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Research article

Exponential spatial decay estimates for the MGT-Fourier system in semi-infinite cylindrical domains

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Abstract: This paper investigates the spatial decay estimates for a coupled Moore-Gibson-Thompson (MGT) and Fourier heat system defined on a semi-infinite cylindrical domain in $\mathbb{R}^3$. Under homogeneous initial-boundary conditions and suitable decay assumptions at infinity, we derive a second-order differential inequality satisfied by the total energy functional. By transforming this inequality into a first-order form via operator factorization, we establish explicit exponential decay estimates of the energy along the axial direction. The result generalizes the classical Saint-Venant principle to higher-order coupled dissipative systems and provides a quantitative characterization of spatial decay for MGT-Fourier-type equations in unbounded domains. The findings contribute to the mathematical theory of thermoviscoelasticity and offer a foundation for further studies on structural stability and long-time behavior in such coupled systems.

Keywords: Moore-Gibson-Thompson equation; Fourier heat equation; spatial decay; Saint-Venant principle; semi-infinite cylinder; energy functional
Mathematics Subject Classification: 35B30, 35K55, 35Q35

1. Introduction

The spatial behaviour of solutions to partial differential equations in unbounded domains has been a recurring theme in applied mathematics. Among the various principles that describe such behavior, the Saint-Venant principle occupies a distinguished position. Originating from elasticity theory, it asserts that localized disturbances in a slender body produce effects that decay rapidly with distance from the region of excitation. Over the years, this principle has been systematically refined and extended to a wide class of dissipative systems. For a detailed historical account and mathematical formulations, the reader is referred to the comprehensive surveys [1, 2] and the monograph [3]. In particular, for problems set on semi-infinite strips or cylindrical domains, it is now well-established that the total energy decays exponentially as the axial coordinate increases, provided the solution is known to decay sufficiently fast at infinity [4–6].

A distinct but related line of research concerns the Phragmén-Lindelöf alternative. Unlike the Saint-Venant approach, which imposes decay conditions a priori, the Phragmén-Lindelöf theory investigates possible asymptotic scenarios without assuming any decay beforehand. Within this framework, one proves that as the spatial variable tends to infinity, the associated energy must either grow or decay exponentially, and these two possibilities are mutually exclusive. This idea has been fruitfully applied to higher-order partial differential equations. For instance, Li and Chen [7] derived alternative results for time-dependent double-diffusive Darcy flows, and Lin and Payne [8] treated a class of quasilinear second-order parabolic problems. Further contributions include the analysis of the Laplace equation with dynamic boundary conditions [9], Stokes flow in semi-infinite cylinders [10], and harmonic functions subject to nonlinear boundary conditions [11].

The Moore-Gibson-Thompson (MGT) equation has emerged as a fundamental model in nonlinear acoustics, particularly for describing high-intensity ultrasound propagation. Its derivation traces back to early studies of second-sound phenomena in viscous thermally relaxing fluids [12–14]. The equation is third-order in time and captures both damping and thermal relaxation effects. Seminal contributions include the proof of well-posedness and exponential stability for the nonlinear Jordan-Moore-Gibson-Thompson equation [15], the incorporation of quadratic gradient nonlinearities [16], the discovery of chaotic dynamics in MGT-type systems [17], and the development of an abstract semigroup framework [18]. The long-time dynamics of the MGT equation in dissipative settings have been rigorously analyzed in [19], and the Cauchy problem and asymptotic profiles have been investigated in [20] and [21], respectively. For the most recent advances on the MGT equation, we refer the reader to [22, 23].

When thermal effects are incorporated, the MGT equation must be coupled with a heat conduction law. The simplest and most widely used choice is the Fourier law, leading to the MGT-Fourier system. Such coupled models are mathematically challenging, especially when the spatial domain is unbounded, because mechanical and thermal dissipation mechanisms interact in a nontrivial way. Recent studies have addressed various aspects of these systems, including global existence, decay estimates, and spatial behavior [24–26]. The specific system under consideration in this paper reads

$$u_{,ttt} + au_{,tt} - b\Delta u_{,t} - c\Delta u = -d\Delta\theta, \quad \theta_{,t} - k\Delta\theta = d\Delta u_{,tt} + \alpha d\Delta u_{,t}, \tag{1.1}$$

where u denotes the displacement field, θ is the temperature variation, and a, b, c, d, k, α are positive constants. The first equation is an MGT-type wave equation that includes a third-order time derivative accounting for thermal relaxation and a viscous dissipation term. The second equation is the Fourier heat conduction law modified by mechanical source terms that describe heat generation due to deformation. The system therefore captures the mutual interaction between mechanical waves and heat transfer in a thermoviscoelastic medium. The supercritical (antidissipative) regime of this system was examined in [27], where the authors analyzed energy transfer mechanisms.

The present work investigates the spatial decay properties of solutions to (1.1) on a semi-infinite cylinder in $\mathbb{R}^3$. Our attention is focused on the initial-boundary problem (1.1) in the space-time region $R \times (0, \infty)$, where

$$R = \{(x_1, x_2, x_3) | (x_1, x_2) \in D, \quad x_3 \geq 0\},$$

with the arbitrary cross section D being a bounded simply connected region in the (x_1, x_2)-plane with piecewise smooth boundary ∂D. We also use the notations

$$R_z = \{(x_1, x_2, x_3)|(x_1, x_2) \in D, \quad x_3 > z \geq 0\},$$

$$D_z = \{(x_1, x_2, x_3)|(x_1, x_2) \in D, \quad x_3 = z \geq 0\}$$

for fixed z. Thus, in particular $R_0 = R$, $D_0 = D$.

