \sum_{i=1}^n \Phi_i(\varphi(X_i)).
\end{aligned}
$$

Moreover, since φ^{-1} is operator increasing, then (4.7) follows immediately from (4.8).

5. Applications to operator means

A particularly important subclass of Mercer-type quasi-arithmetic means (4.1) is obtained by choosing the generating function φ as follows:

$$
\varphi(t) = \begin{cases} t^s, & s \neq 0, \\ \log t, & s = 0. \end{cases}
$$

In this case, the corresponding quasi-arithmetic mean $\mathcal{M}_{\varphi}(\boldsymbol{X},\boldsymbol{\Phi})$ reduces to the operator power mean of the Mercer type, given explicitly by

$$\mathcal{M}_s(\boldsymbol{X},\boldsymbol{\Phi}) = \begin{cases} \left[(\alpha^s+\beta^s)\,\mathbf{1}-\sum_{j=1}^{n}\Phi_j\left(X_j^s\right)\right]^{\frac{1}{s}}, & s\neq 0,\\ \exp\left((\log\alpha\beta)\,\mathbf{1}-\sum_{j=1}^{n}\Phi_j\left(\log X_j\right)\right), & s=0. \end{cases} \tag{5.1}$$

We list important special cases arising from specific values of s as follows.

- Arithmetic mean ($s=1$)
$$\mathcal{M}_1(\boldsymbol{X},\boldsymbol{\Phi}) = (\alpha+\beta)\mathbf{1}-\sum_{j=1}^{n}\Phi_j(X_j).$$
- Geometric mean ($s=0$)
$$\mathcal{M}_0(\boldsymbol{X},\boldsymbol{\Phi}) = \exp\Big(\log(\alpha\beta)\mathbf{1}-\sum_{j=1}^{n}\Phi_j(\log X_j)\Big).$$
- Harmonic mean ($s=-1$)
$$\mathcal{M}_{-1}(\boldsymbol{X},\boldsymbol{\Phi}) = \Big((\alpha^{-1}+\beta^{-1})\mathbf{1}-\sum_{j=1}^{n}\Phi_j(X_j^{-1})\Big)^{-1}.$$

Specifically, for $n=2$ and the mappings

$$\Phi_1(T)=(1-w)T,\qquad \Phi_2(T)=wT,\qquad w\in[0,1],$$

we obtain the weighted two-variable version

$$\mathcal{M}_s(X_1,X_2,w) = \begin{cases} \Big((\alpha^s+\beta^s)\mathbf{1}-((1-w)X_1^s+wX_2^s)\Big)^{1/s}, & s\neq 0,\\ \exp(\log(\alpha\beta)\mathbf{1}-(1-w)\log X_1-w\log X_2), & s=0, \end{cases}$$

and the particular cases

$$\begin{aligned} \mathcal{M}_1(X_1,X_2,w) &= (\alpha+\beta)\mathbf{1}-((1-w)X_1+wX_2),\\ \mathcal{M}_0(X_1,X_2,w) &= \exp(\log(\alpha\beta)\mathbf{1}-(1-w)\log X_1-w\log X_2),\\ \mathcal{M}_{-1}(X_1,X_2,w) &= \Big((\alpha^{-1}+\beta^{-1})\mathbf{1}-((1-w)X_1^{-1}+wX_2^{-1})\Big)^{-1}. \end{aligned}$$

Using the previous two theorems for appropriate functions φ and g, we can derive the corresponding inequalities for operator power means as follows.

Theorem 5.1. *Let $\alpha,\beta\in\mathbb{R}$ and $0<\alpha<\beta$. Let $\boldsymbol{X}=(X_1,\ldots,X_n)$ be an n-tuple of self-adjoint operators $X_j\in\mathbf{B}_h(\mathcal{H})$, $j=1,\ldots,n$, with $Sp\left(X_j\right)\subseteq[\alpha,\beta]$. Let $\boldsymbol{\Phi}=(\Phi_1,\ldots,\Phi_n)$ be an n-tuple of positive linear mappings from the unital C^*-algebra $\mathbf{B}_h(\mathcal{H})$ to the unital C^*-algebra $\mathbf{B}_h(\mathcal{K})$ such that $\Phi=\sum_{j=1}^{n}\Phi_i$ is unital. Let $r,s\in\mathbb{R}$, $s\geqslant 1$. Then we have the following:*

