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Research article

Unified sufficient conditions for generalized classes of analytic functions associated with the Rabotnov fractional operator

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Abstract: In this paper, we investigate the analytical landscape of the normalized Rabotnov function together with certain linear and integral operators acting on subclasses of analytic and univalent functions with negative coefficients. By leveraging convolution techniques and coefficient-based criteria, we derive innovative sufficient requirements for the function to be categorized under the subclasses $P^*_{\alpha_3}(\alpha_1, \alpha_2)$ and $\wp^*_{\alpha_3}(\alpha_1, \alpha_2)$. Particular attention is given to the role of symmetry in the associated operators, which provides additional structural insight into the behavior of these analytic functions. The study specifically explores the interaction between Rabotnov-derived operators and analytic functions characterized by real-part inequalities, thus providing a comprehensive analysis of their inclusion properties. We present a series of corollaries that extend these properties to classical starlike and convex functions of order α_1, thus broadening the scope of previous investigations. Furthermore, we provide analogous geometric results for a specialized integral operator, thus reinforcing the potential for these techniques to be applied to broader families of special functions in mathematical physics.

Keywords: analytic; univalent; geometric function; Rabotnov function
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1. Introduction

Special functions play a central role in the geometric function theory, particularly in the study of analytic and univalent functions defined on the open unit disk $\Omega = \{\sigma \in \mathbb{C} : |\sigma| < 1\}$. The celebrated resolution of the Bieberbach conjecture by de Branges highlighted the deep connections between coefficient problems and the geometric behavior of analytic functions [1]. Since then, considerable attention has been devoted to investigating the coefficient estimates, growth, and distortion properties of various subclasses of analytic functions; see [2, 3]. Related inclusion results and operator-based approaches can also be found in [4–6]. In addition, symmetry considerations in complex analyses and the operator theory have emerged as an important tool in understanding invariant properties of analytic mappings, especially in connection with linear operators, convolution structures, and transformation techniques [7]. These symmetry principles often provide deeper insight into the structural behavior of analytic functions and their associated operators. Recent developments include applications involving generalized Bessel functions, Wright functions, and Mittag–Leffler-type functions; see [8–10]. Further related investigations may be found in [11, 12].

In recent years, there has been growing interest in exploring the geometric properties of functions arising from fractional calculus and integral transforms [13, 14]. Such functions often exhibit more flexible and generalized behaviors compared to classical analytic functions, thus allowing for deeper insights into both theoretical and applied problems. In particular, fractional operators and special functions associated with them have been successfully used to define and analyze new subclasses of univalent and multivalent functions, as well as to establish inclusion relationships and coefficient bounds.

In the context of applied mathematics, the Rabotnov fractional exponential function naturally arises in creep mechanics and viscoelastic material modeling [15, 16]. Introduced by Rabotnov in 1948 [17], this function provides an effective description of time-dependent deformation processes and has found important applications in material science, engineering, and rheology. Its formulation involves a fractional-type behavior that captures memory effects in materials, thus making it particularly useful in modeling hereditary phenomena. Despite its practical significance, the geometric aspects of the Rabotnov function have only recently begun to attract attention within the geometric function theory.

Existing studies on the Rabotnov function mainly focus on investigating properties such as starlikeness, convexity, and related geometric characteristics under certain parameter restrictions; see [18–20]. While these contributions have provided valuable initial insights, the available results are often limited to specific cases and do not fully address the broader behavior of the function within generalized subclasses of analytic functions. In particular, there remains a need for a systematic treatment of sufficient conditions that ensure the membership of the Rabotnov function in classes defined via real-part inequalities and negative coefficients.

Moreover, the interplay between the Rabotnov function, convolution (Hadamard product) operators, and integral transforms has not been thoroughly investigated [21–23]. Such operators play a fundamental role in the geometric function theory, as they provide a unified framework to generate new function classes and establish inclusion relations. Studying these interactions can lead to a deeper understanding of the structural properties of analytic functions and open new avenues for further research.

Motivated by these observations, the main objective of this paper is to establish new sufficient conditions for the normalized Rabotnov function and its associated operators that belong to certain generalized subclasses of analytic functions. Specifically, we consider the classes $P^*_{\alpha_3}(\alpha_1,\alpha_2)$ and $\wp^*_{\alpha_3}(\alpha_1,\alpha_2)$ and derive conditions that ensure membership in these classes.

Our approach is based on the use of coefficient estimates, classical inequalities that involve the Gamma function, and convolution techniques. The obtained results not only generalize several known results in the literature, but also yield a variety of corollaries for well-known subclasses, including starlike and convex functions of order α_1. Furthermore, the methods developed in this paper can be applied to other special functions that arise in fractional calculus, thus providing a useful framework for future investigations.