The initial boundary conditions are

$$\theta(x_1, x_2, x_3, t) = u(x_1, x_2, x_3, t) = u_{,t}(x_1, x_2, x_3, t) = u_{,tt}(x_1, x_2, x_3, t) = 0, \ \ on \ \ \partial D \times (0, \infty), \quad t > 0. \tag{1.2}$$

$$\theta(x_1, x_2, 0, t) = g_1(x_1, x_2, t), \quad (x_1, x_2) \in D, \quad t > 0, \tag{1.3}$$

$$u(x_1, x_2, 0, t) = g_2(x_1, x_2, t), \quad (x_1, x_2) \in D, \quad t > 0, \tag{1.4}$$

$$u_{,t}(x_1, x_2, 0, t) = g_3(x_1, x_2, t), \quad (x_1, x_2) \in D, \quad t > 0, \tag{1.5}$$

$$u_{,tt}(x_1, x_2, 0, t) = g_4(x_1, x_2, t), \quad (x_1, x_2) \in D, \quad t > 0, \tag{1.6}$$

and

$$\theta(x_1, x_2, x_3, 0) = u(x_1, x_2, x_3, 0) = u_{,t}(x_1, x_2, x_3, 0) = u_{,tt}(x_1, x_2, x_3, 0) = 0. \tag{1.7}$$

We further impose the following asymptotic decay assumptions on the solution as $x_3 \to \infty$ uniformly in x_1 and x_2:

$$\theta(x_1, x_2, x_3, t), u(x_1, x_2, x_3, t), u_{,tt}(x_1, x_2, x_3, t), u_{,t}(x_1, x_2, x_3, t) \to 0 \quad as \quad x_3 \to \infty. \tag{1.8}$$

In this work, a comma-based subscript notation is adopted to signify partial differentiation. Specifically, differentiation with respect to the spatial coordinate x_k is denoted as k, and thus, $v_{,i}$ denotes $\partial v/\partial x_i$, and $\theta_{,t}$ denotes $\partial\theta/\partial t$. The usual summation convention is employed with repeated Latin subscripts i summed from 1 to 3. Hence, $u_{,i}u_{,i} = u_{,1}^2 + u_{,2}^2 + u_{,3}^2$.

The methodology employed here differs from that used in earlier works on similar models. Whereas previous investigations often rely on first-order integral-differential inequalities and the introduction of auxiliary functions, we adopt a different technical route. Specifically, we construct an energy functional $E(z, t)$ that incorporates a linear spatial weighting factor $(\xi - z)$. This weighting serves two purposes: It regularizes the energy, and more importantly, it leads naturally to a second-order differential inequality for $E(z, t)$. By applying an operator factorization technique, we transform this second-order inequality into an equivalent first-order form, which can be integrated directly. This approach avoids the construction of auxiliary integral functions and yields a compact derivation of the exponential decay estimate. The exponential decay of mechanical disturbances (displacement) and thermal disturbances (temperature) along the axial direction implies that perturbations localized near the excited end of a cylindrical body attenuate rapidly with increasing distance from that end. This result provides a quantitative characterization of the Saint-Venant principle for thermoviscoelastic materials, demonstrating that end effects are confined to a boundary layer whose decay rate is governed by the material's dissipative properties.

The main result of the paper is an explicit bound of the form

$$G(z, t) \leq E(0, t)\, e^{-bz},$$

where $G(z,t)$ is a non-negative quantity that controls various norms of the solution and its derivatives, and $b > 0$ is a decay rate determined by the system parameters and the weighting constant ω. This estimate confirms that the generalized Saint-Venant principle holds for the coupled MGT-Fourier system in three dimensions.

The paper is organized as follows. Section 2 introduces the weighted energy functionals E_1, E_2, E_3, and their combination $E = E_1 + E_2 + E_3$, and it provides explicit representations. Section 3 derives the fundamental inequalities, including lower bounds for $-\partial E/\partial z$ and upper bounds for the remaining boundary terms, culminating in a second-order differential inequality for E. Section 4 applies operator factorization to reduce this inequality to the first order and obtains the exponential decay estimate. Section 5 concludes the paper with a discussion of possible extensions, including structural stability and the incorporation of biharmonic operators.

2. The energy functional $E(z,t)$

Before constructing the energy functionals, we note that exponential weighting in the axial direction is essential to capture the spatial decay properties of the coupled MGT-Fourier system. The following propositions introduce three auxiliary functionals, whose combination leads to a second-order differential inequality governing the spatial evolution of the total energy. In the following, A is an area element on $x_1 - x_2$ plane. $dA = dx_2 dx_1$.

Proposition 2.1. *Let* (u,θ) *be the classical solution of Eq* (1.1) *and satisfy the initial boundary value problems* (1.2)–(1.8). *We define a function*

$$
\begin{aligned}
E_1(z,t) = & -b\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta\eta}u_{,3\eta}dxd\eta - c\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta\eta}u_{,3}dxd\eta \\
& + d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta\eta}\theta_{,3}dxd\eta.
\end{aligned} \tag{2.1}
$$

$E_1(z,t)$ can also be expressed as

$$
\begin{aligned}
E_1(z,t) = & \left(\frac{\omega}{2}+a\right)\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)(u_{,\eta\eta})^2dxd\eta \\
& + \left(\frac{b\omega}{2}-c\right)\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta}u_{,i\eta}dxd\eta \\
& + \frac{c\omega^2}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i}u_{,i}dxd\eta + \frac{1}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)(u_{,tt})^2dx \\
& + \frac{b}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,it}u_{,it}dx + c\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,it}u_{,i}dx \\
& - d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta}\theta_{,i}dxd\eta + \frac{c\omega}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,i}u_{,i}dx.
\end{aligned} \tag{2.2}
$$

Proof. Multiplying $(1.1)_1$ by $\exp(-\omega\eta)(\xi-z)u_{,\eta\eta}$ and integrating, we have

$$
0 = \int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,\eta\eta}(u_{,\eta\eta\eta}+au_{,\eta\eta}-bu_{,ii\eta}-cu_{,ii}+d\theta_{,ii})dxd\eta. \tag{2.3}
$$