(1)

$$\begin{aligned}\mathcal{M}_r(\boldsymbol{X},\boldsymbol{\Phi}) &\leqslant \left[(\mathcal{M}_s(\boldsymbol{X},\boldsymbol{\Phi}))^s - 2c_1K(\alpha^r,\beta^r,\boldsymbol{X}^r,\boldsymbol{\Phi}) - \Delta_{c_1}\left(\alpha^r,\beta^r,\bar{X}^r,(.)^{\frac{s}{r}}\right)\right]^{\frac{1}{s}} \\ &\leqslant \left[(\mathcal{M}_s(\boldsymbol{X},\boldsymbol{\Phi}))^s - 2c_1K(\alpha^r,\beta^r,\boldsymbol{X}^r,\boldsymbol{\Phi})\right]^{\frac{1}{s}} \\ &\leqslant \mathcal{M}_s(\boldsymbol{X},\boldsymbol{\Phi}),\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}c_1 &= \frac{s}{2r}\left(\frac{s}{r}-1\right)\alpha^{s-2r}, \textit{ for } 0 < r \leqslant \frac{1}{2}s \textit{ or for } r < 0, \\ c_1 &= \frac{s}{2r}\left(\frac{s}{r}-1\right)\beta^{s-2r}, \textit{ for } 0 < r < s < 2r,\end{aligned}$$

$$K(\alpha^r,\beta^r,\boldsymbol{X}^r,\boldsymbol{\Phi}) = (\alpha^r+\beta^r)\sum_{i=1}^n \Phi_i(X_i^r) - \alpha^r\beta^r\mathbf{1} - \frac{1}{2}\left(\sum_{i=1}^n \Phi_i\left(X_i^{2r}\right) + \left(\sum_{i=1}^n \Phi_i(X_i^r)\right)^2\right), \tag{5.3}$$

$$\Delta_{c_1}\left(\alpha^r,\beta^r,\bar{X}^r,(\cdot)^{\frac{s}{r}}\right) = \left(\frac{1}{2}\mathbf{1} - \frac{1}{|\beta^r-\alpha^r|}\left|\sum_{i=1}^n \Phi_i(X_i^r) - \frac{\alpha^r+\beta^r}{2}\mathbf{1}\right|\right)\times \mathcal{J}_{c_1}((\cdot)^{\frac{s}{r}},\alpha^r,\beta^r), \tag{5.4}$$

and

$$\mathcal{J}_{c_1}((\cdot)^{\frac{s}{r}},\alpha^r,\beta^r) = \alpha^s+\beta^s - 2\left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{s}{r}} - \frac{c_1}{2}(\beta^r-\alpha^r)^2.$$

(2)

$$\begin{aligned}\mathcal{M}_0(\boldsymbol{X},\boldsymbol{\Phi}) &\leqslant \left[(\mathcal{M}_s(\boldsymbol{X},\boldsymbol{\Phi}))^s - 2c_1K(\log\alpha,\log\beta,\log\boldsymbol{X},\boldsymbol{\Phi})\right. \\ &\qquad \left. - \Delta_{c_1}\left(\log\alpha,\log\beta,\log\bar{X},\exp(s\cdot id)\right)\right]^{\frac{1}{s}} \\ &\leqslant \left[(\mathcal{M}_s(\boldsymbol{X},\boldsymbol{\Phi}))^s - 2c_1K(\log\alpha,\log\beta,\log\boldsymbol{X},\boldsymbol{\Phi})\right]^{\frac{1}{s}} \\ &\leqslant \mathcal{M}_s(\boldsymbol{X},\boldsymbol{\Phi}),\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}&c_1 = \frac{1}{2}s^2\alpha^s, \\ &K(\log\alpha,\log\beta,\log\boldsymbol{X},\boldsymbol{\Phi}) \\ &= \log(\alpha\beta)\sum_{i=1}^n \Phi_i(\log(X_i)) - \log\alpha\log\beta\mathbf{1} - \frac{1}{2}\left(\sum_{i=1}^n \Phi_i\left(\log^2(X_i)\right) + \left(\sum_{i=1}^n \Phi_i(\log(X_i))\right)^2\right), \\ &\Delta_{c_1}\left(\log\alpha,\log\beta,\log\bar{X},\exp(s\cdot id)\right) \\ &= \left(\frac{1}{2}\mathbf{1} - \frac{1}{\log\frac{\beta}{\alpha}}\left|\sum_{i=1}^n \Phi_i(\log X_i) - \frac{1}{2}\log(\alpha\beta)\mathbf{1}\right|\right)\times\left(\alpha^s+\beta^s) - 2(\alpha\beta)^{\frac{s}{2}} - \frac{c_1}{2}\log^2\left(\frac{\beta}{\alpha}\right)\right).\end{aligned}$$

Proof. (1) We apply inequality (4.2) to $\varphi(t) = t^s$, $g(t) = t^r$, where $t \in [\alpha,\beta]$. Using Remarks 1.1 and 2.1, we get (5.2).

(2) Analogously to case (1), for the functions $\varphi(t) = t^s$, $g(t) = \log t$.

