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Research article

Time-optimal control problem for a heat conduction equation with variable thermal conductivity

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Abstract: This paper investigates a time-optimal boundary control problem for the heat equation governed by a uniformly elliptic operator in a bounded multidimensional domain. Robin boundary conditions model passive heat exchange, while the control acts through a prescribed heat flux on a portion of the boundary. The admissible controls are assumed to be bounded measurable functions. The analysis relies on the spectral properties of the associated elliptic operator, including the discreteness of the spectrum and the positivity of the first eigenfunction. These properties are used to derive an optimal estimate for the minimal time required to achieve a prescribed average temperature in the domain.

Keywords: heat conduction equation; thermal conductivity; Volterra integral equation; admissible control; minimal time
Mathematics Subject Classification: 35K05, 35K15

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a piecewise smooth boundary $\partial\Omega$. We introduce the linear elliptic operator

$$\mathcal{A}u := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right),$$

where the coefficients $a_{ij}(x)$ satisfy the symmetry condition

$$a_{ij}(x) = a_{ji}(x), \qquad a_{ij} \in L^\infty(\Omega)$$

and the uniform ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \gamma|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad \text{a.e. } x \in \Omega,$$

with a constant $\gamma > 0$.

The present paper is concerned with the following initial-boundary value problem for the heat equation:

$$\frac{\partial u(x,t)}{\partial t} + \mathcal{A}u(x,t) = 0, \quad x \in \Omega, \quad t > 0, \tag{1.1}$$

equipped with the Robin-type boundary condition on the uncontrolled part of the boundary

$$\frac{\partial u(x,t)}{\partial \nu_{\mathcal{A}}} + h(x)\,u(x,t) = 0, \quad x \in \partial\Omega \setminus \Gamma, \quad t > 0, \tag{1.2}$$

and the Neumann-type control condition on the remaining part

$$\frac{\partial u(x,t)}{\partial \nu_{\mathcal{A}}} = a(x)\,\mu(t), \quad x \in \Gamma, \quad t > 0, \tag{1.3}$$

together with the homogeneous initial condition

$$u(x,0) = 0, \quad x \in \Omega. \tag{1.4}$$

Here, $\nu_{\mathcal{A}}$ denotes the outward unit normal vector to the boundary $\partial\Omega$ associated with the elliptic operator $\mathcal{A}$, which characterizes the direction of heat flux through the boundary. The set $\Gamma \subset \partial\Omega$ represents an open portion of the boundary where external thermal control is applied, such as heating or cooling devices. Physically, this means that the temperature exchange with the environment is allowed only through Γ, while the remaining part of the boundary is subject to passive heat transfer.

We assume that the control region Γ has positive surface measure,

$$\text{mes}_{n-1}(\Gamma) > 0,$$

which ensures that the applied boundary control has a non-negligible physical influence on the thermal evolution of the system. Here, mes_{n-1} denotes the $(n-1)$-dimensional surface measure on $\partial\Omega$. From a physical point of view, the operator $\mathcal{A}$ characterizes the diffusion mechanism of heat within the medium. The matrix of coefficients $\{a_{ij}(x)\}$ represents the thermal conductivity tensor, which allows the model to account for spatial heterogeneity and anisotropic heat transfer effects. Such a formulation is appropriate for describing heat propagation in complex materials, including composites and heterogeneous solids. The functions $h(x)$ and $a(x)$ are assumed to be piecewise smooth, nonnegative, and not identically zero. The boundary condition (1.2) models passive heat exchange with the environment, while condition (1.3) describes a controlled heat flux on Γ, regulated by a time-dependent control function $\mu(t)$. For convenience, the boundary conditions can be written in a unified form by extending $h(x)$ and $a(x)$ to the whole boundary $\partial\Omega$ as follows:

$$h(x) = 0 \text{ for } x \in \Gamma, \qquad a(x) = 0 \text{ for } x \in \partial\Omega \setminus \Gamma.$$

Then (1.2) and (1.3) are equivalently rewritten as

$$\frac{\partial u(x,t)}{\partial \nu_{\mathcal{A}}} + h(x)u(x,t) = a(x)\mu(t), \quad x \in \partial\Omega, \quad t > 0. \tag{1.5}$$

Equation (1.1) represents the law of conservation of thermal energy in the domain Ω. The term $\partial u/\partial t$ describes the temporal variation of the temperature field, while the elliptic operator $\mathcal{A}u$ corresponds to the divergence of the heat flux governed by Fourier's law. In the absence of internal heat sources, this balance equation models pure heat diffusion within the medium, where thermal energy is redistributed due to spatial temperature gradients and the material properties encoded in the coefficients $a_{ij}(x)$.

Remark 1.1. *Although we assume $u(x,0) = 0$ for simplicity, the analysis extends directly to nonzero initial conditions. Indeed, by linearity, the solution decomposes into a homogeneous part (zero control) and a controlled part (zero initial data). The homogeneous part contributes a known term to the averaged temperature, which modifies the right-hand side of the Volterra equation but does not alter the structure of the control problem. However, we emphasize that the presence of a nonzero initial condition would affect the value of the minimal time $T_{\min}$ and the optimality of the constant control on the interval $[0, T_{\min}]$. A detailed analysis of the nonzero initial condition case is beyond the scope of the present work and is left for future investigation.*

Parabolic equations appear naturally in the mathematical description of heat conduction and diffusion phenomena, and their control has attracted considerable research interest over many years. In this direction, Friedman [1] contributed early results on optimal control problems using variational techniques. Later, fundamental advances in controllability theory for parabolic equations in one space dimension were made in the works [2, 3]. For problems set in higher-dimensional domains, null-controllability was established through the celebrated contributions of Lebeau and Robbiano [4] and Fursikov and Imanuvilov [5], both of which rely on Carleman-type estimates. These ideas were subsequently extended to various geometries, boundary conditions, and classes of operators. The bang-bang property for time-optimal control problems associated with heat equations involving time-dependent potentials was studied in [6], where the authors characterized the attainable subspaces and provided a sufficient condition for the bang-bang property via adjoint equation techniques. A comprehensive treatment of optimal control theory for distributed parameter systems, including parabolic equations, is available in the monographs by Lions and Fursikov [7, 8]. A finite-dimensional version of a related output controllability problem in a long-time horizon was studied in [9], where conditions for steering a linear output functional to a prescribed target are established.