The first term on the right-hand side of (2.3) can be expressed as

$$\begin{aligned}
&\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,\eta\eta}u_{,\eta\eta\eta}dxd\eta\\
=&\frac{\omega}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)(u_{,\eta\eta})^2dxd\eta+\frac{1}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)(u_{,tt})^2dx.
\end{aligned}\tag{2.4}$$

The third term on the right-hand side of (2.3) can be expressed as

$$\begin{aligned}
&-b\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,\eta\eta}u_{,ii\eta}dxd\eta\\
=&\,b\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta\eta}u_{,i\eta}dxd\eta+b\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta\eta}u_{,3\eta}dxd\eta\\
=&\frac{b\omega}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta}u_{,i\eta}dxd\eta+\frac{b}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,it}u_{,it}dx\\
&+b\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta\eta}u_{,3\eta}dxd\eta.
\end{aligned}\tag{2.5}$$

The fourth term on the right-hand side of (2.3) can be expressed as

$$\begin{aligned}
&-c\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,\eta\eta}u_{,ii}dxd\eta\\
=&\,c\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta\eta}u_{,i}dxd\eta+c\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta\eta}u_{,3}dxd\eta\\
=&-c\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta}u_{,i\eta}dxd\eta+c\omega\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta}u_{,i}dxd\eta\\
&+c\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,it}u_{,i}dx+c\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta\eta}u_{,3}dxd\eta\\
=&-c\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta}u_{,i\eta}dxd\eta+\frac{c\omega^2}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i}u_{,i}dxd\eta\\
&+\frac{c\omega}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,i}u_{,i}dx+c\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,it}u_{,i}dx\\
&+c\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta\eta}u_{,3}dxd\eta.
\end{aligned}\tag{2.6}$$

The last term on the right-hand side of (2.3) can be expressed as

$$\begin{aligned}
&d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,\eta\eta}\theta_{,ii}dxd\eta\\
=&-d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta\eta}\theta_{,i}dxd\eta-d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta\eta}\theta_{,3}dxd\eta.
\end{aligned}\tag{2.7}$$

We define $E_1(z,t)$ as (2.1). Combining (2.3)–(2.7), we can easily get that $E_1(z,t)$ has another expression (2.2).

Proposition 2.2. *Let* (u, θ) *be the classical solution of Eq* (1.1) *and satisfy the initial boundary value problems* (1.2)–(1.8). *We define a function*

$$\begin{aligned} E_2(z,t) = & \frac{b}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(u_{,3})^2dx - c\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta}u_{,3}dxd\eta \\ & + d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta}\theta_{,3}dxd\eta. \end{aligned} \tag{2.8}$$

$E_2(z,t)$ *can also be expressed as*

$$\begin{aligned} & E_2(z,t) \\ = & -\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi - z)(u_{,\eta\eta})^2dxd\eta + \frac{\omega^2}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi - z)(u_{,\eta})^2dxd\eta \\ & + \left(\frac{\omega}{2} + \frac{a}{2}\right)\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi - z)(u_{,t})^2dx + \frac{\omega a}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi - z)(u_{,\eta})^2dxd\eta \\ & + b\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi - z)u_{,i\eta}u_{,i\eta}dxd\eta + \frac{c\omega}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi - z)u_{,i}u_{,i}dxd\eta \\ & + \frac{c}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi - z)u_{,i}u_{,i}dx - d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi - z)u_{,i\eta}\theta_{,i}dxd\eta \\ & + \int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi - z)u_{,t}u_{,tt}dx. \end{aligned} \tag{2.9}$$

Proof. Multiplying $(1.1)_1$ by $\exp(-\omega\eta)(\xi - z)u_{,\eta}$ and integrating, we have

$$0 = \int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi - z)u_{,\eta}(u_{,\eta\eta\eta} + au_{,\eta\eta} - bu_{,ii\eta} - cu_{,ii} + d\theta_{,ii})dxd\eta. \tag{2.10}$$

The first term on the right-hand side of (2.10) can be written as

$$\begin{aligned} & \int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi - z)u_{,\eta}u_{,\eta\eta\eta}dxd\eta \\ = & -\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi - z)(u_{,\eta\eta})^2dxd\eta + \omega\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi - z)u_{,\eta}u_{,\eta\eta}dxd\eta \\ & + \int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi - z)u_{,t}u_{,tt}dx \\ = & -\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi - z)(u_{,\eta\eta})^2dxd\eta + \frac{\omega^2}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi - z)(u_{,\eta})^2dxd\eta \\ & + \frac{\omega}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi - z)(u_{,t})^2dx + \int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi - z)u_{,t}u_{,tt}dx. \end{aligned} \tag{2.11}$$

The second term on the right-hand side of (2.10) can be written as

$$\begin{aligned} & a\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi - z)u_{,\eta}u_{,\eta\eta}dxd\eta \\ = & \frac{\omega a}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi - z)(u_{,\eta})^2dxd\eta + \frac{a}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi - z)(u_{,t})^2dx. \end{aligned} \tag{2.12}$$

The third term on the right-hand side of (2.10) can be written as

$$\begin{aligned}&-b\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,\eta}u_{,ii\eta}dxd\eta\\&=b\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta}u_{,i\eta}dxd\eta+b\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta}u_{,3\eta}dxd\eta.\end{aligned}\tag{2.13}$$

The fourth term on the right-hand side of (2.10) can be written as

$$\begin{aligned}&-c\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,\eta}u_{,ii}dxd\eta\\&=c\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta}u_{,i}dxd\eta+c\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta}u_{,3}dxd\eta\\&=\frac{c\omega}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i}u_{,i}dxd\eta+\frac{c}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,i}u_{,i}dx\\&\quad+c\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta}u_{,3}dxd\eta.\end{aligned}\tag{2.14}$$

The last term on the right-hand side of (2.10) can be written as

$$\begin{aligned}&d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,\eta}\theta_{,ii}dxd\eta\\&=-d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta}\theta_{,i}dxd\eta-d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta}\theta_{,3}dxd\eta.\end{aligned}\tag{2.15}$$

We define $E_2(z,t)$ as (2.8). Combining (2.10)–(2.15), we can also get another expression of $E_2(z,t)$ (2.9).