2. Preliminaries and definitions

Let Δ be the class of all univalent and analytic functions on the open unit disk Ω of the following form:

$$\hbar(\sigma) = \sigma + \sum_{s=2}^{\infty} C_s\sigma^s, \quad \sigma\in\Omega, \tag{2.1}$$

which are normalized by $\hbar(0) = \hbar'(0) - 1 = 0$. Additionally, denote by H the subclass of Δ consisting of functions with negative coefficients of the following form:

$$\hbar(\sigma) = \sigma - \sum_{s=2}^{\infty} |C_s|\,\sigma^s, \quad \sigma\in\Omega. \tag{2.2}$$

We consider the following subclasses of analytic functions studied by Ali et al. [24] and Murugusundaramoorthy et al. [25], respectively.

For some $\alpha_1 (0 \le \alpha_1 < 1)$, $\alpha_2(\alpha_2 \ge 0)$ and $\alpha_3(0 \le \alpha_3 \le 1)$. Let the subclass $P^*_{\alpha_3}(\alpha_1,\alpha_2)$ includes of functions in Δ satisfying the criteria

$$\mathrm{Re}\left(\frac{\sigma\hbar'(\sigma)}{(1-\alpha_3)\sigma+\alpha_3\hbar(\sigma)} - \alpha_1\right) > \alpha_2\left|\frac{\sigma\hbar'(\sigma)}{(1-\alpha_3)\sigma+\alpha_3\hbar(\sigma)} - 1\right|, \ \sigma\in\Omega$$

and the subclass $\wp^*_{\alpha_3}(\alpha_1,\alpha_2)$ includes of functions in Δ satisfying the criteria

$$\mathrm{Re}\left(\frac{\sigma\hbar'(\sigma)+\sigma^2\hbar''(\sigma)}{(1-\alpha_3)\sigma+\alpha_3\sigma\hbar'(\sigma)} - \alpha_1\right) > \alpha_2\left|\frac{\sigma\hbar'(\sigma)+\sigma^2\hbar''(\sigma)}{(1-\alpha_3)\sigma+\alpha_3\sigma\hbar'(\sigma)} - 1\right|, \ \sigma\in\Omega.$$

Example 2.1. [26, 27] For some $\alpha_1(0 \le \alpha_1 < 1)$, $\alpha_2(\alpha_2 \ge 0)$, and choosing $\alpha_3 = 1$ and $\hbar(\sigma)$ of the form (2.1), let the subclass $P^*_1(\alpha_1,\alpha_2)$ includes of functions in Δ satisfying the criteria

$$\mathrm{Re}\left(\frac{\sigma\hbar'(\sigma)}{\hbar(\sigma)} - \alpha_1\right) > \alpha_2\left|\frac{\sigma\hbar'(\sigma)}{\hbar(\sigma)} - 1\right|, \ \sigma\in\Omega,$$

and the subclass $\wp^*_1(\alpha_1,\alpha_2)$ includes of functions in Δ satisfying the criteria

$$\mathrm{Re}\left(\frac{\sigma\hbar''(\sigma)}{\hbar'(\sigma)} + 1 - \alpha_1\right) > \alpha_2\left|\frac{\sigma\hbar''(\sigma)}{\hbar'(\sigma)}\right|, \ \sigma\in\Omega.$$

Example 2.2. [25] For some $\alpha_1(0 \leq \alpha_1 < 1)$, $\alpha_2(\alpha_2 \geq 0)$, and choosing $\alpha_3 = 0$ and $\hbar(\sigma)$ of the form (2.1), let the subclass $P_0^*(\alpha_1, \alpha_2)$ includes of functions in Δ satisfying the criteria

$$\mathrm{Re}\left(\hbar'(\sigma) - \alpha_1\right) > \alpha_2 \left|\hbar'(\sigma) - 1\right|, \ \sigma \in \Omega,$$

and the subclass $\wp_0^*(\alpha_1, \alpha_2)$ includes of functions in Δ satisfying the criteria

$$\mathrm{Re}\left((\sigma\hbar'(\sigma))' - \alpha_1\right) > \alpha_2 \left|(\sigma\hbar'(\sigma))' - 1\right|, \ \sigma \in \Omega.$$

Example 2.3. [28] For some $\alpha_1(0 \leq \alpha_1 < 1)$, and choosing $\alpha_2 = 0$, $\alpha_3 = 1$, and $\hbar(\sigma)$ of the form (2.1), let the subclass $P_1^*(\alpha_1, 0) \equiv S^*(\alpha_1)$ includes of functions in Δ satisfying the criteria

$$\mathrm{Re}\left(\frac{\sigma\hbar'(\sigma)}{\hbar(\sigma)}\right) > \alpha_1, \ \sigma \in \Omega,$$

and the subclass $\wp_1^*(\alpha_1, 0) \equiv \mathcal{K}(\alpha_1)$ includes of functions in Δ satisfying the criteria

$$\mathrm{Re}\left(\frac{\sigma\hbar''(\sigma)}{\hbar'(\sigma)} + 1\right) > \alpha_1, \ \sigma \in \Omega.$$