The time-optimal control problem for the heat equation, together with the characterization of optimal controls, continued to attract significant attention in subsequent investigations. In particular, the bang-bang property for time-varying optimal time control of null-controllable heat equations was established in [10], where the equivalence between optimal norm control and optimal time control was first demonstrated. Various numerical and implementation aspects of optimal control for heat equations were discussed in [11], while boundary control strategies for heat conduction models were developed in [12]. The bang-bang principle for time-optimal boundary control problems was further examined in [13], where it was shown that bang-bang controls remain optimal for arbitrary target temperature distributions. Boundary control problems for heat equations in three-dimensional domains were studied in [14], where admissibility of the control function was established using geometric arguments. A time-optimal boundary control problem for a heat equation with involution in a one-dimensional domain was analyzed in [15], where the problem was reformulated as a Volterra integral equation of the first kind and solved using the Laplace transform method, leading to an estimate for the minimal time required to reach a prescribed average temperature. The time-optimal control of a fourth-order parabolic equation describing thin film growth was investigated in [16], where it was found that the optimal time depends on the growth parameters when the average interface height approaches a critical value. A boundary control problem for a heat equation with involution in a two-dimensional domain was considered in [17], where the problem was again reduced to a Volterra integral equation of the first kind and solved using the Laplace transform method to establish the existence of an admissible control. The time-optimal boundary control of a fourth-order parabolic equation in a two-dimensional domain was studied in [18], where a similar Volterra integral equation approach was employed to derive an estimate for the minimal time to reach a prescribed average temperature.

Initial studies of time-optimal control problems for parabolic equations date back to the pioneering work of Fattorini [19] in the 1970s; a comprehensive treatment of the subject can also be found in his monograph [20]. Later contributions by Albeverio and Alimov [21] provided sharp explicit estimates for the minimal time in the specific context of heat exchange processes in bounded domains. Time-optimal boundary control problems for parabolic equations under general state constraints were studied by Mackenroth [22], who established an existence theorem, a maximum principle, and a generalized bang-bang principle for a special case. More recently, Huang [23] derived finite element error estimates for time-optimal control problems governed by the heat equation, obtaining convergence results for optimal controls, states, and adjoint states without requiring additional regularity assumptions. The connection between minimal time and minimal norm control problems for heat equations with internal controls was established by Wang and Zuazua [24], who provided a variational characterization of both the minimal time and the corresponding optimal control. The qualitative behavior of singular parabolic equations with logarithmic nonlinearities was examined by Wu, Zhao, and Yang [25], where local and global solvability, decay estimates, and blow-up phenomena were studied using potential well methods and Hardy–Sobolev inequalities. An inverse problem for parabolic equations, involving the determination of multiple time-dependent coefficients from heat moment observations, was addressed in [26] using finite difference methods and nonlinear least-squares optimization. The boundary control problem for a pseudo-parabolic equation was investigated in [27], where a control function was constructed to achieve a desired temperature profile in an inhomogeneous thin plate. Huntul et al. [28] examined an inverse source problem for linear pseudoparabolic equations with memory, establishing existence, uniqueness, and stability of strong solutions and proposing a numerical scheme based on extended cubic B-splines combined with Tikhonov regularization.

In a recent work [29], a time-optimal control problem for the heat equation involving an involution operator in a multidimensional parallelepiped domain was studied, and an optimal estimate for the minimal time to reach a prescribed average temperature was obtained. Recent developments have also addressed controllability from a spectral and observability perspective. In [30], spectral inequalities for the Laplace–Beltrami operator were derived, leading to observability and null-controllability results for the spherical heat equation. Boundary control problems for generalized heat equations involving Schrödinger-type operators were considered in [31]. Numerical aspects of boundary control and discretization schemes for parabolic equations have been extensively developed, with foundational contributions by Boyer, Hubert, and Le Rousseau [32–34], Fernández-Cara and Münch [35], and Zuazua [36], among others. More recent advances in this direction can be found in [37–39]. Applications to fluid dynamics and thermally coupled systems were studied in [40]. Null controllability of the heat equation with a Wentzell-type boundary condition and Dirichlet control was investigated by Chorfi, Ismailov, Maniar, and Oner [41], who reformulated the problem within the framework of boundary control systems and solved it using a spectral moment method, with numerical validation via a penalized Hilbert uniqueness method (HUM) approach. Optimal control problems for parabolic equations with smooth boundary controls subject to integral constraints were studied in [42], where necessary optimality conditions and an iterative improvement method for admissible controls were derived. Control of nonlinear parabolic partial differential equations (PDEs) with Volterra-type nonlinearities was investigated in the two-part work of Vazquez and Krstic [43,44].

In the present work, we investigate a boundary control problem for the heat conduction equation governed by a uniformly elliptic operator. The remainder of the paper is organized as follows. Section 2 presents the formulation of the time-optimal boundary control problem, introduces the admissible control class, defines the weighted average temperature, and states the main problem. Section 3 is devoted to deriving a Green function representation of the solution and reducing the control problem to a Volterra integral equation of the first kind; the kernel of this equation is then analyzed in detail, with emphasis on positivity, monotonicity, regularity, and integrability. In Section 4, we establish a sharp lower bound for the kernel and use it to derive an explicit upper bound on the minimal time. We also prove the time-optimality of the constant control $\mu \equiv M$ via the comparison principle and show the existence of an admissible control that keeps the desired temperature over a finite interval. Finally, Section 5 contains concluding remarks and discusses several open problems, including the possibility of smooth controls and extensions to nonzero initial conditions.