Proposition 2.3. *Let* (u,θ) *be the classical solution of Eq* (1.1) *and satisfy the initial boundary value problems* (1.2)–(1.8)*. We define a function*

$$\begin{aligned}E_3(z,t)=&-k\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)\theta\theta_{,3}dxd\eta-d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta\eta}\theta_{,3}dxd\eta\\&+d\int_0^t\int_{D_z}\exp(-\omega\eta)u_{,\eta\eta}\theta dAd\eta-ad\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)\theta u_{,3\eta}dxd\eta.\end{aligned}\tag{2.16}$$

$E_3(z,t)$ can also be expressed as

$$\begin{aligned}E_3(z,t)=&\frac{\omega}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)\theta^2dxd\eta+\frac{1}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)\theta^2dx\\&+k\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)\theta_{,i}\theta_{,i}dxd\eta\\&+d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta\eta}\theta_{,i}dxd\eta\\&+ad\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)\theta_{,i}u_{,i\eta}dxd\eta.\end{aligned}\tag{2.17}$$

Proof. Multiplying $(1.1)_2$ by $\exp(-\omega\eta)(\xi-z)\theta$ and integrating, we have

$$0=\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)\theta(\theta_{,\eta}-k\theta_{,ii}-du_{,ii\eta\eta}-adu_{,ii\eta})dxd\eta. \tag{2.18}$$

The first term on the right-hand side of (2.18) can be expressed as

$$\begin{aligned}&\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)\theta\theta_{,\eta}dxd\eta\\ =&\frac{\omega}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)\theta^2dxd\eta+\frac{1}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)\theta^2dx.\end{aligned} \tag{2.19}$$

The second term on the right-hand side of (2.18) can be expressed as

$$\begin{aligned}&-k\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)\theta\theta_{,ii}dxd\eta\\ =&k\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)\theta_{,i}\theta_{,i}dxd\eta+k\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)\theta\theta_{,3}dxd\eta.\end{aligned} \tag{2.20}$$

The third term on the right-hand side of (2.18) can be expressed as

$$\begin{aligned}&-d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)\theta u_{,ii\eta\eta}dxd\eta\\ =&d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta\eta}\theta_{,i}dxd\eta+d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,3\eta\eta}\theta dxd\eta\\ =&d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta\eta}\theta_{,i}dxd\eta+d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta\eta}\theta_{,3}dxd\eta\\ &-d\int_0^t\int_{D_z}\exp(-\omega\eta)u_{,\eta\eta}\theta dAd\eta.\end{aligned} \tag{2.21}$$

The last term term on the right-hand side of (2.18) can be expressed as

$$\begin{aligned}&-ad\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)\theta u_{,ii\eta}dxd\eta\\ =&ad\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)\theta_{,i}u_{,i\eta}dxd\eta+ad\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)\theta u_{,3\eta}dxd\eta.\end{aligned} \tag{2.22}$$

If we define $E_3(z,t)$ as (2.16), a combination of (2.18)–(2.22) gives another expression of $E_3(z,t)$ as (2.17).

Proposition 2.4. *If we define $E(z,t)=E_1(z,t)+E_2(z,t)+E_3(z,t)$, we have*

$$\begin{aligned}E(z,t)=&\frac{b}{2}\int_z^\infty\int_{D_z}\exp(-\omega\eta)(u_{,3})^2dAd\eta+d\int_0^t\int_{D_z}\exp(-\omega\eta)u_{,\eta\eta}\theta dAd\eta\\&-b\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta\eta}u_{,3\eta}dxd\eta-c\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta\eta}u_{,3}dxd\eta\\&-c\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta}u_{,3}dxd\eta+d\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,\eta}\theta_{,3}dxd\eta\\&-k\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)\theta\theta_{,3}dxd\eta-ad\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)\theta u_{,3\eta}dxd\eta.\end{aligned}\tag{2.23}$$

We can also get

$$\begin{aligned}E(z,t)=&\left(\frac{\omega}{2}+a-1\right)\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)(u_{,\eta\eta})^2dxd\eta\\&+\left(\frac{b\omega}{2}+b-c\right)\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta}u_{,i\eta}dxd\eta\\&+\frac{c\omega^2}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i}u_{,i}dxd\eta+\frac{1}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)(u_{,tt})^2dx\\&+\frac{b}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,it}u_{,it}dx+c\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,it}u_{,i}dx\\&+\left(\frac{\omega^2}{2}+\frac{\omega a}{2}\right)\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)(u_{,\eta})^2dxd\eta\\&+\left(\frac{\omega}{2}+\frac{a}{2}\right)\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)(u_{,t})^2dx\\&+\frac{c\omega}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i}u_{,i}dxd\eta\\&+\left(\frac{c\omega}{2}+\frac{c}{2}\right)\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,i}u_{,i}dx\\&+(ad-d)\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta}\theta_{,i}dxd\eta\\&+\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,t}u_{,tt}dx+\frac{\omega}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)\theta^2dxd\eta\\&+\frac{1}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)\theta^2dx+k\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)\theta_{,i}\theta_{,i}dxd\eta.\end{aligned}\tag{2.24}$$

Proof. A combination of Propositions 2.1–2.3 can give the desired results of Proposition 2.4.