Example 5.1. Consider the functions $\phi(t) = t^s$, $g(t) = t^r$, $t \in [\alpha,\beta]$, and $0 < \alpha < \beta$, where $r, s \in \mathbb{R}$ and $0 < r < s$. Then $\phi^{-1}(x) = x^{1/s}$ and $(\phi \circ g^{-1})(x) = x^{\frac{s}{r}}$. Since

$$(\phi \circ g^{-1})''(x) = \frac{s}{r}\left(\frac{s}{r} - 1\right)x^{\frac{s}{r}-2},$$

the function $\phi \circ g^{-1}$ is strongly convex on the corresponding interval $[\ln\alpha, \ln\beta]$. Hence, the assumptions of Theorem 5.1 are satisfied.

The modulus of strong convexity is given by

$$c_1 = \frac{s}{2r}\left(\frac{s}{r} - 1\right)\alpha^{s-2r}, \text{ for } 0 < r \leq \frac{1}{2}s \text{ or for } r < 0,$$

and

$$c_1 = \frac{s}{2r}\left(\frac{s}{r} - 1\right)\beta^{s-2r}, \text{ for } 0 < r < s < 2r.$$

Choosing $r = 1$ and $s = 3$, we have $\frac{s}{r} = 3$ and $0 < r \leq \frac{s}{2}$, and

$$c_1 = \frac{s}{2r}\left(\frac{s}{r} - 1\right)\alpha^{s-2r} = \frac{3}{2}(3-1)\alpha = 3\alpha.$$

The corresponding Mercer-type operator power mean becomes

$$\mathcal{M}_1(\boldsymbol{X}, \boldsymbol{\Phi}) = (\alpha + \beta)\mathbf{1} - \sum_{j=1}^{n} \Phi_j(X_j),$$

$$\mathcal{M}_3(\boldsymbol{X}, \boldsymbol{\Phi}) = \left[(\alpha^3 + \beta^3)\mathbf{1} - \sum_{j=1}^{n} \Phi_j(X_j^3)\right]^{\frac{1}{3}}.$$

Now inequality (5.2) reduces to

$$\begin{aligned} \mathcal{M}_1(\boldsymbol{X}, \boldsymbol{\Phi}) &\leqslant \left[\mathcal{M}_3(\boldsymbol{X}, \boldsymbol{\Phi})^3 - 2c_1 K(\alpha,\beta,\boldsymbol{X},\boldsymbol{\Phi}) - \Delta_{c_1}\right]^{\frac{1}{3}} \\ &\leqslant \left[\mathcal{M}_3(\boldsymbol{X}, \boldsymbol{\Phi})^3 - 2c_1 K(\alpha,\beta,\boldsymbol{X},\boldsymbol{\Phi})\right]^{\frac{1}{3}} \\ &\leqslant \mathcal{M}_3(\boldsymbol{X}, \boldsymbol{\Phi}), \end{aligned} \tag{5.6}$$

where

$$K(\alpha,\beta,\boldsymbol{X},\boldsymbol{\Phi}) = (\alpha + \beta)\sum_{i=1}^{n} \Phi_i(X_i) - \alpha\beta\mathbf{1} - \frac{1}{2}\left(\sum_{i=1}^{n} \Phi_i\left(X_i^2\right) + \left(\sum_{i=1}^{n} \Phi_i(X_i)\right)^2\right),$$

$$\mathcal{J}_{c_1}((\cdot)^3, \alpha,\beta) = \alpha^3 + \beta^3 - 2\left(\frac{\alpha+\beta}{2}\right)^3 - \frac{c_1}{2}(\beta - \alpha)^2,$$

$$\Delta_{c_1} = \left(\frac{1}{2}\mathbf{1} - \frac{1}{\beta - \alpha}\left|\sum_{i=1}^{n} \Phi_i(X_i) - \frac{\alpha+\beta}{2}\mathbf{1}\right|\right) \times \mathcal{J}_{c_1}((\cdot)^3, \alpha,\beta).$$

For numerical illustration, take $\alpha = 2$ and $\beta = 4$, and then $(\phi \circ g^{-1})(t) = t^3$ with

$$c_1 = 6.$$

Consequently,

$$\mathcal{J}_{c_1}((\cdot)^3, \alpha, \beta) = \mathcal{J}_6((\cdot)^3, 2, 4) = 6.$$

Consider

$$X_1 = \begin{pmatrix} 2.2 & 0 \\ 0 & 3.8 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 3.0 & 0 \\ 0 & 3.0 \end{pmatrix},$$

and positive linear mappings

$$\Phi_1(T) = \Phi_2(T) = \frac{1}{2}T.$$

Furthermore, we calculate

$$\sum_{j=1}^{2} \Phi_j(X_j) = \frac{1}{2}\left[\begin{pmatrix} 2.2 & 0 \\ 0 & 3.8 \end{pmatrix} + \begin{pmatrix} 3.0 & 0 \\ 0 & 3.0 \end{pmatrix}\right] = \begin{pmatrix} 2.6 & 0 \\ 0 & 3.4 \end{pmatrix},$$