2. Problem statement

The present section is dedicated to a rigorous formulation of the boundary control problem and to a brief discussion of its physical interpretation. The problem is motivated by thermal processes in which the boundary heat flux is used to regulate the overall thermal behavior of the medium, described here by the spatially averaged temperature. Let $M > 0$ be a given constant. A measurable function $\mu(t)$ defined for $t \geq 0$ is called an admissible control if it fulfills the uniform bound

$$|\mu(t)| \leq M, \quad t \geq 0.$$

We also introduce a weight function $\rho \in L^1(\Omega)$ such that

$$\rho(x) > 0 \quad \text{a.e. in } \Omega, \qquad \int_\Omega \rho(x)\, dx = 1,$$

and there exists a constant $\alpha > 0$ such that

$$\rho(x) \geq \alpha\varphi_1(x) \quad \text{a.e. in } \Omega,$$

where φ_1 is the first positive eigenfunction of $\mathcal{A}$ normalized in $L^2(\Omega)$. The central goal of this paper is to study a time-optimal boundary control problem for the system (1.1)–(1.4).

Time-optimal problem. *Given a prescribed constant $\theta > 0$, find the smallest time $T_{\min} \geq 0$ for which there exists an admissible control $\mu(t)$ such that the corresponding solution $u(x,t)$ of the problem* (1.1)–(1.4) *satisfies*

$$\frac{1}{|\Omega|}\int_\Omega \rho(x)u(x,t)\, dx = \theta, \quad t \in [T_{\min}, T_1] \tag{2.1}$$

for some $T_1 > T_{\min}$. This condition is physically meaningful: It requires that the spatially averaged temperature in Ω, weighted by $\rho(x)$, attains the target value θ and remains at that level throughout the interval $[T_{\min}, T_1]$. From the perspective of control theory, such an averaged constraint permits non-uniform temperature distributions while guaranteeing the desired overall thermal effect.

The weighted average temperature $\frac{1}{|\Omega|}\int_\Omega \rho(x)u(x,t)dx$ represents a global thermal characteristic of the system, which is often more relevant in practical applications than pointwise temperature values. For instance, in thermal processing of composite materials, the overall energy stored in the material determines the curing quality; in climate control of enclosed spaces, the average temperature is the key regulated quantity; in biological tissues during hyperthermia treatment, the total heat dose matters more than the exact temperature distribution. The weight function $\rho(x)$ allows flexibility in modeling non-uniform sensor distributions or importance weighting in heterogeneous media. The standard average $\rho(x) \equiv 1/|\Omega|$ corresponds to the total thermal energy in the domain.

It is worth mentioning that the corresponding time-optimal boundary control problem for the classical Laplace operator in n-dimensional domains was studied earlier in [21], where precise estimates for the minimal time and admissible controls were established. The present work generalizes those findings to the setting of a uniformly elliptic operator $\mathcal{A}$ with spatially varying coefficients and Robin-type boundary conditions. We consider the spectral problem associated with the elliptic operator $\mathcal{A}$, which will be used for constructing solution representations via eigenfunction expansions and Green functions.

The eigenvalue problem is

$$\mathcal{A}\varphi(x) = \lambda\varphi(x), \quad x \in \Omega, \tag{2.2}$$

subject to the homogeneous Robin boundary condition

$$\frac{\partial\varphi}{\partial\nu_{\mathcal{A}}} + h(x)\varphi = 0, \quad x \in \partial\Omega. \tag{2.3}$$

This spectral problem is understood in the weak sense: We seek $\lambda \in \mathbb{R}$ and a nonzero function $\varphi \in H^1(\Omega)$ such that

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x)\frac{\partial\varphi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\,dx + \int_{\partial\Omega} h(x)\varphi\psi\,d\sigma = \lambda\int_\Omega \varphi\psi\,dx, \quad \forall\psi \in H^1(\Omega),$$

where $d\sigma$ denotes the $(n-1)$-dimensional surface measure on $\partial\Omega$.

Under the above assumptions on the coefficients $a_{ij}(x)$ and the boundary function $h(x)$, the operator $\mathcal{A}$ is self-adjoint, positive definite, and has a compact resolvent in $L^2(\Omega)$ [45]. Hence, its spectrum consists entirely of real eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots, \qquad \lambda_k \to +\infty \quad \text{as } k \to \infty,$$

each having finite multiplicity. The corresponding eigenfunctions $\{\varphi_k\}_{k=1}^\infty$ form a complete orthonormal basis of $L^2(\Omega)$. The smallest eigenvalue λ_1 is simple, and its associated eigenfunction φ_1 may be chosen to be strictly positive a.e. in Ω; this is a consequence of the strong maximum principle for elliptic operators and the Krein–Rutman theorem [46, 47]. These spectral facts allow us to represent the solution of the parabolic control problem via eigenfunction expansions and Green function representations.

Let σ denote the surface measure on Γ induced by the Lebesgue measure. In particular,

$$\int_{\Gamma} d\sigma(x) = \text{mes}_{n-1}(\Gamma).$$

Set

$$\Phi_k = \int\limits_{\partial\Omega} \varphi_k(x)\, a(x)\, d\sigma(x), \quad k \in \mathbb{N}. \tag{2.4}$$

The main theoretical outcome of this paper is summarized in the following result, which furnishes an explicit estimate for the minimal time needed to reach the prescribed averaged temperature by means of a boundary control.

Theorem 2.1. *Let the prescribed average temperature θ satisfy*

$$0 < \theta < \frac{\alpha\, \Phi_1\, M}{\lambda_1\, |\Omega|},$$

where Φ_1 is defined in (2.4)*, λ_1 is the first eigenvalue of $\mathcal{A}$, M is the maximal allowable control amplitude, and $|\Omega|$ is the volume of the domain. Set*

$$T_0 = -\frac{1}{\lambda_1} \ln\left(1 - \frac{\theta\, \lambda_1\, |\Omega|}{\alpha\, \Phi_1\, M}\right).$$

Then a solution T_{min} of the time-optimal control problem exists, and the estimate $T_{min} \le T_0$ is valid.