3. Some basic inequalities

Having constructed the energy functionals, we proceed to derive the fundamental inequalities that relate the spatial derivative of $E(z,t)$ to the dissipative terms in the system. Through a combination of integration by parts, weighted estimates, and elementary inequalities, we obtain lower bounds for $-\partial E/\partial z$ and upper bounds for the remaining boundary contributions. The results are summarized in the propositions below.

Proposition 3.1. *For the functional $E(z,t)$ defined in* (2.24)*, we have the following estimates:*

$$\begin{aligned}
-\frac{\partial E(z,t)}{\partial z} \geq & \left(\frac{\omega}{2}+a-1\right)\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(u_{,\eta\eta})^2dxd\eta \\
&+\left(\frac{b\omega}{2}+b-c-\frac{(ad+d)^2}{2k}\right)\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,i\eta}u_{,i\eta}dxd\eta \\
&+\frac{c\omega^2}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,i}u_{,i}dxd\eta+\frac{1}{4}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(u_{,tt})^2dx \\
&+\frac{b}{4}\int_z^\infty\int_{D_\xi}\exp(-\omega t)u_{,it}u_{,it}dx+\left(\frac{\omega^2}{2}+\frac{\omega a}{2}\right)\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(u_{,\eta})^2dxd\eta \\
&+\left(\frac{\omega}{2}+\frac{a}{2}-1\right)\int_z^\infty\int_{D_\xi}\exp(-\omega t)(u_{,t})^2dx \\
&+\frac{c\omega}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)u_{,i}u_{,i}dxd\eta \\
&+\left(\frac{c\omega}{2}+\frac{c}{2}-\frac{c^2}{b}\right)\int_z^\infty\int_{D_\xi}\exp(-\omega t)u_{,i}u_{,i}dx \\
&+\frac{\omega}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)\theta^2dxd\eta+\frac{1}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega t)\theta^2dx \\
&+\frac{k}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)\theta_{,i}\theta_{,i}dxd\eta,
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
\frac{\partial^2 E(z,t)}{\partial z^2} \geq & \left(\frac{\omega}{2}+a-1\right)\int_0^t\int_{D_z}\exp(-\omega\eta)(u_{,\eta\eta})^2dAd\eta \\
&+\left(\frac{b\omega}{2}+b-c-\frac{(ad+d)^2}{2k}\right)\int_0^t\int_{D_z}\exp(-\omega\eta)u_{,i\eta}u_{,i\eta}dAd\eta \\
&+\frac{c\omega^2}{2}\int_0^t\int_{D_z}\exp(-\omega\eta)u_{,i}u_{,i}dAd\eta+\frac{1}{4}\int_{D_z}\exp(-\omega t)(u_{,tt})^2dA \\
&+\frac{b}{4}\int_{D_z}\exp(-\omega t)u_{,it}u_{,it}dA+\left(\frac{\omega^2}{2}+\frac{\omega a}{2}\right)\int_0^t\int_{D_z}\exp(-\omega\eta)(u_{,\eta})^2dAd\eta \\
&+\left(\frac{\omega}{2}+\frac{a}{2}-1\right)\int_{D_z}\exp(-\omega t)(u_{,t})^2dA+\frac{c\omega}{2}\int_0^t\int_{D_z}\exp(-\omega\eta)u_{,i}u_{,i}dAd\eta \\
&+\left(\frac{c\omega}{2}+\frac{c}{2}-\frac{c^2}{b}\right)\int_{D_z}\exp(-\omega t)u_{,i}u_{,i}dA+\frac{\omega}{2}\int_0^t\int_{D_z}\exp(-\omega\eta)\theta^2dAd\eta \\
&+\frac{1}{2}\int_{D_z}\exp(-\omega t)\theta^2dA+\frac{k}{2}\int_0^t\int_{D_z}\exp(-\omega\eta)\theta_{,i}\theta_{,i}dAd\eta.
\end{aligned} \tag{3.2}$$

Proof. We now bound terms in (2.24). Using the Schwarz inequality, we have

$$\begin{aligned}\left|c\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,it}u_{,i}dx\right|\leq{}&\frac{b}{4}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,it}u_{,it}dx\\&+\frac{c^2}{b}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,i}u_{,i}dx,\end{aligned}\tag{3.3}$$

$$\begin{aligned}&\left|(ad-d)\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta}\theta_{,i}dxd\eta\right|\\&\leq\frac{k}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)\theta_{,i}\theta_{,i}dxd\eta\\&\quad+\frac{(ad+d)^2}{2k}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta}u_{,i\eta}dxd\eta,\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}\left|\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,t}u_{,tt}dx\right|\leq{}&\frac{1}{4}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)(u_{,tt})^2dx\\&+\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,t}^2dx.\end{aligned}\tag{3.5}$$

Combining (2.24) and (3.3)–(3.5), we obtain

$$\begin{aligned}E(z,t)\geq{}&\left(\frac{\omega}{2}+a-1\right)\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)(u_{,\eta\eta})^2dxd\eta\\&+\left(\frac{b\omega}{2}+b-c-\frac{(ad+d)^2}{2k}\right)\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i\eta}u_{,i\eta}dxd\eta\\&+\frac{c\omega^2}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i}u_{,i}dxd\eta+\frac{1}{4}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)(u_{,tt})^2dx\\&+\frac{b}{4}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,it}u_{,it}dx+\left(\frac{\omega^2}{2}+\frac{\omega a}{2}\right)\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)(u_{,\eta})^2dxd\eta\\&+\left(\frac{\omega}{2}+\frac{a}{2}-1\right)\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)(u_{,t})^2dx\\&+\frac{c\omega}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)u_{,i}u_{,i}dxd\eta\\&+\left(\frac{c\omega}{2}+\frac{c}{2}-\frac{c^2}{b}\right)\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)u_{,i}u_{,i}dx\\&+\frac{\omega}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)\theta^2dxd\eta+\frac{1}{2}\int_z^\infty\int_{D_\xi}\exp(-\omega t)(\xi-z)\theta^2dx\\&+\frac{k}{2}\int_0^t\int_z^\infty\int_{D_\xi}\exp(-\omega\eta)(\xi-z)\theta_{,i}\theta_{,i}dxd\eta\\={}&G(z,t).\end{aligned}\tag{3.6}$$

If we choose

$$\omega > \max\left\{2,\ \frac{2b + 2c + \frac{(ad+d)^2}{k}}{b},\ \frac{2c}{b}\right\}, \tag{3.7}$$

we can easily get that all coefficients on the right-hand side of (3.6) are positive.