Remark 2.1. *Both subclasses* $\mathrm{P}_1^*(\alpha_1, 0) \equiv \mathrm{S}^*(\alpha_1)$ *and* $\wp_1^*(\alpha_1, 0) \equiv \mathcal{K}(\alpha_1)$ *are the well known subclasses of starlike and convex functions of order* α_1, *respectively (see* [28]*).*

Now, for a function $\hbar \in H$, we provide some sufficient and necessary requirements for functions in the aforementioned subclasses.

Lemma 2.1. [25] Let $\hbar \in H$ be of the form (2.2). A function $\hbar \in P_{\alpha_3}^*(\alpha_1, \alpha_2)$ if and only if

$$\sum_{s=2}^{\infty} \left[s\,(\alpha_2 + 1) - \alpha_3(\alpha_1 + \alpha_2)\right] |C_s| \leq 1 - \alpha_1, \tag{2.3}$$

and $\hbar \in \wp_{\alpha_3}^*(\alpha_1, \alpha_2)$ if and only if

$$\sum_{s=2}^{\infty} s \left[s\,(\alpha_2 + 1) - \alpha_3(\alpha_1 + \alpha_2)\right] |C_s| \leq 1 - \alpha_1. \tag{2.4}$$

Lemma 2.2. [26, 27] Let $\hbar \in H$ be of the form (2.2). A function $\hbar \in P_1^*(\alpha_1, \alpha_2)$ if and only if

$$\sum_{s=2}^{\infty} \left[s\,(\alpha_2 + 1) - (\alpha_1 + \alpha_2)\right] |C_s| \leq 1 - \alpha_1,$$

and $\hbar \in \wp_1^*(\alpha_1, \alpha_2)$ if and only if

$$\sum_{s=2}^{\infty} s \left[s\,(\alpha_2 + 1) - (\alpha_1 + \alpha_2)\right] |C_s| \leq 1 - \alpha_1.$$

Lemma 2.3. [25] Let $\hbar \in H$ be of the form (2.2). A function $\hbar \in P_0^*(\alpha_1, \alpha_2)$ if and only if

$$\sum_{s=2}^{\infty} s(\alpha_2 + 1)\,|C_s| \leq 1 - \alpha_1,$$

and $\hbar \in \wp_0^*(\alpha_1, \alpha_2)$ if and only if

$$\sum_{s=2}^{\infty} s^2(\alpha_2 + 1)\,|C_s| \leq 1 - \alpha_1.$$

The significance of special functions in the geometric function theory is widely acknowledged. This is particularly evident in de Branges answer to the renowned Bieberbach conjecture. Numerous studies have investigated the geometric properties of various special functions from different perspectives. Recent contributions include applications to q-analogues, generalized Bessel functions, and Lucas-balancing polynomials; for example, see [29–31]. Related coefficient estimates and geometric investigations can also be found in [32, 33].

In 1948 [17], Rabotnov introduced the Rabotnov fractional exponential function (Rabotnov function) as follows:

$$\Xi_{\eta,\delta}(\sigma) = \sigma^{\eta} \sum_{s=0}^{\infty} \frac{\delta^s \sigma^{s(1+\eta)}}{\Gamma((s+1)(1+\eta))}, \quad (\eta, \delta, \sigma \in \mathbb{C}\ ). \tag{2.5}$$

It is clear that $\Xi_{\eta,\delta}(\sigma) \notin \Delta$. Consequently, we examine the Rabotnov functions in the following manner:

$$\begin{aligned}\mathcal{R}_{\eta,\delta}(\sigma) &= \sigma^{\frac{1}{1+\eta}}\Gamma(1+\eta)\,\Xi_{\eta,\delta}(\sigma^{\frac{1}{1+\eta}}) \\ &= \sigma + \sum_{s=2}^{\infty} \frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)\,s)}\sigma^s,\ \ \sigma \in \Omega.\end{aligned} \tag{2.6}$$

Table 1 below presents the calculated values for the first few coefficients

$$a_s = \frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)\,s)}$$

across different values of the fractional parameter η with $\delta = 1$. These demonstrate the rapid decay of coefficients.