Physically, the condition $\theta < \frac{\alpha\, \Phi_1\, M}{\lambda_1\, |\Omega|}$ ensures that the desired average temperature is attainable given the maximal boundary control $\mu(t)$ of amplitude M. It reflects the natural limitation that the control cannot instantaneously or infinitely raise the domains temperature: The prescribed average temperature θ must be compatible with the total heating power that the boundary control can provide. In other words, the inequality guarantees that the system can be driven to the target thermal state within a finite time using admissible controls.

The main contributions of the paper are as follows. First, we derive an explicit upper bound for the minimal attainable time. Second, we prove that the constant control $\mu(t) \equiv M$ is time-optimal, i.e., no admissible control can achieve the prescribed average temperature in a shorter time. The present work differs from the above-mentioned studies in several essential aspects. First, the analysis is carried out for a general uniformly elliptic operator with variable coefficients, rather than for special model operators such as the Laplacian. This significantly increases the level of generality and allows the results to be applied to a broader class of diffusion processes. Second, the control objective is formulated in terms of a weighted spatial average involving a prescribed weight function $\rho(x)$ in the integral condition. This extension introduces additional analytical difficulties, since the spectral representation and the associated kernel of the resulting Volterra integral equation explicitly depend on the weight function. As a consequence, we derive a minimal time estimate that remains valid in this general elliptic setting and under the weighted integral constraint, thereby extending and refining previously known results.

3. Volterra integral equation

This section is devoted to constructing the solution of the initial-boundary value problem using the Green function. This representation provides a convenient way to convert the boundary control problem into a Volterra integral equation of the first kind. The kernel of this integral operator is then studied in depth, and sharp estimates are derived. These estimates are essential for establishing the existence of an admissible control and for obtaining an explicit upper bound on the minimal time.

For the proof of Theorem 2.1, we define the Green function as

$$G(x,y,t) = \sum_{k=1}^{\infty} e^{-\lambda_k t}\,\varphi_k(x)\,\varphi_k(y), \quad x,y\in\Omega, \quad t>0. \tag{3.1}$$

This function is known to solve the initial-boundary value problem

$$G_t(x,y,t) + \mathcal{A}_y G(x,y,t) = 0, \quad x,y\in\Omega, \quad t>0,$$

with the Robin boundary condition

$$\frac{\partial G(x,y,t)}{\partial\nu_{\mathcal{A}}(y)} + h(y)G(x,y,t) = 0, \quad x\in\Omega,\ y\in\partial\Omega, \quad t>0,$$

and the initial condition

$$G(x,y,0) = \delta(x-y),$$

where δ is the Dirac delta.

Here and throughout, the notation $\mathcal{A}_y$ means that the operator $\mathcal{A}$ is applied with respect to the spatial variable $y\in\Omega$, while x is treated as a parameter. This convention removes any ambiguity in expressions involving the Green function $G(x,y,t)$.

Since h is not identically zero on $\partial\Omega$, the maximum principle for uniformly parabolic equations with Robin boundary conditions implies (see [50])

$$G(x,y,t)\geq 0, \quad x,y\in\Omega, \quad t>0.$$

Moreover, G satisfies the standard Gaussian upper bound

$$0\leq G(x,y,t)\leq C\,t^{-n/2}\exp\left(-\frac{|x-y|^2}{4\kappa t}\right), \quad x,y\in\Omega,\ t>0,$$

where the constants $C>0$ and $\kappa>0$ depend only on the ellipticity constant γ and the L^∞-norm of the coefficients a_{ij} (see, e.g., [48, 49]).

Lemma 3.1. *Let $f\in C^2(\Omega\times[0,T])\cap C^1(\overline{\Omega}\times[0,T])$. Then for every $x\in\Omega$ and $t>0$, the following identity holds:*

$$\begin{aligned} f(x,t) &= \int_\Omega f(y,0)\,G(x,y,t)\,dy \\ &+ \int_0^t\int_\Omega\left(\frac{\partial f(y,s)}{\partial s} + \mathcal{A}f(y,s)\right)G(x,y,t-s)\,dy\,ds \\ &+ \int_0^t\int_{\partial\Omega} G(x,y,t-s)\left(\frac{\partial f(y,s)}{\partial\nu_{\mathcal{A}}} + h(y)f(y,s)\right)d\sigma(y)\,ds, \end{aligned} \tag{3.2}$$

where $G(x,y,t)$ is the Green function associated with the operator $\partial_t+\mathcal{A}$ under the Robin boundary condition and

$$\frac{\partial f}{\partial\nu_{\mathcal{A}}} := \sum_{i,j=1}^{n} a_{ij}(y)\frac{\partial f}{\partial x_j}(y)\nu_i(y)$$

denotes the conormal derivative.