Differentiating (2.24) with respect to z and using the same procedure as deriving (3.6), we can also obtain the desired results (3.1) and (3.2).

Proposition 3.2. *For the functional $E(z,t)$ defined in* (2.24)*, we have the following estimate:*

$$E(z,t) \le k_1 \frac{\partial^2 E(z,t)}{\partial z^2} - k_2 \frac{\partial E(z,t)}{\partial z}, \tag{3.8}$$

where k_1 and k_2 are positive constants to be determined later.

Proof. We now bound terms in (2.24). Using the Schwarz inequality, we have

$$\begin{aligned} d\int_0^t \int_{D_z} \exp(-\omega\eta) u_{,\eta\eta}\theta dA d\eta \le& \frac{d}{2}\int_0^t \int_{D_z} \exp(-\omega\eta)(u_{,\eta\eta})^2 dA d\eta \\ &+ \frac{d}{2}\int_0^t \int_{D_z} \exp(-\omega\eta)\theta^2 dA d\eta. \end{aligned} \tag{3.9}$$

We thus obtain

$$\frac{b}{2}\int_0^t \int_{D_z} \exp(-\omega\eta)(u_{,3})^2 dA d\eta + d\int_0^t \int_{D_z} \exp(-\omega\eta) u_{,\eta\eta}\theta dA d\eta \le k_1 \frac{\partial^2 E(z,t)}{\partial z^2}, \tag{3.10}$$

with

$$k_1 = \max\left\{\frac{d}{\omega + 2a - 2},\ \frac{d}{\omega},\ \frac{b}{c\omega + c}\right\}.$$

Following the same procedures, we can also set

$$\begin{aligned} &- b\int_0^t \int_z^\infty \int_{D_\xi} \exp(-\omega\eta) u_{,\eta\eta} u_{,3\eta} dx d\eta - c\int_0^t \int_z^\infty \int_{D_\xi} \exp(-\omega\eta) u_{,\eta\eta} u_{,3} dx d\eta \\ &- c\int_0^t \int_z^\infty \int_{D_\xi} \exp(-\omega\eta) u_{,\eta} u_{,3} dx d\eta + d\int_0^t \int_z^\infty \int_{D_\xi} \exp(-\omega\eta) u_{,\eta}\theta_{,3} dx d\eta \\ &- k\int_0^t \int_z^\infty \int_{D_\xi} \exp(-\omega\eta)\theta\theta_{,3} dx d\eta - ad\int_0^t \int_z^\infty \int_{D_\xi} \exp(-\omega\eta)\theta u_{,3\eta} dx d\eta \\ \le& k_2\left(-\frac{\partial E(z,t)}{\partial z}\right), \end{aligned} \tag{3.11}$$

with

$$k_2 = \max\left\{\frac{b+c}{\omega + 2a - 2}, \frac{b + ad}{b\omega + 2b - 2c - \frac{(ad+d)^2}{k}}, \frac{2}{\omega^2}, \frac{c+d}{\omega^2 + \omega a}, \frac{d+k}{k}, \frac{k + ad}{\omega}\right\}.$$

Combining (2.24), (3.1), (3.2), (3.10), and (3.11), we can obtain the desired result (3.8).

4. Spatial decay estimate

In this section, we aim to establish a spatial decay estimate for the energy functional $E(z,t)$ defined in (2.24). To this end, we first rewrite inequality (3.8) in a more convenient form. After rearranging the terms, we obtain the following second-order differential inequality:

$$\frac{\partial^2 E(z,t)}{\partial z^2} - \frac{k_2}{k_1}\frac{\partial E(z,t)}{\partial z} - \frac{1}{k_1}E(z,t) \geq 0. \tag{4.1}$$

This inequality is not directly solvable in its present form. However, because first-order differential inequalities are generally easier to handle, we seek to transform (4.1) into an equivalent first-order system. By applying an operator factorization technique, we can rewrite the above inequality in a form that allows direct integration, as shown below.

Applying operator theory, we observe that inequality (4.1) can be expressed in the following operator formulation:

$$\left(\frac{\partial}{\partial z} - a\right)\left(\frac{\partial E(z,t)}{\partial z} + bE(z,t)\right) \geq 0. \tag{4.2}$$

For inequalities (4.1) and (4.2) to be equivalent, the parameters a and b are required to satisfy the following relationships:

$$b - a = -\frac{k_2}{k_1},\ ab = \frac{1}{k_1}. \tag{4.3}$$

The solutions to Eq (4.3) can be readily obtained:

$$a = \frac{\sqrt{(\frac{k_2}{k_1})^2 + \frac{4}{k_1}} + \frac{k_2}{k_1}}{2},\ b = \frac{\sqrt{(\frac{k_2}{k_1})^2 + \frac{4}{k_1}} - \frac{k_1}{k_1}}{2}.$$

Inequality (4.2) may be equivalently expressed as follows:

$$\frac{\partial}{\partial z}\left\{e^{-az}\left(\frac{\partial E(z,t)}{\partial z} + bE(z,t)\right)\right\} \geq 0. \tag{4.4}$$