Table 1. Numerical values of coefficients $a_s = \frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)s)}$ for the Rabotnov function with $\delta = 1$.

s	$\eta = 0.25$	$\eta = 0.50$	$\eta = 0.75$	$\eta = 1.00$
2	0.70422	0.37613	0.19894	0.16667
3	0.36815	0.08333	0.02114	0.00833
4	0.15542	0.01389	0.00118	0.00019
5	0.05429	0.00182	0.00004	0.000003

The convolution or Hadamard product of two functions $\jmath$ and ℓ in Δ of the form

$$\jmath(\sigma) = \sigma + \sum_{s=2}^{\infty} A_s \sigma^s \ \text{ and } \ \ell(\sigma) = \sigma + \sum_{s=2}^{\infty} B_s \sigma^s,$$

is the function $(\jmath * \ell)$ defined by

$$(\jmath * \ell)(\sigma) = \sigma + \sum_{s=2}^{\infty} A_s B_s \sigma^s.$$

Now, by convolution, consider a linear operator $\mathcal{R}_{\eta,\delta}(\sigma)\hbar : \Delta \to \Delta$ as follows:

$$\begin{aligned}\mathcal{R}_{\eta,\delta}(\sigma)\hbar &= \mathcal{R}_{\eta,\delta}(\sigma) * \hbar(\sigma)\\ &= \sigma + \sum_{s=2}^{\infty} \frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)s)} C_s \sigma^s.\end{aligned}$$

The current work aims to create new sufficient conditions for the normalized Rabotnov function to be in various subclasses of analytic univalent functions. This study is motivated by recent investigations on the geometric properties of the Rabotnov function, its operator-theoretic aspects, and sufficient conditions for special functions in geometric function theory; for example, see [34–36].

3. Main results

In this section, we will give sufficient conditions for the normalized Rabotnov function to be in the aforementioned subclasses.

In our first results, we need the following lemma given by Eker and Ece [37].

Lemma 3.1. [37] For $s \in \mathbb{N}$ and $\eta \geq 0$,

$$(\eta+1)^{s-1}(s-1)!\Gamma(\eta+1) \leq \Gamma((\eta+1)s).$$

Additionally, from this lemma, we can write the following:

$$\frac{\Gamma(\eta+1)}{\Gamma((\eta+1)s)} \leq \frac{1}{(\eta+1)^{s-1}(s-1)!}. \tag{3.1}$$

In this work, we will assume that $0 \leq \alpha_1 < 1$, $\alpha_2 \geq 0$, and $0 \leq \alpha_3 \leq 1$, unless otherwise stated.

Theorem 3.1. *If* $\eta \geq 0$ *and* $\delta > 0$, *then* $\mathcal{R}_{\eta,\delta}(\sigma) \in \mathrm{P}^*_{\alpha_3}(\alpha_1, \alpha_2)$ *if the following condition is satisfied:*

$$\frac{\delta(\alpha_2+1)}{\eta+1} e^{\frac{\delta}{\eta+1}} + (\alpha_2 - \alpha_3(\alpha_1+\alpha_2)+1)\left(e^{\frac{\delta}{\eta+1}} - 1\right) \leq 1 - \alpha_1.$$

Proof. Via the Rabotnov function provided by (2.6) and Eq (2.3), it suffices to show that

$$\sum_{s=2}^{\infty} [s(\alpha_2+1) - \alpha_3(\alpha_1+\alpha_2)] \frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)s)} \leq 1 - \alpha_1.$$

By writing $s = (s-1)+1$, we have the following:

$$\sum_{s=2}^{\infty} [s(\alpha_2+1) - \alpha_3(\alpha_1+\alpha_2)] \frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)s)}$$

$$= \quad (\alpha_2+1)\sum_{s=2}^{\infty}(s-1)\frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)s)} + (\alpha_2-\alpha_3(\alpha_1+\alpha_2)+1)\sum_{s=2}^{\infty}\frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)s)}.$$

Under Eq (3.1), we get the following:

$$\begin{aligned}
&\sum_{s=2}^{\infty}[s(\alpha_2+1)-\alpha_3(\alpha_1+\alpha_2)]\frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)s)}\\
&\leq (\alpha_2+1)\sum_{s=2}^{\infty}(s-1)\frac{\delta^{s-1}}{(\eta+1)^{s-1}(s-1)!} + (\alpha_2-\alpha_3(\alpha_1+\alpha_2)+1)\sum_{s=2}^{\infty}\frac{\delta^{s-1}}{(\eta+1)^{s-1}(s-1)!}\\
&= \frac{\delta(\alpha_2+1)}{\eta+1}e^{\frac{\delta}{\eta+1}} + (\alpha_2-\alpha_3(\alpha_1+\alpha_2)+1)\left(e^{\frac{\delta}{\eta+1}}-1\right).
\end{aligned} \tag{3.2}$$