Proof. The proof follows from the Green formula for second-order uniformly elliptic operators in divergence form. Applying integration by parts in space to the operator $\mathcal{A}$ and integration by parts in time on $(0, t)$, and using the definition of the Green function $G(x, y, t)$ corresponding to the operator $\partial_t + \mathcal{A}$ with the Robin boundary condition, we obtain identity (3.2). The symmetry and uniform ellipticity of the coefficients $a_{ij}(x)$ ensure the validity of the Green representation formula under the stated regularity assumptions on f. □

Lemma 3.2. *Let $f \in C^2(\Omega \times [0, T]) \cap C^1(\overline{\Omega} \times [0, T])$ be a solution of the heat equation*

$$\frac{\partial f(x,t)}{\partial t} + \mathcal{A} f(x,t) = 0, \quad x \in \Omega,\ t > 0,$$

with the initial condition

$$f(x, 0) = 0, \quad x \in \Omega.$$

Then for every $x \in \Omega$ and $t > 0$, the following representation holds:

$$f(x,t) = \int_0^t \int_{\partial\Omega} G(x, y, t-s) \left(\frac{\partial f(y,s)}{\partial \nu_{\mathcal{A}}} + h(y) f(y,s) \right) d\sigma(y)\, ds. \tag{3.3}$$

Proof. Substituting the relation

$$\frac{\partial f}{\partial t} + \mathcal{A} f = 0 \quad \text{and} \quad f(\cdot, 0) = 0$$

into the identity (3.2) of Lemma 3.1, the volume integral and the initial term vanish identically. As a result, representation (3.3) follows immediately. □

It is well known that the solution of the initial-boundary value problem (1.1)–(1.4) admits a representation in terms of the Green function associated with the operator $\partial_t + \mathcal{A}$.

Lemma 3.3. *Let $u(x, t)$ denote the solution of the initial-boundary value problem* (1.1)–(1.4). *Then, for each $x \in \Omega$ and $t > 0$, the following representation formula holds:*

$$u(x,t) = \int_0^t \mu(s) \left(\int_\Gamma G(x, y, t-s) a(y) d\sigma(y) \right) ds. \tag{3.4}$$

Proof. In view of Lemma 3.2, the solution u can be expressed through the boundary integral

$$u(x,t) = \int_0^t \int_{\partial\Omega} G(x, y, t-s) \left(\frac{\partial u(y,s)}{\partial \nu_{\mathcal{A}}} + h(y) u(y,s) \right) d\sigma(y)\, ds.$$

Making use of the boundary conditions (1.2)–(1.3), we have

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}} + h(x) u = a(x) \mu(t) \quad \text{on } \partial\Omega,$$

with the convention that $a(x) = 0$ on $\partial\Omega \backslash \Gamma$. Substituting this relation into the boundary integral reduces the integration to the subset Γ, which directly yields the desired identity (3.4). □

We now introduce the auxiliary function

$$W(x,t) := \int_{\Omega} G(x,y,t)\,\rho(y)\,dy, \quad x \in \Omega, \quad t > 0. \tag{3.5}$$

This quantity accounts for the contribution at point x and time t from the initial data weighted by $\rho(y)$. It is straightforward to verify that W satisfies the heat equation

$$\frac{\partial W}{\partial t} + \mathcal{A}W = 0, \quad x \in \Omega, \quad t > 0,$$

the Robin boundary condition

$$\frac{\partial W}{\partial \nu_{\mathcal{A}}} + h(x)W = 0, \quad x \in \partial\Omega, \quad t > 0,$$

and the initial condition

$$W(x,0) = \rho(x), \quad x \in \Omega.$$

Lemma 3.4. *Assume that there exists a constant $\alpha > 0$ such that*

$$\rho(x) \geq \alpha\varphi_1(x) \quad \text{a.e. in } \Omega,$$

where φ_1 is the first positive eigenfunction of $\mathcal{A}$ normalized in $L^2(\Omega)$. Then the auxiliary function

$$W(x,t) = \int_{\Omega} G(x,y,t)\rho(y)\,dy, \quad x \in \Omega, \quad t > 0$$

satisfies the estimate

$$W(x,t) \geq \alpha e^{-\lambda_1 t}\varphi_1(x), \quad x \in \Omega, \quad t > 0, \tag{3.6}$$

where λ_1 is the first eigenvalue of $\mathcal{A}$.

Proof. Define

$$L(x,t) := W(x,t) - \alpha e^{-\lambda_1 t}\varphi_1(x).$$

By construction, $L(x,t)$ satisfies the same parabolic equation and Robin boundary condition as $W(x,t)$. The initial condition is

$$L(x,0) = W(x,0) - \alpha\varphi_1(x) = \rho(x) - \alpha\varphi_1(x) \geq 0 \quad \text{a.e. in } \Omega$$

by the assumption $\rho(x) \geq \alpha\varphi_1(x)$. Using the Green function representation

$$L(x,t) = \int_{\Omega} L(y,0)\,G(x,y,t)\,dy$$

and noting that $G(x,y,t) \geq 0$, we obtain

$$L(x,t) \geq 0, \quad x \in \Omega, \ t > 0.$$

Therefore, the estimate (3.6) holds for all $(x,t) \in \Omega \times (0,\infty)$. □

Next, define

$$K(t) := \int_{\partial\Omega} W(y,t)\, a(y)\, d\sigma(y), \tag{3.7}$$

which represents the cumulative contribution of the boundary control weighted by $a(y)$. Using the boundary integral representation of the solution $u(x,t)$ to problem (1.1)–(1.4), we obtain the Volterra integral equation for the control function $\mu(t)$:

$$\frac{1}{|\Omega|} \int_0^t K(t-\tau)\, \mu(\tau)\, d\tau = g(t), \quad t > 0, \tag{3.8}$$

where $g(t) = \theta$ for $t \in [T_{\min}, T_1]$. This reformulation provides an effective way to convert the time-optimal boundary control problem into a solvable integral equation that explicitly involves the Green function, the weight function $\rho(x)$, and the boundary data. The resulting equation is a Volterra integral equation of the first kind, which plays a central role in our analysis. Since the associated integral operator is compact and quasinilpotent, a thorough investigation of its kernel (3.7) alone is insufficient; its derivatives must also be carefully studied in order to understand the solvability and regularity properties of the control problem.