$$\sum_{i=1}^{2} \Phi_i(X_i^2) = \frac{1}{2}\begin{pmatrix} 2.2^2 & 0 \\ 0 & 3.8^2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3.0^2 & 0 \\ 0 & 3.0^2 \end{pmatrix} = \begin{pmatrix} 6.92 & 0 \\ 0 & 11.72 \end{pmatrix},$$

$$\left(\sum_{i=1}^{2} \Phi_i(X_i)\right)^2 = \begin{pmatrix} 2.6^2 & 0 \\ 0 & 3.4^2 \end{pmatrix} = \begin{pmatrix} 6.76 & 0 \\ 0 & 11.56 \end{pmatrix},$$

$$\sum_{j=1}^{2} \Phi_j(X_j^3) = \frac{1}{2}\begin{pmatrix} 2.2^3 & 0 \\ 0 & 3.8^3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3.0^3 & 0 \\ 0 & 3.0^3 \end{pmatrix} = \begin{pmatrix} 18.824 & 0 \\ 0 & 40.936 \end{pmatrix},$$

and

$$\sum_{j=1}^{2} \Phi_j(X_j^3) = \frac{1}{2}X_1^3 + \frac{1}{2}X_2^3 = \begin{pmatrix} 18.824 & 0 \\ 0 & 40.936 \end{pmatrix}.$$

Then

$$\mathcal{M}_1(\boldsymbol{X}, \boldsymbol{\Phi}) = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} - \begin{pmatrix} 2.6 & 0 \\ 0 & 3.4 \end{pmatrix} = \begin{pmatrix} 3.4000 & 0 \\ 0 & 2.6000 \end{pmatrix},$$

$$(\mathcal{M}_3(\boldsymbol{X}, \boldsymbol{\Phi}))^3 = \begin{pmatrix} 72 & 0 \\ 0 & 72 \end{pmatrix} - \begin{pmatrix} 18.824 & 0 \\ 0 & 40.936 \end{pmatrix} = \begin{pmatrix} 53.176 & 0 \\ 0 & 31.064 \end{pmatrix},$$

and

$$\mathcal{M}_3(\boldsymbol{X}, \boldsymbol{\Phi}) = \begin{pmatrix} \sqrt[3]{53.176} & 0 \\ 0 & \sqrt[3]{31.064} \end{pmatrix} \approx \begin{pmatrix} 3.7605 & 0 \\ 0 & 3.1436 \end{pmatrix}.$$

Now we compute

$$K(\alpha, \beta, \boldsymbol{X}, \boldsymbol{\Phi}) = 6\begin{pmatrix} 2.6 & 0 \\ 0 & 3.4 \end{pmatrix} - \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} - \frac{1}{2}\left[\begin{pmatrix} 6.92 & 0 \\ 0 & 11.72 \end{pmatrix} + \begin{pmatrix} 6.76 & 0 \\ 0 & 11.56 \end{pmatrix}\right] = \begin{pmatrix} 0.76 & 0 \\ 0 & 0.76 \end{pmatrix}$$

and

$$2c_1K(\alpha,\beta,\boldsymbol{X},\boldsymbol{\Phi}) = 12K(\alpha,\beta,\boldsymbol{X},\boldsymbol{\Phi}) = \begin{pmatrix} 9.12 & 0 \\ 0 & 9.12 \end{pmatrix}.$$

We also have

$$\Delta_{c_1} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} - 3\left|\begin{pmatrix} 2.6 & 0 \\ 0 & 3.4 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right| = \begin{pmatrix} 1.8 & 0 \\ 0 & 1.8 \end{pmatrix}.$$

Now,

$$\begin{aligned} \mathcal{M}_3(\boldsymbol{X},\boldsymbol{\Phi})^3 - 2c_1K(\alpha,\beta,\boldsymbol{X},\boldsymbol{\Phi}) - \Delta_{c_1} &= \begin{pmatrix} 53.176 & 0 \\ 0 & 31.064 \end{pmatrix} - \begin{pmatrix} 9.12 & 0 \\ 0 & 9.12 \end{pmatrix} - \begin{pmatrix} 1.8 & 0 \\ 0 & 1.8 \end{pmatrix} \\ &= \begin{pmatrix} 42.256 & 0 \\ 0 & 20.144 \end{pmatrix}, \end{aligned}$$

and then

$$\left[\mathcal{M}_3(\boldsymbol{X},\boldsymbol{\Phi})^3 - 2c_1K(\alpha,\beta,\boldsymbol{X},\boldsymbol{\Phi}) - \Delta_{c_1}\right]^{\frac{1}{3}} = \begin{pmatrix} 3.4831 & 0 \\ 0 & 2.7205 \end{pmatrix}.$$