However, Expression (3.2) is bounded above by $1-\alpha_1$; thus,

$$\frac{\delta(\alpha_2+1)}{\eta+1}e^{\frac{\delta}{\eta+1}} + (\alpha_2-\alpha_3(\alpha_1+\alpha_2)+1)\left(e^{\frac{\delta}{\eta+1}}-1\right) \leq 1-\alpha_1.$$

□

Theorem 3.2. *If* $\eta \geq 0$ *and* $\delta > 0$, *then* $\mathcal{R}_{\eta,\delta}(\sigma) \in \wp^{*}_{\alpha_3}(\alpha_1,\alpha_2)$ *if the following condition is satisfied:*

$$\begin{aligned}
&\frac{\delta^2(\alpha_2+1)}{(\eta+1)^2}e^{\frac{\delta}{\eta+1}} + \frac{\delta(3(\alpha_2+1)-\alpha_3(\alpha_1+\alpha_2))}{\eta+1}e^{\frac{\delta}{\eta+1}}\\
&+(\alpha_2-\alpha_3(\alpha_1+\alpha_2)+1)\left(e^{\frac{\delta}{\eta+1}}-1\right)\\
\leq\ & 1-\alpha_1.
\end{aligned}$$

Proof. Via the Rabotnov function provided by (2.6) and Eq (2.4), it suffices to show that

$$\sum_{s=2}^{\infty}s[s(\alpha_2+1)-\alpha_3(\alpha_1+\alpha_2)]\frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)s)} \leq 1-\alpha_1.$$

By writing $s=(s-1)+1$ and $s^2=(s-1)(s-2)+3(s-1)+1$, we have the following:

$$\begin{aligned}
&\sum_{s=2}^{\infty}s[s(\alpha_2+1)-\alpha_3(\alpha_1+\alpha_2)]\frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)s)}\\
=\ &(\alpha_2+1)\sum_{s=2}^{\infty}(s-1)(s-2)\frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)s)}\\
&+(3(\alpha_2+1)-\alpha_3(\alpha_1+\alpha_2))\sum_{s=2}^{\infty}(s-1)\frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)s)}\\
&+(\alpha_2-\alpha_3(\alpha_1+\alpha_2)+1)\sum_{s=2}^{\infty}\frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)s)}.
\end{aligned}$$

Under Eq (3.1), we get the following:

$$\sum_{s=2}^{\infty}s[s(\alpha_2+1)-\alpha_3(\alpha_1+\alpha_2)]\frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)s)}$$

$$
\begin{aligned}
&\leq (\alpha_2+1)\sum_{s=3}^{\infty}\frac{\delta^{s-1}}{(\eta+1)^{s-1}(s-3)!}\\
&+(3(\alpha_2+1)-\alpha_3(\alpha_1+\alpha_2))\sum_{s=2}^{\infty}\frac{\delta^{s-1}}{(\eta+1)^{s-1}(s-2)!}\\
&+(\alpha_2-\alpha_3(\alpha_1+\alpha_2)+1)\sum_{s=2}^{\infty}\frac{\delta^{s-1}}{(\eta+1)^{s-1}(s-1)!}\\
&=\frac{\delta^2(\alpha_2+1)}{(\eta+1)^2}e^{\frac{\delta}{\eta+1}}+\frac{\delta\,(3(\alpha_2+1)-\alpha_3(\alpha_1+\alpha_2))}{\eta+1}e^{\frac{\delta}{\eta+1}}\\
&+(\alpha_2-\alpha_3(\alpha_1+\alpha_2)+1)\left(e^{\frac{\delta}{\eta+1}}-1\right). \qquad (3.3)
\end{aligned}
$$

However, Expression (3.3) is bounded above by $1-\alpha_1$; thus,

$$
\begin{aligned}
&\frac{\delta^2(\alpha_2+1)}{(\eta+1)^2}e^{\frac{\delta}{\eta+1}}+\frac{\delta\,(3(\alpha_2+1)-\alpha_3(\alpha_1+\alpha_2))}{\eta+1}e^{\frac{\delta}{\eta+1}}\\
&+(\alpha_2-\alpha_3(\alpha_1+\alpha_2)+1)\left(e^{\frac{\delta}{\eta+1}}-1\right)\\
\leq\ & 1-\alpha_1.
\end{aligned}
$$

□

Corollary 3.1. *If* $\eta\geq 0$ *and* $\delta>0$, *then* $\mathcal{R}_{\eta,\delta}(\sigma)\in \mathrm{P}_1^*(\alpha_1,\alpha_2)$ *if*

$$\frac{\delta(\alpha_2+1)}{\eta+1}e^{\frac{\delta}{\eta+1}}+(1-\alpha_1)\left(e^{\frac{\delta}{\eta+1}}-1\right)\leq 1-\alpha_1.$$