Remark 3.1. *For the Laplace operator on simple domains, the eigenfunctions and eigenvalues are explicitly known, and the Green function admits a closed-form representation. In contrast, for a general uniformly elliptic operator $\mathcal{A}$ with variable coefficients $a_{ij}(x)$, no such explicit formulas are available. The analysis therefore relies entirely on qualitative spectral properties, namely the simplicity and strict positivity of the first eigenfunction φ_1, established via the Krein–Rutman theorem and the strong maximum principle, together with the Gaussian upper bound for $G(x,y,t)$. These constitute the new analytical challenges of the present generalization.*

By the maximum principle applied to the solution $W(x,t)$ of the parabolic problem, we have

$$W(x,t) > 0 \quad \text{for all } x \in \Omega,\ t > 0.$$

Consequently, since $a(x) > 0$ on Γ, the kernel $K(t)$ defined by (3.7) satisfies

$$K(t) > 0 \quad \text{for all } t > 0.$$

Moreover, differentiating $K(t)$ with respect to t and using the parabolic equation satisfied by $W(x,t)$, we obtain

$$K'(t) = \int_\Gamma \partial_t W(y,t)\, a(y)\, d\sigma(y) \le 0,$$

which shows that $K(t)$ is a non-increasing function. Here, the inequality follows from the maximum principle: Since $W(x,t) \ge 0$ and its spatial maximum is non-increasing in time, the integral against the nonnegative weight $a(y)$ is also non-increasing. In particular, evaluating $K(t)$ at $t = 0$ yields

$$\begin{aligned} K(0) &= \int_\Gamma W(y,0)\, a(y)\, d\sigma(y) \\ &= \int_\Gamma \rho(y)\, a(y)\, d\sigma(y) > 0. \end{aligned}$$

Let $\chi(x)$ be the solution of the boundary value problem

$$\mathcal{A}\chi(x) = \rho(x), \quad x \in \Omega,$$

subject to the Robin boundary condition

$$\frac{\partial \chi}{\partial \nu_{\mathcal{A}}} + h(x)\chi(x) = 0, \quad x \in \partial\Omega.$$

Then the following identity holds:

$$\int_0^\infty K(t)\, dt = \int_\Gamma \chi(y)\, a(y)\, d\sigma(y). \tag{3.9}$$

Indeed, since $W(x,t) = \int_\Omega G(x,y,t)\rho(y)dy$ and $\int_0^\infty G(x,y,t)dt = \chi(x)$ (the Green function integrated in time gives the solution of the stationary problem $\mathcal{A}\chi = \rho$), we have

$$\begin{aligned}\int_0^\infty K(t)dt &= \int_0^\infty \int_{\partial\Omega} W(y,t)a(y)d\sigma(y)dt \\ &= \int_{\partial\Omega} \left(\int_0^\infty W(y,t)dt \right) a(y)d\sigma(y) \\ &= \int_\Gamma \chi(y)a(y)d\sigma(y),\end{aligned}$$

which proves (3.9). Moreover, according to (3.7) and (3.9),

$$\int_0^\infty |K'(s)|\, ds = -\int_0^\infty K'(s)\, ds = K(0).$$

The function $W(x,t)$ solves a uniformly parabolic equation with smooth coefficients. By standard parabolic regularity theory (see [48, 49]), $W(x,t)$ is infinitely differentiable in x and t for any $t > 0$. Consequently, differentiation under the integral sign is justified, and

$$K'(t) = \int_{\partial\Omega} \frac{\partial W}{\partial t}(y,t)\, a(y)\, d\sigma(y)$$

is continuous on $(0,\infty)$. Moreover, $K'(t)$ is bounded on every interval $[\varepsilon,\infty)$ with $\varepsilon > 0$. This regularity will be used in the analysis of the Volterra equation in Lemma 4.4.

4. Estimate of the minimal time

In this section, we derive an explicit upper bound on the minimal time required to reach a prescribed averaged temperature in the domain. The desired estimate is obtained by establishing a lower bound for the kernel of the associated Volterra integral equation, which plays a central role in the time-optimal control analysis.

Lemma 4.1. *The following lower bound holds for all $t > 0$:*

$$K(t) \geq \alpha \Phi_1 e^{-\lambda_1 t}, \quad t > 0, \tag{4.1}$$

where λ_1 denotes the smallest eigenvalue of the operator $\mathcal{A}$ and the constant Φ_1 is given by (2.4) *for $k = 1$.*

Proof. We start from the definition of the kernel

$$K(t) = \int_{\partial\Omega} W(y,t)a(y)d\sigma(y).$$

By Lemma 3.4, we have the pointwise lower bound

$$W(y,t) \geq \alpha e^{-\lambda_1 t}\varphi_1(y), \quad y \in \Omega,\ t > 0.$$

Inserting this estimate into the expression for $K(t)$, we get

$$\begin{aligned} K(t) &\geq \alpha e^{-\lambda_1 t} \int_{\partial\Omega} \varphi_1(y)a(y)d\sigma(y) \\ &= \alpha\Phi_1 e^{-\lambda_1 t}, \end{aligned}$$

which establishes the desired inequality (4.1). □

We now define the cumulative function

$$H(t) := \frac{1}{|\Omega|}\int_0^t K(t-\tau)\,d\tau = \frac{1}{|\Omega|}\int_0^t K(\tau)\,d\tau, \quad t > 0, \tag{4.2}$$

where the equality of the two integrals is obtained via the substitution $\eta = t - \tau$. The function $H(t)$ has a clear physical meaning: It quantifies the total (time-integrated) averaged temperature response of the domain Ω up to time t resulting from a unit input applied at the boundary. From the properties of the kernel $K(t)$ established above, it follows that

$$H(0) = 0, \qquad H'(t) = \frac{1}{|\Omega|}K(t) \geq 0,$$

and therefore $H(t)$ is nonnegative, continuous, and monotonically increasing for all $t > 0$. Moreover, since $K(t)$ is nonnegative, monotonically decreasing, and integrable over $(0, +\infty)$, we may define the finite limit

$$H^* := \frac{1}{|\Omega|}\int_0^\infty K(\tau)\,d\tau < +\infty. \tag{4.3}$$

Physically, the quantity H^* represents the maximal total accumulated average temperature that can be achieved in the domain Ω by a sustained unit boundary action applied over an infinite time horizon. In other words, H^* characterizes the intrinsic thermal capacity of the system under the given boundary configuration and operator $\mathcal{A}$. Consequently, for any admissible boundary control $\mu(t)$ satisfying $|\mu(t)| \leq M$, the accumulated averaged temperature generated by the control cannot exceed the value MH^*.