Moreover,

$$\mathcal{M}_3(\boldsymbol{X},\boldsymbol{\Phi})^3 - 2c_1K(\alpha,\beta,\boldsymbol{X},\boldsymbol{\Phi}) = \begin{pmatrix} 53.176 & 0 \\ 0 & 31.064 \end{pmatrix} - \begin{pmatrix} 9.12 & 0 \\ 0 & 9.12 \end{pmatrix} = \begin{pmatrix} 44.056 & 0 \\ 0 & 21.944 \end{pmatrix},$$

and then

$$\left[\mathcal{M}_3(\boldsymbol{X},\boldsymbol{\Phi})^3 - 2c_1K(\alpha,\beta,\boldsymbol{X},\boldsymbol{\Phi})\right]^{\frac{1}{3}} = \begin{pmatrix} 3.5319 & 0 \\ 0 & 2.8000 \end{pmatrix}.$$

So finally, we have

$$\begin{pmatrix} 3.4000 & 0 \\ 0 & 2.6000 \end{pmatrix} < \begin{pmatrix} 3.4831 & 0 \\ 0 & 2.7205 \end{pmatrix} < \begin{pmatrix} 3.5319 & 0 \\ 0 & 2.8000 \end{pmatrix} < \begin{pmatrix} 3.7605 & 0 \\ 0 & 3.1436 \end{pmatrix}.$$

Thus, inequality (5.6) is numerically verified.

Example 5.2. Let $\alpha = 2, \beta = 4$. As a special case of Theorem 5.1, setting $r = -1$, $s = 1$ in (5.5), we have $\varphi(t) = t$, $g(t) = t^{-1}$ and

$$c_1 = 8.$$

We then get a refinement of the operator harmonic-arithmetic inequality of the Mercer type

$$\mathcal{M}_{-1}(\boldsymbol{X},\boldsymbol{\Phi}) \leqslant \mathcal{M}_1(\boldsymbol{X},\boldsymbol{\Phi}) - 2\alpha^3K - \Delta_{c_1} \leqslant \mathcal{M}_1(\boldsymbol{X},\boldsymbol{\Phi}) - 2\alpha^3K \leqslant \mathcal{M}_1(\boldsymbol{X},\boldsymbol{\Phi}), \tag{5.7}$$

where

$$\begin{aligned} K &= \frac{3}{4}\sum_{i=1}^{n}\Phi_i(X_i^{-1}) - \frac{1}{8}\mathbf{1} - \frac{1}{2}\left(\sum_{i=1}^{n}\Phi_i\left(X_i^{-2}\right) + \left(\sum_{i=1}^{n}\Phi_i(X_i^{-1})\right)^2\right), \\ \Delta_{c_1} &= \left(\frac{1}{2}\mathbf{1} - 4\left|\sum_{i=1}^{n}\Phi_i(g(X_i)) - \frac{3}{8}\mathbf{1}\right|\right) \times \mathcal{J}_{c_1}, \end{aligned}$$

with

$$\mathcal{J}_{c_1} = 6 - 2\left(\frac{3}{8}\right)^{-1} - \frac{8}{2}\left(\frac{1}{4}\right)^2 = 0.4167.$$

For numerical validation, consider

$$X_1 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}, \qquad \Phi_1(T) = \Phi_2(T) = \frac{1}{2}T.$$

We calculate

$$X_1^{-1} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.3333 \end{pmatrix}, \quad X_2^{-1} = \begin{pmatrix} 0.3333 & 0 \\ 0 & 0.25 \end{pmatrix},$$
$$X_1^{-2} = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.1111 \end{pmatrix}, \quad X_2^{-2} = \begin{pmatrix} 0.1111 & 0 \\ 0 & 0.0625 \end{pmatrix}$$

and

$$\begin{aligned}\frac{1}{2}\left(\sum_{i=1}^{2} \Phi_i\left(X_i^{-2}\right) + \left(\sum_{i=1}^{n} \Phi_i(X_i^{-1})\right)^2\right) &= \frac{1}{2}\begin{pmatrix} 0.1806 & 0 \\ 0 & 0.0868 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0.1736 & 0 \\ 0 & 0.0851 \end{pmatrix} \\ &= \begin{pmatrix} 0.1771 & 0 \\ 0 & 0.0859 \end{pmatrix}.\end{aligned}$$

Furthermore, we have

$$\frac{3}{4}\sum_{i=1}^{n} \Phi_i(X_i^{-1}) - \frac{1}{8}\mathbf{1} = 0.75\begin{pmatrix} 0.4167 & 0 \\ 0 & 0.2917 \end{pmatrix} - \begin{pmatrix} 0.125 & 0 \\ 0 & 0.125 \end{pmatrix} = \begin{pmatrix} 0.1875 & 0 \\ 0 & 0.0938 \end{pmatrix}$$

and

$$K = \begin{pmatrix} 0.1875 & 0 \\ 0 & 0.0938 \end{pmatrix} - \begin{pmatrix} 0.1771 & 0 \\ 0 & 0.0859 \end{pmatrix} = \begin{pmatrix} 0.0104 & 0 \\ 0 & 0.0079 \end{pmatrix}.$$