Corollary 3.2. *If* $\eta\geq 0$ *and* $\delta>0$, *then* $\mathcal{R}_{\eta,\delta}(\sigma)\in \wp_1^*(\alpha_1,\alpha_2)$ *if*

$$\frac{\delta^2(\alpha_2+1)}{(\eta+1)^2}e^{\frac{\delta}{\eta+1}}+\frac{\delta\,(2\alpha_2-\alpha_1+3)}{\eta+1}e^{\frac{\delta}{\eta+1}}+(1-\alpha_1)\left(e^{\frac{\delta}{\eta+1}}-1\right)\leq 1-\alpha_1.$$

Corollary 3.3. *If* $\eta\geq 0$ *and* $\delta>0$, *then* $\mathcal{R}_{\eta,\delta}(\sigma)\in \mathrm{P}_0^*(\alpha_1,\alpha_2)$ *if*

$$\frac{\delta(\alpha_2+1)}{\eta+1}e^{\frac{\delta}{\eta+1}}+(\alpha_2+1)\left(e^{\frac{\delta}{\eta+1}}-1\right)\leq 1-\alpha_1.$$

Corollary 3.4. *If* $\eta\geq 0$ *and* $\delta>0$, *then* $\mathcal{R}_{\eta,\delta}(\sigma)\in \wp_0^*(\alpha_1,\alpha_2)$ *if*

$$\frac{\delta^2(\alpha_2+1)}{(\eta+1)^2}e^{\frac{\delta}{\eta+1}}+\frac{3\delta\,(\alpha_2+1)}{\eta+1}e^{\frac{\delta}{\eta+1}}+(\alpha_2+1)\left(e^{\frac{\delta}{\eta+1}}-1\right)\leq 1-\alpha_1.$$

Corollary 3.5. *If* $\eta\geq 0$ *and* $\delta>0$, *then* $\mathcal{R}_{\eta,\delta}(\sigma)\in \mathrm{S}^*(\alpha_1)$ *if*

$$e^{\frac{\delta}{\eta+1}}\left(\frac{\delta}{\eta+1}+1-\alpha_1\right)\leq 2\,(1-\alpha_1)\,.$$

Corollary 3.6. *If* $\eta\geq 0$ *and* $\delta>0$, *then* $\mathcal{R}_{\eta,\delta}(\sigma)\in \mathcal{K}(\alpha_1)$ *if*

$$e^{\frac{\delta}{\eta+1}}\left(\frac{\delta^2}{(\eta+1)^2}+\frac{\delta\,(3-\alpha_1)}{\eta+1}+1-\alpha_1\right)\leq 2\,(1-\alpha_1)\,.$$

Remark 3.1. *The results in Corollaries 3.5 and 3.6 were obtained by Eker et al. [35], in Corollaries 1 and 2, respectively.*

4. Inclusion properties

For $0 < \lambda_1 \leq 1$, $\lambda_2 < 1$, and $\lambda_3 \in \mathbb{C} - \{0\}$, a function $\hbar \in \Delta$ is said to be in the subclass $\mathcal{K}^{\lambda_3}(\lambda_1, \lambda_2)$ if it satisfies the following relation:

$$\left| \frac{(1-\lambda_1)\frac{\hbar(\sigma)}{\sigma} + \lambda_1 \hbar'(\sigma) - 1}{2\lambda_3(1-\lambda_2) + (1-\lambda_1)\frac{\hbar(\sigma)}{\sigma} + \lambda_1 \hbar'(\sigma) - 1} \right| < 1, \ \sigma \in \Omega.$$

Lemma 4.1. [36] Let $\hbar \in H$ be of the form (2.2). A function $\hbar \in \mathcal{K}^{\lambda_3}(\lambda_1, \lambda_2)$ if it satisfies the following condition:

$$|C_s| \leq \frac{2\,|\lambda_3|\,(1-\lambda_2)}{\lambda_1(s-1)+1}, \quad s \in \mathbb{N} - \{1\}.$$

Using the previous lemma, we will study the action of the Rabotnov function $\mathcal{R}_{\eta,\delta}(\sigma)\hbar$ on the subclass $\wp^*_{\alpha_3}(\alpha_1, \alpha_2)$.

Theorem 4.1. *For $\hbar \in \mathcal{K}^{\lambda_3}(\lambda_1, \lambda_2)$, if the inequality*

$$\frac{\delta(\alpha_2+1)}{\eta+1} e^{\frac{\delta}{\eta+1}} + (\alpha_2 - \alpha_3(\alpha_1+\alpha_2)+1)\left(e^{\frac{\delta}{\eta+1}} - 1\right) \leq \frac{\lambda_1\,(1-\alpha_1)}{2\,|\lambda_3|\,(1-\lambda_2)} \tag{4.1}$$

is satisfied, then $\mathcal{R}_{\eta,\delta}(\sigma)\hbar \in \wp^*_{\alpha_3}(\alpha_1, \alpha_2)$.