Integrating (4.4) with respect to z from z to ∞ and invoking the property that the solution vanishes at infinity, we arrive at

$$-\left\{e^{-az}\left(\frac{\partial E(z,t)}{\partial z} + bE(z,t)\right)\right\} \geq 0. \tag{4.5}$$

From (4.5), we can easily obtain

$$\frac{\partial E(z,t)}{\partial z} + bE(z,t) \leq 0. \tag{4.6}$$

Solving inequality (4.6), we easily get

$$E(z,t) \leq E(0,t)e^{-bz}. \tag{4.7}$$

From (3.6), we have

$$G(z,t) \leq E(0,t)\mathrm{e}^{-bz}. \tag{4.8}$$

Inequality (4.8) implies that the energy can decay as $z \to \infty$. We thus have proved the following:
Theorem 4.1. *Assuming that* (u,θ) *constitutes the classical solution to the initial-boundary value problems governed by Eqs* (1.1)–(1.8)*, we proceed to establish that the corresponding decay estimates for the energy* $G(z,t)$ *as defined in* (3.6) *satisfy the following estimate:*

$$G(z,t) \leq E(0,t)\mathrm{e}^{-\frac{\sqrt{(\frac{k_2}{k_1})^2+\frac{4}{k_1}}-\frac{k_2}{k_1}}{2}z}. \tag{4.9}$$

Remark 4.1. *With the exponential decay estimate (4.9) established, we have achieved the main goal of this paper, namely, to verify the generalized Saint-Venant principle for the coupled MGT-Fourier system in a semi-infinite cylinder. The coupling introduces additional dissipation channels: Mechanical waves generate heat via the* $d\Delta u_{,tt}$ *and* $d\Delta u_{,t}$ *source terms, and temperature gradients feed back into the MGT equation through* $d\Delta\theta$*. This mutual interaction enhances the overall decay rate compared to decoupled systems, as both mechanical viscosity (*$b\Delta u_t$*) and thermal diffusivity (*$k\Delta\theta$*) jointly contribute to energy attenuation. The decay rate depends on the weighting parameter* ω *and the system constants, reflecting the combined dissipative mechanisms of mechanical and thermal origin. The weighting parameter* ω*is a mathematical tool that selects which spatial modes contribute most to the energy functional. In practice, it allows engineers to tune the decay estimate by emphasizing near-field versus far-field behavior. Larger* ω *accelerates the apparent decay but requires stricter positivity conditions, offering flexibility in designing numerical simulations or experimental validations.*
Remark 4.2. *We clarify that* $G(z,t)$ *is not an artificial auxiliary quantity but an explicit weighted energy functional defined directly from the solution* (u,θ) *and its derivatives (Eq (3.6)), controlling physically meaningful norms such as weighted* L_2*-integrals of* $|\nabla u|$, $|u_{,t}|$ θ*, and* $|\nabla\theta|$ *. Because we prove* $E(z,t) \geq G(z,t)$ *and* $E(z,t) \leq E(0,t)e^{-bz}$*, the decay of* $G(z,t)$ *directly implies the corresponding spatial decay of the original solution. The decay rate* b *is explicitly determined by the system parameters and the weight* ω*, transparently reflecting the combined mechanical and thermal dissipation.*

5. Conclusions

In this paper, we establish a generalized Saint-Venant principle for the coupled MGT-Fourier system on a semi-infinite cylinder in $\mathbb{R}^3$. Under homogeneous initial-boundary conditions and decay assumptions at infinity, we construct a weighted energy functional and prove that it satisfies a second-order differential inequality. By applying operator factorization, we reduce this inequality to first-order form and obtain an explicit exponential decay estimate $G(z,t) \leq E(0,t)e^{-bz}$, where the decay rate b depends explicitly on the system parameters and the weighting constant. To the best of our knowledge, this is the first decay result of its kind for such coupled higher-order thermoviscoelastic models in unbounded domains.

This work makes three theoretical contributions. First, it extends the classical Saint-Venant principle to a higher-order coupled dissipative system involving both mechanical and thermal effects. Second, the weighted energy functional method and the second-order differential inequality technique provide a systematic framework adaptable to other coupled partial differential equations with similar dissipative structures. Third, the explicit decay rate quantifies how the combined dissipation from the MGT equation and the Fourier heat equation influences spatial decay. Consider the propagation of high-intensity ultrasound in a thermoviscoelastic rod. The MGT-Fourier system governs the coupled mechanical vibrations and heat generation arising from acoustic attenuation. The exponential decay estimate establishes that both the displacement amplitude and the temperature rise attenuate monotonically along the axial direction. This result is of practical significance for transducer design, thermal hotspot mitigation, and the safe operation of medical and industrial ultrasonic devices.

Future research directions include unweighted energy estimates; structural stability analysis of the coupled system in unbounded domains; and extensions to biharmonic operators, higher-gradient theories, and nonlinear versions of the system. These directions will be addressed in forthcoming work.

Author contributions

Jincheng Shi: Conceptualization, methodology design, derivation of the main conclusions; Zijun Cheng: Writing–original draft, manuscript revision, and verification of the conclusions. All authors have read and approved the final version of the manuscript for publication. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. C. O. Horgan, Recent developments concerning Saint-Venant's principle: an update, *Appl. Mech. Rev.*, **42** (1989), 295–303. https://doi.org/10.1115/1.3152414
2. C. O. Horgan, Recent developments concerning Saint-Venant's principle: a second update, *Appl. Mech. Rev.*, **49** (1996), S101–S111. https://doi.org/10.1115/1.3101961
3. C. O. Horgan, J. K. Knowles, Recent developments concerning Saint-Venant's principle, *Adv. Appl. Mech.*, **23** (1983), 179–269. https://doi.org/10.1016/S0065-2156(08)70244-8
4. M. C. Leseduarte, R. Quintanilla, Spatial behavior in high-order partial differential equations, *Math. Methods Appl. Sci.*, **41** (2018), 2480–2493. https://doi.org/10.1002/mma.4753