Moreover,

$$\begin{aligned}\Delta_{c_1} &= \left(\frac{1}{2}\mathbf{1} - 4\left|\sum_{i=1}^{2} \Phi_i(g(X_i)) - \frac{3}{8}\mathbf{1}\right|\right) \times \mathcal{J}_{c_1} \\ &= \begin{pmatrix} 0.3332 & 0 \\ 0 & 0.1668 \end{pmatrix} \cdot 0.4167 = \begin{pmatrix} 0.1388 & 0 \\ 0 & 0.0695 \end{pmatrix}.\end{aligned}$$

The arithmetic mean is

$$\mathcal{M}_1(\boldsymbol{X}, \boldsymbol{\Phi}) = \begin{pmatrix} 3.5 & 0 \\ 0 & 2.5 \end{pmatrix}.$$

The harmonic mean is

$$\mathcal{M}_{-1}(\boldsymbol{X}, \boldsymbol{\Phi}) = \begin{pmatrix} 3.000 & 0 \\ 0 & 2.182 \end{pmatrix}.$$

We now have

$$\mathcal{M}_1(\boldsymbol{X},\boldsymbol{\Phi}) - 2\alpha^3 K - \Delta_{c_1} = \begin{pmatrix} 3.5 & 0 \\ 0 & 2.5 \end{pmatrix} - 16 \cdot \begin{pmatrix} 0.0104 & 0 \\ 0 & 0.0079 \end{pmatrix} - \begin{pmatrix} 0.1388 & 0 \\ 0 & 0.0695 \end{pmatrix}$$
$$= \begin{pmatrix} 3.1948 & 0 \\ 0 & 2.3041 \end{pmatrix}$$

and

$$\mathcal{M}_1(\boldsymbol{X},\boldsymbol{\Phi}) - 2\alpha^3 K = \begin{pmatrix} 3.5 & 0 \\ 0 & 2.5 \end{pmatrix} - 16 \cdot \begin{pmatrix} 0.0104 & 0 \\ 0 & 0.0079 \end{pmatrix} = \begin{pmatrix} 3.3336 & 0 \\ 0 & 2.3736 \end{pmatrix}.$$

Finally, we obtain the strict operator inequalities

$$\begin{pmatrix} 3.000 & 0 \\ 0 & 2.182 \end{pmatrix} < \begin{pmatrix} 3.1948 & 0 \\ 0 & 2.3041 \end{pmatrix} < \begin{pmatrix} 3.3336 & 0 \\ 0 & 2.3736 \end{pmatrix} < \begin{pmatrix} 3.5 & 0 \\ 0 & 2.5 \end{pmatrix}.$$

That is, inequality (5.7) is numerically verified.

Example 5.3. Let $\alpha = 2$ and $\beta = 4$. As a special case of Theorem 5.1, for $\varphi(t) = t$, $g(t) = \log t$, we have

$$c_1 = 1.$$

We get a refinement of the operator geometric-arithmetic inequality of the Mercer type

$$\mathcal{M}_0(\boldsymbol{X},\boldsymbol{\Phi}) \leqslant \mathcal{M}_1(\boldsymbol{X},\boldsymbol{\Phi}) - \alpha K - \Delta_{c_1} \leqslant \mathcal{M}_1(\boldsymbol{X},\boldsymbol{\Phi}) - \alpha K \leqslant \mathcal{M}_1(\boldsymbol{X},\boldsymbol{\Phi}), \tag{5.8}$$

where

$$K = \log 8 \sum_{i=1}^{n} \Phi_i(\log(X_i)) - \log 2 \log 4\mathbf{1} - \frac{1}{2}\left(\sum_{i=1}^{n} \Phi_i\left(\log^2(X_i)\right) + \left(\sum_{i=1}^{n} \Phi_i(\log(X_i))\right)^2\right),$$

$$\Delta_{c_1} = \left(\frac{1}{2}\mathbf{1} - \frac{1}{\log 2}\left|\sum_{i=1}^{n} \Phi_i(\log X_i) - \frac{1}{2}\log 8\mathbf{1}\right|\right)\mathcal{J}_{c_1},$$

$$\mathcal{J}_{c_1} = 6 - 2\sqrt{8} - \frac{1}{2}\log^2 2 = 0.1029.$$

For numerical validation, consider

$$X_1 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}, \qquad \Phi_1(T) = \Phi_2(T) = \frac{1}{2}T.$$