Lemma 4.2. *Let*

$$0 < \theta < MH^*.$$

Then there exists a time $T_{\min} > 0$ and a measurable control function $\mu(t)$ such that $|\mu(t)| \leq M$ and

$$\frac{1}{|\Omega|}\int_0^{T_{\min}} K(T_{\min} - \tau)\,\mu(\tau)\,d\tau = \theta. \tag{4.4}$$

Proof. Choose the admissible control $\mu(t) \equiv M$. Then, for all $t > 0$, we have

$$\frac{1}{|\Omega|} \int_0^t K(t-\tau)\,\mu(\tau)\,d\tau = \frac{M}{|\Omega|} \int_0^t K(t-\tau)\,d\tau = M\,H(t).$$

Since $H(t)$ is continuous, strictly increasing, and satisfies $\lim_{t\to+\infty} H(t) = H^*$, the condition $0 < \theta < MH^*$ guarantees the existence of a unique time $T_{\min} > 0$ such that

$$M\,H(T_{\min}) = \theta. \tag{4.5}$$

This proves the existence of a solution to (4.4) with an admissible control. □

The central goal of this paper is to establish a quantitative estimate for the minimal time $T_{\min}$ needed to achieve the prescribed averaged temperature θ.

Lemma 4.3. *Let*

$$0 < \theta < \frac{\alpha\,\Phi_1\,M}{\lambda_1\,|\Omega|}. \tag{4.6}$$

Then there exists a minimal time $T_{\min} > 0$ such that $T_{\min} \le T_0$ and the equality (4.4) *holds, where T_0 is defined by*

$$T_0 = -\frac{1}{\lambda_1} \ln\left(1 - \frac{\theta\,\lambda_1\,|\Omega|}{\alpha\,\Phi_1\,M}\right).$$

Proof. To obtain an explicit estimate for the minimal time, we make use of the lower bound for the kernel $K(t)$ established in Lemma 4.1. In particular, for all $t > 0$, we have

$$K(t) \ge \alpha\,\Phi_1\,e^{-\lambda_1 t}. \tag{4.7}$$

Integrating inequality (4.7) over the interval $(0, t)$ yields the following estimate for the cumulative function $H(t)$:

$$\begin{aligned} H(t) = \frac{1}{|\Omega|} \int_0^t K(\tau)\,d\tau &\ge \frac{\alpha\,\Phi_1}{|\Omega|} \int_0^t e^{-\lambda_1 \tau}\,d\tau \\ &= \frac{\alpha\,\Phi_1}{\lambda_1\,|\Omega|}\left(1 - e^{-\lambda_1 t}\right). \end{aligned}$$

Now consider the equation defining the minimal time $T_{\min}$,

$$M\,H(T_{\min}) = \theta.$$

From the above inequality, it follows that equation (4.5) admits a solution provided that

$$0 < \theta < \frac{\alpha\,\Phi_1\,M}{\lambda_1\,|\Omega|}. \tag{4.8}$$

Next, we introduce the auxiliary time $T_0 > 0$ as the unique solution of the equation

$$\frac{\alpha\,\Phi_1\,M}{\lambda_1\,|\Omega|}\left(1 - e^{-\lambda_1 T_0}\right) = \theta. \tag{4.9}$$

Solving (4.9) explicitly, we obtain

$$T_0 = -\frac{1}{\lambda_1} \ln\left(1 - \frac{\theta\,\lambda_1\,|\Omega|}{\alpha\,\Phi_1\,M}\right).$$

Finally, since $H(t)$ is continuous and strictly increasing on $(0, +\infty)$, from (4.5) and (4.9), we conclude that there exists a unique minimal time $T_{\min}$ satisfying

$$0 < T_{\min} \le T_0,$$

which completes the proof. □

Lemma 4.4. *Let $T_{\min} > 0$ be the minimal time satisfying equality* (4.5) *together with condition* (4.6). *Then there exists a time $T_1 > T_{\min}$ and a measurable real-valued control function $\mu(t)$ such that*

$$|\mu(t)| \le M \quad \text{a.e. on } (0, T_1),$$

and the integral condition (2.1) *is fulfilled.*

Proof. The averaged temperature at time t is expressed through the convolution

$$\frac{1}{|\Omega|}\int_0^t K(t-\tau)\,\mu(\tau)\,d\tau.$$

Thus, achieving the prescribed level θ for all $t \in [T_{\min}, T_1]$ reduces to the solvability of

$$\frac{1}{|\Omega|}\int_0^t K(t-\tau)\,\mu(\tau)\,d\tau = g(t), \quad t \in [0, T_1], \tag{4.10}$$

where the function $g(t)$ is defined by

$$g(t) = \begin{cases} M\,H(t), & 0 \le t \le T_{\min}, \\ \theta, & T_{\min} \le t \le T_1. \end{cases} \tag{4.11}$$

Due to condition (4.5), the function $g(t)$ is continuous and piecewise continuously differentiable. We define the control $\mu(t)$ in the form

$$\mu(t) = \begin{cases} M, & 0 \le t \le T_{\min}, \\ \mu_1(t), & T_{\min} \le t \le T_1, \end{cases} \tag{4.12}$$

where $\mu_1(t)$ is determined from the integral equation

$$\frac{M}{|\Omega|}\int_0^{T_{\min}} K(t-\tau)\,d\tau + \frac{1}{|\Omega|}\int_{T_{\min}}^t K(t-\tau)\,\mu_1(\tau)\,d\tau = \theta, \quad t \in [T_{\min}, T_1]. \tag{4.13}$$