5. X. J. Chen, Y. F. Li, Spatial properties and the influence of the Soret coefficient on the solutions of time-dependent double-diffusive Darcy plane flow, *Electron. Res. Arch.*, **31** (2023), 421–441. https://doi.org/10.3934/era.2023021

6. J. R. Fernández, R. Quintanilla, Analysis of a higher order problem within the second gradient theory, *Appl. Math. Lett.*, **154** (2024), 109086. https://doi.org/10.1016/j.aml.2024.109086

7. Y. F. Li, X. J. Chen, Phragmén-Lindelöf alternative results in time-dependent double-diffusive Darcy plane flow, *Math. Methods Appl. Sci.*, **45** (2022), 6982–6997. https://doi.org/10.1002/mma.8220

8. C. H. Lin, L. E. Payne, A Phragmén-Lindelöf alternative for a class of quasilinear second order parabolic problems, *Differ. Integral Equ.*, **8** (1995), 539–551. https://doi.org/10.57262/die/1369316504

9. M. C. Leseduarte, R. Quintanilla, Phragmén-Lindelöf of alternative for the Laplace equation with dynamic boundary conditions, *J. Appl. Anal. Comput.*, **7** (2017), 1323–1335. https://doi.org/10.11948/2017081

10. Y. Liu, C. H. Lin, Phragmén-Lindelöf type alternative results for the stokes flow equation, *Math. Inequal. Appl.*, **9** (2006), 671–694.

11. C. O. Horgan, L. E. Payne, Phragmén-Lindelöf type results for harmonic functions with nonlinear boundary conditions, *Arch. Ration. Mech. Anal.*, **122** (1993), 123–144. https://doi.org/10.1007/BF00378164

12. P. M. Jordan, Second-sound phenomena in inviscid, thermally relaxing gases, *Discrete Contin. Dyn. Syst. Ser. B*, **19** (2014), 2189–2205. https://doi.org/10.3934/dcdsb.2014.19.2189

13. D. T. Blackstock, *Approximate equations governing finite-amplitude sound in thermoviscous fluids*, Technical Report, Rochester, New York, 1963.

14. M. F. Hamilton, D. T. Blackstock, *Nonlinear acoustics*, 3 Eds., San Diego: Academic Press, 1998.

15. B. Kaltenbacher, I. Lasiecka, M. K. Pospieszalska, Well-posedness and exponential decay of the energy in the nonlinear Jordan-Moore-Gibson-Thompson equation arising in high intensity ultrasound, *Math. Models Methods Appl. Sci.*, **22** (2012), 1250035. https://doi.org/10.1142/S0218202512500352

16. B. Kaltenbacher, V. Nikolić, The Jordan-Moore-Gibson-Thompson equation: well-posedness with quadratic gradient nonlinearity and singular limit for vanishing relaxation time, *Math. Models Methods Appl. Sci.*, **29** (2019), 2523–2556. https://doi.org/10.1142/S0218202519500532

17. J. A. Conejero, C. Lizama, F. Rodenas, Chaotic behaviour of the solutions of the Moore-Gibson-Thompson equation, *Appl. Math. Inf. Sci.*, **9** (2015), 2233–2238.

18. R. Marchand, T. McDevitt, R. Triggiani, An abstract semigroup approach to the third-order Moore-Gibson-Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability, *Math. Methods Appl. Sci.*, **35** (2012), 1896–1929. https://doi.org/10.1002/mma.1576

19. M. Pellicer, R. Quintanilla, Continuous dependence and convergence for Moore-Gibson-Thompson heat equation, *Acta Mech.*, **234** (2023), 3241–3257. https://doi.org/10.1007/s00707-023-03537-y

20. W. H. Chen, Large time asymptotic behavior for the dissipative Timoshenko system and its application, *Anal. Appl.*, **24** (2026), 385–418. https://doi.org/10.1142/S0219530525500241
21. W. H. Chen, H. Takeda, Large time behavior for the nonlinear dissipative Boussinesq equation, *Math. Nachr.*, **298** (2025), 2770–2793. https://doi.org/10.1002/mana.70015
22. A. E. Abouelregal, S. S. Alsaeed, Advanced thermo-viscoelastic modeling of nanoscale materials under laser radiation and magnetic fields using memory-enhanced MGT theory, *Contin. Mech. Thermodyn.*, **37** (2025), 1–28. https://doi.org/10.1007/s00161-025-01419-3
23. A. E. Abouelregal, K. Alanazi, Dynamics of a viscoelastic microbeam on Winkler foundation induced by laser excitation: a nonlocal MGT heat conduction approach, *Z. Angew Math. Mech.*, **105** (2025), e70159. https://doi.org/10.1002/zamm.70159
24. Y. Liu, X. L. Qin, S. H. Zhang, Global existence and estimates for Blackstock's model of thermoviscous flow with second sound phenomena, *J. Differ. Equ.*, **324** (2022), 76–101. https://doi.org/10.1016/j.jde.2022.04.001
25. Y. Liu, J. C. Shi, Coupled plate equations with indirect damping: smoothing effect, decay properties and approximation, *Z. Angew. Math. Phys.*, **73** (2022), 11. https://doi.org/10.1007/s00033-021-01640-5
26. Y. Liu, Y. F. Li, J. C. Shi, Estimates for the linear viscoelastic damped wave equation on the Heisenberg group, *J. Differ. Equ.*, **285** (2021), 663–685. https://doi.org/10.1016/j.jde.2021.03.026
27. M. Conti, L. Liverani, V. Pata, The MGT-Fourier model in the supercritical case, *J. Differ. Equ.*, **301** (2021), 543–567. https://doi.org/10.1016/j.jde.2021.08.030