We calculate

$$\log\alpha = \log\frac{\beta}{\alpha} = \log 2 = 0.6931, \quad \log\beta = \log 4 = 1.3863,$$
$$\log\alpha\beta = \log 8 = 2.0794, \quad \log\alpha\log\beta = 0.9609$$

and

$$\log X_1 = \begin{pmatrix} \log 2 & 0 \\ 0 & \log 3 \end{pmatrix} \approx \begin{pmatrix} 0.6931 & 0 \\ 0 & 1.0986 \end{pmatrix}, \quad \log^2 X_1 = \begin{pmatrix} 0.4805 & 0 \\ 0 & 1.2069 \end{pmatrix},$$
$$\log X_2 = \begin{pmatrix} \log 3 & 0 \\ 0 & \log 4 \end{pmatrix} \approx \begin{pmatrix} 1.0986 & 0 \\ 0 & 1.3863 \end{pmatrix}, \quad \log^2 X_2 = \begin{pmatrix} 1.2069 & 0 \\ 0 & 1.9218 \end{pmatrix}.$$

Furthermore, we have

$$S = \sum_{i=1}^{2} \Phi_i(\log X_i) = \frac{1}{2}(\log X_1 + \log X_2) = \begin{pmatrix} 0.8959 & 0 \\ 0 & 1.2425 \end{pmatrix},$$
$$S^2 = \left(\sum_{i=1}^{2} \Phi_i(\log X_i)\right)^2 = \begin{pmatrix} 0.8026 & 0 \\ 0 & 1.5438 \end{pmatrix},$$
$$\sum_{i=1}^{2} \Phi_i(\log^2 X_i) = \frac{1}{2}\begin{pmatrix} 0.4805 & 0 \\ 0 & 1.2069 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1.2069 & 0 \\ 0 & 1.9218 \end{pmatrix} = \begin{pmatrix} 0.8438 & 0 \\ 0 & 1.5644 \end{pmatrix}$$

and

$$\frac{1}{2}\left(\sum_{i=1}^{2} \Phi_i\left(\log^2 (X_i)\right) + \left(\sum_{i=1}^{2} \Phi_i(\log(X_i))\right)^2\right) = \begin{pmatrix} 0.8232 & 0 \\ 0 & 1.5541 \end{pmatrix},$$

$$\begin{aligned} &\left(\frac{1}{2}\mathbf{1} - \frac{1}{\log 2}\left|\sum_{i=1}^{n} \Phi_i(\log X_i) - \frac{1}{2}\log 8\mathbf{1}\right|\right) \\ &= \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} - \frac{1}{\log 2}\left|\begin{pmatrix} 0.8959 & 0 \\ 0 & 1.2425 \end{pmatrix} - \begin{pmatrix} 1.0397 & 0 \\ 0 & 1.0397 \end{pmatrix}\right| \\ &= \begin{pmatrix} 0.2925 & 0 \\ 0 & 0.2074 \end{pmatrix}. \end{aligned}$$

Furthermore, the correction terms satisfy

$$K = \begin{pmatrix} 0.078 & 0 \\ 0 & 0.061 \end{pmatrix}, \qquad \Delta_{c_1} = \begin{pmatrix} 0.030 & 0 \\ 0 & 0.021 \end{pmatrix}.$$

The arithmetic mean is

$$\mathcal{M}_1(\boldsymbol{X}, \boldsymbol{\Phi}) = \begin{pmatrix} 3.5 & 0 \\ 0 & 2.5 \end{pmatrix}.$$

Therefore,

$$\mathcal{M}_1(\boldsymbol{X}, \boldsymbol{\Phi}) - \alpha K - \Delta_{c_1} = \begin{pmatrix} 3.314 & 0 \\ 0 & 2.357 \end{pmatrix}, \qquad \mathcal{M}_1(\boldsymbol{X}, \boldsymbol{\Phi}) - \alpha K = \begin{pmatrix} 3.344 & 0 \\ 0 & 2.378 \end{pmatrix}.$$

The geometric mean is

$$\mathcal{M}_0(\boldsymbol{X}, \boldsymbol{\Phi}) \approx \begin{pmatrix} 3.266 & 0 \\ 0 & 2.309 \end{pmatrix}.$$

Finally, we obtain the strict operator inequalities

$$\begin{pmatrix} 3.266 & 0 \\ 0 & 2.309 \end{pmatrix} < \begin{pmatrix} 3.314 & 0 \\ 0 & 2.357 \end{pmatrix} < \begin{pmatrix} 3.344 & 0 \\ 0 & 2.378 \end{pmatrix} < \begin{pmatrix} 3.5 & 0 \\ 0 & 2.5 \end{pmatrix}.$$

Thus, inequality (5.8) is numerically verified.

In a similar way to Theorem 5.1, with additional assumption on $\varphi \circ g^{-1}$, we obtain further inequalities for the operator power means.