Differentiating (4.13) with respect to t, we obtain the Volterra equation of the second kind

$$K(0)\,\mu_1(t) + \int_{T_{\min}}^t K'(t-\tau)\,\mu_1(\tau)\,d\tau = M\big(K(t-T_{\min}) - K(t)\big). \tag{4.14}$$

From (3.7) and $W(x,0) = \rho(x)$, we have $K(0) = \int_{\partial\Omega} \rho(x)a(x)\,d\sigma(x) > 0$. As established in Section 3, $K'(t)$ is continuous on $(0,\infty)$. Therefore, (4.14) is a linear Volterra equation of the second kind with a continuous kernel and $K(0) > 0$, which guarantees the existence of a unique continuous solution $\mu_1(t)$ on any finite interval $[T_{\min}, T_1]$ (see, e.g., [51]). In particular,

$$0 < \mu_1(T_{\min}) = M\left(1 - \frac{K(T_{\min})}{K(0)}\right) < M,$$

and hence $|\mu_1(T_{\min})| < M$. By continuity, there exists $\delta > 0$ such that $|\mu_1(t)| \le M$ for $t \in [T_{\min}, T_{\min} + \delta]$. We define T_1 as the supremum of all $t > T_{\min}$ for which $|\mu_1(s)| \le M$ for all $s \in [T_{\min}, t]$. Thus, $T_1 > T_{\min}$ and $|\mu_1(t)| \le M$ on $[T_{\min}, T_1]$. Therefore, μ defined by (4.12) belongs to $L^\infty(0,T_1)$ with $\|\mu\|_{L^\infty} \le M$, and solves (4.10). Hence, (2.1) holds. □

Remark 4.1. *The control function $\mu(t)$ exhibits a jump discontinuity at $t = T_{\min}$, which corresponds to the transition from the maximal heating phase to the regulation phase. This discontinuity does not compromise admissibility, since $\mu \in L^\infty(0,T_1)$, and the corresponding weak solution of the heat equation remains well-defined.*

The proof of Theorem 2.1 now follows easily from Lemmas 4.3 and 4.4.

Remark 4.2. (Optimality of the control) Let $\widetilde{\mu}(t)$ be an arbitrary admissible control satisfying $|\widetilde{\mu}(t)| \le M$, and let $\widetilde{u}(x,t)$ be the corresponding solution of problem (1.1)–(1.4). Denote by $\underline{u}(x,t)$ the solution corresponding to the constant control $\underline{\mu}(t) \equiv M$. Define $w(x,t) = \underline{u}(x,t) - \widetilde{u}(x,t)$. Then w satisfies

$$\frac{\partial w}{\partial t} + \mathcal{A}w = 0 \quad \text{in } \Omega,$$

$$\frac{\partial w}{\partial \nu_{\mathcal{A}}} + h(x)w = (M - \widetilde{\mu}(t))a(x) \quad \text{on } \partial\Omega,$$

and the initial condition $w(x,0) = 0$. Since $|\widetilde{\mu}(t)| \le M$ and $a(x) \ge 0$ on $\partial\Omega$, we have $(M - \widetilde{\mu}(t))a(x) \ge 0$. Therefore, by the comparison principle for parabolic equations with Robin boundary conditions (see [50]), it follows that $w(x,t) \ge 0$ for all $x \in \Omega$, $t > 0$. Hence,

$$\underline{u}(x,t) \ge \widetilde{u}(x,t), \qquad x \in \Omega,\ t > 0.$$

Since $\rho(x) \ge 0$, we obtain

$$\frac{1}{|\Omega|}\int_\Omega \rho(x)\underline{u}(x,t)\,dx \ge \frac{1}{|\Omega|}\int_\Omega \rho(x)\widetilde{u}(x,t)\,dx.$$

Consequently, the weighted average temperature corresponding to an arbitrary admissible control cannot exceed that generated by the control $\mu(t) \equiv M$. Therefore, the prescribed temperature level θ cannot be attained earlier than the time $T_{\min}$ determined by $MH(T_{\min}) = \theta$. Thus, the control $\mu(t) \equiv M$ is time-optimal, and the corresponding time $T_{\min}$ is the minimal attainable time among all admissible controls.

5. Conclusions

In this work, we have investigated a time-optimal boundary control problem for the classical heat equation governed by a uniformly elliptic operator in a bounded domain. The control mechanism acts through a portion of the boundary under a prescribed amplitude constraint, while the control objective is formulated in terms of the spatially averaged temperature. By applying the method of separation of variables and utilizing the spectral decomposition with respect to the eigenfunctions of the associated elliptic operator, we have reduced the initial-boundary value problem to an integral representation involving a convolution kernel. This reduction transforms the time-optimal control problem into a Volterra integral equation of the first kind with a positive and monotonically decreasing kernel. A detailed analysis of this kernel, based on the spectral properties of the elliptic operator and the exponential decay of the generated semigroup has allowed us to establish sharp lower bounds that are essential for the solvability analysis. Moreover, the existence of an admissible control is proved, and the time-optimality of the maximal control is established via the comparison principle. As a result, an explicit estimate for the minimal time required to reach a prescribed average temperature is derived. This estimate provides a quantitative description of the heating process and highlights the fundamental role of the spectral characteristics of the elliptic operator in the structure of the time-optimal boundary control. Finally, some open problems and directions for future research, such as the construction of smooth controls and the extension to nonzero initial conditions, are also discussed. It would also be of interest to investigate whether a smooth admissible control can achieve the same control objective, avoiding the discontinuity at $t = T_{\min}$ inherent in the bang-bang strategy. Such a construction would naturally involve a trade-off between smoothness of the control and precision of the temperature tracking.

Author contributions

M. Huntul: Writing-review & editing, Methodology, Formal analysis, Validation; F. Dekhkonov: Writing-original draft, Writing-review & editing, Methodology, Formal analysis; M. Nematjonova: Writing-review & editing, Methodology, Validation. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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