Proof. Let $\hbar \in \mathcal{K}^{\lambda_3}(\lambda_1, \lambda_2)$ and given by (2.2). By virtue of Lemma 2.1, it suffices to show that

$$\sum_{s=2}^{\infty} s\,[s\,(\alpha_2+1) - \alpha_3(\alpha_1+\alpha_2)]\,\frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)\,s)}\,|C_s| \leq 1-\alpha_1.$$

Since $\hbar \in \mathcal{K}^{\lambda_3}(\lambda_1, \lambda_2)$, then by Lemma 4.1,

$$\begin{aligned}
&\sum_{s=2}^{\infty} s\,[s\,(\alpha_2+1) - \alpha_3(\alpha_1+\alpha_2)]\,\frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)\,s)}\,|C_s| \\
\leq\ & 2\,|\lambda_3|\,(1-\lambda_2)\sum_{s=2}^{\infty} \frac{s\,[(s(\alpha_2+1) - \alpha_3(\alpha_1+\alpha_2)]}{\lambda_1(s-1)+1}\,\frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)\,s)}.
\end{aligned}$$

Since $\lambda_1(s-1)+1 \geq s\lambda_1$ for $0 < \lambda_1 \leq 1$, then

$$\begin{aligned}
&\sum_{s=2}^{\infty} s\,[s\,(\alpha_2+1) - \alpha_3(\alpha_1+\alpha_2)]\,\frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)\,s)}\,|C_s| \\
\leq\ & \frac{2\,|\lambda_3|\,(1-\lambda_2)}{\lambda_1}\sum_{s=2}^{\infty} [(s(\alpha_2+1) - \alpha_3(\alpha_1+\alpha_2)]\,\frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)\,s)}.
\end{aligned}$$

Proceeding as in Theorem 3.1, we obtain the following:

$$\begin{aligned}
&\sum_{s=2}^{\infty} s\,[s\,(\alpha_2+1) - \alpha_3(\alpha_1+\alpha_2)]\,\frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)\,s)}\,|C_s| \\
\leq\ & \frac{2\,|\lambda_3|\,(1-\lambda_2)}{\lambda_1}\left[\frac{\delta(\alpha_2+1)}{\eta+1} e^{\frac{\delta}{\eta+1}} + (\alpha_2 - \alpha_3(\alpha_1+\alpha_2)+1)\left(e^{\frac{\delta}{\eta+1}} - 1\right)\right].
\end{aligned} \tag{4.2}$$

However, (4.2) is bounded above by $1-\alpha_1$, which concludes Expression (4.1). $\square$

Corollary 4.1. *For $\hbar \in \mathcal{K}^{\lambda_3}(\lambda_1, \lambda_2)$, if the inequality*

$$\frac{\delta(\alpha_2+1)}{\eta+1}e^{\frac{\delta}{\eta+1}} + (1-\alpha_1)\left(e^{\frac{\delta}{\eta+1}} - 1\right) \leq \frac{\lambda_1(1-\alpha_1)}{2|\lambda_3|(1-\lambda_2)}$$

is satisfied, then $\mathcal{R}_{\eta,\delta}(\sigma)\hbar \in \wp_1^*(\alpha_1, \alpha_2)$.

5. An integral operator

This section is devoted to achieving comparable outcomes with respect to a particular integral operator.

Theorem 5.1. *Let $\eta \geq 0$ and $\delta > 0$. If the integral operator $\mathcal{L}_{\eta,\delta}$ is given by*

$$\mathcal{L}_{\eta,\delta}(\sigma) = \int_0^\sigma \frac{\mathcal{R}_{\eta,\delta}(\kappa)}{\kappa} d\kappa, \ \sigma \in \Omega, \tag{5.1}$$

then $\mathcal{L}_{\eta,\delta} \in \wp_{\alpha_3}^*(\alpha_1, \alpha_2)$ if

$$\frac{\delta(\alpha_2+1)}{\eta+1}e^{\frac{\delta}{\eta+1}} + (\alpha_2 - \alpha_3(\alpha_1+\alpha_2)+1)\left(e^{\frac{\delta}{\eta+1}} - 1\right) \leq 1 - \alpha_1. \tag{5.2}$$