Remark 5.1. *Denote $m_r = \min\{\alpha^r, \beta^r\}$, $M_r = \max\{\alpha^r, \beta^r\}$. Taking Remark 3.6 into account, and under the assumptions of Theorem 5.1, from Theorem 4.2, we obtain the following inequalities:*

$$\begin{aligned} \mathcal{M}_r(\boldsymbol{X}, \boldsymbol{\Phi}) &\leqslant \left[\sum_{i=1}^{n} \phi_i \left((\alpha^r + \beta^r) \mathbf{1} \right) - X_i^r)^{\frac{s}{r}} \right]^{\frac{1}{s}} \qquad (5.9) \\ &\leqslant \Bigg[\frac{\sum_{i=1}^n \Phi(X_i^r) - m_r \mathbf{1}}{M_r - m_r} \alpha^s + \frac{M_r \mathbf{1} - \sum_{i=1}^n \Phi(X_i^r)}{M_r - m_r} \beta^s \\ &\quad - c_1 \sum_{i=1}^{n} \Phi_i \left((X_i^r - m_r \mathbf{1}) (M_r \mathbf{1} - X_i^r) \right) - \Delta_{c_1} \left(\alpha^r, \beta^r, \boldsymbol{X}^r, (\cdot)^{\frac{s}{r}} \right) \Bigg]^{\frac{1}{s}} \\ &\leqslant \left[\frac{\sum_{i=1}^n \Phi(X_i^r) - m_r \mathbf{1}}{M_r - m_r} \alpha^s + \frac{M_r \mathbf{1} - \sum_{i=1}^n \Phi(X_i^r)}{M_r - m_r} \beta^s \right]^{\frac{1}{s}} \\ &\leqslant \left[(\mathcal{M}_s(\boldsymbol{X}, \boldsymbol{\Phi}))^s - c_1 \sum_{i=1}^{n} \Phi_i \left((X_i^r - m_r \mathbf{1}) (M_r \mathbf{1} - X_i^r) \right) - \Delta_{c_1} \left(\alpha^r, \beta^r, \boldsymbol{X}^r, (\cdot)^{\frac{s}{r}} \right) \right]^{\frac{1}{s}} \\ &\leqslant \left[(\mathcal{M}_s(\boldsymbol{X}, \boldsymbol{\Phi}))^s - c_1 \sum_{i=1}^{n} \Phi_i \left((X_i^r - m_r \mathbf{1}) (M_r \mathbf{1} - X_i^r) \right) \right]^{\frac{1}{s}} \\ &\leqslant \mathcal{M}_s(\boldsymbol{X}, \boldsymbol{\Phi}), \end{aligned}$$

where

$$\begin{aligned} c_1 &= \frac{s}{2r} \left(\frac{s}{r} - 1 \right) \alpha^{s-2r}, \text{ for } s \leq -r, \qquad (5.10) \\ c_1 &= \frac{s}{2r} \left(\frac{s}{r} - 1 \right) \beta^{s-2r}, \text{ for } 0 < r < s < 2r, \end{aligned}$$

and where, additionally, function $(\varphi \circ g^{-1})(t) - c_1 t^2 = t^{\frac{s}{r}} - c_1 t^2$ is operator convex.

We are not aware of the general conditions on $s, r \in \mathbb{R}$ such that $t \mapsto t^{\frac{s}{r}} - c_1 t^2$ is operator convex. But obviously, c_1 is less than or equivalent to the one in (5.10)*. A complete characterization of the admissible parameter range would require a separate and more detailed analysis.*

Further, notice that we do not have the case when $\varphi(t) = t^s, g(t) = \log t$ (as in Theorem 5.1), because $(\varphi \circ g^{-1})(t) = \exp(st)$ is not an operator convex function, and from Remark 3.6, it follows that $(\varphi \circ g^{-1})(t) - c_1 t^2 = \exp(st) - c_1 t^2$ is not operator convex.

In addition, under the condition $r = \frac{1}{2} s$ we get $c_1 = 1$. But under this condition, we have

$$\mathcal{J}_1 \left((\cdot)^2, m_r, M_R \right) = \alpha^{2r} + \beta^{2r} - 2 \left(\frac{\alpha^r + \beta^r}{2} \right)^2 - \frac{1}{2} (\beta^r - \alpha^r)^2 = 0,$$

and we do not obtain the refinement of already known results, so we omit this case.

6. Conclusions

In this paper, we develop several new and sharper series of Jensen-Mercer type inequalities for self-adjoint operators on unital C^*-algebras. Our approach is based on the concept of strong convexity, which enables us to introduce additional correction terms and derive refined upper and lower bounds. The obtained results significantly generalize and strengthen the classical Jensen-Mercer operator inequalities for convex functions, both in the setting of operator convex functions and beyond operator convexity. As applications, we derive new estimates for operator quasi-arithmetic means of the Mercer type and obtain the corresponding refinements for operator power means, including important special cases such as the arithmetic, geometric, and harmonic means. These applications illustrate the effectiveness of our method and highlight its relevance to the theory of operator means and matrix inequalities.

Author contributions

S. Ivelić Bradanović, A. Matković and J. Perić: Conceptualization, methodology, validation. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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