Proof. Since

$$\mathcal{L}_{\eta,\delta}(\sigma) = \sigma + \sum_{s=2}^{\infty} \frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)s)} \frac{\sigma^s}{s}, \tag{5.3}$$

from the function provided by (5.3) and Eq (2.4), it suffices to show that

$$\sum_{s=2}^{\infty} s\left[s(\alpha_2+1) - \alpha_3(\alpha_1+\alpha_2)\right] \frac{\delta^{s-1}\Gamma(1+\eta)}{s\Gamma((1+\eta)s)} \leq 1 - \alpha_1,$$

that is,

$$\sum_{s=2}^{\infty} \left[s(\alpha_2+1) - \alpha_3(\alpha_1+\alpha_2)\right] \frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)s)} \leq 1 - \alpha_1.$$

As a proceeding in the proof of Theorem 3.1, we get

$$\begin{aligned} &\sum_{s=2}^{\infty} \left[s(\alpha_2+1) - \alpha_3(\alpha_1+\alpha_2)\right] \frac{\delta^{s-1}\Gamma(1+\eta)}{\Gamma((1+\eta)s)} \\ &\leq \frac{\delta(\alpha_2+1)}{\eta+1}e^{\frac{\delta}{\eta+1}} + (\alpha_2 - \alpha_3(\alpha_1+\alpha_2)+1)\left(e^{\frac{\delta}{\eta+1}} - 1\right). \end{aligned} \tag{5.4}$$

However, (5.4) is bounded above by $1 - \alpha_1$, which concludes Expression (5.2). □

Remark 5.1. *The integral operator introduced in this section provides a mechanism to generate new members of the class while preserving the geometric properties established in Theorem 5.1. Consequently, it may be employed to derive further inclusion relationships, coefficient estimates, and radius problems for subclasses associated with the generalized Mittag–Leffler-type Poisson distribution.*

Corollary 5.1. *Let $\eta \geq 0$ and $\delta > 0$. If the integral operator $\mathcal{L}_{\eta,\delta}$ is given (5.1), then $\mathcal{L}_{\eta,\delta} \in \wp_1^*(\alpha_1, \alpha_2)$ if*

$$\frac{\delta(\alpha_2+1)}{\eta+1} e^{\frac{\delta}{\eta+1}} + (1-\alpha_1)\left(e^{\frac{\delta}{\eta+1}} - 1\right) \leq 1 - \alpha_1.$$

Corollary 5.2. *Let $\eta \geq 0$ and $\delta > 0$. If the integral operator $\mathcal{L}_{\eta,\delta}$ is given (5.1), then $\mathcal{L}_{\eta,\delta} \in \wp_0^*(\alpha_1, \alpha_2)$ if*

$$\frac{\delta(\alpha_2+1)}{\eta+1} e^{\frac{\delta}{\eta+1}} + (\alpha_2+1)\left(e^{\frac{\delta}{\eta+1}} - 1\right) \leq 1 - \alpha_1.$$

6. Conclusions

In this paper, we investigated the geometric behavior of the normalized Rabotnov function $\mathcal{R}_{\eta,\delta}(\sigma)$ and its associated linear and integral operators, $\mathcal{R}_{\eta,\delta}(\sigma)\hbar$ and $\mathcal{L}_{\eta,\delta}$, within several subclasses of analytic functions with negative coefficients. By employing coefficient inequalities, convolution techniques, and properties of the Gamma function, we derived new sufficient conditions that ensure membership in the classes $P_{\alpha_3}^*(\alpha_1, \alpha_2)$ and $\wp_{\alpha_3}^*(\alpha_1, \alpha_2)$.

The obtained results significantly generalize and extend a number of known results in the literature, thus naturally leading to several corollaries for classical subclasses such as starlike and convex functions of order α_1. In addition, the inclusion relationships established in this study provide a deeper understanding of the structural behavior of analytic functions defined via real-part inequalities and negative coefficients. Moreover, the analysis highlights the role of operator-based methods, particularly convolution and integral transforms, in generating new subclasses and preserving geometric properties.

Moreover, the results presented here reveal certain symmetry features in the behavior of Rabotnov-related operators, which contribute to a better understanding of invariant properties within these analytic function classes. Such observations may serve as a foundation for further investigations into symmetry-preserving transformations and their applications in the geometric function theory.

The techniques developed in this work are sufficiently flexible and can be applied to other special functions which arise in fractional calculus and mathematical physics. In particular, they may be extended to study functions defined via different fractional operators, as well as to explore more general subclasses of analytic and bi-univalent functions.

Future research directions include the determination of sharp coefficient bounds, the study of radius problems, and the investigation of extremal functions associated with the obtained results. Additionally, it would be of interest to consider q-analogues, higher-order operators, and symmetry-based approaches in the operator theory. Such extensions may further enrich the theory and open new avenues for applications in both pure and applied mathematics.

Author contributions

Feras Yousef and Tariq Al-Hawary: Conceptualization , methodology; Basem Aref Frasin: Supervision, critical revisions; Ibtisam Aldawish: Research, formal analysis, write-original draft. All authors reviewed and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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